On the evaluation of some infinite series

Zihong Chen

Abstract

This paper deals with some general methods of evaluating infinite series, among which the Poisson summation formula and the residue theorem are the main tools. The idea was initially inspired by the well-known problem of finding the zeta value at even integers using Fourier series, which reveals a recurrence relation, and the body of this paper focuses on explicit formulas of the shifted value, a modified form of the famous Dirichlet series. Hence, the first main result of this paper is evaluating explicitly the shifted zeta function at all even integers. The rest of the paper seeks a similar formula for a more general type of rational functions, and illustrates some of its applications.

Keywords: series evaluation, (shifted)zeta function, Poisson summation formula, residue theorem

Contents

1 Introduction

Tracing back to Basel's problem of finding the sum of the reciprocal of all perfect squares, the exact computation of some infinite series have been in itself an appealing topic in mathematics.

This paper starts with evaluating the zeta function at even integers by Fourier series, which reveals a recurrence relation. Motivated towards a more straightforward result, we define the 'shifted zeta function' with a parameter t and turn to Fourier transform, together with the Poisson summation formula. The immediate goal is to find all even values of shifted zeta function(and other shifted functions may be computed likewise). However, the fact that the type two shifted zeta function is not of moderate decrease, a property that excludes itself from the subject of Fourier analysis, drives us to overcome this challenge by some tricks in contour integration. Two methods are introduced at this point(the first of which is inspired by a proof in Professor Elias Stein's *Complex Analysis*), and the latter is extended to a range of rational function whose poles are simple.

After deriving such general formula, we illustrated some applications of the formula by two examples of summing over points in the complex plane. The latter example is particularly worth noticing since it deals with irrational functions in general, while avoiding the trouble involved in defining a non-integral power, i.e. to have a branch cut.

The paper ends with a final section that extends the summation formula to a larger set of rational functions, posing no restriction on its poles. We represent the values of these infinite series in terms of the numbers $A(n, x)$, which is defined in analogue to the Bernoulli numbers.

Some definitions are stated here.

Definition 1.1 The **zeta function** is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{for } s > 1,$$

the **Dirichlet Lambda function** is defined by:

$$\lambda(s) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s}, \quad \text{for } s > 1,$$

the **Dirichlet eta function** is defined by:

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}, \quad \text{for } s > 0,$$

and the **Dirichlet Beta function** is defined by:

$$\beta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^s}, \quad \text{for } s > 0.$$

Propsition 1.2 Two basic identities of the Dirichlet functions:

(1) $\zeta(s) = \dfrac{2^s}{2^s - 1}\lambda(s)$;

(2) $\zeta(s) = \dfrac{2^s}{2^s - 2}\eta(s)$.

Proof. For (1),

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} + \frac{1}{2^s}\sum_{n=1}^{\infty} \frac{1}{n^s} = \lambda(s) + \frac{1}{2^s}\zeta(s).$$

For (2),

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} - \frac{1}{2^s} \sum_{n=1}^{\infty} \frac{1}{n^s} = \lambda(s) - \frac{1}{2^s}\zeta(s).$$

Also, $\lambda(s) = \dfrac{2^s - 1}{2^s}\zeta(s)$, hence the result.

Definition 1.3 If f is an integrable function given on an interval $[a, b]$ of length L, then the n^{th} **Fourier coefficient** of f is defined by

$$\hat{f}(n) = \frac{1}{L} \int_a^b f(x) e^{-\frac{2\pi i n x}{L}} \, dx, \quad n \in \mathbb{Z}.$$

The **Fourier series** of f is given by

$$f(x) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{\frac{2\pi i n x}{L}}.$$

2 A recurrence formula for $\zeta(2k)$

At the beginning of this section, let us consider a simple function defined on $[-\pi, \pi]$ by $f(\theta) = |\theta|$. By the Dirichlet condition, the Fourier series of this function converges to itself. A simple calculation yields:

$$\hat{f}(n) = \begin{cases} \frac{\pi}{2}, & n = 0, \\ \frac{-1+(-1)^n}{\pi n^2}, & n \neq 0. \end{cases}$$

Hence, $f(\theta) = \dfrac{\pi}{2} e^{in\theta} + \displaystyle\sum_{n \neq 0} \frac{-1+(-1)^n}{\pi n^2} e^{in\theta}$. Plugging in $\theta = 0$,

$$\sum_{n \neq 0} \frac{-1+(-1)^n}{\pi n^2} = -\frac{\pi}{2} \Rightarrow \sum_{n \geq 1,\, odd} \frac{-4}{\pi n^2} = -\frac{\pi}{2} \Rightarrow \lambda(2) = \frac{\pi^2}{8}.$$

By the identity $\zeta(s) = \dfrac{2^s}{2^s - 1}\lambda(s)$, we obtain that $\zeta(2) = \dfrac{\pi^2}{6}$. An interesting question arises at this point: as we see that $f(\theta) = |\theta|$ yields the value of $\zeta(2)$, how about a more general function $f(\theta) = |\theta^{2k+1}|$?

Let f be the function defined on $[-\pi, \pi]$ by $f(\theta) = |\theta^{2k+1}|$. The nth Fourier coefficient of this function is given by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} \, d\theta = \frac{1}{2\pi} \left(\int_0^{\pi} \theta^{2k+1} e^{-in\theta} - \int_{-\pi}^{0} \theta^{2k+1} e^{-in\theta} \, d\theta \right).$$

Integrate by part and obtain

$$\hat{f}(n)=\frac{1}{2\pi}\left[\frac{\theta^{2k+1}e^{-in\theta}}{-in}-\cdots\cdots-\frac{P_{2k+1}^{2k+1}e^{-in\theta}}{(-in)^{2k+2}}\right]_0^{\pi}-\frac{1}{2\pi}\left[\frac{\theta^{2k+1}e^{-in\theta}}{-in}-\cdots\cdots-\frac{P_{2k+1}^{2k+1}e^{-in\theta}}{(-in)^{2k+2}}\right]_{-\pi}^{0}$$

$$=-\frac{1}{2\pi}\sum_{r=0}^{2k+1}\left[\frac{P_{2k+1}^{r}\theta^{2k+1-r}e^{-in\theta}}{(in)^{r+1}}\right]_0^{\pi}+\frac{1}{2\pi}\sum_{r=0}^{2k+1}\left[\frac{P_{2k+1}^{r}\theta^{2k+1-r}e^{-in\theta}}{(in)^{r+1}}\right]_{-\pi}^{0}$$

$$=-\frac{1}{2\pi}\left[\sum_{r=0}^{2k+1}\frac{P_{2k+1}^{r}\pi^{2k+1-r}(-1)^n}{(in)^{r+1}}-\frac{P_{2k+1}^{2k+1}}{(in)^{2k+2}}\right]+\frac{1}{2\pi}\left[-\sum_{r=0}^{2k+1}\frac{P_{2k+1}^{r}(-\pi)^{2k+1-r}(-1)^n}{(in)^{r+1}}+\frac{P_{2k+1}^{2k+1}}{(in)^{2k+2}}\right]$$

$$=\frac{P_{2k+1}^{2k+1}\big(1-(-1)^n\big)}{\pi(-1)^{k+1}n^{2k+2}}-\frac{1}{2\pi}\sum_{r=0}^{2k}\frac{P_{2k+1}^{r}\pi^{2k+1-r}\big((-1)^n-(-1)^{n+2k-r}\big)}{(in)^{r+1}},$$

where $P_n^m=n!/(n-m)!$ stands for permutation. We need to calculate the value of $\hat{f}(0)$ separately because the above calculation requires that $n\neq 0$, but the process is trivial. Hence, we have

$$\hat{f}(n)=\begin{cases}\frac{\pi^{2k+1}}{2k+2}, & n=0,\\ \frac{P_{2k+1}^{2k+1}\left(1-(-1)^n\right)}{\pi(-1)^{k+1}n^{2k+2}}-\frac{1}{2}\sum_{r=0}^{2k}\frac{P_{2k+1}^{r}\pi^{2k-r}\left((-1)^n-(-1)^{n+2k-r}\right)}{(in)^{r+1}}, & n\neq 0.\end{cases}$$

Let $\theta=0$, then

$$\begin{aligned}f(0)\quad&\sim\quad\sum_{n\neq 0}\hat{f}(n)+\frac{\pi^{2k+1}}{2k+2}\\&=\quad\sum_{n\neq 0}\frac{P_{2k+1}^{2k+1}\big(1-(-1)^n\big)}{\pi(-1)^{k+1}n^{2k+2}}-\frac{1}{2}\sum_{n\neq 0}\sum_{r=0}^{2k}\frac{P_{2k+1}^{r}\pi^{2k-r}\big((-1)^n-(-1)^{n+2k-r}\big)}{(in)^{r+1}}+\frac{\pi^{2k+1}}{2k+2}\\&=\quad 0.\end{aligned}$$

Now, let's take two steps by order.

1) $$\sum_{n\neq 0}\frac{P_{2k+1}^{2k+1}\big(1-(-1)^n\big)}{\pi(-1)^{k+1}n^{2k+2}}=2\,P_{2k+1}^{2k+1}\sum_{n>0,\,odd}\frac{2}{\pi(-1)^{k+1}n^{2k+2}}$$
$$=(-1)^{k+1}P_{2k+1}^{2k+1}\frac{4}{\pi}\,\lambda(2k+2).$$

2) $$\frac{1}{2}\sum_{n\neq 0}\sum_{r=0}^{2k}\frac{P_{2k+1}^{r}\pi^{2k-r}\big((-1)^n-(-1)^{n+2k-r}\big)}{(in)^{r+1}}=\frac{1}{2}\sum_{r=0}^{2k}P_{2k+1}^{r}\pi^{2k-r}\sum_{n\neq 0}\frac{(-1)^n-(-1)^{n-r}}{(in)^{r+1}}.$$

If r is even, then the latter sum is definitely zero. Therefore, we shall only consider r at odd integers.

$$\begin{aligned}\frac{1}{2}\sum_{r=0}^{2k} P_{2k+1}^{r}\pi^{2k-r}\sum_{n\neq 0}\frac{(-1)^n-(-1)^{n-r}}{(in)^{r+1}} &= \frac{1}{2}\sum_{j=1}^{k} P_{2k+1}^{2j-1}\pi^{2k-2j+1}\sum_{n\neq 0}\frac{2(-1)^n}{(in)^{2j}}\\ &= 2\sum_{j=1}^{k} P_{2k+1}^{2j-1}\pi^{2k-2j+1}\sum_{n=1}^{\infty}\frac{(-1)^{n-j}}{n^{2j}}\\ &= 2\sum_{j=1}^{k}(-1)^{j+1}P_{2k+1}^{2j-1}\pi^{2k-2j+1}\eta(2j)\\ &= 2\sum_{j=1}^{k}(-1)^{j+1}P_{2k+1}^{2j-1}\pi^{2k-2j+1}\left(\frac{2^{2j}-2}{2^{2j}-1}\right)\lambda(2j).\end{aligned}$$

Summing up the result of 1) and 2), we get

$$(-1)^{k+1}P_{2k+1}^{2k+1}\frac{4}{\pi}\lambda(2k+2) = 2\sum_{j=1}^{k}(-1)^{j+1}P_{2k+1}^{2j-1}\pi^{2k-2j+1}\left(\frac{2^{2j}-2}{2^{2j}-1}\right)\lambda(2j) - \frac{\pi^{2k+1}}{2k+2}$$

$$\Rightarrow \lambda(2k+2) = \frac{\pi^{2k+2}}{2P_{2k+1}^{2k+1}}\left(\sum_{j=1}^{k}(-1)^{k+j}P_{2k+1}^{2j-1}\pi^{-2j}\left(\frac{2^{2j}-2}{2^{2j}-1}\right)\lambda(2j) - \frac{(-1)^{k+1}}{4k+4}\right). \tag{1}$$

To complete the formula for $\zeta(2k)$, we use the identity $\zeta(s) = \dfrac{2^s}{2^s-1}\lambda(s)$ and substitute k for $k+1$. This leads to

$$\zeta(2k) = \frac{(2\pi)^{2k}}{2(2^{2k}-1)P_{2k-1}^{2k-1}}\left(\sum_{j=1}^{k-1}(-1)^{k+j+1}P_{2k-1}^{2j-1}\pi^{-2j}\left(\frac{2^{2j}-2}{2^{2j}}\right)\zeta(2j) - \frac{(-1)^k}{4k}\right), \tag{2}$$

which is a recurrence formula for zeta function at even integers.

3 The shifted zeta function

3.1 Some tools

This subsection will provide some preliminaries and tools we will use to pursue our main results. Let $\mathcal{M}(\mathbb{R})$ denote the set of functions of **moderate decrease** in the sense that f is continuous and there exists a constant $A > 0$ so that

$$|f(x)| \le \frac{A}{1+x^2}, \quad \text{for all } x \in \mathbb{R}.$$

For a function in $\mathcal{M}(\mathbb{R})$, we define its **Fourier transform** by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\,dx.$$

The **Fourier inversion** is defined by

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i x\xi}\,d\xi.$$

Indeed, for any function in $\mathcal{M}(\mathbb{R})$, its Fourier inversion is the function itself. (More often we allow Fourier transform and inversion to functions of the **Schwartz space**, but an extension can be readily made to functions of moderate decrease. A brief reasoning can be found in [1], Chapter 5. section 1.7.)

