A GENERALIZATION OF G-PARKING FUNCTIONS AND RELATED ALGEBRAS

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Abstract. It is known that for all graphs G, the G-parking functions biject to the spanning trees of G. Given a graph G and a family of subsets Σ of $\{1, \ldots, n\}$ that is downward closed by inclusion, we define the notion of a (G, Σ)-parking function, which generalizes the G-parking functions; the G-parking functions correspond to the case $\Sigma = \{\{\emptyset\}\}$. We show that the (G, Σ)-parking functions are in bijection with the forests of G that have a certain property determined by Σ.

Given G and Σ, we also define algebras $\mathcal{A}_{G,\Sigma}$ and $\mathcal{B}_{G,\Sigma}$. These algebras are the quotients of the polynomial ring modulo two ideals; one of these is a monomial ideal and the other, an ideal generated by powers of linear forms, is its deformation. We prove that these algebras have the same dimension, which is equal to the number of (G, Σ)-parking functions. Moreover, we prove that these two algebras have the same Hilbert series, thereby establishing a class of monomial ideals and their deformations with equal Hilbert series.

1. Introduction

The classical parking functions are sequences of n nonnegative integers $(b_1, \ldots, b_n)$ whose decreasing rearrangements are termwise less than $(n, n-1, \ldots, 1)$. The parking functions have several interesting combinatorial and geometric interpretations; for instance, they count the number of spanning trees of the complete graph on $n+1$ vertices [9] and the number of regions of the Shi hyperplane arrangement [14]. More properties of classical parking functions may be found in [9], [14], and [15].

Various generalizations of the parking functions have been studied in the literature. The ρ-parking functions, studied by Pitman and Stanley [11] and Yan [16], are sequences $(b_1, \ldots, b_n)$ whose decreasing rearrangements are termwise less than a nonincreasing sequence $\rho = (\rho_1, \ldots, \rho_n)$. These will be discussed in Section 8.

The G-parking functions, introduced by Postnikov and Shapiro [12], are another robust generalization of the parking functions, which generalize the parking functions from the complete graph K_{n+1} to an arbitrary digraph G. The G-parking functions have many interpretations in combinatorics and physics, and are related to chip-firing games [2] and the abelian sandpile model [7] [10] introduced by Dhar. For example, if G is a symmetric digraph, Gabrielov showed [8] that the G-parking functions biject to the *recurrent states* of the abelian sandpile model.

Given a graph G, Postnikov and Shapiro [12] defined two algebras $\mathcal{A}_G$ and $\mathcal{B}_G$, which are the quotients of the polynomial ring by a monotone monomial ideal and its deformation, a power ideal. They proved that a set of monomials corresponding to the G-parking functions are a basis for both algebras, showing that these algebras have the same dimension and Hilbert series. These are a case of a monotone monomial ideal and its deformation having equal Hilbert series; however, this equality is not true for all monotone monomial ideals.

For graphs G, Desjardins [6] proved that two pairs of algebras closely related to the algebras $\mathcal{A}_G$ and $\mathcal{B}_G$ also have same monomial bases. In each pair of algebras, one algebra is the quotient of the polynomial ring by a monomial ideal and the other is the quotient of the polynomial ring by a power ideal, its deformation. This result provides two other instances of a monotone monomial ideal and its deformation having equal Hilbert series.

In this paper, for all graphs G and families of subsets Σ of $\{1, \ldots, n\}$ that are downward closed by inclusion, we define the (G, Σ)-parking functions, which generalize the G-parking functions. We also define the algebras $\mathcal{A}_{G,\Sigma}$ and $\mathcal{B}_{G,\Sigma}$, which are the quotients of the polynomial ring by a monotone monomial ideal and its deformation. We prove that these algebras share a monomial basis that corresponds to the (G, Σ)-parking functions and have the same dimension and Hilbert series; consequently, we describe a class of monotone monomial ideals and their deformations with equal Hilbert series that generalizes the work of Postnikov-Shapiro and Desjardins.

The remainder of this paper is organized as follows: in Sections 2 and 3, we review definitions and already-known results on G-parking functions and monotone monomial ideals. These include the aforementioned results of Postnikov-Shapiro and Desjardins, as well as Theorem 3.3, which relates the Hilbert series of a monotone monomial ideal to that of its deformation. In Section 4, we define the (G, Σ)-parking functions and the algebras $\mathcal{A}_{G,\Sigma}$ and $\mathcal{B}_{G,\Sigma}$; we also state our two main results, Theorems 4.2 and 4.4. Theorem 4.2 states that the (G, Σ)-parking functions biject to a class of oriented forests of G with a property determined by Σ, providing a combinatorial interpretation of the (G, Σ)-parking functions; Theorem 4.4 states that the algebras $\mathcal{A}_{G,\Sigma}$ and $\mathcal{B}_{G,\Sigma}$ share a monomial basis corresponding to the (G, Σ)-parking functions, and consequently share a Hilbert series. We show that certain known results on monotone monomial ideals and deformations with equal Hilbert series are special cases of Theorem 4.4. In Section 5 we illustrate Theorem 4.4 and the concept of a (G, Σ)-parking function with examples. In Section 6 we prove Theorem 4.2 by formulating and proving a bijection between the (G, Σ)-parking functions and a class of oriented forests of G that generalizes the bijection of Chebikin and Pylyavskyy [5] between the G-parking functions and the oriented spanning trees of G. In Section 7 we finish the proof of Theorem 4.4. While the proofs of Theorems 4.2 and 4.4 are inspired by the work of Chebkin-Pylyavskyy [5] and Postnikov-Shapiro [12], respectively, the combinatorial details of these proofs are different, due to the added generality of these results. In Section 8 we address the ρ-parking functions and define two algebras, $\mathcal{A}_\rho$ and $\mathcal{B}_\rho$, also the quotients of the polynomial ring by a monotone monomial ideal and its deformation. We discuss the implications of this work on the question of when $\mathcal{A}_\rho$ and $\mathcal{B}_\rho$ have equal dimension and Hilbert series. Finally, in Section 9 we state some conjectures related to this work and suggest possibilities for future study.

2. G-Parking Functions, Monomial Algebras, and Power Algebras

The G-parking functions [12] are a broad generalization of the classical parking functions. Let G be a directed graph on the vertices $\{0, 1, \ldots, n\}$. We allow G to have multiple edges, but not loops[1]. If G is an undirected graph, we may define the G-parking functions by treating G as a symmetric directed graph and replacing each undirected edge with a pair of directed edges, one in each direction. For a nonempty $I \subseteq \{1, \ldots, n\}$ and a vertex $i \in I$, let $d_I(i)$ denote the number of edges from i to vertices outside I. A *G-parking function* is a sequence of nonnegative integers $(b_1, \ldots, b_n)$ with the property that for each nonempty $I \subseteq \{1, \ldots, n\}$, there exists a vertex $i \in I$ such that $b_i < d_I(i)$.

Remark 2.1. The K_{n+1}-parking functions are the classical parking functions of size n.

An *oriented subtree* of a digraph G is a subgraph $T \subseteq G$ such that for every vertex $i \in T$, there exists a directed path in T from i to 0. A *oriented spanning tree* of G is an oriented subtree of G that includes every vertex of G. If G is an undirected graph, the oriented spanning trees of G correspond to the ordinary spanning trees of G.

Theorem 2.2. *[12] The number of G-parking functions equals the number of oriented spanning trees of G.*

Theorem 2.2 implies that the number of G-parking functions can be computed by the *Matrix-Tree Theorem*, which gives the number of oriented spanning trees of a digraph G in terms of its Laplacian matrix L_G [15].

Observe that setting $G = K_{n+1}$ recovers the fact that the classical parking functions of size n count the spanning trees of K_{n+1}.

In [5], Chebikin and Pylyavskyy proved Theorem 2.2 combinatorially by establishing a bijection between the G-parking functions the oriented spanning trees of G. We will generalize this bijection to a larger class of parking functions in Section 6.

We now reformulate the G-parking functions algebraically. Let $\mathbb{K}$ be a field with characteristic 0. In the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$, define the monomial

$$m_I = \prod_{i \in I} x_i^{d_I(i)}$$

for all nonempty $I \subseteq \{1, \ldots, n\}$, and let $\mathcal{I}_G = \langle m_I \rangle$ be the ideal in $\mathbb{K}[x_1, \ldots, x_n]$ generated by all such m_I. Define the algebra $\mathcal{A}_G = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}_G$. Then, a sequence $(b_1, \ldots, b_n)$ is a G-parking function if and only if $\prod_i x_i^{b_i}$ does not equal 0 in $\mathcal{A}_G$.

The set of monomials not in $\mathcal{I}_G$ form a linear basis of $\mathcal{A}_G$, known as its *standard monomial basis*. These monomials correspond to the G-parking functions; thus, the number of G-parking functions equals $\dim \mathcal{A}_G$.

Suppose that G is a graph on $\{0, 1, \ldots, n\}$. For a nonempty $I \subseteq \{1, \ldots, n\}$, define D_I as the number of edges from vertices in I to vertices outside I. In other words, $D_I = \sum_{i \in I} d_I(i)$. For all nonempty $I \subseteq \{1, \ldots, n\}$, define

$$p_I = \left(\sum_{i \in I} x_i \right)^{D_I}$$

and let $\mathcal{J}_G = \langle p_I \rangle$ be the ideal generated in $\mathbb{K}[x_1, \ldots, x_n]$ by all such p_I. Define the algebra $\mathcal{B}_G = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{J}_G$.

