Research on Tian Ji Horse Racing

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Abstract

In this thesis, we generalize Tian Ji Horse Racing from the eponymous ancient Chinese story by changing the number of races from three into arbitrary many and considering all possible combinations of participating horses from both sides. As a result, we derive a recurrence relation for race results, generating Tian Ji's triangle, which is analogous to Pascal's triangle and Stirling's triangle. After scrutinizing some properties of Tian Ji's triangle, we prove that the limit of Tian Ji's chance of winning exists as the number of races approaches infinity and discuss this result from a philosophical perspective. We also define and study the Tian Ji distribution, which may have applications in real life.

Keywords

Pascal's triangle, Tian Ji's triangle, the limit of Tian Ji's chance of winning, bivariate generating functions, Tian Ji distribution, Stirling number

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1 Introduction

Tian Ji Horse Racing is a famous story from the Records of the Grand Historian by Sima Qian. The following is an excerpt from the book regarding the event:

Tian Ji was Racing Horse with the King for a prize. Sunzi, who was a bright friend of Tian Ji, saw that Tian Ji's horses was less outstanding than the King's, and there were 3 kinds of horses – Best, Medium and Faint. (If two horse in a level race together, King's horse win, while in different levels, higher level win, without considering the owner of horse.) The advisor Sunzi told Tian Ji: ' You just use your faint one, racing with his best one, while the medium one racing with his faint one, and your best one racing with his medium one. ' Then Tian Ji won the whole game. **[1]**

What will happen if we change three horses into four or five or even more? In addition, if we suppose Tian Ji hadn't met the brilliant advisor Sunzi and placed all the horses in arbitrary order to race, then what would have happened to Tian Ji's chance of winning? In this thesis, we scrutinize both situations closely.

Throughout the thesis, we assume that Tian Ji couldn't know the speed of opponent's horses beforehand so that he had no way to use the winning strategy derived in other research.

2 Tian-King Racing Model

We use H_1、H_3、……、H_{2n-1} and H_2、H_4、……、H_{2n} to denote Tian Ji's horses and King's ones, respectively. The horses with larger index run faster and the results of each race between two horses H_i and H_j ($i \neq j$, $i, j \in \{1, 2, ..., 2n\}$) is given by i, j . There are n races in a horse racing event with each horse from both sides participating in only one race against one opponent.

Then we fix the King's sequence of participating horse for each race as H_2、H_4、……、H_{2n}. Thus we have $n!$ possible horse racing events in total as the sequence of Tian Ji's horses has $n!$ permutations.

We use $A_{i,j}$ to denote the number of horse racing events with Tian Ji winning *i* races and the King winning *j*.

For instance, when $n = 3$, there are 6 races. Among them, we have $A_{0,3} = 1$, $A_{1,2} = 4$, $A_{2,1} = 1$ and $A_{3,0} = 0$.

For any n, apparently we have $A_{n,0} = 0$, or equivalently the index of Tian Ji's horses cannot be larger than that of the King's ones in each event.

Now we should derive the recurrence relation for $A_{i,j}$.

Consider a horse-racing event with n horses. We now add two fastest horses H_{2n+1}, H_{2n+2} to Tian Ji's team and the King's team. First, we let H_{2n+1} race against H_{2n+2}, with all other n races preserving the original participants. Apparently, Tian Ji will lose one more race. Now we exchange H_{2n+1} with any other Tian Ji's horse H_{2k+1} which races against H_{2j}. If $2k+1<2j$, then Tian Ji will win one more race. Conversely, if $2k+1>2j$, Tian Ji will lose one more race. As all $A_{i,j}$ can be written as $A_{(i-1)+1,j}$ or $A_{i,(j-1)+1}$, we have:

$$A_{i,j+1} = (i+1)\cdot A_{i,j}$$
$$A_{i+1,j} = j\cdot A_{i,j}$$

Thus the recurrence relation for $A_{i,j}$ is:

$$A_{i,j} = (i+1)\cdot A_{i,j-1} + j\cdot A_{i-1,j}$$

We can calculate $A_{i,j}$ directly when n is small, as listed in the following table.

	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1
2	1	4	11	26	57	120	247	502	1013
3	1	11	66	302	1191	4293	14608	47840	152637
4	1	26	302	2416	15619	88234	455192	2203488	10187685
5	1	57	1191	15619	156190	1310354	9738114	66318474	423281535
6	1	120	4293	88234	1310354	15724248	162512286	1505621508	12843262863
7	1	247	14608	455192	9738114	162512286	2275172004	27971176092	311387598411
8	1	502	47840	2203488	66318474	1505621508	27971176092	447538817472	6382798925475

Table (Tian Ji's score is in the row, King's score is in the column)

Tian Ji's total number of winning is in the first row while King's one is in the first column. For example, if we have five horse races for each racing event, then 26 events will end with Tian Ji winning three out of five and the King winning only two.