Poisson Summation Formula: If $f \in \mathcal{M}(\mathbb{R})$, then

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi inx}.$$

Proof. Define $g(x) = \sum_{n=-\infty}^{\infty} f(x+n)$ and let

$$g(x) = \sum_{n=-\infty}^{\infty} \hat{g}(n)e^{2\pi inx},$$

which is the Fourier series of g. As $g(x)$ is clearly of period 1,

$$\begin{aligned}
\hat{g}(n) &\sim \int_0^1 g(x)e^{-2\pi inx}\,dx \\
&= \int_0^1 \sum_{k=-\infty}^{\infty} f(x+k)e^{-2\pi inx}\,dx \\
&= \sum_{k=-\infty}^{\infty} \int_0^1 f(x+k)e^{-2\pi inx}\,dx \\
&= \int_{-\infty}^{\infty} f(x)e^{-2\pi inx}\,dx \\
&= \hat{f}(n).
\end{aligned}$$

This completes the proof.

3.2 Definition

In this section, we are going to focus on some types of Dirichlet series with a parameter t. Let's begin with the definition:

Definition 3.1 For $s > 1$ and $t > 0$, the shifted zeta function by t is defined as

$$\zeta_t(s) = \sum_{n=1}^{\infty} \frac{1}{n^s + t^s}.$$

Define similarly for other Dirichlet L-series.

It seems unclear at present why we need to have this new definition. But observe that once we add a non-zero parameter t, the function $1/z^k$, whose pole is of order k, becomes $1/(z^k + t^k)$, whose poles are all simple. This property will make the calculation a lot simpler as we will see later in this section. Our goal here is to find a formula when s is an even integer and $t \in \mathbb{R}$, though our computation holds in some cases where t isn't real.

3.3 The shifted values at $s = 2$

To solve the shifted zeta function at $s = 2$, we shall meet with a special function named the **Poisson kernel**, which is given by

$$\mathcal{P}_y(x) = \frac{y}{\pi(x^2 + y^2)}, \quad \text{for } x \in \mathbb{R} \text{ and } y > 0.$$

Poisson kernel has its significance in physics since it is a solution to the steady-state heat equation in the upper half plane. However, at this point, we are to explore how this special function relates to our first shifted zeta function, the $\zeta_t(2)$.

We claim that the Fourier transform of $\mathcal{P}_y(x)$ is:

$$\int_{-\infty}^{\infty} \mathcal{P}_y(x)e^{-2\pi i x\xi}\, dx = e^{-2\pi|\xi|y}.$$

Proof. Firstly, we observe that $\mathcal{P}_y(x)$ is of moderate decrease, so the Fourier transform make sense. We now use Fourier inversion to prove this.

$$\int_{-\infty}^{\infty} e^{-2\pi|\xi|y}e^{2\pi i\xi x}\, d\xi = \mathcal{P}_y(x).$$

Split this integral into $-\infty$ to 0 and 0 to ∞. Then we have

$$\int_{0}^{\infty} e^{-2\pi\xi y}e^{2\pi i\xi x}\, d\xi = \int_{0}^{\infty} e^{2\pi i(x+iy)\xi}\, d\xi = \left[\frac{e^{2\pi i(x+iy)\xi}}{2\pi i(x+iy)}\right]_0^{\infty} = -\frac{1}{2\pi i(x+iy)}.$$

and similarly,

$$\int_{-\infty}^{0} e^{2\pi\xi y}e^{2\pi i\xi x}\, d\xi = \frac{1}{2\pi i(x-iy)}.$$

Therefore

$$\int_{-\infty}^{\infty} e^{-2\pi|\xi|y}e^{2\pi i\xi x}\, d\xi = -\frac{1}{2\pi i(x+iy)} + \frac{1}{2\pi i(x-iy)} = \frac{y}{\pi(x^2+y^2)}.$$

In order to obtain $\zeta_t(2)$, we apply the Poisson summation to the Poisson kernel.

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{y}{\pi\big((x+n)^2+y^2\big)} &= \sum_{n=-\infty}^{\infty} e^{-2\pi y|n|}e^{2\pi inx} \\ &= \sum_{n=0}^{\infty} e^{-2\pi yn}e^{2\pi inx} + \sum_{n=-\infty}^{0} e^{2\pi yn}e^{2\pi inx} - 1 \\ &= \frac{1}{1-e^{2\pi i(x+iy)}} - \frac{e^{2\pi i(x-iy)}}{1-e^{2\pi i(x-iy)}} - 1 \\ &= \frac{e^{2\pi i(x+iy)} - e^{2\pi i(x-iy)}}{1-\left(e^{2\pi i(x+iy)}+e^{2\pi i(x-iy)}\right)+e^{4\pi ix}} \\ &= \frac{e^{4\pi y}-1}{e^{4\pi y}-2\cos(2\pi x)e^{2\pi y}+1}.\end{aligned}$$

Where we've used the usual geometric series sum. Substitute t for y,

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+x)^2+t^2} = \frac{\pi(e^{4\pi t}-1)}{t(e^{4\pi t}-2e^{2\pi t}\cos(2\pi x)+1)}.$$

Let $x=0$, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2+t^2} = \frac{1}{2}\left(\frac{\pi(e^{2\pi t}+1)}{t(e^{2\pi t}-1)} - \frac{1}{t^2}\right) = \frac{t\pi(e^{2\pi t}+1)-e^{2\pi t}+1}{2t^2(e^{2\pi t}-1)}. \tag{3}$$

Which is the formula for $\zeta_t(2)$.

By a simple observation we have

$$\begin{aligned}\eta_t(2) &= \zeta_t(2) - \frac{1}{2}\zeta_{\frac{t}{2}}(2)\\ &= \frac{t\pi(e^{2\pi t}+1) - e^{2\pi t} + 1}{2t^2(e^{2\pi t}-1)} - \frac{1}{2}\frac{t\pi(e^{\pi t}+1) - 2e^{\pi t} + 2}{t^2(e^{\pi t}-1)}\\ &= \frac{e^{2\pi t} - 2\pi t e^{\pi t} - 1}{2t^2(e^{2\pi t}-1)}.\end{aligned}$$

In fact, there is another way by which we may directly derive $\eta_t(2)$. In this case, we return to the Fourier series for periodic functions. Consider the 2π-periodic even function on the interval $[-\pi,\pi]$ defined by

$$f(\theta) = \begin{cases} e^{-t\theta}, & [0,\pi],\\ e^{t\theta}, & [-\pi,0).\end{cases}$$

The Fourier coefficient of this function is

$$\begin{aligned}\hat{f}(n) &= \frac{1}{2\pi}\int_0^{\pi} e^{-t\theta}e^{-in\theta}\,d\theta + \frac{1}{2\pi}\int_{-\pi}^{0} e^{t\theta}e^{-in\theta}\,d\theta\\ &= \frac{1}{2\pi}\left[\frac{e^{-(t+in)\theta}}{-(t+in)}\right]_0^{\pi} + \frac{1}{2\pi}\left[\frac{e^{(t-in)\theta}}{t-in}\right]_{-\pi}^{0}\\ &= \frac{1}{2\pi}\left(\frac{e^{-t\pi}e^{-in\pi}}{-(t+in)} + \frac{1}{t+in}\right) + \frac{1}{2\pi}\left(\frac{1}{t-in} - \frac{e^{-t\pi}e^{in\pi}}{t-in}\right)\\ &= \frac{\big(1-e^{-t\pi}(-1)^n\big)t}{\pi(n^2+t^2)}.\end{aligned}$$

Fortunately, this holds for each n, so we don't need to separate the case when $n=0$.

$$\begin{aligned}f(\theta) &\sim \sum_{n\neq 0}\frac{\big(1-e^{-t\pi}(-1)^n\big)t}{\pi(n^2+t^2)} + \frac{1-e^{-t\pi}}{\pi t}\\ &= \sum_{n=1}^{\infty}\frac{\big(1-e^{-t\pi}(-1)^n\big)t}{\pi(n^2+t^2)}\cdot 2\cos(n\theta) + \frac{1-e^{-t\pi}}{\pi t}.\end{aligned}$$

Let $\theta = \frac{\pi}{2}$, then

$$\begin{aligned} 2\sum_{n=1}^{\infty}\frac{1-e^{-t\pi}(-1)^n}{\pi(n^2+t^2)}\cos(\frac{n\pi}{2}) &= \frac{e^{-\frac{t\pi}{2}}}{t} - \frac{1-e^{-t\pi}}{\pi t^2}\\ 2\sum_{n=1}^{\infty}\frac{1-e^{-t\pi}}{\pi((2n)^2+t^2)}\cos(n\pi) &= \frac{e^{-\frac{t\pi}{2}}}{t} - \frac{1-e^{-t\pi}}{\pi t^2}\\ \sum_{n=1}^{\infty}\frac{1-e^{-t\pi}}{(n^2+(\frac{t}{2})^2)}(-1)^n &= 2\pi\Big(\frac{e^{-\frac{t\pi}{2}}}{t} - \frac{1-e^{-t\pi}}{\pi t^2}\Big)\\ \Rightarrow \qquad \eta_{\frac{t}{2}}(2) &= \frac{2\pi}{1-e^{-t\pi}}\Big(\frac{e^{-\frac{t\pi}{2}}}{t} - \frac{1-e^{-t\pi}}{\pi t^2}\Big) = \frac{2(e^{t\pi} - t\pi e^{\frac{t\pi}{2}} - 1)}{t^2(e^{t\pi}-1)}.\end{aligned}$$

Substitute t for $\frac{t}{2}$, we will obtain the expression for $\eta_t(2)$:

$$\eta_t(2) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(n^2+t^2)} = \frac{e^{2\pi t} - 2\pi t e^{\pi t} - 1}{2t^2(e^{2\pi t}-1)}. \tag{4}$$

Finally, we come to the shifted value of the Lambda function.

$$\begin{aligned}
\lambda_t(2) &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2+t^2} \\
&= \frac{1}{2}(\zeta_t(2)+\eta_t(2)) \\
&= \frac{1}{2}\left(\frac{t\pi(e^{2\pi t}+1)-e^{2\pi t}+1}{2t^2(e^{2\pi t}-1)} + \frac{e^{2\pi t}-2\pi t e^{\pi t}-1}{2t^2(e^{2\pi t}-1)}\right) \\
&= \frac{\pi e^{2\pi t}-2\pi e^{\pi t}+\pi}{4t(e^{2\pi t}-1)}. \qquad (5)
\end{aligned}$$

3.4 The value of $\zeta(2), \eta(2)$ and $\lambda(2)$ as a limit

In fact, the function $\sum_{n=1}^{\infty} 1/(n^2+t^2)$ is continuous on $\mathbb{R}$. Thus, we can check our result by letting $t \to 0$, expecting those values tend exactly to the common Dirichlet series. The computation is rather simple:

Let's come first with the $\zeta_t(2)$.

$$\begin{aligned}
\lim_{t\to 0}\zeta_t(2) &= \lim_{t\to 0}\frac{t\pi(e^{2\pi t}+1)-e^{2\pi t}+1}{2t^2(e^{2\pi t}-1)} \\
&= \lim_{t\to 0}\frac{\pi(e^{2\pi t}+1)+2\pi^2 t e^{2\pi t}-2\pi e^{2\pi t}}{4t(e^{2\pi t}-1)+4\pi t^2 e^{2\pi t}} \\
&= \lim_{t\to 0}\frac{4\pi^3 t e^{2\pi t}}{4(e^{2\pi t}-1)+16\pi t e^{2\pi t}+8\pi^2 t^2 e^{2\pi t}} \\
&= \lim_{t\to 0}\frac{4\pi^3 e^{2\pi t}+8\pi^4 t e^{2\pi t}}{24\pi e^{2\pi t}+48\pi^2 t e^{2\pi t}+16\pi^3 t^2 e^{2\pi t}} \\
&= \frac{\pi^2}{6}.
\end{aligned}$$

Where we have used the L'Hospital's rule three times. Similarly,

$$\begin{aligned}
\lim_{t\to 0}\eta_t(2) &= \lim_{t\to 0}\frac{e^{2\pi t}-2\pi t e^{\pi t}-1}{2t^2(e^{2\pi t}-1)} \\
&= \lim_{t\to 0}\frac{2\pi e^{2\pi t}-2\pi e^{\pi t}-2\pi^2 t e^{\pi t}}{4t(e^{2\pi t}-1)+4\pi t^2 e^{2\pi t}} \\
&= \lim_{t\to 0}\frac{4\pi^2 e^{2\pi t}-4\pi^2 e^{\pi t}-2\pi^3 t e^{\pi t}}{4(e^{2\pi t}-1)+16\pi t e^{2\pi t}+8\pi^2 t^2 e^{2\pi t}} \\
&= \lim_{t\to 0}\frac{8\pi^3 e^{2\pi t}-6\pi^3 e^{\pi t}-2\pi^4 t e^{2\pi t}}{24\pi e^{2\pi t}+48\pi^2 t e^{2\pi t}+16\pi^3 t^2 e^{2\pi t}} \\
&= \frac{\pi^2}{12},
\end{aligned}$$

and

$$\begin{aligned}\lim_{t\to 0}\lambda_t(2) &= \lim_{t\to 0}\frac{\pi e^{2\pi t}-2\pi e^{\pi t}+\pi}{4t(e^{2\pi t}-1)}\\ &= \lim_{t\to 0}\frac{2\pi^2 e^{2\pi t}-2\pi^2 e^{\pi t}}{4(e^{2\pi t}-1)+8\pi t e^{2\pi t}}\\ &= \lim_{t\to 0}\frac{4\pi^3 e^{2\pi t}-2\pi^3 e^{\pi t}}{16\pi e^{2\pi t}+16\pi^2 t e^{2\pi t}}\\ &= \frac{\pi^2}{8}.\end{aligned}$$

All these limits agree to the original Dirichlet series. From another point of view, we have found a new way to evaluate the Dirichlet series at 2, that is, by seeing them as the limits of the shifted values.