[1]For the remainder of this paper, we will use the terms "directed graph" and "digraph" to refer to directed graphs and "undirected graph" and "graph" to refer to undirected graphs. We allow all directed and undirected graphs to have multiple edges, but not loops.

Theorem 2.3. *[12] The monomials $\prod_i x_i^{b_i}$, as $(b_1, \dots, b_n)$ ranges over all G-parking functions, form a basis of $\mathcal{B}_G$, and*

$$\dim \mathcal{A}_G = \dim \mathcal{B}_G = N_G$$

where N_G is the number of spanning trees of G. Moreover, Hilb $\mathcal{A}_G$ = Hilb $\mathcal{B}_G$.

In fact, we can refine the above theorem. For any graph G, fix a total order on the edges of G. Given a spanning tree $T \subseteq G$, say that an edge $e \in T$ is *internally active* if there does not exist an edge $e' \in G \setminus T$ such that e' is smaller than e and $T \cup e'$ contains a cycle that includes e; say that an edge $e \in G \setminus T$ is *externally active* if it is the smallest edge of the unique cycle in $T \cup e$. The *internal activity* of T is the number of edges $e \in T$ that are internally active, and the *external activity* of T is the number of edges $e \in G \setminus T$ that are externally active. Although the external and internal activities of any spanning tree are dependent on the order of the edges, the number of spanning trees with each pair of internal and external activities (i, j) is independent of the edge ordering; these are given by the coefficients of the graph's *Tutte polynomial*. For more on tree activities and the Tutte polynomial, see for example [4].

Theorem 2.3 (continuing from p. 4)**.** *The k^{th} graded components $\mathcal{A}_G^k$ and $\mathcal{B}_G^k$ obey*

$$\dim \mathcal{A}_G^k = \dim \mathcal{B}_G^k = N_G^{|G|-n-k}$$

where $N_G^{|G|-n-k}$ is the number of spanning trees of G with external activity $|G| - n - k$.

Theorem 2.3 implies that the number of G-parking functions with sum k equals the number of spanning trees of G with external activity $|G| - n - k$. Benson, Chakrabarty, and Tetali proved this fact combinatorially in [3] by finding an external activity-preserving bijection from the G-parking functions to the spanning trees of G.

Two results related to Theorem 2.3 were proved by Desjardins in [6]:

Let G be a graph on $\{0, 1, \dots, n\}$. For all nonempty $I = \{i_1 < \cdots < i_r\} \subseteq \{1, \dots, n\}$, define

$$m_I = x_{i_1} \prod_{i \in I} x_i^{d_I(i)}$$

and

$$p_I = \left(\sum_{i \in I} x_i \right)^{D_I + 1}.$$

Let $\mathcal{I}_{G,1} = \langle m_I \rangle$ and $\mathcal{J}_{G,1} = \langle p_I \rangle$ be the ideals generated by all such m_I and p_I, respectively. Define the algebras $\mathcal{A}_{G,1} = \mathbb{K}[x_1, \dots, x_n]/\mathcal{I}_{G,1}$ and $\mathcal{B}_{G,1} = \mathbb{K}[x_1, \dots, x_n]/\mathcal{J}_{G,1}$.

Define a *forest* of a graph G as an acyclic edge set of G. Fix a total order on the edges of G. For a forest $F \subseteq G$, define an edge $e \in G \setminus F$ to be *externally active* if $F \cup E$ contains a cycle and e is the smallest edge of this cycle. The *external activity* of a forest is the number of edges $e \in G \setminus F$ that are externally active. The number of forests with each external activity is independent of the chosen edge ordering.

Theorem 2.4. *[6] The standard monomial basis of $\mathcal{A}_{G,1}$ is a basis of $\mathcal{B}_{G,1}$. Furthermore, the k^{th} graded components $\mathcal{A}_{G,1}^k$ and $\mathcal{B}_{G,1}^k$ have dimension equal to the number of forests F of G with external activity $|G| - |F| - k$.*

In [13], Postnikov, Shapiro, and Shapiro proved Theorem 2.4 for the case $G = K_{n+1}$. In this case, Postnikov, Shapiro, and Shapiro showed that the algebra $\mathcal{B}_{G,1}$ is isomorphic to an algebra generated by curvature forms on the complete flag manifold.

Next, suppose G is a graph on $\{0, 1, \ldots, n\}$ with the property that there is at least one edge between any two vertices. For every nonempty $I = \{i_1 < \cdots < i_r\} \subseteq \{1, \ldots, n\}$, let

$$m_I = x_{i_1}^{d_I(i_1)-1} \prod_{\substack{i \in I \\ i \neq i_1}} x_i^{d_I(i)}$$

and

$$p_I = \left(\sum_{i \in I} x_i \right)^{D_I - 1}.$$

Define $\mathcal{I}_{G,-1} = \langle m_I \rangle$ and $\mathcal{J}_{G,-1} = \langle p_I \rangle$ as the ideals generated by all such m_I and p_I, respectively. Define the algebras $\mathcal{A}_{G,-1} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}_{G,-1}$ and $\mathcal{B}_{G,-1} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{J}_{G,-1}$.

Theorem 2.5. *[6] The standard monomial basis of $\mathcal{A}_{G,-1}$ is a basis of $\mathcal{B}_{G,-1}$. The k^{th} graded components $\mathcal{A}_{G,-1}^k$ and $\mathcal{B}_{G,-1}^k$ have dimension equal to the number of spanning trees of G with internal activity 0 and external activity $|G| - n - k$.*

Ideals generated by powers of linear forms, such as $\mathcal{B}_G$, $\mathcal{B}_{G,1}$, and $\mathcal{B}_{G,-1}$ have also been studied extensively in the context of hyperplane arrangements and their Tutte polynomials; see for example [1] and [6].

3. Monotone Monomial Ideals and their Deformations

Consider a set of monomials $\{m_I\}$ in the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$, one for each nonempty subset $I \subseteq \{1, \ldots, n\}$. Such a set is a *monotone monomial family* [12] if:

- For all I, m_I is a monomial in the variables x_i, $i \in I$, and
- If $J \subset I$ and $i \in J$, then $\deg_{x_i} m_J \geq \deg_{x_i} m_I$.

A *monotone monomial ideal* is the ideal generated in $\mathbb{K}[x_1, \ldots, x_n]$ by a monotone monomial family.

If we let $I = \{i_1, \ldots, i_r\}$, then a homogenous polynomial p_I in the variables $x_{i_1}, \ldots, x_{i_r}$ is an *I-deformation of m_I* if $\deg(p_I) = \deg(m_I)$ and

$$\mathbb{K}[x_{i_1}, \ldots, x_{i_r}] = \langle R_{m_I} \rangle \oplus (p_I)$$

where $\langle R_{m_I} \rangle$ denotes the linear span of monomials not divisible by m_I and (p_I) denotes the ideal in $\mathbb{K}[x_{i_1}, \ldots, x_{i_r}]$ generated by p_I. Furthermore, if p_I is an I-deformation of m_I for all nonempty $I \subseteq \{1, \ldots, n\}$ and $\mathcal{I} = \langle m_I \rangle$ is a monotone monomial ideal, then we say that $\mathcal{J} = \langle p_I \rangle$ is a *deformation* of $\mathcal{I}$.

Lemma 3.1. *[12] Suppose $I = \{i_1, \ldots, i_r\}$ and m_I is a monomial in $x_{i_1}, \ldots, x_{i_r}$. If $\alpha_{i_1}, \ldots, \alpha_{i_r}$ are nonzero elements of $\mathbb{K}$, then*

$$(\alpha_{i_1} x_{i_1} + \cdots + \alpha_{i_r} x_{i_r})^{\deg(m_I)}$$

is an I-deformation of m_I.

Remark 3.2. Observe that for all graphs G, $\mathcal{I}_G$ is a monotone monomial ideal and $\mathcal{J}_G$ is its deformation. Moreover, $\mathcal{J}_{G,1}$ is a deformation of the monotone monomial ideal $\mathcal{I}_{G,1}$, and $\mathcal{J}_{G,-1}$ is a deformation of the monotone monomial ideal $\mathcal{I}_{G,-1}$.

The following result is an important property of monotone monomial ideals that will be important to showing that certain pairs of algebras have equal Hilbert series.