3 Tian Ji's Triangle

To make the results more visually symmetrical, we rotate the whole table 45 degrees clockwisely and conceal the second row in the original one to form a triangle.

Definition If a triangle has a first row with 1 , and second row with 1,1, the third row with $1\left(A_{0,3}\right)$, $4\left(A_{1,2}\right)$, $1\left(A_{2,1}\right)$, and for any other items

$$A_{i,j}=\left(i+1\right)\cdot A_{i,j-1}+j\cdot A_{i-1,j}$$

,then it is called **Tian Ji's triangle.**

The Tian Ji's triangle can facilitate the study of racing results.

1

1 1

1 4 1

1 11 11 1

1 26 66 26 1

1 57 302 302 57 1

......

Graph 2 (Tian Ji's triangle)

3.1 Symmetry of Tian Ji's triangle

Theorem 1(Symmetry) Tian Ji's triangle is symmetrical, $A_{i,j}=A_{j\text{-}1,i+1}$.

Proof We use mathematical induction here.

First, we have $A_{3,2}=A_{1,4}$.

If $A_{i,j}=A_{j\text{-}1,i+1}$, and $A_{i+1,j}=j\cdot A_{i,j}$, then

$$A_{i,j+1}=\left(i+1\right)\cdot A_{i,j}\Leftrightarrow A_{\mathrm{j-1,i+1+1}}=\left(j-1+1\right)\cdot A_{\mathrm{j-1,i+1}}=j\cdot_{\mathrm{j-1,i+1}}$$

Thus, by mathematical induction, we have $A_{i+1,j}=A_{j\text{-}1,i+2}$ for all $i\in N$. Similar method can be used for proof regarding all $j\in N$.

In conclusion, for any $i,j\in N$, $A_{i,j}=A_{j\text{-}1,i+1}$ is true.

Statement 1 Number of racing events whose result are $i:j$ equals to that of $j-1:i+1$.

Explanation The Theorem is much more significant than Statement 1 since it reveals the similarity between Tian

Ji's triangle and Pascal's triangle.

3.2 Property of Correspondence

Every number in the table is linked to many races, so that we can do some correspondence work between some races of then.

First, we can observe the ' 4 horses race'. Fix the order of Tian Ji's horses, and just observe the result of 24 traversal order of the King. (In the table, we numbered the horses by its index)

1		2	6	2	6	2	6	6	2	2	6	4	8	4	8	8	4	8	4	8	4	2	6	8	4
3	V	4	8	6	2	8	4	4	8	4	8	6	2	2	6	6	2	4	8	4	8	4	8	2	6
5	S	8	4	8	4	6	2	8	4	6	2	2	6	6	2	4	8	6	2	2	6	6	2	4	8
7		6	2	4	8	4	8	2	6	8	4	8	4	8	4	2	6	2	6	6	2	8	4	6	2

Graph3 ('1:3's are in yellow column, last four columns are the special situations)

We find that except the last 4 columns (2,4,6,8 & 8,2,4,6) whose results are 0:4 and 3:1(corresponding with 2:2 and 1:3), all the other groups are $2:2$ (in yellow column) and $1:3$ (in white column). Among, two adjacent 'yellow – white' (not 'white - yellow') groups correspond with each other. The law of correspondence is :

Place one number, plus 4, get its remainder in mod8(write '0' as '8'), you can get the other group.

For instance, the group 2, 4, 8, 6($1:3$) and 6, 8, 4, 2($2:2$) are two corresponding groups, $2+4=6$, $4+4=8$, $8+4=4+8\times1$, $6+4=2+8\times1$, you can change (2,4,8,6) into (6,8,4,2).

In special situations, 0:4 and 3:1, they can correspondence with two groups of $2:2$ and $1:3$.

Then we observe the '6 horses race', after traversal verification, we can find it also satisfy the similar rule:

Place one number, plus 6, get its remainder in mod12 (write '0' as '12'), you can get the other group.

$$
\begin{aligned}
&0:6\rightarrow3:3 &&: (2,4,6,8,10,12)\rightarrow(8,10,12,2,4,6)\\
&5:1\rightarrow2:4(3:3) &&: (12,2,4,6,8,10)\rightarrow(6,8,10,12,2,4)\\
&1:5\rightarrow2:4(3:3) &&: (2,4,12,8,10,6)\rightarrow(8,10,6,2,4,12)\\
&3:3\rightarrow2:4 &&: (8,10,2,12,6,4)\rightarrow(2,4,8,6,12,10)
\end{aligned}
$$