3.5 The shifted zeta function at 2^k

In this subsection, we are going to present a method to calculate the shifted zeta function at natural powers of 2 and $t \in \mathbb{R}$. In fact, we extend our definition of the shifted value as followed.

Definition 3.2 For $s > 1$ and $t \in \mathbb{C}$, the type 1 shifted zeta function by t is defined as

$$\zeta_t^{+}(s) = \sum_{n=1}^{\infty}\frac{1}{n^s+t^s}$$

and the type 2 shifted zeta function by t is defined as

$$\zeta_t^{-}(s) = \sum_{n=1}^{\infty}\frac{1}{n^s-t^s}.$$

For convenience, we may omit the + in $\zeta_t^{+}(s)$.

Our first step forward is to extend our previous formula

$$\zeta_t(2) = \sum_{n=1}^{\infty}\frac{1}{n^2+t^2} = \frac{t\pi(e^{2\pi t}+1)-e^{2\pi t}+1}{2t^2(e^{2\pi t}-1)} \quad , \quad t>0$$

to $t \in \mathbb{C}$. However, viewed as a function in t for $t \in \mathbb{C}$, the right hand side above is a meromorphic function except for poles at $\pm i, \pm 2i, \cdots$. Since the series on the left also converges absolutely except at these points, it follows immediately from analytic continuation that the equation hold for all $t \in \mathbb{C}$ which is not a nonzero multiple of i. Substitute it for t where t is not a nonzero integer in the above equation, we obtain

$$\zeta_t^{-}(2) = \frac{1}{2t^2} - \frac{\pi}{2t\tan \pi t}. \tag{6}$$

Now, we are able to obtain a recurrence formula for $\zeta_t^{-}(2^k)$.

$$\begin{aligned}\zeta_t^{-}(2^{k+1}) &= \sum_{n=1}^{\infty}\frac{1}{n^{2^{k+1}}-t^{2^{k+1}}}\\ &= \frac{1}{2t^{2^k}}\sum_{n=1}^{\infty}\Big(\frac{1}{n^{2^k}-t^{2^k}}-\frac{1}{n^{2^k}+t^{2^k}}\Big)\\ &= \frac{1}{2t^{2^k}}\Big(\zeta_t^{-}(2^k)-\zeta_{\omega_k(t)}^{-}(2^k)\Big).\end{aligned} \tag{7}$$

where $\omega_k(t) = e^{(i\frac{\pi}{2^k})}t$ and t is not a nonzero multiple of any number in the set $\{e^{i\frac{\pi}{2^n}} \mid n = 0, 1, 2, \cdots, k\}$. We've also used the simple fact that

$$\zeta_t(2^k) = \zeta_{\omega_k(t)}^{-}(2^k), \quad \text{for} \quad k \in \mathbb{N}^*.$$

In fact, the recurrence formula

1. $$\zeta_t^-(2^{k+1}) = \frac{1}{2t^{2^k}}\left(\zeta_t^-(2^k) - \zeta_{\omega_k(t)}^-(2^k)\right)$$

2. $$\zeta_t^-(2) = \frac{1}{2t^2} - \frac{\pi}{2t\tan \pi t} \quad \text{(initial condition)} \tag{8}$$

holds for all complex t that is not a nonzero multiple of any number in the set $\{e^{i\frac{\pi}{2^n}} \mid n = 0, 1, 2, \cdots, k\}$. Hence, we may compute the value of $\zeta_t^-(2^k)$ for all real numbers t that is not a nonzero integer(notice that the poles of the shifted zeta function of type 2 are exactly the nonzero integers). To compute $\zeta_t(2^k)$, use the identity $\zeta_t(2^k) = \zeta_{\omega_k(t)}^-(2^k)$.

3.6 The shifted zeta function at even integers

Now, we are heading for our mission stated at the beginning of this section, to find the shifted zeta function at $s = 2k$ and $t \in \mathbb{R}$. Previously, we used Fourier series to evaluate the shifted eta function at $s = 2$; however, for $s > 2$, the Fourier series is no longer useful. In the general case, we turn to the application of the Poisson summation formula, which requires us to know at first the Fourier transform of a function. A simple yet powerful tool we will use throughout the rest of this paper, and in particular, to compute a wide range of Fourier transform at this moment, is the residue theorem stated as follows:

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{N} \mathrm{res}_{z_k} f, \quad \text{when the orientation of } \gamma \text{ is positive;}$$

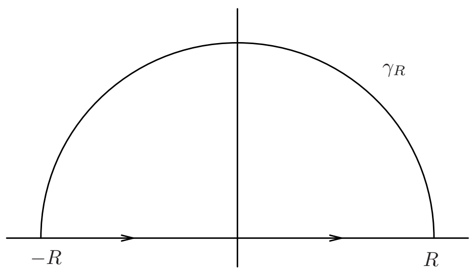
$$\int_\gamma f(z)\,dz = -2\pi i \sum_{k=1}^{N} \mathrm{res}_{z_k} f, \quad \text{when the orientation of } \gamma \text{ is negative.}$$

Here, positive means counterclockwise and negative means clockwise.

Now, we'll move on to the calculation of

$$\int_{-\infty}^{\infty} \frac{1}{x^{2k} + y^{2k}}\, e^{-2\pi i x\xi}\, dx, \quad y, \xi \in \mathbb{R}.$$

Consider the function $f(z) = \dfrac{1}{z^{2k} + y^{2k}}\, e^{-2\pi i z\xi}$ and choose the contour consisting of an oriented semicircle in the upper half plane for $\xi < 0$.



The oriented semicircle in the upper half plane

Figure 1

Denote the half circle by γ_R, and we will find that in the upper half plane, the poles of f are $z = y\, e^{\frac{(2n-1)}{2k}\, i\pi}$ $, n \in \{1, 2, \cdots, k\}$. Let $\omega_n = y\, e^{\frac{(2n-1)}{2k}\, i\pi}$. By the residue formula:

$$\int_{-R}^{R} \frac{1}{x^{2k} + y^{2k}}\, e^{-2\pi i x\xi}\, dx + \int_{\gamma_R} f(z)\, dz = 2\pi i \sum_{n=1}^{k} \mathrm{res}_{\omega_n} f.$$

(1). Consider the second integral on the left:

$$\begin{aligned}\Big|\int_{\gamma_R} f(z)\, dz\Big| &= \Big|\int_0^{\pi} \frac{e^{-2\pi i(R\cos\theta + iR\sin\theta)\xi}}{(Re^{i\theta})^{2k} + y^{2k}}\, (iRe^{i\theta}) d\theta\Big| \\ &\leq \int_0^{\pi} \Big|\frac{e^{2\pi\xi R\sin\theta}}{R^{2k-1} + O(\frac{1}{R})}\Big|\, d\theta \\ &\leq \frac{\pi}{R^{2k-1} + O(\frac{1}{R})}.\end{aligned}$$

Let R tend to infinity, then this integral clearly tends to zero.

(2). To calculate the residue, note that:

$$(z - \omega_n) f(z) = \frac{z - \omega_n}{z^{2k} + y^{2k}}\, e^{-2\pi i z\xi}$$

and

$$\lim_{z\to\omega_n} \frac{z - \omega_n}{z^{2k} + y^{2k}}\, e^{-2\pi i z\xi} = \lim_{z\to\omega_n} \frac{e^{-2\pi i z\xi} - 2\pi i\xi\, e^{-2\pi i z\xi}(z - \omega_n)}{2kz^{2k-1}} = \frac{e^{-2\pi i\omega_n\xi}}{2k\omega_n^{2k-1}}.$$

In fact, f have poles of order 1 (simple poles):

$$\mathrm{res}_{\omega_n} f = \frac{e^{-2\pi i\omega_n\xi}}{2k\omega_n^{2k-1}}.$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{1}{x^{2k} + y^{2k}}\, e^{-2\pi i x\xi}\, dx = 2\pi i \sum_{n=1}^{k} \mathrm{res}_{\omega_n} f = \frac{i\pi}{k} \sum_{n=1}^{k} \frac{e^{-2\pi i\omega_n\xi}}{\omega_n^{2k-1}}.$$

as the radius R tends to infinity.

For $\xi > 0$, we use the negatively oriented semicircle in the lower half plane. The calculation is similar and the trivial differences are that:
(1). $z(\theta) = Re^{-i\theta}$ with regard to the negative orientation.
(2). The poles of f become $\overline{\omega}_n$. So we jump to conclusion that:

$$\int_{-\infty}^{\infty} \frac{1}{x^{2k} + y^{2k}}\, e^{-2\pi i x\xi}\, dx = -\frac{i\pi}{k} \sum_{n=1}^{k} \frac{e^{-2\pi i\overline{\omega}_n\xi}}{\overline{\omega}_n^{2k-1}}.$$

Let's see what can be done further. Denote $\overline{\omega}_n$ by $a_n - b_n i$. Take the complex conjugate of the right hand side of the above equation.

$$\begin{aligned}\overline{-\frac{i\pi}{k} \sum_{n=1}^{k} \frac{e^{-2\pi i\overline{\omega}_n\xi}}{\overline{\omega}_n^{2k-1}}} &= \frac{i\pi}{k} \sum_{n=1}^{k} \frac{\overline{e^{-2\pi i(a_n - b_n i)\xi}}}{\omega_n^{2k-1}} \\ &= \frac{i\pi}{k} \sum_{n=1}^{k} \frac{e^{2\pi i(a_n + b_n i)\xi}}{\omega_n^{2k-1}} \\ &= \frac{i\pi}{k} \sum_{n=1}^{k} \frac{e^{2\pi i\omega_n\xi}}{\omega_n^{2k-1}}.\end{aligned}$$

And we also known that the Fourier transform of a real function is real. Hence,

$$-\frac{i\pi}{k}\sum_{n=1}^{k}\frac{e^{-2\pi i\overline{\omega}_n\xi}}{\overline{\omega}_n^{2k-1}}=\frac{i\pi}{k}\sum_{n=1}^{k}\frac{e^{2\pi i\omega_n\xi}}{\omega_n^{2k-1}}.$$

To sum up with,

$$\int_{-\infty}^{\infty}\frac{1}{x^{2k}+y^{2k}}\,e^{-2\pi ix\xi}\,dx=\frac{i\pi}{k}\sum_{n=1}^{k}\frac{e^{2\pi i\omega_n|\xi|}}{\omega_n^{2k-1}}\quad\text{where}\quad \omega_n=ye^{\frac{2n-1}{2k}i\pi}. \tag{9}$$

Indeed, the validity of the above result can also be checked by an elementary way using the Fourier inversion. Let's see how this can be done.

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{i\pi}{k}\sum_{n=1}^{k}\frac{e^{2\pi i\omega_n|\xi|}}{\omega_n^{2k-1}}\,e^{2\pi ix\xi}\,d\xi&=\frac{i\pi}{k}\sum_{n=1}^{k}\Big(\int_{0}^{\infty}\frac{e^{2\pi i(x+\omega_n)\xi}}{\omega_n^{2k-1}}\,d\xi+\int_{-\infty}^{0}\frac{e^{2\pi i(x-\omega_n)\xi}}{\omega_n^{2k-1}}\,d\xi\Big)\\&=\frac{i\pi}{k}\sum_{n=1}^{k}\frac{1}{\omega_n^{2k-1}}\Big(\frac{e^{2\pi i(x+\omega_n)\xi}}{2\pi i(x+\omega_n)}\Big|_{0}^{\infty}+\frac{e^{2\pi i(x-\omega_n)\xi}}{2\pi i(x-\omega_n)}\Big|_{-\infty}^{0}\Big)\\&=\frac{i\pi}{k}\sum_{n=1}^{k}\frac{1}{\omega_n^{2k-1}}\Big(-\frac{1}{2\pi i(x+\omega_n)}+\frac{1}{2\pi i(x-\omega_n)}\Big)\\&=\frac{1}{k}\sum_{n=1}^{k}\frac{1}{\omega_n^{2k-2}(x^2-{\omega_n}^2)}.\end{aligned}$$

$$\begin{aligned}\frac{1}{k}\sum_{n=1}^{k}\frac{1}{\omega_n^{2k-2}(x^2-\omega_n^2)}&=\frac{1}{k}\sum_{n=1}^{k}\frac{1}{\omega_n^{2k-2}x^2-\omega_n^{2k}}\\&=\frac{1}{k}\sum_{n=1}^{k}\frac{1}{x^2e^{\frac{(k-1)(2n-1)}{k}i\pi}y^{2k-2}-e^{(2n-1)i\pi}y^{2k}}\\&=\frac{1}{ky^{2k-2}}\sum_{n=1}^{k}\frac{1}{y^2-x^2e^{\frac{1-2n}{k}i\pi}}\\&=\frac{1}{ky^{2k-2}}\sum_{n=1}^{k}\frac{1}{y^2-x^2e^{\frac{2(k-n+1)-1}{k}i\pi}}\\&=\frac{1}{ky^{2k-2}}\sum_{n=1}^{k}\frac{1}{y^2-x^2e^{\frac{2n-1}{k}i\pi}}\\&=\frac{1}{kx^2y^{2k-2}}\sum_{n=1}^{k}\frac{1}{\frac{y^2}{x^2}-e^{\frac{2n-1}{k}i\pi}}.\end{aligned}$$