Theorem 3.3. *[12] Let $\mathcal{I}$ be a monotone monomial ideal in $\mathbb{K}[x_1, \dots, x_n]$, and let $\mathcal{J}$ be a deformation of $\mathcal{I}$. Define the algebras $\mathcal{A} = \mathbb{K}[x_1, \dots, x_n]/\mathcal{I}$ and $\mathcal{B} = \mathbb{K}[x_1, \dots, x_n]/\mathcal{J}$. The standard monomial basis of $\mathcal{A}$ spans $\mathcal{B}$. Consequently, the Hilbert series Hilb $\mathcal{I}$, Hilb $\mathcal{J}$, Hilb $\mathcal{A}$, and Hilb $\mathcal{B}$ obey the termwise inequalities*

$$\text{Hilb } \mathcal{I} \le \text{Hilb } \mathcal{J}$$

and

$$\text{Hilb } \mathcal{A} \ge \text{Hilb } \mathcal{B}.$$

4. (G, Σ)-Parking Functions

In this section we present a generalization of the notion of a G-parking function. Let G be a digraph on $\{0, 1, \dots, n\}$ and Σ be a set of subsets of $\{1, \dots, n\}$ with the *downward inclusion property* that if $I \in \Sigma$ and $J \subset I$, then $J \in \Sigma$. As before, if G is an undirected graph, we may treat it as a symmetric directed graph. Let $\mathcal{I}_{G,\Sigma} = \langle m_I \rangle$ be the ideal in $\mathbb{K}[x_1, \dots, x_n]$ generated by

$$m_I = \begin{cases} x_{i_1} \prod_{i \in I} x_i^{d_I(i)} & I \in \Sigma \\ \prod_{i \in I} x_i^{d_I(i)} & I \notin \Sigma \end{cases}$$

as $I = \{i_1 < \cdots < i_r\}$ ranges over all nonempty subsets of $\{1, \dots, n\}$. Define $\mathcal{A}_{G,\Sigma} = \mathbb{K}[x_1, \dots, x_n]/\mathcal{I}_{G,\Sigma}$. The sequence of nonnegative integers $(b_1, \dots, b_n)$ is a (G, Σ)-*parking function* if and only if $\prod_i x_i^{b_i}$ is nonvanishing in $\mathcal{A}_{G,\Sigma}$. Because the monomials not in $\mathcal{I}_{G,\Sigma}$ comprise the standard monomial basis of $\mathcal{A}_{G,\Sigma}$, the number of (G, Σ)-parking functions equals $\dim \mathcal{A}_{G,\Sigma}$.

Remark 4.1. The $(G, \{\{\emptyset\}\})$-parking functions are the G-parking functions.

Define a *oriented forest* of a directed graph as a collection of vertices, some of which are designated roots, and directed edges among these vertices, such that from each vertex there is a unique path (which may have length 0) to a root. Define a *proper forest* as an oriented forest in which each vertex is rooted at a vertex smaller than or equal to itself. Define a Σ-*proper forest* as a proper forest in which the set of vertices not rooted at 0 is an element of Σ. For undirected graphs, define a Σ-forest as an acyclic edge set in which the set of vertices not connected to 0 is an element of Σ. In the case that a graph is undirected, its Σ-proper forests correspond to its Σ-forests.

The first main result of this paper is:

Theorem 4.2. *For any digraph G and any Σ with the downward inclusion property, the (G, Σ)-parking functions biject to the $(n+1)$-vertex Σ-proper forests of G.*

We will present this bijection in Section 6.

Remark 4.3. When $\Sigma = \{\{\emptyset\}\}$, we recover Theorem 2.2 from Theorem 4.2.

Suppose that G is a graph on $\{0, 1, \ldots, n\}$ and Σ is a set of subsets of $\{1, \ldots, n\}$ with the downward inclusion property. Let $\mathcal{J}_{G,\Sigma} = \langle p_I \rangle$ be the ideal in $\mathbb{K}[x_1, \ldots, x_n]$ generated by

$$p_I = \begin{cases} \left(\sum_{i\in I} x_i\right)^{D_I+1} & I \in \Sigma \\ \left(\sum_{i\in I} x_i\right)^{D_I} & I \notin \Sigma \end{cases}$$

as I ranges over all nonempty subsets of $\{1, \ldots, n\}$. Define the algebra $\mathcal{B}_{G,\Sigma} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{J}_{G,\Sigma}$. Observe that the downward inclusion property ensures that $\mathcal{I}_{G,\Sigma}$ is a monotone monomial ideal, and that $\mathcal{J}_{G,\Sigma}$ is a deformation of $\mathcal{I}_{G,\Sigma}$.

Our second main result is:

Theorem 4.4. *For all undirected graphs G and all Σ with the downward inclusion property, the monomials $\prod_i x_i^{b_i}$, as $(b_1, \ldots, b_n)$ ranges over all (G, Σ)-parking functions, form a basis of $\mathcal{B}_{G,\Sigma}$, and*

$$\dim \mathcal{A}_{G,\Sigma} = \dim \mathcal{B}_{G,\Sigma} = N_{G,\Sigma}$$

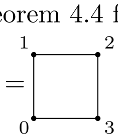
where $N_{G,\Sigma}$ is the number of Σ-forests of G. Furthermore, the k^{th} graded components $\mathcal{A}^k_{G,\Sigma}$ and $\mathcal{B}^k_{G,\Sigma}$ have dimension equal to the number of Σ-forests F of G with external activity $|G| - |F| - k$.

Theorem 4.4 establishes a large class of monotone monomial ideals and their deformations with equal Hilbert series. We will prove this theorem in Sections 6 and 7.

Remark 4.5. When $\Sigma = \{\{\emptyset\}\}$, Theorem 4.4 reduces to Theorem 2.3. When $\Sigma = \mathcal{P}(\{1, \ldots, n\})$, Theorem 4.4 reduces to Theorem 2.4. Theorem 4.4 interpolates between and generalizes these two results.

5. Examples

To demonstrate Theorem 4.4 and the notions of (G, Σ)-parking functions and Σ-forests, we present examples of Theorem 4.4 for the graph

$G =$ 

and various values of Σ.

Example 5.1. Let $\Sigma = \{\{\emptyset\}\}$. In this case, the (G, Σ)-parking functions are the G-parking functions and the Σ-proper forests of G are the spanning trees of G. G has four spanning trees:

We have

$$\mathcal{I}_{G,\Sigma} = \langle x_1^2, x_2^2, x_3^2, x_1x_2, x_1^2x_3^2, x_2x_3, x_1x_2^0x_3 \rangle$$

and

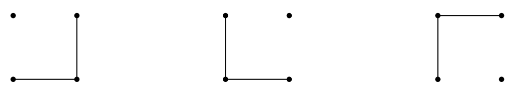
$$\mathcal{J}_{G,\Sigma} = \langle x_1^2, x_2^2, x_3^2, (x_1+x_2)^2, (x_1+x_3)^4, (x_2+x_3)^2, (x_1+x_2+x_3)^2 \rangle.$$

The monomials not in $\mathcal{I}_{G,\Sigma}$ are

$$1, x_1, x_2, x_3.$$

These monomials are a basis for $\mathcal{A}_{G,\Sigma}$ and $\mathcal{B}_{G,\Sigma}$ and give rise to four (G,Σ)-parking functions. The algebras $\mathcal{A}_{G,\Sigma}$ and $\mathcal{B}_{G,\Sigma}$ both have dimension 4, the number of spanning trees of G, and both have Hilbert series $1+3t$.

Example 5.2. Let $\Sigma = \{\{\emptyset\}, \{1\}, \{2\}, \{3\}\}$. The Σ-forests of G are forests of G in which the set of vertices not connected to 0 is $\{\emptyset\}$, $\{1\}$, $\{2\}$, or $\{3\}$. In addition to the four spanning trees above, three more forests of G are Σ-forests for this Σ:



In this case,

$$\mathcal{I}_{G,\Sigma} = \langle x_1^3, x_2^3, x_3^3, x_1x_2, x_1^2x_3^2, x_2x_3, x_1x_2^0x_3 \rangle$$

and

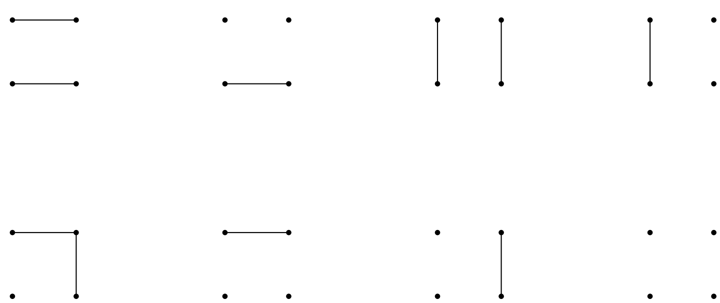
$$\mathcal{J}_{G,\Sigma} = \langle x_1^3, x_2^3, x_3^3, (x_1+x_2)^2, (x_1+x_3)^4, (x_2+x_3)^2, (x_1+x_2+x_3)^2 \rangle.$$

The monomials not in $\mathcal{I}_{G,\Sigma}$ are

$$1, x_1, x_2, x_3, x_1^2, x_2^2, x_3^2$$

which correspond to seven (G,Σ)-parking functions. These monomials form a basis for $\mathcal{A}_{G,\Sigma}$ and $\mathcal{B}_{G,\Sigma}$. The dimension of $\mathcal{A}_{G,\Sigma}$ and $\mathcal{B}_{G,\Sigma}$ is 7, which equals the number of Σ-proper forests of G, and the common Hilbert Series of $\mathcal{A}_{G,\Sigma}$ and $\mathcal{B}_{G,\Sigma}$ is $1+3t+3t^2$.

Example 5.3. Let $G = \mathcal{P}(\{1,2,3\})$. In this case, any forest of G is a Σ-forest. In addition to the seven forests listed above, there are eight more:



We have

$$\mathcal{I}_{G,\Sigma} = \langle x_1^3, x_2^3, x_3^3, x_1^2x_2, x_1^3x_3^2, x_2^2x_3, x_1^2x_2^0x_3 \rangle$$

and

$$\mathcal{J}_{G,\Sigma} = \langle x_1^3, x_2^3, x_3^3, (x_1+x_2)^3, (x_1+x_3)^5, (x_2+x_3)^3, (x_1+x_2+x_3)^3 \rangle.$$

The monomials not in $\mathcal{I}_{G,\Sigma}$, are

$$1, x_1, x_2, x_3, x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3, x_1x_2^2, x_1x_3^2, x_2x_3^2, x_1x_2x_3, x_1x_2x_3^2,$$

which correspond to 15 (G,Σ)-parking functions. These form a basis of $\mathcal{A}_{G,\Sigma}$ and $\mathcal{B}_{G,\Sigma}$. The dimension of $\mathcal{A}_{G,\Sigma}$ and $\mathcal{B}_{G,\Sigma}$ is 15, the number of forests of G, and their Hilbert series is $1+3t+6t^2+4t^3+t^4$.