Substitute s for $\dfrac{y^2}{x^2}$ in the sum part,

$$\frac{1}{kx^2y^{2k-2}}\sum_{n=1}^{k}\frac{1}{s-e^{\frac{2n-1}{k}i\pi}}=\frac{1}{kx^2y^{2k-2}}\frac{\sum_{r=1}^{k}\prod_{n\neq r}(s-e^{\frac{2n-1}{k}i\pi})}{\prod_{n=1}^{k}(s-e^{\frac{2n-1}{k}i\pi})}.$$

Assume

$$\prod_{n=1}^{k}(s-e^{\frac{2n-1}{k}i\pi})=A_1s^k+A_2s^{k-1}+\cdots+A_ks+A_{k+1},$$

$$\sum_{r=1}^{k}\prod_{n\neq r}(s-e^{\frac{2n-1}{k}i\pi})=B_1s^{k-1}+B_2s^{k-2}+\cdots+B_{k-1}s+B_k.$$

A careful observation of their binomial expansions will tell:

$$B_n = \frac{kC_{k-1}^{n-1}}{C_k^{n-1}} A_n = (k-n+1)A_n, \qquad n \in \{1, 2, \cdots, k\}.$$

Also notice that $e^{\frac{2n-1}{k}i\pi}$ is the root of unity of $x^k + 1 = 0$. Hence,

$$\prod_{n=1}^{k}(s - e^{\frac{2n-1}{k}i\pi}) = A_1 s^k + A_2 s^{k-1} + \cdots + A_k s + A_{k+1} = s^k + 1$$

$$\Rightarrow \qquad B_1 = k, \qquad B_2 = B_3 = \cdots = B_k = 0.$$

Therefore,

$$\begin{aligned}\frac{1}{kx^2y^{2k-2}} \sum_{n=1}^{k} \frac{1}{s - e^{\frac{2n-1}{k}i\pi}} &= \frac{1}{kx^2y^{2k-2}} \frac{\sum_{r=1}^{k} \prod_{n\neq r}(s - e^{\frac{2n-1}{k}i\pi})}{\prod_{n=1}^{k}(s - e^{\frac{2n-1}{k}i\pi})} \\ &= \frac{1}{kx^2y^{2k-2}} \frac{ks^{k-1}}{s^k+1} \\ &= \frac{1}{x^{2k}+y^{2k}}.\end{aligned}$$

Which completes the proof.

Apply the Poisson summation formula to (9). Then,

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{1}{n^{2k}+y^{2k}} &= \frac{i\pi}{k} \sum_{n=1}^{k} \sum_{r=-\infty}^{\infty} \frac{e^{2\pi i \omega_n |r|}}{\omega_n^{2k-1}} \\ &= \frac{i\pi}{k} \sum_{n=1}^{k} \Big(\frac{2}{1-e^{2\pi i \omega_n}} - 1\Big)\Big(\frac{1}{\omega_n^{2k-1}}\Big) \\ &= \frac{i\pi}{k} \sum_{n=1}^{k} \frac{1+e^{2\pi i \omega_n}}{(1-e^{2\pi i \omega_n})\omega_n^{2k-1}}.\end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}+y^{2k}} = \frac{1}{2}\Big(\frac{i\pi}{k} \sum_{n=1}^{k} \frac{1+e^{2\pi i \omega_n}}{(1-e^{2\pi i \omega_n})\omega_n^{2k-1}} - \frac{1}{y^{2k}}\Big). \tag{10}$$

This is our formula for the type 1 shifted zeta function at even integers.

Indeed, from (9) we may derive that for $0 \leq a < 1$,

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i a x}}{x^{2k}+y^{2k}} \, e^{-2\pi i x \xi} \, dx = \frac{i\pi}{k} \sum_{n=1}^{k} \frac{e^{2\pi i \omega_n |\xi + a|}}{\omega_n^{2k-1}}.$$

Applying the Poisson summation formula yields

$$\sum_{m=-\infty}^{\infty} \frac{e^{-2\pi i a m}}{m^{2k}+y^{2k}} = \frac{i\pi}{k} \sum_{n=1}^{k} \sum_{m=-\infty}^{\infty} \frac{e^{2\pi i \omega_n |m+a|}}{\omega_n^{2k-1}} = \frac{i\pi}{k} \sum_{n=1}^{k} \frac{e^{2\pi i \omega_n a} + e^{-2\pi i \omega_n a} e^{2\pi i \omega_n}}{(1-e^{2\pi i \omega_n})\omega_n^{2k-1}}. \tag{11}$$

Since the leftmost term equals $2\sum_{n=1}^{\infty} \frac{\cos(2\pi a n)}{n^{2k}+y^{2k}} + \frac{1}{y^{2k}}$. Setting $a = 1/q$, we will obtain a shifted zeta function modulo q.

The value of the type 2 shifted zeta function follows from analytic continuation. However, we want to derive these values directly, which in fact require some more effort. The difficulty lies in the fact that $g(x) = 1/(x^{2k} - y^{2k})$ is not of moderate decrease, and hence, using Fourier transform and Poisson summation is questionable. However, by applying a few tricks in contour integration, we are able to overcome these problems.

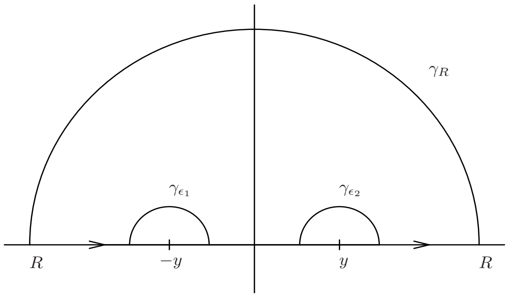


Figure 2

Let $f(z) = e^{-2\pi iz\xi}/(z^{2k} - y^{2k})$. For $\xi < 0$, integrate this function along the contour in figure 2, which consists of a big semicircle γ_R with radius R, centered at origin; two small semicircles γ_{ϵ_1} and γ_{ϵ_2}, with radius ϵ_1 and ϵ_2, centered at $-y$ and y, respectively; and finally, three line segments along the real axis.

Hence

$$\Big(\int_{-R}^{-y-\epsilon_1} + \int_{-y+\epsilon_1}^{y-\epsilon_2} + \int_{y+\epsilon_2}^{R}\Big) \frac{e^{-2\pi ix\xi}}{x^{2k} - y^{2k}}\, dx + \int_{\gamma_{\epsilon_1}} f(z)\, dz + \int_{\gamma_{\epsilon_2}} f(z)\, dz + \int_{\gamma_R} f(z)\, dz = 2\pi i \sum res_{\omega_n} f.$$

Where $\omega_n = ye^{\frac{n}{k}i\pi}$, $n \in \{1, 2, \cdots, k-1\}$. The residue can be computed as before. Thus,

$$2\pi i \sum res_{\omega_n} f = \frac{i\pi}{k} \sum_{n=1}^{k-1} \frac{e^{-2\pi i\omega_n \xi}}{\omega_n^{2k-1}}.$$

Also,

$$\Big|\int_{\gamma_R} f(z)\, dz\Big| \to 0 \quad \text{as} \quad R \to \infty.$$

What's new about the contour is the two small semicircles γ_{ϵ_1} and γ_{ϵ_2}. First look at the integration alone γ_{ϵ_1} , which is parametrized by $z = -y + \epsilon_1 e^{i\theta}$.

$$\int_{\gamma_{\epsilon_1}} f(z)\, dz = \int_{\pi}^{0} \frac{e^{-2\pi i(-y+\epsilon_1 e^{i\theta})}}{(-y + \epsilon_1 e^{i\theta})^{2k} - y^{2k}}\, i\epsilon_1 e^{i\theta}\, d\theta.$$

Let ϵ_1 tends to 0, then the above integral tends to

$$\begin{aligned}
\lim_{\epsilon_1 \to 0} \int_{\gamma_{\epsilon_1}} f(z)\, dz &= \lim_{\epsilon_1 \to 0} \int_{\pi}^{0} \frac{e^{-2\pi i(-y+\epsilon_1 e^{i\theta})}}{(-y + \epsilon_1 e^{i\theta})^{2k} - y^{2k}}\, i\epsilon_1 e^{i\theta}\, d\theta \\
&= i \int_{\pi}^{0} e^{2\pi iy\xi} \Big(\lim_{\epsilon_1 \to 0} \frac{\epsilon_1 e^{i\theta}}{(-y + \epsilon_1 e^{i\theta})^{2k} - y^{2k}}\Big)\, d\theta \\
&= i \int_{0}^{\pi} \frac{e^{2\pi iy\xi}}{2ky^{2k-1}}\, d\theta \\
&= \frac{i\pi e^{2\pi iy\xi}}{2ky^{2k-1}}.
\end{aligned}$$

We may exchange the limit with the integration since the integrand is bounded and integrable for small ϵ_1. Similarly, the integral along γ_{ϵ_2} tends to $-\dfrac{i\pi e^{-2\pi iy\xi}}{2ky^{2k-1}}$.

Therefore, for $\xi < 0$

$$\mathrm{PV}\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{x^{2k}-y^{2k}}\,dx = 2\pi i \sum res_{\omega_n} f - \Big(\int_{\gamma_{\epsilon_1}} f(z)\,dz + \int_{\gamma_{\epsilon_2}} f(z)\,dz + \int_{\gamma_R} f(z)\,dz\Big)$$
$$= \frac{i\pi}{k}\sum_{n=1}^{k-1} \frac{e^{-2\pi i\omega_n\xi}}{\omega_n^{2k-1}} + \frac{\pi\sin(2\pi y\xi)}{ky^{2k-1}}.$$

Note that in the above equation, PV stands for **Cauchy principal value**, and is defined by

$$\mathrm{PV}\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{x^{2k}-y^{2k}}\,dx = \lim_{\substack{\epsilon_1,\epsilon_2\to 0+\\ R\to\infty}} \Big(\int_{-R}^{-y-\epsilon_1} + \int_{-y+\epsilon_1}^{y-\epsilon_2} + \int_{y+\epsilon_2}^{R}\Big) \frac{e^{-2\pi i x\xi}}{x^{2k}-y^{2k}}\,dx.$$

A similar approach for $\xi > 0$, except that we use the contour in the lower plane by symmetry, yields that:

$$\mathrm{PV}\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{x^{2k}-y^{2k}}\,dx = -\frac{i\pi}{k}\sum_{n=1}^{k-1} \frac{e^{-2\pi i\overline{\omega}_n\xi}}{\overline{\omega}_n^{2k-1}} - \frac{\pi\sin(2\pi y\xi)}{ky^{2k-1}}.$$

To sum up, for all $\xi \in \mathbb{R}$

$$\mathrm{PV}\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{x^{2k}-y^{2k}}\,dx = \frac{i\pi}{k}\sum_{n=1}^{k-1} \frac{e^{2\pi i\omega_n|\xi|}}{\omega_n^{2k-1}} - \frac{\pi\sin(2\pi y|\xi|)}{ky^{2k-1}}. \tag{12}$$

These integrals converge in terms of their Cauchy principal values. But neither could be identified as a 'Fourier transform' because the integrand is not of moderate decrease(they are not even Riemann integrable). Indeed, neither Fourier inversion nor Poisson summation formula holds in this case, which is an evident fact since the $\sin z$ function oscillates rapidly when z tends to infinity along the real axis. So our next mission is to find a summation formula that works.

We will use a similar approach to that in the proof of the Poisson summation formula demonstrated in Book [2], page $118-119$.

Construct a function $h(z) = 1/((z^{2k}-y^{2k})(e^{2\pi iz}-1))$, $y \in \mathbb{R}$. The poles of this function in the complex plane consists of all integers and the zeros of $z^{2k}-y^{2k}$. Choose the rectangle contour γ_N shown in figure 3, of length $2N+1$ and width $2b$.

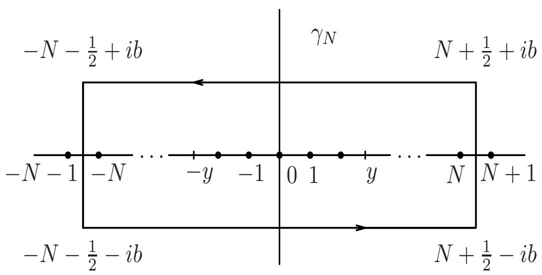


Figure 3

Let b arbitrarily small so that the contour doesn't contain any other zeros of $z^{2k}-y^{2k}$ beside y and $-y$, which are on the real axis. First, let's compute the residue.

1) $res_n h = \lim_{z\to n} \dfrac{z-n}{(z^{2k}-y^{2k})(e^{2\pi iz}-1)} = \dfrac{1}{2\pi i(n^{2k}-y^{2k})}.$

2) $res_y h = \lim_{z\to y} \dfrac{z-y}{(z^{2k}-y^{2k})(e^{2\pi iz}-1)} = \dfrac{1}{2ky^{2k-1}(e^{2\pi iy}-1)}$ and $res_{-y} h = \dfrac{1}{-2ky^{2k-1}(e^{-2\pi iy}-1)}.$

Hence,

$$\sum_{|n|\le N} \frac{1}{n^{2k}-y^{2k}} + \frac{i\pi}{ky^{2k-1}}\Big(\frac{1}{e^{2\pi iy}-1} - \frac{1}{e^{-2\pi iy}-1}\Big) = \int_{\gamma_N} h(z)\,dz.$$

Let N tends to infinity, the first sum on the left hand side becomes $\sum_{n=-\infty}^{\infty} g(n)$, where $g(z) = 1/(z^{2k} - y^{2k})$. Consider the integral along the left vertical side,

$$\Big|\int_{N-\frac{1}{2}+ib}^{N-\frac{1}{2}-ib} h(z)\,dz\Big| \le \int_{-b}^{b} \frac{1}{|(-N-\frac{1}{2}-it)^{2k} - y^{2k}| \cdot |e^{2\pi i(-N-\frac{1}{2}-it)}|}\,dt$$
$$\le \int_{-b}^{b} \frac{A}{|(-N-\frac{1}{2}-it)^{2k} - y^{2k}|}\,dt,$$

from the fact that $1/(e^{2\pi iz} - 1)$ is bounded for $\mathrm{Re}(z) = n + \frac{1}{2}$. The integral tends to zero as N tends to infinity, so does that along the right side. Therefore,

$$\sum_{n=-\infty}^{\infty} g(n) + \frac{i\pi}{ky^{2k-1}}\Big(\frac{1}{e^{2\pi iy}-1} - \frac{1}{e^{-2\pi iy}-1}\Big) = \int_{L_1} h(z)\,dz - \int_{L_2} h(z)\,dz. \tag{13}$$

Where L_1 and L_2 represent the real line shifted by $-b$ and b respectively, both oriented from left to right.

Notice that $h(z) = \dfrac{g(z)}{e^{2\pi iz}-1}$. On L_1, as $|e^{2\pi iz}| > 1$,

$$\frac{1}{e^{2\pi iz}-1} = e^{-2\pi iz}\sum_{n=0}^{\infty} e^{-2\pi inz},$$

and on L_2, as $|e^{2\pi iz}| < 1$,

$$\frac{1}{e^{2\pi iz}-1} = -\sum_{n=0}^{\infty} e^{2\pi inz}.$$

So that

$$\int_{L_1} h(z)\,dz - \int_{L_2} h(z)\,dz = \int_{L_1} g(z)e^{-2\pi iz}\sum_{n=0}^{\infty} e^{-2\pi inz}\,dz + \int_{L_2} g(z)\sum_{n=0}^{\infty} e^{2\pi inz}\,dz$$
$$= \sum_{n=0}^{\infty}\int_{L_1} g(z)e^{-2\pi i(n+1)z}\,dz + \sum_{n=0}^{\infty}\int_{L_2} g(z)e^{2\pi inz}\,dz. \tag{14}$$

To simplify the above, we need another observation. Integrate $f(z) = \dfrac{e^{-2\pi iz\xi}}{z^{2k} - y^{2k}}$, $\xi > 0$ over the contour shown below.

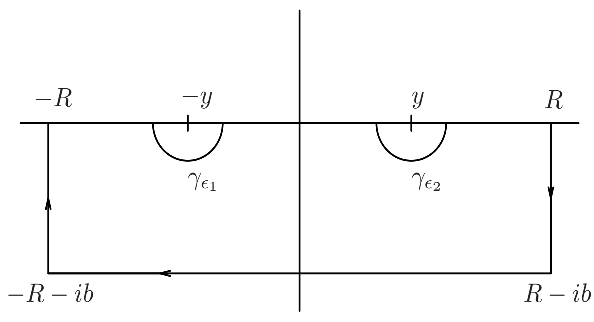


Figure 4

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The integral over the vertical sides tend to 0 as R tends to infinity. And the integral over the two semicircles sum up to $\pi \sin(2\pi y\xi)/ky^{2k-1}$ as ϵ_1 and ϵ_2 tend to 0. So we conclude that

$$\int_{L_1} f(z)\,dz = \mathrm{PV}\int_{-\infty}^{\infty} f(x)\,dx + \frac{\pi\sin(2\pi y\xi)}{ky^{2k-1}}.$$

For $\xi < 0$, integrate along the rectangle in the upper half plane by symmetry and we yield:

$$\int_{L_2} f(z)\,dz = \mathrm{PV}\int_{-\infty}^{\infty} f(x)\,dx - \frac{\pi\sin(2\pi y\xi)}{ky^{2k-1}}.$$

Combining this with (12) and (14), we find that the unpleasant 'sine's are cancelled.

$$\begin{aligned}
\int_{L_1} h(z)\,dz - \int_{L_2} h(z)\,dz &= \sum_{n=0}^{\infty}\int_{L_1} g(z)e^{-2\pi i(n+1)z}\,dz + \sum_{n=0}^{\infty}\int_{L_2} g(z)e^{2\pi inz}\,dz \\
&= \sum_{n=0}^{\infty}\Big(\mathrm{PV}\int_{-\infty}^{\infty} g(x)e^{-2\pi i(n+1)x}\,dx + \frac{\pi\sin(2\pi y(n+1))}{ky^{2k-1}}\Big) \\
&\quad + \sum_{n=0}^{\infty}\Big(\mathrm{PV}\int_{-\infty}^{\infty} g(x)e^{2\pi inx}\,dx - \frac{\pi\sin(2\pi y(-n))}{ky^{2k-1}}\Big) \\
&= \sum_{n=0}^{\infty}\left(\frac{i\pi}{k}\sum_{m=1}^{k-1}\frac{e^{2\pi i\omega_m(n+1)}}{\omega_m^{2k-1}} - \frac{\pi\sin(2\pi y(n+1))}{ky^{2k-1}} + \frac{\pi\sin(2\pi y(n+1))}{ky^{2k-1}}\right) \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{i\pi}{k}\sum_{m=1}^{k-1}\frac{e^{2\pi i\omega_m n}}{\omega_m^{2k-1}} - \frac{\pi\sin(2\pi yn)}{ky^{2k-1}} - \frac{\pi\sin(2\pi y(-n))}{ky^{2k-1}}\right) \\
&= \frac{i\pi}{k}\sum_{m=1}^{k-1}\frac{2\sum_{n=0}^{\infty} e^{2\pi i\omega_m n} - 1}{\omega_m^{2k-1}} \\
&= \frac{i\pi}{k}\sum_{n=1}^{k-1}\frac{1+e^{2\pi i\omega_n}}{(1-e^{2\pi i\omega_n})\omega_n^{2k-1}}.
\end{aligned}$$

By (13),

$$\sum_{n=-\infty}^{\infty} g(n) + \frac{i\pi}{ky^{2k-1}}\Big(\frac{1}{e^{2\pi iy}-1} - \frac{1}{e^{-2\pi iy}-1}\Big) = \frac{i\pi}{k}\sum_{n=1}^{k-1}\frac{1+e^{2\pi i\omega_n}}{(1-e^{2\pi i\omega_n})\omega_n^{2k-1}}. \tag{15}$$

But

$$\begin{aligned}
\frac{i\pi}{ky^{2k-1}}\Big(\frac{1}{e^{2\pi iy}-1} - \frac{1}{e^{-2\pi iy}-1}\Big) &= \frac{\pi}{ky^{2k-1}}\cdot\cot(\pi y) \\
&= -\frac{i\pi}{k}\cdot\frac{1+e^{2\pi i(-y)}}{(1-e^{2\pi i(-y)})(-y)^{2k-1}}.
\end{aligned}$$

Hence we may conclude that

$$\sum_{n=-\infty}^{\infty} g(n) = \frac{i\pi}{k}\sum_{n=1}^{k}\frac{1+e^{2\pi i\omega_n}}{(1-e^{2\pi i\omega_n})\omega_n^{2k-1}},$$

where $\omega_n = ye^{\frac{n}{k}i\pi}$. If we plug in $y = ye^{\frac{i\pi}{2k}}$, the above formula clearly agrees with the formula of $\zeta_y(2k)$,

$$\sum_{n=-\infty}^{\infty}\frac{1}{n^{2k}+y^{2k}} = \frac{i\pi}{k}\sum_{n=1}^{k}\frac{1+e^{2\pi i\omega_n}}{(1-e^{2\pi i\omega_n})\omega_n^{2k-1}},$$

where $\omega_n = ye^{\frac{2n-1}{2k}i\pi}$.

4 A summation formula for functions in $\mathcal{R}$

We define $\mathcal{R}$ to be the set of rational functions

$$R(z) = \frac{P(z)}{Q(z)},$$

where P and Q are polynomials with no common zeros such that $(Degree\ Q) \geq ((Degree\ P) + 2$, and Q's zeros are all distinct. Obviously,

$$g(z) = \frac{1}{z^{2k} - y^{2k}} \in \mathcal{R}.$$

In this section, we seek a general summation formula for $R(z) \in \mathcal{R}$. Such formula can be obtained through a similar argument in the previous section. However, we now present a simpler alternative.

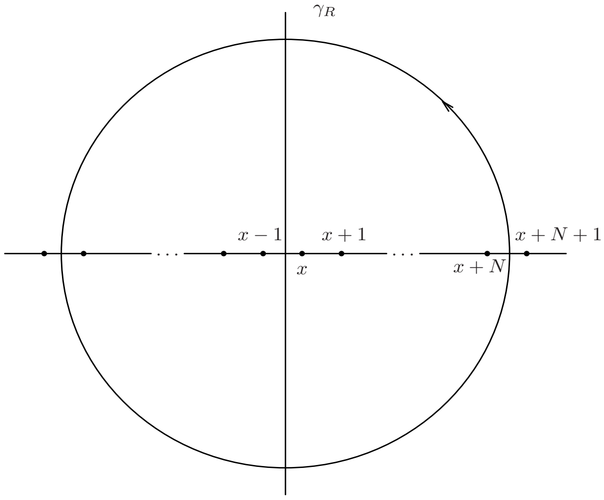


Figure 5

For $R \in \mathcal{R}$, construct a function $g(z) = R(z)/(e^{2\pi iz} - e^{2\pi ix})$ where $0 \leq x < 1$ is a real number such that $R(z)$ has no pole of the form $n + x$, $n \in \mathbb{Z}$.

Integrate $g(z)$ along the positive big circle γ_R. Choose an R so large that the contour contains all zeros of Q(recall that $R(z) = P(z)/Q(z)$) and does not pass through any point of $\{n + x : n \in \mathbb{Z}\}$.Note that the poles of g in the complex plane are $\{n + x : n \in \mathbb{Z}\}$ and the zeros of $Q(z)$.

(1). The residue of g at $\{n + x : n \in \mathbb{Z}\}$ is:

$$res_{(n+x)}g = \lim_{z \to (n+x)} \frac{(z - (n + x))R(z)}{e^{2\pi iz} - e^{2\pi ix}} = \frac{R(n + x)}{2\pi ie^{2\pi ix}}.$$

(2). The residue of g at Q's zeros is:

$$res_{\zeta}g = \lim_{z \to \zeta} \frac{P(z)(z - \zeta)}{Q(z)(e^{2\pi iz} - e^{2\pi ix})} = \frac{P(\zeta)}{Q'(\zeta)(e^{2\pi i\zeta} - e^{2\pi ix})}.$$

(3). Also, by analogue to the previous cases that

$$\begin{aligned}\Big|\int_{\gamma_R} g(z)\,dz\Big| &\le c_1 \int_{\gamma_R} \Big|\frac{P(z)}{Q(z)}\Big|\,dz \\ &\le c_2 \int_0^{2\pi} \Big|\frac{1}{R^2} R\Big|\,d\theta \\ &\to 0 \quad \text{as} \quad R \to \infty.\end{aligned}$$

Hence, by the residue formula:

$$\sum_{|n+x|\le R} \frac{P(n+x)}{Q(n+x)e^{2\pi i x}} = \int_{\gamma_R} g(z)\,dz - 2\pi i \sum_{\zeta} \frac{P(\zeta)}{Q'(\zeta)(e^{2\pi i\zeta} - e^{2\pi i x})}.$$

The right hand sum is taken over all zeros of Q, denoted by ζ . Let R tend to infinity,

$$\sum_{n\in\mathbb{Z}} \frac{P(n+x)}{Q(n+x)} = 2\pi i \sum_{\zeta} \frac{P(\zeta)}{Q'(\zeta)(1 - e^{2\pi i(\zeta - x)})}. \tag{16}$$

Which is our desired result.

We've seen that to apply this summation formula to some function in $\mathcal{R}$, it's not even necessary to compute its Fourier transform. This convenience was facilitated by the use of an auxiliary function whose poles are some(infinite) equidistant points on the real axis.

Translation and dilation

By this, we mean the sum of the form:

$$\sum_{n\in\mathbb{Z}} \frac{P(\delta n + x)}{Q(\delta n + x)}, \quad \text{for } \delta\,,\, x \in \mathbb{C}.$$

In fact, we simply use the auxiliary function

$$g(z) = \frac{1}{e^{\frac{2\pi i z}{\delta}} - e^{\frac{2\pi i x}{\delta}}}, \quad \text{for } \delta\,,\, x \in \mathbb{C}.$$

and integrate

$$f(z) = \frac{P(z)}{Q(z)(e^{\frac{2\pi i z}{\delta}} - e^{\frac{2\pi i x}{\delta}})}$$

along the previous contour. A similar manipulation yields

$$\sum_{n\in\mathbb{Z}} \frac{P(\delta n + x)}{Q(\delta n + x)} = \frac{2\pi i}{\delta} \sum_{\zeta} \frac{P(\zeta)}{Q'(\zeta)(1 - e^{\frac{2\pi i(\zeta - x)}{\delta}})}. \tag{17}$$

The sum on the right is taken over all zeros of $Q(z)$, denoted by ζ, which we assume is not of the form $\delta n + x$.