6. A Bijection from (G,Σ)-Parking Functions to Σ-Proper Forests

In this section we give a bijection from the (G,Σ)-parking functions to the $(n+1)$-vertex Σ-proper forests of G. Let $\mathcal{P}_{G,\Sigma}$ and $\mathcal{F}_{G,\Sigma}$ denote the sets of (G,Σ)-parking functions and $(n+1)$-vertex Σ-proper forests of G, respectively.

Say that an oriented forest F is a *subforest* of an oriented forest F' if the following conditions hold:

(1) The vertices of F are a subset of the vertices of F'
(2) The edges of F are a subset of the edges of F'
(3) The roots of F are a subset of the roots of F'

Observe that by the downward inclusion property, any subforest of a Σ-proper forest must be a Σ-proper forest.

Furthermore, for any oriented forest F and any vertex $i \in F$, let $r_F(i)$ and $e_F(i)$ denote, respectively, the vertex at which i is rooted in F and the edge coming out of i in F, if it exists.

For every Σ-proper forest $F \subseteq G$, we assign a total order $\pi(F)$ to the vertices of F. Let $i >_{\pi(F)} j$ denote that i is larger than j in this order. A set of such orders $\Pi(G,\Sigma)$ is a *proper set of forest orders* if the following conditions hold:

(1) For all F, if $e_F(i) = (i,j)$, then $i >_{\pi(F)} j$.
(2) For all F, if vertices $i,j \in F$ satisfy $r_F(i) > r_F(j)$, then $i >_{\pi(F)} j$.
(3) For all F, if F' is a subforest of F, then the orders $\pi(F)$ and $\pi(F')$ are consistent.

One example of a proper set of forest orders is the breadth-first search order, which is defined as follows: let $h_F(i)$ denote the length of the unique path in F from i to a root; for all F and all $i,j \in F$, let $i >_{\pi(F)} j$ if:

- $r_F(i) > r_F(j)$, or
- $r_F(i) = r_F(j)$ and $h_F(i) > h_F(j)$, or
- $r_F(i) = r_F(j)$ and $h_F(i) = h_F(j)$ and $i > j$.

Fix a proper set of forest orders $\Pi(G,\Sigma)$. If G has multiple edges, fix a total order on each set of multiple edges.

For each Σ-proper forest $F \subseteq G$ and each vertex $i \in G$, we define a total order on the edges from i to vertices in F. If $e = (i, j_1)$ and $e' = (i, j_2)$ are edges from i to vertices in F, let $e >_{\pi(F)} e'$ if $j_1 >_{\pi(F)} j_2$, or if $j_1 = j_2$ and e is larger than e' in the fixed order of multiple edges.

Define the function $\Theta_{\Pi,G,\Sigma} : \mathcal{F}_{G,\Sigma} \to \mathcal{P}_{G,\Sigma}$ as follows: for all F, let $\Theta_{\Pi,G,\Sigma}(F) = (b_1,\ldots,b_n)$, where b_i is:

- the number of edges e from i such that $e <_{\pi(F)} e_F(i)$, if $e_F(i)$ exists, and
- the number of edges from i to vertices j such that $j <_{\pi(F)} i$, otherwise.

Proposition 6.1. *$\Theta_{\Pi,G,\Sigma}$ is a bijection between $\mathcal{P}_{G,\Sigma}$ and $\mathcal{F}_{G,\Sigma}$.*

Remark 6.2. Observe that this bijection preserves Chebikin and Pylyavskyy's bijection [5] between G-parking functions and oriented spanning trees of G.

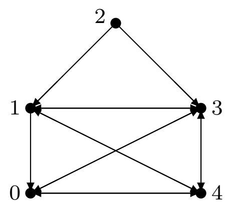
We construct a function $\Phi_{\Pi,G,\Sigma} : \mathcal{P}_{G,\Sigma} \to \mathcal{F}_{G,\Sigma}$, which we claim is the inverse of $\Theta_{\Pi,G,\Sigma}$: let $P \in \mathcal{P}_{G,\Sigma}$. Let the oriented forest F_0 consist of the vertex 0. We construct oriented forests $F_1, \ldots, F_n = \Phi_{\Pi,G,\Sigma}(P)$ by the following algorithm, run for $m = 0, \ldots, n-1$:

Let U_m consist of the set of vertices $i \notin F_m$ with more than b_i outgoing edges to vertices in F_m. We consider two cases:

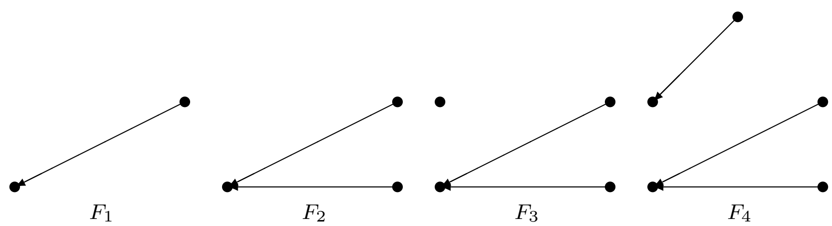
(1) If $|U_m| > 0$: For each vertex $i \in U_m$, let e_i denote the $b_i + 1^{\text{th}}$ smallest edge from i to F_m in the order $\pi(F_m)$. Let f_m be the oriented forest consisting of F_m, all vertices $i \in U$, and all edges e_i, for $i \in U$. Let v_{m+1} be the smallest vertex in U_m in the order $\pi(f_m)$. Construct F_{m+1} by adding v_{m+1} and $e_{v_{m+1}}$ to F_m.

(2) If $|U_m| = 0$: Let v_{m+1} be the numerically smallest vertex not in F_m. Construct F_{m+1} by adding v_{m+1} to F_m without adding an edge.

Example 6.3. Let

$G =$ 

and $\Sigma = \mathcal{P}(\{1, 2, 3, 4\})$. Let our proper set of forest orders $\Pi(G, \Sigma)$ be the breadth-first search order. Consider the (G, Σ)-parking function $P = (3, 2, 0, 0)$. The algorithm for constructing $\Phi_{\Pi,G,\Sigma}(P)$ constructs the following oriented forests:



We have $U_0 = \{3, 4\}$, so f_0 consists of the vertices 0, 3, 4, and the edges $e_3 = (3, 0)$ and $e_4 = (4, 0)$. 3 is smaller than 4 in $\pi(f_0)$, so we construct F_1 by adding the vertex 3 and the edge $(3, 0)$ to F_0.

Then, $U_1 = \{4\}$ and $e_4 = (4, 0)$, so we construct F_2 by adding the vertex 4 and the edge $(4, 0)$ to F_1.

Next, U_2 is empty. Thus, we construct F_3 by adding 1, the smallest vertex outside of F_2, without adding an edge.

Lastly, $U_3 = \{2\}$ and $e_2 = (2, 1)$, so we construct $F_4 = \Phi_{\Pi,G,\Sigma}(P)$ by adding the vertex 2 and the edge $(2, 1)$ to F_3.

Observe that the order $\pi(F_4)$ is $0 <_{\pi(F_4)} 3 <_{\pi(F_4)} 4 <_{\pi(F_4)} 1 <_{\pi(F_4)} 2$. Thus $\Theta_{\Pi,G,\Sigma}(F_4) = (3, 2, 0, 0)$, as expected.

We now prove that this is a bijection. We first show that $\Theta_{\Pi,G,\Sigma}$ and $\Phi_{\Pi,G,\Sigma}$ map the sets $\mathcal{F}_{G,\Sigma}$ and $\mathcal{P}_{G,\Sigma}$ to each other.

Lemma 6.4. *If $F \in \mathcal{F}_{G,\Sigma}$, then $\Theta_{\Pi,G,\Sigma}(F) \in \mathcal{P}_{G,\Sigma}$.*

Proof. Let $\Theta_{\Pi,G,\Sigma}(F) = (b_1, \ldots, b_n)$. Consider any nonempty $I \subseteq \{1, \ldots, n\}$, and let j be the minimal element of I in the order $\pi(F)$. We consider two cases:

(1) $e_F(j)$ exists: By definition of $\Theta_{\Pi,G,\Sigma}$, there are b_j edges smaller than $e_F(j)$ in $\pi(F)$. Because $\Pi(G, \Sigma)$ is a proper set of forest orders and j is minimal

in the order $\pi(F)$, $e_F(j)$ and the b_j edges smaller than it in $\pi(F)$ must go to vertices outside I. Therefore $d_I(j) \geq b_j + 1$, and $\deg_{x_j} m_I \geq d_I(j) > b_j = \deg_{x_j} \prod_i x_i^{b_i}$. So, m_I does not divide $\prod_i x_i^{b_i}$.

(2) $e_F(j)$ does not exist: j must be a root of F. Because $\Pi(G, \Sigma)$ is a proper set of forest orders and j is minimal in $\pi(F)$, we must have $r_F(i) \geq r_F(j) = j$ for all $i \in I$. Because F is a proper forest, all vertices $i \in I$ must satisfy $i \geq r_F(i) \geq j$; thus j is the numerically smallest vertex in I. Moreover, because $r_F(i) \geq r_F(j) = j > 0$ for all $i \in I$, all elements of I are not rooted at 0; hence $I \in \Sigma$. This implies $\deg_{x_j} m_I = d_I(j) + 1$. By definition of $\Theta_{\Pi,G,\Sigma}$, there are b_j edges from j to vertices smaller than j in $\pi(F)$. By minimality of j in $\pi(F)$, all of these edges must go to vertices outside I. Therefore $d_I(j) \geq b_j$. So, $\deg_{x_j} m_I = d_I(j)+1 \geq b_j+1 > b_j = \deg_{x_j} \prod_i x_i^{b_i}$. Thus m_I does not divide $\prod_i x_i^{b_i}$.

Therefore $\prod_i x_i^{b_i}$ is not divisible by any m_I and does not vanish in $\mathcal{A}_{G,\Sigma}$. □

Lemma 6.5. *Let $P \in \mathcal{P}_{G,\Sigma}$. In the algorithm for constructing $\Phi_{\Pi,G,\Sigma}(P)$, if $|U_m| = 0$, then the set of vertices $\{0, 1, \ldots, n\} \setminus F_m \in \Sigma$.*

Proof. Let $P = (b_1, \ldots, b_n)$ and $\{0, 1, \ldots, n\} \setminus F_m = I$. Suppose for sake of contradiction that $I \notin \Sigma$. Then $m_I = \prod_{i \in I} x_i^{d_I(i)}$. Because $P \in \mathcal{P}_{G,\Sigma}$, m_I does not divide $\prod_i x_i^{b_i}$. Thus, there exists $i \in I$ such that $d_I(i) > b_i$. But $d_I(i)$ is the number of edges from i to F_m, so there exists $i \in I$ with more than b_i edges to F_m. This contradicts $|U_m| = 0$. □

Lemma 6.6. *If $P \in \mathcal{P}_{G,\Sigma}$, then $\Phi_{\Pi,G,\Sigma}(P) \in \mathcal{F}_{G,\Sigma}$.*

Proof. Let $\Phi_{\Pi,G,\Sigma}(P) = F$. It is clear that each of the F_m is an oriented forest. Because each F_m has one more vertex than the previous, $F = F_n$ has $n+1$ vertices. F is a proper forest because the roots of F are precisely the vertices that were added to some F_m where $|U_m| = 0$, and each such vertex was the numerically smallest vertex not in that F_m when it was added.

If every vertex in F is rooted at 0, F is clearly Σ-proper. Else, let F_m be such that every vertex in F_m is rooted at 0 and m is maximal. Then $|U_m| = 0$; by Lemma 6.5, the set of vertices $\{0, 1, \ldots, n\} \setminus F_m \in \Sigma$. Hence, the set of vertices of F that are not rooted at 0 is an element of Σ. Therefore, F is an $(n+1)$-vertex Σ-proper forest. □

Next we show that $\Theta_{\Pi,G,\Sigma}$ and $\Phi_{\Pi,G,\Sigma}$ are inverses.

Lemma 6.7. *Let $P \in \mathcal{P}_{G,\Sigma}$ and $F = \Phi_{\Pi,G,\Sigma}(P)$. For $m = 1, \ldots, n$, let v_m be the vertex in F_m but not in F_{m-1}. Then $0 <_{\pi(F)} v_1 <_{\pi(F)} v_2 <_{\pi(F)} \cdots <_{\pi(F)} v_n$.*

Proof. Because $\Pi(G, \Sigma)$ is a proper set of forest orders and 0 is the smallest root of F, 0 is minimal in $\pi(F)$. Therefore $0 <_{\pi(F)} v_1$. We inductively prove that $0 <_{\pi(F)} v_1 <_{\pi(F)} \cdots <_{\pi(F)} v_m$. Suppose $0 <_{\pi(F)} v_1 <_{\pi(F)} \cdots <_{\pi(F)} v_m$; we show that $v_m <_{\pi(F)} v_{m+1}$. We consider three cases:

(1) $|U_m| = 0$: v_{m+1} must be a root of F. As F_m is a subforest of F, $r_F(v_m) = r_{F_m}(v_m) \in F_m$. If $r_F(v_m)$=0, then $r_F(v_m) < v_{m+1} = r_F(v_{m+1})$, so $v_m <_{\pi(F)} v_{m+1}$ because $\Pi(G, \Sigma)$ is a proper set of forest orders. Else, let F_i be such that $r_F(v_m) \in F_i$ and i is minimal. $r_F(v_m)$ must be the

smallest vertex not in F_{i-1}. But, v_{m+1} is not in F_{i-1}, so $r_F(v_m) < v_{m+1} = r_F(v_{m+1})$. Because $\Pi(G,\Sigma)$ is a proper set of forest orders, $v_m <_{\pi(F)} v_{m+1}$.

(2) $|U_m| > 0$, $v_{m+1} \notin U_{m-1}$: v_{m+1} has at most $b_{v_{m+1}}$ edges to F_{m-1} but at least $b_{v_{m+1}}+1$ edges to F_m, so G must have at least one edge from v_{m+1} to v_m. Moreover, by the inductive hypothesis, v_m is the maximal vertex in F_m in the order $\pi(F)$. As F_m is a subforest of F, v_m is also the maximal vertex in F_m in the order $\pi(F_m)$. Thus the $b_{v_{m+1}}+1^{\text{th}}$ smallest edge from v_{m+1} to F_m in the order $\pi(F_m)$ is from v_{m+1} to v_m, and F includes the edge (v_{m+1}, v_m). Because $\Pi(G,\Sigma)$ is a proper set of forest orders, $v_m <_{\pi(F)} v_{m+1}$.

(3) $|U_m| > 0$, $v_{m+1} \in U_{m-1}$: Because $v_{m+1} \in U_{m-1}$, $|U_{m-1}| > 0$. So, $v_m \in U_{m-1}$. Let e_{v_m} and $e_{v_{m+1}}$ denote, respectively, the $b_{v_m}+1^{\text{th}}$ smallest edge from v_m to F_{m-1} in the order $\pi(F_{m-1})$ and the $b_{v_{m+1}}+1^{\text{th}}$ smallest edge from v_{m+1} to F_{m-1} in the order $\pi(F_{m-1})$. By the inductive hypothesis, v_m is the largest vertex in F_m in $\pi(F)$; it is therefore also the largest vertex in F_m in $\pi(F_m)$. So, in $\pi(F_m)$, all edges from v_{m+1} to v_m are larger than edges from v_{m+1} to vertices in F_{m-1}; then, because there are at least $b_{v_{m+1}}+1$ edges from v_{m+1} to F_{m-1} and the orders $\pi(F_{m-1})$ and $\pi(F_m)$ are consistent, the $b_{v_{m+1}}+1^{\text{th}}$ smallest edge from v_{m+1} to F_{m-1} in $\pi(F_{m-1})$ is also the $b_{v_{m+1}}+1^{\text{th}}$ smallest edge from v_{m+1} to F_m in $\pi(F_m)$. Hence F_{m+1} is formed by adding $e_{v_{m+1}}$ and v_{m+1} to F_m. Let f be the oriented forest consisting of F_{m-1}, v_m, v_{m+1}, e_{v_m} and $e_{v_{m+1}}$. f is a subforest of both f_{m-1} and F, so $\pi(f)$ is consistent with both $\pi(f_{m-1})$ and $\pi(F)$. $v_m <_{\pi(f_{m-1})} v_{m+1}$ because v_m is the smallest vertex in U_{m-1} in the order $\pi(f_{m-1})$. Hence $v_m <_{\pi(f)} v_{m+1}$, and $v_m <_{\pi(F)} v_{m+1}$.

□

Lemma 6.8. *Let $P \in \mathcal{P}_{G,\Sigma}$ and $F = \Phi_{\Pi,G,\Sigma}(P)$. If $|U_{m-1}| = 0$, then v_m has exactly b_{v_m} edges to vertices in F_{m-1}.*

Proof. Suppose $|U_{m-1}| = 0$. Then v_m is the numerically smallest vertex not in F_{m-1}. Let I be the set of vertices $\{0,1,\ldots,n\}\setminus F_{m-1}$. By Lemma 6.5, $I \in \Sigma$. Thus $m_I = x_{v_m}\prod_{i\in I} x_i^{d_I(i)}$. Because $|U_{m-1}| = 0$, each vertex $i \in I$ has at most b_i edges to vertices in F_{m-1}. Hence $d_I(i) \le b_i$ for all $i \in I$. In particular, $d_I(v_m) \le b_{v_m}$. But, $\prod_i x_i^{b_i}$ is not divisible by m_I. This is only possible if $\deg_{x_{v_m}} m_I = d_I(v_m)+1 > b_{v_m}$, which requires that $d_I(v_m) = b_{v_m}$. Therefore v_m has exactly b_{v_m} edges to F_{m-1}. □

Lemma 6.9. *Let $P \in \mathcal{P}_{G,\Sigma}$. Then $\Theta_{\Pi,G,\Sigma}(\Phi_{\Pi,G,\Sigma}(P)) = P$.*

Proof. Let $P = (b_1,\ldots,b_n)$, $F = \Phi_{\Pi,G,\Sigma}(P)$, and $\Theta_{\Pi,G,\Sigma}(F) = P' = (b'_1,\ldots,b'_n)$. As before, let $F_1,\ldots,F_n = F$ be the oriented forests made in the construction of $\Phi_{\Pi,G,\Sigma}(P)$, and let v_m $(1 \le m \le n)$ be the vertex in F_m but not F_{m-1}. For each m, we consider two cases:

(1) $e_F(v_m)$ exists: The edge $e_F(v_m)$ must go to a vertex in F_{m-1}. By Lemma 6.7, all edges e from v_m such that $e <_{\pi(F)} e_F(v_m)$ must go to vertices in F_{m-1}. By construction, there are b_{v_m} edges e from v_m to F_{m-1} such that $e <_{\pi(F_{m-1})} e_F(v_m)$. Because the orders $\pi(F_{m-1})$ and $\pi(F)$ are consistent, there are b_{v_m} edges e from v_m such that $e <_{\pi(F)} e_F(v_m)$. Thus $b'_{v_m} = b_{v_m}$.