Transformations with respect to polynomials

Here we examine a more general summation:

$$\sum_{n\in\mathbb{Z}} \frac{P(F(n))}{Q(F(n))}, \quad \text{where } F \text{ is a polynomial.}$$

The proceedings are very straightforward, though. Let $f(z) = \dfrac{P(F(z))}{Q(F(z))(e^{2\pi i z} - 1)}$. Since F tends to infinity with $|z|$ at least as fast as $G(z) = z$,

$$\Big|\int_{\gamma_R} f(z)\,dz\Big| \to 0 \quad \text{as} \quad R \to \infty.$$

And by some simple computation on its residues, we find

$$\sum_{n\in\mathbb{Z}} \frac{P(F(n))}{Q(F(n))} = 2\pi i \sum_{\omega} \frac{P(F(\omega))}{F'(\omega)Q'(F(\omega))(1-e^{2\pi i\omega})}. \tag{18}$$

$\{\omega\}$ denotes the sets of points such that $F(\omega) = \zeta$, where $\{\zeta\}$ are the zeros of Q. We should also assume that $F'(\omega) \neq 0$. In fact, the translation and dilation is a special case of such transformation, where we've let $F(z) = \delta z + x$.

Summation modulo q

The identity (11), which is an example of the shifted zeta function modulo q reminded us with a simple question: given a sequence $\{a_n\}_{n=-\infty}^{\infty}$ such that $a_m = a_n$ if $m \equiv n (\bmod\ q)$, is there a formula for $\sum_{n=1}^{\infty} a_n P(n)/Q(n)$?

We say a sequence $\{a_n\}_{n=-\infty}^{\infty}$ to be odd if $a_{-m} = -a_m$ for $m \in \mathbb{Z}$, and it's even if $a_{-m} = a_m$. The answer is, there is a formula if the sequence $\{a_n\}_{n=-\infty}^{\infty}$ and the function $P(z)/Q(z)$ are both odd or even. In fact, if both the sequence and the function are odd or even, then we have (if assume $Q(0)$ is nonzero.)

$$\sum_{n=1}^{\infty} \frac{a_n P(n)}{Q(n)} = \frac{1}{2}\Big(\sum_{n=-\infty}^{\infty} \frac{a_n P(n)}{Q(n)} - \frac{a_0 P(0)}{Q(0)}\Big).$$

Then,we have

$$\sum_{n=-\infty}^{\infty} \frac{a_n P(n)}{Q(n)} = \sum_{i=1}^{q} \sum_{n=-\infty}^{\infty} \frac{a_i P(qn+i)}{Q(qn+i)}, \tag{19}$$

which could be summed using the translation and dilation formula (17).

5 Two special summations in the complex plane

This section provides some insight into the applications of the results we have obtained, i.e., the translation and dilation formula and the idea of auxiliary function.

5.1 Gradation

The title 'gradation' is not an intimidating mathematics terminology but merely a literal description of the pattern of the set of points:

$$\Lambda_1 = \{ni\pi + \log m : n \in \mathbb{Z}\,,\ m \in \mathbb{N}^*\}.$$

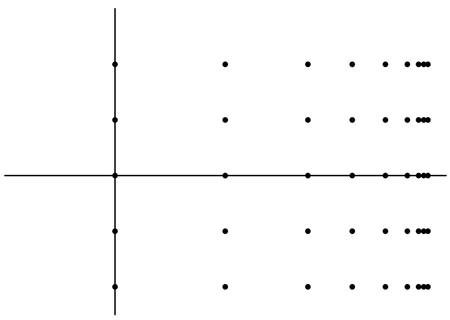


Figure 6

In fact, Λ_1 is the set of points z such that e^z is an integer. A figure of this lattice is given above and, hence its name.

Now, we are going to seek a formula for the sum

$$\sum_{z\in\Lambda_1} \frac{P(z)}{Q(z)}e^{-cz}, \quad \text{where } c \text{ is an integer} \geq 1 \text{ and } \frac{P(z)}{Q(z)} \in \mathcal{R}.$$

We restrict the order of summation to first vertically, then horizontally; this is important since the series itself doesn't converge absolutely. First we shall see that

$$\begin{aligned}\sum_{z\in\Lambda_1} \frac{P(z)}{Q(z)}e^{-cz} &= \sum_{n\in\mathbb{Z},m\in\mathbb{N}^*} \frac{P(i\pi n+\log m)}{Q(i\pi n+\log m)}\cdot\frac{(-1)^{cn}}{m^c} \\ &= \sum_{m=1}^{\infty}\frac{1}{m^c}\sum_{n=-\infty}^{\infty}\frac{(-1)^{cn}P(i\pi n+\log m)}{Q(i\pi n+\log m)}.\end{aligned}$$

Then, we separate two cases in terms of the parity of c.

1. If c is even, then $(-1)^{cn}=1$. Hence

$$\begin{aligned}\sum_{z\in\Lambda_1} \frac{P(z)}{Q(z)}e^{-cz} &= \sum_{m=1}^{\infty}\frac{1}{m^c}\sum_{n=-\infty}^{\infty}\frac{P(i\pi n+\log m)}{Q(i\pi n+\log m)} \\ &= \sum_{m=1}^{\infty}\frac{1}{m^c}\cdot\frac{2i\pi}{i\pi}\sum_{\omega}\frac{P(\omega)}{Q'(\omega)(1-e^{\frac{2i\pi(\omega-\log m)}{i\pi}})} \\ &= 2\sum_{m=1}^{\infty}\frac{1}{m^c}\sum_{\omega}\frac{P(\omega)}{Q'(\omega)(1-\frac{e^{2\omega}}{m^2})} \\ &= 2\sum_{\omega}\frac{P(\omega)}{Q'(\omega)}\sum_{m=1}^{\infty}\frac{1}{m^c-e^{2\omega}m^{c-2}}.\end{aligned}$$

where we have applied the translation and dilation formula, and $\{\omega\}$ denotes the zeros of Q.

(1). If $c=2$, we find the sum is simply $2\sum_{\omega}\frac{P(\omega)\zeta^-_{e^\omega}(2)}{Q'(\omega)}$. Recall that $\zeta^-_t(2)$ is the type two shifted zeta function.

(2). If $c>2$, let's first consider the sum $\sum_{m=1}^{\infty}\frac{1}{m^c-e^{2\omega}m^{c-2}}$.

Let $f(z)=\frac{1}{(z^c-e^{2\omega}z^{c-2})(e^{2\pi iz}-1)}$. Integrate this function over the big circle γ_R as before, but we should be careful with the residue since this time, $Q(z)=z^c-e^{2\omega}z^{c-2}$ whose zeros are not all simple.

The function f has simple poles at $z=n, n\in\mathbb{Z}-\{0\}$ with residue

$$res_{n\neq 0}f = \lim_{z\to n}\frac{(z-n)}{(z^c-e^{2\omega}z^{c-2})(e^{2\pi iz}-1)} = \frac{1}{2\pi i(n^c-e^{2\omega}n^{c-2})}.$$

and at $z=e^\omega$ or $e^{-\omega}$ with residue

$$res_{e^\omega}f = \frac{1}{2e^{(c-1)\omega}(e^{2\pi ie^\omega}-1)} \quad \text{and} \quad res_{-e^\omega}f = -\frac{1}{2e^{(c-1)\omega}(e^{-2\pi ie^\omega}-1)}.$$

It also has a pole at $z=0$ of order $c-1$ because $z^c-e^{2\omega}z^{c-2}$ and $e^{2\pi iz}-1$ has zero at 0 of order $c-2$ and 1 respectively. By the general residue formula,

$$res_{z=0}f = \lim_{z\to 0}\frac{1}{(c-2)!}\Big(\frac{d}{dz}\Big)^{c-2}\frac{z}{(z^2-e^{2\omega})(e^{2\pi iz}-1)}.$$

However, calculating the $c-2$ th derivative could be a lot of work; so we think of simplifying the residue through some other approach. An important tool we will use is the generating function for Bernoulli numbers,

$$\frac{z}{e^z-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}z^n, \quad \text{where } |z|<2\pi.$$

For z in a neighborhood of zero,

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\frac{1}{z^{c+1}-e^{2\omega}z^{c-1}}\frac{2\pi iz}{e^{2\pi iz}-1}\\
&= -\frac{1}{2\pi ie^{2\omega}}\frac{1}{(1-\frac{z^2}{e^{2\omega}})z^{c-1}}\sum_{n=0}^{\infty}\frac{B_n}{n!}(2\pi iz)^n\\
&= -\frac{1}{2\pi ie^{2\omega}}\sum_{m=1}^{\infty}\Big(\frac{z^2}{e^{2\omega}}\Big)^m\sum_{n=0}^{\infty}\frac{(2\pi i)^nB_n}{n!}z^{n-c+1}\\
&= -\frac{1}{2\pi ie^{2\omega}}\sum_{r=1-c}^{\infty}\sum_{2m+n=r+c-1}\frac{(2\pi i)^nB_n}{n!e^{2m\omega}}z^r.
\end{aligned}$$

We've used the Cauchy product of the two series since they converge absolutely. The residue is now equivalent to the coefficient of the term $r=-1$ multiplies $-\frac{1}{2\pi ie^{2\omega}}$, which is

$$-\sum_{\substack{n=0\\ even}}^{c-2}\frac{(2\pi i)^{n-1}B_n}{n!e^{(c-n)\omega}}.$$

By the residue theorem,

$$\sum_{n\neq 0}\frac{1}{(n^c-e^{2\omega}n^{c-2})}-\sum_{\substack{n=0\\ even}}^{c-2}\frac{(2\pi i)^nB_n}{n!e^{(c-n)\omega}}+2\pi i(\frac{1}{2e^{(c-1)\omega}(e^{2\pi ie^{\omega}}-1)}-\frac{1}{2e^{(c-1)\omega}(e^{-2\pi ie^{\omega}}-1)})=0.$$

Since c is even, we have

$$\sum_{m=1}^{\infty}\frac{1}{m^c-e^{2\omega}m^{c-2}}=\frac{1}{2}\sum_{\substack{n=0\\ even}}^{c-2}\frac{(2\pi i)^nB_n}{n!e^{(c-n)\omega}}-\frac{\pi\cot(\pi e^{\omega})}{2e^{(c-1)\omega}}.$$

Therefore

$$\sum_{z\in\Lambda_1}\frac{P(z)}{Q(z)}e^{-cz}=\sum_{\omega}\frac{P(\omega)}{Q'(\omega)}\left(\sum_{\substack{n=0\\ even}}^{c-2}\frac{(2\pi i)^nB_n}{n!e^{(c-n)\omega}}-\frac{\pi\cot(\pi e^{\omega})}{e^{(c-1)\omega}}\right). \tag{20}$$

2. If c is odd, then $(-1)^{cn}=(-1)^n$.

$$\sum_{z\in\Lambda_1}\frac{P(z)}{Q'(z)}e^{-cz}=\sum_{m=1}^{\infty}\frac{1}{m^c}\sum_{n=-\infty}^{\infty}(-1)^n\frac{P(i\pi n+\log m)}{Q(i\pi n+\log m)}.$$

We observe that

$$\sum_{z\in\mathbb{Z}}(-1)^n\frac{P(\delta n+x)}{Q(\delta n+x)}=2\sum_{n\in\mathbb{Z}}\frac{P(2\delta n+x)}{Q(2\delta n+x)}-\sum_{n\in\mathbb{Z}}\frac{P(\delta n+x)}{Q(\delta n+x)}.$$

Hence,

$$\sum_{m=1}^{\infty}\frac{1}{m^c}\sum_{n=-\infty}^{\infty}(-1)^n\frac{P(i\pi n+\log m)}{Q(i\pi n+\log m)}=\sum_{m=1}^{\infty}\frac{1}{m^c}\left(2\cdot\frac{2i\pi}{2i\pi}\sum_{\omega}\frac{P(\omega)}{Q'(\omega)(1-e^{\frac{2i\pi(\omega-\log m)}{2i\pi}})}\right.$$
$$\left.-\frac{2i\pi}{i\pi}\sum_{\omega}\frac{P(\omega)}{Q'(\omega)(1-e^{\frac{2i\pi(\omega-\log m)}{i\pi}})}\right)$$
$$=2\sum_{m=1}^{\infty}\frac{1}{m^c}\sum_{\omega}\frac{P(\omega)}{Q'(\omega)}\left(\frac{1}{1-\frac{e^{\omega}}{m}}-\frac{1}{1-\frac{e^{2\omega}}{m^2}}\right)$$
$$=2\sum_{\omega}\frac{P(\omega)}{Q'(\omega)}\sum_{m=1}^{\infty}\frac{e^{\omega}}{m^{c-1}(m^2-e^{2\omega})}.$$

We find that the result is so similar to the case when c is even.

(1). If $c=1$, then the sum is $2\sum_{\omega}\frac{P(\omega)e^{\omega}\zeta^{-}_{e^{\omega}}(2)}{Q'(\omega)}$.