(2) $e_F(v_m)$ does not exist: Then $|U_{m-1}| = 0$. By Lemma 6.8, v_m has exactly b_{v_m} edges to vertices in F_{m-1}. By Lemma 6.7, these are precisely the edges from v_m to vertices j such that $j <_{\pi(F)} v_m$. Therefore $b'_{v_m} = b_{v_m}$.

It follows that $b'_{v_m} = b_{v_m}$ for all m. Therefore $P' = P$. □

Lemma 6.10. *Let $F \in \mathcal{F}_{G,\Sigma}$. Then $\Phi_{\Pi,G,\Sigma}(\Theta_{\Pi,G,\Sigma}(F)) = F$.*

Proof. Let $P = (b_1, \dots, b_n) = \Theta_{\Pi,G,\Sigma}(F)$ and $F' = \Phi_{\Pi,G,\Sigma}(P)$. Let $F_1, \dots, F_n = F'$ be the oriented forests made in the construction of $\Phi_{\Pi,G,\Sigma}(P)$, and let v_m $(1 \le m \le n)$ be the vertex in F_m but not F_{m-1}. We prove by induction on m that F_m is a subforest of F whose vertices are the $m+1$ smallest vertices of F in $\pi(F)$. 0 is the smallest vertex in the order $\pi(F)$, so the claim is true for $m = 0$.

Assume that F_{m-1} is a subforest of F whose vertices are the m smallest vertices of F in $\pi(F)$. Let v'_m be the $m+1^{\text{th}}$ smallest vertex of F in $\pi(F)$. We consider two cases:

(1) $e_F(v'_m)$ exists: Let $e_F(v'_m) = (v'_m, v)$. Because $\Pi(G, \Sigma)$ is a proper set of forest orders, $v <_{\pi(F)} v'_m$. Thus $v \in F_{m-1}$. Because F_{m-1} consists of the m smallest vertices of F in $\pi(F)$, if an edge e from v'_m satisfies $e <_{\pi(F)} e_F(v'_m)$, then e is to a vertex in F_{m-1}. By definition of $\Theta_{\Pi,G,\Sigma}$, there are $b_{v'_m}$ edges from v'_m such that $e <_{\pi(F)} e_F(v'_m)$. These edges and the edge $e_F(v'_m)$ all go from v'_m to vertices in F_{m-1}; hence $v'_m \in U_{m-1}$.

For each $i \in U_{m-1}$, let e_i be the $b_i + 1^{\text{th}}$ smallest edge from i to F_{m-1} in the order $\pi(F_{m-1})$. Because F_{m-1} consists of the m smallest vertices of F in $\pi(F)$, all edges e coming out of $i \in U_{m-1}$ and satisfying $e <_{\pi(F)} e_i$ must go to a vertex in F_{m-1}. Moreover, because the orders $\pi(F_{m-1})$ and $\pi(F)$ are consistent, an edge from i satisfies $e <_{\pi(F_{m-1})} e_i$ if and only if it satisfies $e <_{\pi(F)} e_i$. Thus, for each $i \in U_{m-1}$ there are exactly b_i edges e from i satisfying $e <_{\pi(F)} e_i$. By choice of b_i, e_i is an edge in F. It follows that f_{m-1}, the oriented forest consisting of F_{m-1}, all $i \in U_{m-1}$, and all e_i for $i \in U_{m-1}$, is a subforest of F. So, the orders $\pi(f_{m-1})$ and $\pi(F)$ are consistent. Because $v'_m \in U_{m-1}$ and v'_m is the smallest vertex not in F_m in the order $\pi(F)$, v'_m is the smallest vertex in U_{m-1} in the order $\pi(f_{m-1})$. Therefore F_m consists of F_{m-1}, v'_m, and $e_F(v'_m)$ and is a subforest of F whose vertices are the $m+1$ smallest vertices of F in $\pi(F)$.

(2) $e_F(v'_m)$ does not exist: v'_m is a root of F. Because v'_m is the smallest vertex in $\pi(F)$ not in F_{m-1}, no edges in F go from a vertex outside F_{m-1} to a vertex in F_{m-1}. Because $\Pi(G, \Sigma)$ is a proper set of forest orders, v'_m must be the numerically smallest root of F outside of F_{m-1}; moreover, because F is a proper forest, v'_m must be the numerically smallest vertex of F outside of F_{m-1}. By definition of $\Theta_{\Pi,G,\Sigma}$, each $i \notin F_{m-1}$ has at most b_i edges to F_{m-1}; hence $|U_{m-1}| = 0$. Then F_m consists of F_{m-1} and v'_m; therefore F_m is a subforest of F whose vertices are the $m+1$ smallest vertices of F in $\pi(F)$.

This implies that $F_n = F'$ is a subforest of F whose vertices are the vertices of F. Thus $F' = F$. □

Proof of Proposition 6.1: Proposition 6.1 follows from Lemmas 6.4, 6.6, 6.9, and 6.10. □

Hence the (G, Σ)-parking functions biject to the Σ-proper forests of G, as claimed by Theorem 4.2.

7. Σ-Forest Algebras

Let G be a graph on $\{0, \ldots, n\}$ and Σ be a set of subsets of $\{1, \ldots, n\}$ with the downward inclusion property.

For each nonempty $I \subseteq \{1, \ldots, n\}$, let H_I denote the set of edges between vertices in I and vertices in $\{0, 1, \ldots, n\} \setminus I$.

Associate with each edge $e \in G$ a commutative variable ϕ_e, and let $\Phi_{G,\Sigma}$ be the algebra over $\mathbb{K}$ generated by the ϕ_e and obeying

$$\phi_e^2 = 0 \quad \text{For all edges } e \in G$$

and

$$\prod_{e \in H_I} \phi_e = 0 \quad \text{For all nonempty } I \notin \Sigma.$$

Define a set of edges $H \subseteq G$ to be Σ-*good* if $\prod_{e \in H} \phi_e$ does not vanish in $\Phi_{G,\Sigma}$. Equivalently, H is Σ-good if the set of vertices not connected to 0 in $G \setminus H$ is an element of Σ.

For $i = 1, \ldots, n$, define

$$X_i = \sum_{\substack{e=(i,j) \in G \\ i<j}} \phi_e - \sum_{\substack{e=(i,j) \in G \\ i>j}} \phi_e$$

and let $\mathcal{C}_{G,\Sigma}$ be the subalgebra of $\Phi_{G,\Sigma}$ generated by $X_1, \ldots, X_n$.

Proposition 7.1. *For all graphs G and all Σ,*

$$\dim \mathcal{C}_{G,\Sigma} = N_{G,\Sigma}$$

where $N_{G,\Sigma}$ equals the number of Σ-forests of G. Moreover, the k^{th} graded component $\mathcal{C}^k_{G,\Sigma}$ has dimension equal to the number of Σ-forests F of G with external activity $|G| - |F| - k$.

Define $\mathcal{S}_{G,\Sigma}$ as the subspace of $\mathbb{K}[y_1, \ldots, y_n]$ linearly spanned by

$$\alpha_H = \prod_{e \in H} (\alpha_e)$$

as H ranges over all Σ-good subgraphs of G, where $\alpha_e = y_i - y_j$ for $e = (i,j)$ with $0 < i < j$ and $\alpha_e = -y_j$ for $e = (0, j)$.

Lemma 7.2. *For any Σ-good edge set $H \subseteq G$ and any sequence $a = (a_1, \ldots, a_n)$ with sum $|H|$, the coefficient of $\prod_{e \in H} \phi_e$ in the expansion $\frac{1}{a_1! \cdots a_n!} X_1^{a_1} \cdots X_n^{a_n}$ equals the coefficient of $y_1^{a_1} \cdots y_n^{a_n}$ in the expansion α_H.*

Proof. For fixed H and a, define an (H,a)-*valid assignment* as an assignment of each edge of H to one of its endpoints such that each vertex $i \in \{1, \ldots, n\}$ has a_i edges assigned to it. In each (H,a)-valid assignment, let the value of an edge be $+1$ if it is assigned to its smaller endpoint, and -1 if it is assigned to its larger endpoint. Define the value of an (H,a)-valid assignment to be the product of the values of its edges. Finally define $f(H,a)$ as the sum of the values of all (H,a)-valid assignments.