(2). If $c>1$,

$$\sum_{z\in\Lambda_1}\frac{P(z)}{Q(z)}e^{-cz}=\sum_{\omega}\frac{P(\omega)}{Q'(\omega)}\left(\sum_{\substack{n=0\\ even}}^{c-1}\frac{(2\pi i)^n B_n}{n!e^{(c-n)\omega}}-\frac{\pi\cot(\pi e^{\omega})}{e^{(c-1)\omega}}\right).\tag{21}$$

Now, consider the lattice given by:

$$\Lambda_2=\{ni\pi-\log m : n\in\mathbb{Z}\,,\, m\in\mathbb{N}^*\}$$

which is symmetric to the previous one along the imagery axis. We may proceed the sum $\sum_{z\in\Lambda_2}\frac{P(z)}{Q(z)}e^{-cz}$ by a similar approach. But an interesting fact is that this sum converges (when first summed in n, then in m) for $c\geq 0$. And the lower-bound case as $c=0$ is:

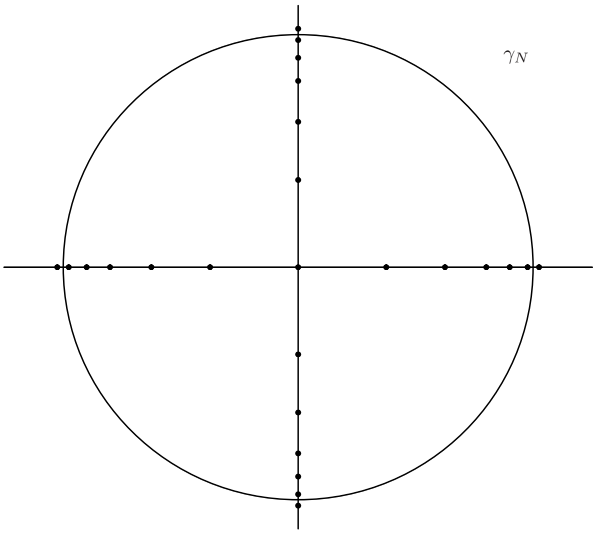
$$\sum_{z\in\Lambda_2}\frac{P(z)}{Q(z)}=\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\frac{P(i\pi n-\log m)}{Q(i\pi n-\log m)}$$
$$=\sum_{m=1}^{\infty}\frac{2i\pi}{i\pi}\sum_{\omega}\frac{P(\omega)}{Q'(\omega)(1-e^{\frac{2i\pi(\omega+\log m)}{i\pi}})}$$
$$=2\sum_{m=1}^{\infty}\sum_{\omega}\frac{P(\omega)}{Q'(\omega)(1-e^{2\omega}m^2)}$$
$$=-2\sum_{\omega}\frac{P(\omega)\zeta^{-}_{e^{-\omega}}(2)}{Q'(\omega)e^{2\omega}}.$$

5.2 Radiation

In this subsection, we will explore a new type of auxiliary function. Consider the set of points given by:

$$\Lambda_3=\{z=e^{\frac{mi\pi}{k}}n^{\frac{1}{k}} : m=1,2,\cdots,2k\,,\ n\in\mathbb{N}\},$$

which are the poles of the function $g(z)=1/(e^{2\pi i z^k}-1)$, $k\in\mathbb{Z}$ and $k\geq 1$(we will assume $k>1$ later since $k=1$ was already discussed in the previous section). The pattern gives the sense of particles illuminating from the origin, but what is unlike a normal radiation, on each ray, points get denser when they are more distant from the center.



The lattice Λ_3 when $k = 2$

Figure 7

For $P(z)/Q(z) \in \mathcal{R}$ whose zeros or poles are not in Λ_3, construct the function $f(z) = P(z)/[Q(z)(e^{2\pi i z^k} - 1)]$. Integrate this function along the circle γ_N centered at the origin with radius $[N^{\frac{1}{k}} + (N+1)^{\frac{1}{k}}]/2$ where N is an integer so large that γ_N contains all the zeros of Q , denoted by $\{\zeta\}$. (Here we also assume that Q's zeros do not coincide with those of $e^{2\pi i z^k} - 1$.) Let's first compute the residue of f.

(1). $res_\zeta f = \lim\limits_{z\to\zeta} \dfrac{P(z)(z-\zeta)}{Q(z)(e^{2\pi i z^k} - 1)} = \dfrac{P(\zeta)}{Q'(\zeta)(e^{2\pi i \zeta^k} - 1)}$.
(2). $res_{z_0\in\Lambda_3 - \{0\}} f = \lim\limits_{z\to z_0} \dfrac{P(z)(z-z_0)}{Q(z)(e^{2\pi i z^k} - 1)} = \dfrac{1}{2\pi i} \cdot \dfrac{P(z_0)}{k z_0^{k-1} Q(z_0)}$.
(3). The pole of f at 0 is of order k as we observe the first derivative of $e^{2\pi i z^k} - 1$, which is $2\pi i k z^{k-1} e^{2\pi i z^k}$, has a zero at 0 of order $k-1$. Hence,

$$res_{z=0} f = \lim_{z\to 0} \frac{1}{(k-1)!} \Big(\frac{d}{dz}\Big)^{k-1} \frac{P(z) z^k}{Q(z)(e^{2\pi i z^k} - 1)}.$$

Indeed, we may simplify this result to a much nicer form. First, let's come back to the definition of residue and prove that

$$res_{z=0}\Big(\frac{P(z)}{Q(z)(e^{2\pi i z^k} - 1)}\Big) = \frac{1}{2\pi i} res_{z=0}\Big(\frac{P(z)}{Q(z) z^k}\Big).$$

Here, we invoke again the Bernoulli numbers given by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n, \quad \text{where } |z| < 2\pi,$$

and obtain

$$\begin{aligned}\frac{P(z)}{Q(z)(e^{2\pi i z^k}-1)} &= \frac{P(z)}{Q(z)}\sum_{n=0}^{\infty}\frac{B_n}{n!}(2\pi i z^k)^{n-1}\\ &= \frac{P(z)}{Q(z)}\Big(\frac{1}{2\pi i z^k}+\sum_{n=1}^{\infty}\frac{(2\pi i)^{n-1}B_n}{n!}z^{k(n-1)}\Big)\\ &= \frac{1}{2\pi i}\frac{P(z)}{Q(z)z^k}+\frac{P(z)}{Q(z)}\sum_{n=0}^{\infty}\frac{(2\pi i)^n B_{n+1}}{(n+1)!}z^{kn}.\end{aligned}$$

Where $|z|<1$ and we've used the fact that $B_0=1$. Since the last sum denotes a meromorphic function whose poles are non-zero, we may conclude that

$$res_{z=0}\Big(\frac{P(z)}{Q(z)(e^{2\pi i z^k}-1)}\Big)=\frac{1}{2\pi i}res_{z=0}\Big(\frac{P(z)}{Q(z)z^k}\Big).$$

Next, we prove a lemma:

Lemma 5.1 Let A denote the set of zeros of $Q(z)$. Then, for $z\notin A$,

$$\frac{P(z)}{Q(z)}=\sum_{\zeta\in A}\frac{P(\zeta)}{Q'(\zeta)(z-\zeta)} \tag{22}$$

whenever $P(z)/Q(z)$ is a proper rational function and Q has distinct zeros.

Proof. Let $f(z)=P(z)/Q(z)(z-x)$, where x is a complex number not in A. Hence, $f(z)\in\mathcal{R}$. Integrate $f(z)$ along γ_R, the circle centered at the origin with radius R that contains x. Obviously,

$$\Big|\int_{\gamma_R}f(z)\,dz\Big|\to 0\quad\text{as}\quad R\to 0.$$

Also, $res_{z=x}f=\dfrac{P(x)}{Q(x)}$ by definition, and

$$res_{z=\zeta}f=\lim_{z\to\zeta}\frac{P(z)(z-\zeta)}{Q(z)(z-x)}=\frac{P(\zeta)}{Q'(\zeta)(\zeta-x)}.$$

Hence,

$$\frac{P(x)}{Q(x)}=\sum_{\zeta\in A}\frac{P(\zeta)}{Q'(\zeta)(x-\zeta)}$$

by the residue theorem and our lemma is proved.
Therefore,

$$\begin{aligned}2\pi i\cdot res_{z=0}f &= res_{z=0}\Big(\frac{P(z)}{Q(z)z^k}\Big)\\ &= \frac{1}{(k-1)!}\lim_{z\to 0}\Big(\frac{d}{dz}\Big)^{k-1}\frac{P(z)}{Q(z)}\\ &= \frac{1}{(k-1)!}\lim_{z\to 0}\Big(\frac{d}{dz}\Big)^{k-1}\sum_{\zeta}\frac{P(\zeta)}{Q'(\zeta)(z-\zeta)}\\ &= \frac{1}{(k-1)!}\lim_{z\to 0}\sum_{\zeta}\frac{P(\zeta)}{Q'(\zeta)}(-1)^{k-1}\frac{(k-1)!}{(z-\zeta)^k}\\ &= -\sum_{\zeta}\frac{P(\zeta)}{Q'(\zeta)\zeta^k}.\end{aligned}$$

Now, we shall move on to the integration over γ_N. Intuitively we would think this integral tends to zero, but the reasoning is a little trickier. Since we find that on each branch, the circle γ_N always goes in between two adjacent poles. Our concern is, as N tends infinitely large, the distance between adjacent poles becomes infinitely small, and hence the contour gets infinitely close to some poles. So it is necessary to examine the value(or growth, if no value at all) at the intersection of γ_N with each branch. But it suffices to compute the limit of $g(z)$ at $z = [N^{\frac{1}{k}} + (N+1)^{\frac{1}{k}}]/2$ as N tends to infinity.

$$
\begin{aligned}
\lim_{N\to\infty} z^k &= \lim_{N\to\infty} \frac{\sum_{i=1}^{k} C_k^i \sqrt[k]{N^i(N+1)^{k-i}}}{2^k} \\
&= \lim_{N\to\infty} \frac{\sum_{i=1}^{k} C_k^i \sqrt[k]{N^k + (k-i)N^{k-1} + O(N^{k-2})}}{2^k} \\
&= \lim_{N\to\infty} \frac{\sum_{i=1}^{k} C_k^i \sqrt[k]{(N + \frac{k-i}{k})^k + O(N^{k-2})}}{2^k} \\
&= \frac{\sum_{i=1}^{k} C_k^i (N + 1 - \frac{i}{k})}{2^k} \\
&= \frac{(N+1)2^k - 2^{k-1}}{2^k} \\
&= N + \frac{1}{2}.
\end{aligned}
$$

where we've used two basic combinatoric identities $\sum_{i=1}^{k} C_k^i = 2^k$ and $\frac{i}{k} C_k^i = C_{k-1}^{i-1}$. Thus,

$$\lim_{N\to\infty} g(z) = \frac{1}{e^{2\pi i (N+\frac{1}{2})} - 1} = -\frac{1}{2}.$$

which means that g is bounded. Now that we may easily conclude that

$$\left| \int_{\gamma_N} f(z)\, dz \right| = 0 \quad \text{as} \quad N \to \infty.$$

By the residue theorem,

$$\sum_{z_0 \in \Lambda_3 - \{0\}} \frac{P(z_0)}{k z_0^{k-1} Q(z_0)} + 2\pi i \sum_{\zeta} \frac{P(\zeta)}{Q'(\zeta)(e^{2\pi i \zeta^k} - 1)} - \sum_{\zeta} \frac{P(\zeta)}{Q'(\zeta)\zeta^k} = 0.$$

which is equivalent to

$$\sum_{m=1}^{2k} \sum_{n=1}^{\infty} \frac{P(e^{\frac{mi\pi}{k}} n^{\frac{1}{k}})}{k e^{\frac{(k-1)mi\pi}{k}} n^{1-\frac{1}{k}} Q(e^{\frac{mi\pi}{k}} n^{\frac{1}{k}})} = 2\pi i \sum_{\zeta} \frac{P(\zeta)}{Q'(\zeta)(1 - e^{2\pi i \zeta^k})} + \sum_{\zeta} \frac{P(\zeta)}{Q'(\zeta)\zeta^k}. \tag{23}$$

This formula enables us to evaluate explicitly a range of infinite series containing non-integral powers without having trouble with the definition of complex powers(i.e. to have a branch cut).

6 Extending the summation formula to $\mathcal{R}^*$

In the previous section, we have encountered several occasions where computing the residue of a higher-order pole is required; they are solved with the help of Bernoulli numbers. In light of those results, we extend our summation formula in 4.3 to functions with arbitrary poles.

Let $\mathcal{R}^*$ denotes the set of rational function with real coefficients

$$R(z) = \frac{P(z)}{Q(z)}, \quad z \in \mathbb{C}$$

such that $(Degree\ Q) \geq (Degree\ Q) + 2$ and P, Q have no common zeros. These functions differ from those from $\mathcal{R}$ only by the absence of restriction on the zeros of Q. However, this, as we will see later, could make the calculation of

$$\sum_{n\in\mathbb{Z}} \frac{P(\delta n + x)}{Q(\delta n + x)}, \quad \delta, x \in \mathbb{C}$$

a lot more complicated.

Let $f(z) = \dfrac{P(z)}{Q(z)(e^{\frac{2\pi i}{\delta}(z-x)} - 1)}$ where δ and x are complex constants. Integrate f along the the circle centered at the origin of radius R, denoted by γ_R. Obviously, we can argue as before that

$$\Big|\int_{\gamma_R} f(z)\, dz\Big| \to 0 \qquad as \qquad R \to \infty.$$

Now we only need to concern its residue. Denote the zeros of Q by $S = \{\zeta_1, \zeta_2, \cdots, \zeta_q\}$ and their order by $a_1, a_2, \cdots, a_q$, respectively.
(1). f has simple poles at the $\delta n + x$ if $\delta n + x \notin S$, $n \in \mathbb{Z}$. Hence,

$$\begin{aligned} res_{\delta n + x \notin S} f &= \lim_{z \to \delta n + x} \frac{P(z)(z - \delta n - x)}{Q(z)(e^{\frac{2\pi i}{\delta}(z-x)} - 1)} \\ &= \frac{\delta}{2\pi i} \frac{P(\delta n + x)}{Q(\delta n + x)} \end{aligned}$$

(2). f also has poles at the zeros of Q with corresponding order. The residue formula tells us

$$res_{\zeta_i} f = \lim_{z\to\zeta_i} \frac{1}{(a_i - 1)!} \Big(\frac{d}{dz}\Big)^{a_i - 1} \frac{P(z)(z - \zeta_i)}{Q(z)(e^{\frac{2\pi i}{\delta}(z-x)} - 1)}$$

for ζ_i not of the form $\delta n + x$. For $a_i > 1$ in general, we have little idea about this higher-derivative. Therefore, we seek f's residue at ζ_m, $m \in \{1, 2, \cdots, q\}$ in a different approach, in analogue to the cases in section 5, though the details are more delicate.