The coefficient of $\prod_{e \in H} \phi_e$ in the expansion $\frac{1}{a_1! \cdots a_n!} X_1^{a_1} \cdots X_n^{a_n}$ and the coefficient of $y_1^{a_1} \cdots y_n^{a_n}$ in the expansion α_H both count $f(H,a)$ - the first by choosing edges to assign to each vertex, and the second by choosing the vertex to which each edge is assigned. Therefore these coefficients are equal. □

Lemma 7.3. *For all G, Σ and all k, the k^{th} graded components $\mathcal{C}^k_{G,\Sigma}$ and $\mathcal{S}^k_{G,\Sigma}$ obey* $\dim \mathcal{C}^k_{G,\Sigma} = \dim \mathcal{S}^k_{G,\Sigma}$.

Proof. Define $b_{H,a} = f(H, a)$, and let the matrix $B = (b_{H,a})$, as H ranges over all Σ-good sets of k edges and $a = (a_1, \ldots, a_n)$ ranges over all sequences of length n with sum k. Then, by Lemma 7.2, the dimensions of the k^{th} graded components of $\mathcal{C}_{G,\Sigma}$ and $\mathcal{S}_{G,\Sigma}$ both equal the rank of B. Therefore $\dim \mathcal{C}^k_{G,\Sigma} = \dim \mathcal{S}^k_{G,\Sigma}$. $\square$

Fix an order on the edges of G. For all Σ-forests F in G, let F^+ be the graph consisting of F and all externally active edges.

Lemma 7.4. *As F ranges over all Σ-forests of G, the $\alpha_{G\setminus F^+}$ linearly span $\mathcal{S}_{G,\Sigma}$.*

Proof. Suppose for sake of contradiction that there exists a Σ-good edge set H such that α_H cannot be expressed as a linear combination of the $\alpha_{G\setminus F^+}$. Out of all such edge sets, let H be lexicographically maximal with respect to the order of G's edges. Observe that because H is Σ-good, all spanning forests of $G \setminus H$ are Σ-forests. We consider two cases:

(1) No edge $e \in H$ is an externally active edge of any spanning forest $F \subseteq G\setminus H$: We claim that $G \setminus H$ has a spanning forest F such that F^+ includes all edges of $G \setminus H$. We may construct such an F by starting with an arbitrary spanning forest f and repeatedly applying the following algorithm: if $f^+ = G \setminus H$, stop; otherwise, let $e \in G \setminus H$ be an edge not in f^+. Because e is not externally active with respect to f, there exists an edge e' in the cycle in $f \cup e$ that is smaller than e. Modify f by replacing e' with e. This algorithm must terminate because it replaces an edge in f by a larger edge at each step. So, there exists F such that $F^+ = G \setminus H$. Consequently $H = G \setminus F^+$, and $\alpha_H = \alpha_{G\setminus F^+}$ is a contradiction.

(2) There exists an edge $e \in H$ that is externally active in a spanning forest $F \subseteq G \setminus H$: Let $e, e_1, e_2, \ldots, e_k$ be a cycle in G such that e is the minimal edge in this cycle and $e_1, \ldots, e_k \in G \setminus H$. Then, $\alpha_e = -(\alpha_{e_1} + \cdots + \alpha_{e_n})$. Let $H_1, H_2, \ldots, H_n$ be the Σ-good edge sets obtained from H by replacing e with $e_1, e_2, \ldots, e_n$, respectively. These are lexicographically larger than H, so $\alpha_{H_1}, \alpha_{H_2}, \ldots, \alpha_{H_n}$ are all expressible as linear combinations of the $\alpha_{G\setminus F^+}$. But now $\alpha_H = -(\alpha_{H_1} + \cdots + \alpha_{H_n})$ is a contradiction.

$\square$

Lemma 7.5. *As F ranges over all Σ-forests of G, the $\alpha_{G\setminus F^+}$ form a linear basis of $\mathcal{S}_{G,\Sigma}$.*

Proof. By Lemma 7.4, it suffices to prove $\dim \mathcal{S}_{G,\Sigma} = N_{G,\Sigma}$, where $N_{G,\Sigma}$ denotes the number of Σ-forests of G. We induct on the number of edges in G.

Say a Σ-forest F is a *minimal Σ-forest* if the forest produced by removing any edge $e \in F$ from F is not a Σ-forest. If G is a minimal Σ-forest, then $\dim \mathcal{S}_{G,\Sigma} = 1 = N_{G,\Sigma}$. If G is a forest that is not a Σ-forest, then $\dim \mathcal{S}_{G,\Sigma} = 0 = N_{G,\Sigma}$. This proves the induction's base case.

If G has at least one edge, choose an edge $e = (i, j)$ where $i < j$. For all $I \in \Sigma$, define

$$f_e(I) = \begin{cases} I \setminus \{j\} & j \in I \\ I \setminus \{i\} & j \notin I, i \in I \\ I & \text{otherwise} \end{cases}$$

and let $\Sigma_e = \{f_e(I) | I \in \Sigma\}$. It is clear that Σ_e also has the downward inclusion property.

Let $G - e$ be G with e removed; let G/e be G with e contracted and i and j both relabeled as i. The Σ-forests of G that do not include e are the Σ-forests of $G - e$, and the Σ-forests of G that include e biject to the Σ_e-forests of G/e by contraction of e. Thus $N_{G,\Sigma} = N_{G-e,\Sigma} + N_{G/e,\Sigma_e}$. By the inductive hypothesis, $\dim \mathcal{S}_{G-e,\Sigma} = N_{G-e,\Sigma}$ and $\dim \mathcal{S}_{G/e,\Sigma_e} = N_{G/e,\Sigma_e}$.

Let $\mathcal{S}'_{G,\Sigma}$ denote the span of the α_H, where H is Σ-good and $e \in H$. Let $\mathcal{S}''_{G,\Sigma}$ denote the span of the α_H, where H is Σ-good and $e \notin H$. We have $\dim \mathcal{S}'_{G,\Sigma} = \dim \mathcal{S}_{G-e,\Sigma}$ because these spaces are isomorphic as vector spaces via multiplication by α_e. Let p be the vector space homomorphism that takes elements of $\mathcal{S}_{G,\Sigma}$ modulo $y_i - y_j$. Then $p(\mathcal{S}''_{G,\Sigma}) = \mathcal{S}_{G/e,\Sigma_e}$. Thus

$$\dim \mathcal{S}''_{G,\Sigma} = \dim \mathcal{S}_{G/e,\Sigma_e} + \dim \ker(p).$$

But, $\mathcal{S}'_{G,\Sigma} \cap \mathcal{S}''_{G,\Sigma} \subseteq \ker(p)$. Hence

$$\dim \mathcal{S}''_{G,\Sigma} \geq \dim \mathcal{S}_{G/e,\Sigma_e} + \dim(\mathcal{S}'_{G,\Sigma} \cap \mathcal{S}''_{G,\Sigma}).$$

Because $\mathcal{S}'_{G,\Sigma}$ and $\mathcal{S}''_{G,\Sigma}$ together span $\mathcal{S}_{G,\Sigma}$, we have

$$\dim \mathcal{S}_{G,\Sigma} = \dim \mathcal{S}'_{G,\Sigma} + \dim \mathcal{S}''_{G,\Sigma} - \dim(\mathcal{S}'_{G,\Sigma} \cap \mathcal{S}''_{G,\Sigma}).$$

Summing the last two relations yields

$$\dim \mathcal{S}_{G,\Sigma} \geq \dim \mathcal{S}'_{G,\Sigma} + \dim \mathcal{S}_{G/e,\Sigma_e} = \dim \mathcal{S}_{G-e,\Sigma} + \dim \mathcal{S}_{G/e,\Sigma_e}$$

By induction, the last quantity equals $N_{G-e,\Sigma} + N_{G/e,\Sigma_e} = N_{G,\Sigma}$, so $\dim \mathcal{S}_{G,\Sigma} \geq N_{G,\Sigma}$. But Lemma 7.4 implies $\dim \mathcal{S}_{G,\Sigma} \leq N_{G,\Sigma}$. Thus $\dim \mathcal{S}_{G,\Sigma} = N_{G,\Sigma}$. □

Proof of Proposition 7.1: Proposition 7.1 follows from Lemmas 7.3 and 7.5. □

Recall that $\mathcal{B}_{G,\Sigma} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{J}_{G,\Sigma}$, where $\mathcal{J}_{G,\Sigma} = \langle p_I \rangle$ and

$$p_I = \begin{cases} \left(\sum_{i \in I} x_i\right)^{D_I+1} & I \in \Sigma \\ \left(\sum_{i \in I} x_i\right)^{D_I} & I \notin \Sigma \end{cases}$$

as I ranges over all nonempty subsets of $\{1, \ldots, n\}$.

Lemma 7.6. *$\mathcal{C}_{G,\Sigma}$ is a subalgebra of $\mathcal{B}_{G,\Sigma}$.*

Proof. For all $I \in \Sigma$,

$$\left(\sum_{i \in I} X_i\right)^{D_I+1} = \left(\sum_{e \in H_I} \pm\phi_e\right)^{D_I+1} = 0$$

because each term of the expansion $\left(\sum_{e \in H_I} \pm\phi_e\right)^{D_I+1}$ is divisible by the square of some ϕ_e. For all $I \notin \Sigma$,

$$\left(\sum_{i \in I} X_i\right)^{D_I} = \left(\sum_{e \in H_I} \pm\phi_e\right)^{D_I} = 0$$

because the only square-free term of the expansion $\left(\sum_{e \in H_I} \pm\phi_e\right)^{D_I}$ is $\prod_{e \in H} \phi_e$, which is 0 because $I \notin \Sigma$. □

We can now prove Theorem 4.4.

Proof of Theorem 4.4: By Theorem 3.3, we have the termwise inequality of Hilbert series

$$\text{Hilb } \mathcal{A}_{G,\Sigma} \geq \text{Hilb } \mathcal{B}_{G,\Sigma}$$

because $\mathcal{J}_{G,\Sigma}$ is a deformation of $\mathcal{I}_{G,\Sigma}$. By Lemma 7.6,

$$\text{Hilb } \mathcal{B}_{G,\Sigma} \geq \text{Hilb } \mathcal{C}_{G,\Sigma}.$$

Therefore

$$\text{Hilb } \mathcal{A}_{G,\Sigma} \geq \text{Hilb } \mathcal{B}_{G,\Sigma} \geq \text{Hilb } \mathcal{C}_{G,\Sigma}.$$

But, by Theorem 4.2 and Proposition 7.1,

$$\dim \mathcal{A}_{G,\Sigma} = N_{G,\Sigma} = \dim \mathcal{C}_{G,\Sigma}.$$

Thus we in fact have

$$\dim \mathcal{A}_{G,\Sigma} = \dim \mathcal{B}_{G,\Sigma} = \dim \mathcal{C}_{G,\Sigma} = N_{G,\Sigma}$$

and

$$\text{Hilb } \mathcal{A}_{G,\Sigma} = \text{Hilb } \mathcal{B}_{G,\Sigma} = \text{Hilb } \mathcal{C}_{G,\Sigma}.$$

Moreover, by Proposition 7.1, $\dim \mathcal{A}^k_{G,\Sigma} = \dim \mathcal{B}^k_{G,\Sigma} = \dim \mathcal{C}^k_{G,\Sigma}$ equals the number of Σ-forests F of G with external activity $|G| - |F| - k$, and the theorem is proved. $\square$

8. ρ-Parking Functions

The ρ-parking functions are another generalization of the classical parking functions developed in [11] and [16]. Let $\rho = (\rho_1, \ldots, \rho_n)$ be a nonincreasing sequence of positive integers. A sequence $(b_1, \ldots, b_n)$ is a *ρ-parking function* if and only if its decreasing rearrangement is termwise less than ρ. Equivalently, for all nonempty $I \subseteq \{1, \ldots, n\}$, define

$$m_I = \left(\prod_{i \in I} x_i\right)^{\rho_{|I|}}$$

$$p_I = \left(\sum_{i \in I} x_i\right)^{|I|\rho_{|I|}}.$$

Let $\mathcal{I}_\rho = \langle m_I \rangle$ and $\mathcal{J}_\rho = \langle p_I \rangle$ be ideals in $\mathbb{K}[x_1, \ldots, x_n]$ generated by all such m_I and p_I, respectively. Define the algebras $\mathcal{A}_\rho = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}_\rho$ and $\mathcal{B}_\rho = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{J}_\rho$. A sequence $(b_1, \ldots, b_n)$ is a ρ-parking function if and only if $\prod_i x_i^{b_i}$ is nonvanishing in $\mathcal{I}_\rho$.

Remark 8.1. When $\rho = (n, n-1, \ldots, 1)$, the ρ-parking functions are the classical parking functions of size n.

Observe that $\mathcal{A}_\rho$ is a monotone monomial ideal and $\mathcal{B}_\rho$ is its deformation. While it is not true that Hilb $\mathcal{A}_\rho$ and Hilb $\mathcal{B}_\rho$ are always equal, Theorem 3.3 implies that:

Proposition 8.2. *The monomials $\prod_i x_i^{b_i}$, as $(b_1, \ldots, b_n)$ ranges over all ρ-parking functions, span $\mathcal{B}_\rho$, and*

$$\textit{Hilb } \mathcal{A}_\rho \geq \textit{Hilb } \mathcal{B}_\rho.$$

Furthermore,

Proposition 8.3. *When $\rho = (l + (n-1)k, l + (n-2)k, \ldots, l)$ is a decreasing arithmetic sequence, the Hilbert series Hilb $\mathcal{A}_\rho$ and Hilb $\mathcal{B}_\rho$ are equal.*

This is because when $\rho = (l + (n-1)k, l + (n-2)k, \ldots, l)$, $\mathcal{I}_\rho$ and $\mathcal{J}_\rho$ are $\mathcal{I}_G$ and $\mathcal{J}_G$, where $G = K_{n+1}^{k,l}$, the graph with k edges between any two nonzero vertices and l edges between any nonzero vertex and 0. Hence Theorem 2.3 implies that $\mathcal{A}_\rho$ and $\mathcal{B}_\rho$ have equal Hilbert series.

We claim that Hilb $\mathcal{A}_\rho$ and Hilb $\mathcal{B}_\rho$ for another class of ρ as well:

Proposition 8.4. *When $\rho = (l + (n-1)k + 1, l + (n-2)k, \ldots, l)$ is a decreasing arithmetic sequence whose largest term is increased by 1, the Hilbert series Hilb $\mathcal{A}_\rho$ and Hilb $\mathcal{B}_\rho$ are equal.*

Observe that when $\rho = (l + (n-1)k + 1, l + (n-2)k, \ldots, l)$, $\mathcal{I}_\rho$ and $\mathcal{J}_\rho$ are $\mathcal{I}_{G,\Sigma}$ and $\mathcal{J}_{G,\Sigma}$, where $G = K_{n+1}^{k,l}$ and $\Sigma = \{\{\emptyset\}, \{1\}, \{2\}, \ldots, \{n\}\}$. Thus, as a corollary to Theorem 4.4 we have that Hilb $\mathcal{A}_\rho =$ Hilb $\mathcal{B}_\rho$.

9. Future Work

Suppose a graph G on $\{0, 1, \ldots, n\}$ has the property that there is at least one edge between any two vertices. Let Σ be a family of subsets of $\{1, \ldots, n\}$ with the downward inclusion property. Then, for each nonempty subset $I = \{i_1 < \cdots < i_r\}$ of $\{1, \ldots, n\}$, define

$$m_I = \begin{cases} x_{i_1}^{d_I(i_1)-1} \prod_{\substack{i \in I \\ i \neq i_1}} x_i^{d_I(i)} & I \in \Sigma \\ \prod\limits_{i \in I} x_i^{d_I(i)} & I \notin \Sigma \end{cases}$$

and

$$p_I = \begin{cases} \left(\sum\limits_{i \in I} x_i \right)^{D_I - 1} & I \in \Sigma \\ \left(\sum\limits_{i \in I} x_i \right)^{D_I} & I \notin \Sigma \end{cases}$$

and let $\mathcal{I}'_{G,\Sigma} = \langle m_I \rangle$ and $\mathcal{J}'_{G,\Sigma} = \langle p_I \rangle$ be the ideals generated by all such m_I and p_I, respectively. Observe that because there is at least one edge between any two vertices of G, $\mathcal{I}'_{G,\Sigma}$ is a monotone monomial ideal and $\mathcal{J}'_{G,\Sigma}$ is its deformation. Define the algebras $\mathcal{A}'_{G,\Sigma} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}'_{G,\Sigma}$ and $\mathcal{B}'_{G,\Sigma} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{J}'_{G,\Sigma}$. Computer experiments suggest that the following conjecture is true:

Conjecture 9.1. *For all G and Σ, the standard monomial basis of $\mathcal{A}'_{G,\Sigma}$ is a basis of $\mathcal{B}'_{G,\Sigma}$. Consequently, $\mathcal{A}'_{G,\Sigma}$ and $\mathcal{B}'_{G,\Sigma}$ have equal dimension and Hilbert series.*

When $\Sigma = \{\{\emptyset\}\}$, this conjecture reduces to Theorem 2.3. Moreover, when $\Sigma = \mathcal{P}(\{1, \ldots, n\})$, this conjecture reduces to Theorem 2.5. If proven, this conjecture would interpolate between these two known results and establish another large class of monotone monomial ideals and their deformations with equal Hilbert series.

In the context of ρ-algebras and ρ-parking functions, setting $G = K_{n+1}^{k,l}$ and $\Sigma = \{\{\emptyset\}, \{1\}, \{2\}, \ldots, \{n\}\}$ in Conjecture 9.1 implies:

Conjecture 9.2. *When $\rho = (l + (n-1)k - 1, l + (n-2)k, \ldots, l)$ is a decreasing arithmetic sequence whose largest term is decreased by 1, the Hilbert series Hilb $\mathcal{A}_\rho$ and Hilb $\mathcal{B}_\rho$ are equal.*

It would be interesting to characterize all ρ for which Hilb $\mathcal{A}_\rho$ and Hilb $\mathcal{B}_\rho$ are equal. Though Proposition 8.3, Proposition 8.4, and Conjecture 9.2 describe a large class of ρ for which Hilb $\mathcal{A}_\rho =$ Hilb $\mathcal{B}_\rho$, these are not the only cases of equality; for instance, Hilb $\mathcal{A}_\rho =$ Hilb $\mathcal{B}_\rho$ for $\rho = (5, 5, 3)$ and $\rho = (8, 6, 5, 3)$. Nonetheless, the author does not know of a strictly decreasing ρ satisfying Hilb $\mathcal{A}_\rho =$ Hilb $\mathcal{B}_\rho$ that is not of the form:

- $(l + (n-1)k + c, l + (n-2)k, \ldots, l)$, where $c \in \{-1, 0, 1\}$, or
- $(l + 3k + c, l + 2k + c, l + k, l)$, where $c \in \{-1, 1\}$.

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