Write $1/Q(z)$ in the form $c_0^{-1} \prod_{i=1}^{q} (z - \zeta)^{-a_i}$, where c_0 is the leading coefficient. We first expand each of its factor $(z - \zeta_i)^{-a_i}$, $i \neq m$ to power series around ζ_m. Since

$$\Big(\frac{d}{dz}\Big)^n (z - \zeta_i)^{-a_i} = (-1)^n (a_i)_n (z - \zeta_i)^{-a_i - n},$$

where $(a)_n$ is the shifted factorial:$(a)_n = a(a+1)\cdots(a+n-1)$, we have its Taylor expansion as

$$\begin{aligned} \frac{1}{(z - \zeta_i)^{a_i}} &= \sum_{n=0}^{\infty} \frac{(-1)^n (a_i)_n}{n!(\zeta_m - \zeta_i)^{a_i + n}} (z - \zeta_m)^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \binom{a_i + n - 1}{a_i - 1}}{(\zeta_m - \zeta_i)^{a_i + n}} (z - \zeta_m)^n \end{aligned}$$

for z near ζ_m. Also,

$$P(z) = \sum_{k=0}^{\infty} \frac{P^{(k)}(\zeta_m)}{k!} (z - \zeta_m)^k,$$

which is actually a finite sum. Next, we want to expand $\dfrac{1}{e^{\frac{2\pi i}{\delta}(z-x)} - 1}$ around ζ_m. In order to do so, we define $A(n, x)$ by the generating function

$$\frac{1}{e^z - 1} = \sum_{n=0}^{\infty} \frac{A(n, x)}{n!} (z - x)^n,$$

where $x \in \mathbb{C}$ and $x \neq 2\pi i n$, $n \in \mathbb{Z}$. For z in a neighborhood of x,

$$\begin{aligned}
1 &= (e^z - 1)\sum_{n=0}^{\infty} \frac{A(n,x)}{n!}(z-x)^n \\
&= \Big(\sum_{m=0}^{\infty} \frac{e^x}{m!}(z-x)^m - 1\Big)\Big(\sum_{n=0}^{\infty} \frac{A(n,x)}{n!}(z-x)^n\Big) \\
&= \sum_{r=0}^{\infty} \sum_{n+m=r} \frac{A(n,x)}{n!m!}(z-x)^r - \sum_{n=0}^{\infty} \frac{A(n,x)}{n!}(z-x)^n \\
&= \sum_{r=0}^{\infty} \frac{e^x}{r!} \sum_{n=0}^{r} A(n,x)\binom{r}{n}(z-x)^r - \sum_{n=0}^{\infty} \frac{A(n,x)}{n!}(z-x)^n .
\end{aligned}$$

We've proceeded the Cauchy product of two series despite the uncertainty about the convergence of the series involving $A(n,x)$; but we will prove it later. Equating the coefficients leads to

1. $$A(0,x) = \frac{1}{e^x - 1}$$

2. $$A(n,x) = \frac{e^x}{1-e^x} \sum_{k=0}^{n-1} \binom{n}{k} A(k,x). \tag{24}$$

if $x \neq 2\pi i n$. We also define $A(-1,x) = 0$ in this case. Regarding the problem of convergence of $\sum_{n=0}^{\infty} \frac{A(n,x)}{n!}(z-x)^n$, we only prove, though something quite intuitive, that its radius of convergence is larger than zero for $x \neq 2\pi i n$, i.e. the series doesn't diverge everywhere. By the ratio test,

$$\lim_{n\to\infty} \Big| \frac{A(n+1,x)(z-x)}{A(n,x)(n+1)} \Big| < 1,$$

and from the observation that

$$\begin{aligned}
\frac{A(n+1,x)}{A(n,x)} &= \frac{\sum_{m=0}^{n} \binom{n+1}{m} A(m,x)}{\sum_{k=0}^{n-1} \binom{n}{k} A(k,x)} \\
&= \frac{(n+1)A(n,x) + \sum_{m=0}^{n-1} \binom{n+1}{m} A(m,x)}{\sum_{k=0}^{n-1} \binom{n}{k} A(k,x)} \\
&= \frac{\frac{e^x}{1-e^x} \sum_{m=0}^{n-1} (n+1)\binom{n}{m} A(m,x) + \sum_{m=0}^{n-1} \binom{n+1}{m} A(m,x)}{\sum_{k=0}^{n-1} \binom{n}{k} A(k,x)} \\
&< \frac{\big(\frac{e^x}{1-e^x} + 1\big)(n+1) \sum_{m=0}^{n-1} \binom{n}{m} A(m,x)}{\sum_{k=0}^{n=1} \binom{n}{k} A(k,x)} \\
&= \frac{n+1}{1-e^x},
\end{aligned}$$

because $\binom{n+1}{m} < (n+1)\binom{n}{m}$, we may conclude that the radius of convergence is at least $|1-e^x|$. Hence it converges absolutely for z in a neighborhood of x if $x \neq 2\pi i n$.

On the other hand, if $x = 2\pi i m \,, m \in \mathbb{Z}$,

$$\frac{1}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!}(z-x)^{n-1} = \sum_{n=-1}^{\infty} \frac{B_{n+1}}{(n+1)!}(z-x)^n$$

by the definition of Bernoulli numbers. Its radius of convergence is 2π, which is a well known fact. Therefore, if x is an integral multiple of $2\pi i$, we define

$$A(n,x) = \frac{B_{n+1}}{n+1} \quad \text{for } n \geq 0 \quad \text{and} \quad A(-1,x) = 1. \tag{25}$$

(24) and (25) together completes our definition for $A(n,x)$, $n \geq -1$ and $x \in \mathbb{C}$. Now, the generating function can be expressed as

$$\frac{1}{e^z-1} = \sum_{n=-1}^{\infty} \frac{A(n,x)}{|n|!}(z-x)^n.$$

Let $z = \frac{2\pi i}{\delta}(z-x)$ and $x = \frac{2\pi i}{\delta}(\zeta_m - x)$, we obtain

$$\begin{aligned}\frac{1}{e^{\frac{2\pi i}{\delta}(z-x)}-1} &= \sum_{n=-1}^{\infty} \frac{A(n,\frac{2\pi i}{\delta}(\zeta_m - x))}{|n|!}\big(\frac{2\pi i}{\delta}(z-\zeta_m)\big)^n \\ &= \sum_{n=-1}^{\infty} \frac{(2\pi i)^n A(n,\frac{2\pi i}{\delta}(\zeta_m - x))}{|n|!\delta^n}(z-\zeta_m)^n.\end{aligned}$$

We need to decide the definition of $A(n,x)$ in terms of whether $\frac{2\pi i}{\delta}(\zeta_m - x)$ is a multiple of $2\pi i$, or equivalently, whether ζ_m is in the form of $\delta n + x$, $n \in \mathbb{Z}$.

Summing up the above results gives

$$\begin{aligned}f(z) &= \frac{1}{c_0(z-\zeta_m)^{a_m}}\Big(\sum_{k=0}^{\infty}\frac{P^{(k)}(\zeta_m)}{k!}(z-\zeta_m)^k\Big)\prod_{i\neq m}\Big(\sum_{m_i=0}^{\infty}\frac{(-1)^n\binom{a_i+m_i-1}{a_i-1}}{(\zeta_m-\zeta_i)^{a_i+m_i}}(z-\zeta_m)^{m_i}\Big) \\ &\quad \Big(\sum_{n=-1}^{\infty}\frac{(2\pi i)^n A(n,\frac{2\pi i}{\delta}(\zeta_m - x))}{|n|!\delta^n}(z-\zeta_m)^n\Big) \\ &= \frac{1}{c_0(z-\zeta_m)^{a_m}}\sum_{r=0}^{\infty}\sum_{n+k+\sum m_i = r}\frac{(2\pi i)^n A(n,\frac{2\pi i}{\delta}(\zeta_m - x))P^{(k)}(\zeta_m)}{\delta^n|n|!k!}\prod_{i\neq m}\frac{(-1)^{m_i}\binom{a_i+m_i-1}{a_i-1}}{(\zeta_m-\zeta_i)^{a_i+m_i}}\,(z-\zeta_m)^r \\ &= \frac{1}{c_0}\sum_{r=0}^{\infty}\left(\sum_{n+k+\sum m_i = r}\frac{(2\pi i)^n A(n,\frac{2\pi i}{\delta}(\zeta_m - x))P^{(k)}(\zeta_m)}{\delta^n|n|!k!}\prod_{i\neq m}\frac{(-1)^{m_i}\binom{a_i+m_i-1}{a_i-1}}{(\zeta_m-\zeta_i)^{a_i+m_i}}\right)(z-\zeta_m)^{r-a_m}.\end{aligned}$$

By its definition, the residue of f at ζ_m is the coefficient of $(z-\zeta_m)^{-1}$. In the present case, it's the coefficient of the term with $r = a_m - 1$, which is

$$\frac{1}{c_0\prod_{i\neq m}(\zeta_m-\zeta_i)^{a_i}}\sum_{n+k+\sum m_i = a_m-1}\frac{(-1)^{\sum m_i}(2\pi i)^n A(n,\frac{2\pi i}{\delta}(\zeta_m - x))P^{(k)}(\zeta_m)}{\delta^n|n|!k!}\prod_{i\neq m}\frac{\binom{a_i+m_i-1}{a_i-1}}{(\zeta_m-\zeta_i)^{m_i}}, \tag{26}$$

with $k \geq 0$, $n \geq -1$ and $m_i \geq 0$. It's indeed complicated! In particular, if we let ζ_m be simple, i.e $a_m = 1$ for all m, and require that ζ_m is not of the form $\delta n + x$, which implies the $n = -1$ term vanishes, (26) becomes

$$\frac{1}{c_0\prod_{i\neq m}(\zeta_m-\zeta_i)^{a_i}}P(\zeta_m)A(0,\frac{2\pi i}{\delta}(\zeta_m - x)) = \frac{P(\zeta_m)}{Q'(\zeta_m)(e^{\frac{2\pi i}{\delta}(\zeta_m-x)}-1)}$$

by definition of $A(n,x)$ in (24). This is identical to the result we obtained in 4.3. The general formula in terms of residue is

$$\sum_{\substack{n\in\mathbb{Z}\\ \delta n+x\notin S}}\frac{P(\delta n+x)}{Q(\delta n+x)} = -\frac{2\pi i}{\delta}\sum_{\zeta\in S} res_{z=\zeta}f.$$

Where S is the set of zeros of Q, and the residue can be computed through (26). To end this paper, we illustrate its application by a simple case of finding the value of $\zeta(2s)$(s is an integer) using complex analysis.

Let $f(z) = 1/[z^s(e^{2\pi iz}-1)]$. Its residue at $z = n$, $n \neq 0$ is $1/2\pi i n^k$. Since the zero of z^s coincides with that of $1/(e^{2\pi iz}-1)$ at 0, we use the definition of $A(n,x)$ in (25). Also, $P(z) = 1$

and thus $P^{(k)}(z)$ vanishes for $k > 0$. Finally, $1/z^s$ has only one pole so there is no ζ_i with $i \neq m$. So (26) reduces to one term:

$$\frac{(2\pi i)^s A(s-1,0)}{(s-1)!},$$

because $c_0 = 1$, $\delta = 1$ and $x = 0$. By the residue theorem,

$$\sum_{n\in\mathbb{Z}} \frac{1}{n^s} = -\frac{(2\pi i)^s A(s-1,0)}{(s-1)!}.$$

Substitute $2s$ for s, we obtain

$$\zeta(2s) = \frac{(-1)^{s+1} 2^{2s-1} \pi^{2s} B_{2s}}{(2s)!},$$

using $A(2s-1,0) = B_{2s}/2s$ in (25), a beautiful formula first obtained by Euler almost three centuries ago.

Acknowledgements

I would thank Ms. Wen for her valuable suggestions on this paper as well as her commitment in guiding me through the study in mathematics beyond. I would also thank Mr. Wang Jieliang for instructing me in mathematical writing last term.

References

[1] Elias M.Stein & Rami Shakarchi, *Fourier Analysis An introduction, page* $150-154$, 278, Princeton University Press, 2003.

[2] Elias M.Stein & Rami Shakarchi, *Complex Analysis, page* $114-120$, Princeton University Press, 2003.

[3] Anthony Sofo, *Summing Series Using Residues*, http://vuir.vu.edu.au/15695/1/Sofo_1998compressed.pdf, page $16-20$

[4] Adrian Down, *Summation of Series*, http://people.duke.edu/~ad159/files/m185/23.pdf

[5] George E.Andrews & Richard Askey & Ranjan Roy, *Special Functions*, Cambridge University Press, 2000.

[6] Robin Chapmam, *Evaluatuing* $\zeta(2)$, University of Exeter, 30 April 1999.

[7] Roman J.Dwilewicz & Jan Minac, *Values of the Riemann zeta function at integers*, Universitat Autonoma de Barcelona, Volumn 2009, treball no.6,26 pp.