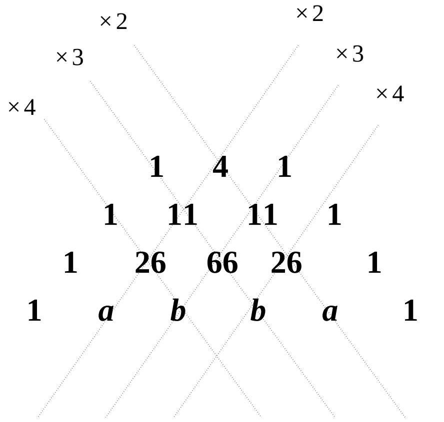
Statement For even row $(2n)$, all the numbers in two sides can be corresponded with the medial numbers, in the rule of 'Plus n, get it remainder in $\bmod 2n$. Write '0' as '$2n$'.'

Note As we can easily find that the event of $0:2n$ correspondents with the event of $n:n$, so there must be a **basket** in this particular row, the numbers out of which can correspondent with the numbers in it.

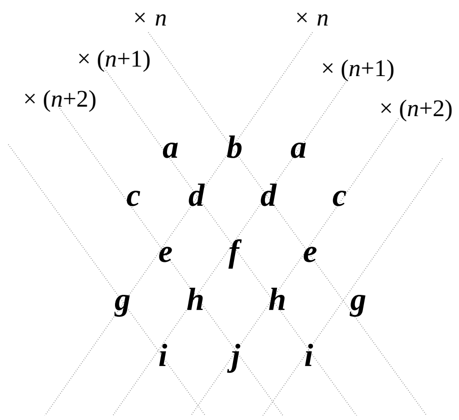
3.3 Monotonicity

Theorem 2 $f(n)=\dfrac{A_{n,n+1}}{(2n+1)!}$ decreases as n increases.

Proof



Graph 4



Graph 5

In two graphs of Tian Ji's triangle, every number in the bottom is obtained by adding the two numbers above with weights determined by which line they belong to. For instance:

$$e=n\cdot d+(n+2)\cdot c$$

The 'e, f, e' row in the Graph5 is an odd row representing $2n+1$ horse races, so we can know that

$j=(2n+4)\cdot h$, $h=(n+2)e+(n+1)f$.

First notice that $(n+2)^2 e<(n+1)^2 f \Leftrightarrow \dfrac{f}{(2n+1)!}>\dfrac{j}{(2n+3)!}$.

To prove $\dfrac{A_{n,n+1}}{(2n+1)!}<\dfrac{A_{n+1,n+2}}{(2n+3)!}$ by mathematical induction, we should derive the following first:

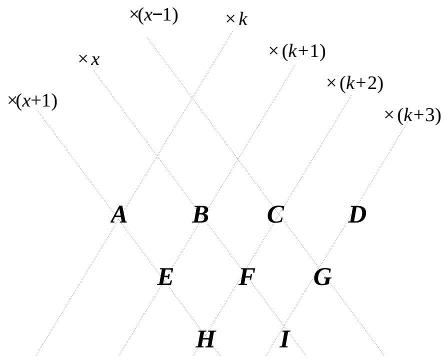
$$(n+2)^2 e<(n+1)^2 f$$

If we use a,b to represent e, f , we should introduce a new parameter (the one the right of b). Then we should introduce more and more parameters. So we should prove

$$(n+2)^2 e<(n+1)^2 f<n^2\cdot m<\ldots<1^2\cdot z$$

Among, $m,\ldots,z$ are in the same line with e, f ,but on the right side of e, f .

So we just consider an abstract graph like this:



Graph 6

We would like to prove

$$H\cdot(k+2)^2 > I\cdot(k+3)^2 . \qquad (*)$$

by the hypothesis: $A\cdot k^2 > B\cdot(k+1)^2 > C\cdot(k+2)^2 > D\cdot(k+3)^2$

(*)

$$\Leftrightarrow (k+2)^2\cdot\left\{(x+1)\left[(x+1)\cdot A+(k+1)\cdot B\right]+\left[x\cdot B+(k+2)\cdot C\right](k+2)\right\}$$

$$>(k+3)^2\cdot\left\{x\left[x\cdot B+(k+2)\cdot C\right]+\left[(x-1)\cdot C+(k+3)\cdot D\right](k+3)\right\}$$

$$\Leftrightarrow (k+2)^2\cdot(x+1)^2\cdot A+\left[(k+2)^2\cdot(2xk+3x+k+1)-(k+3)^2\cdot x^2\right]\cdot B$$

$$+\left[(k+2)^4-(k+3)^2\cdot(2kx+5x-k-3)\right]\cdot C-(k+3)^4\cdot D>0$$

We just consider the situation in $k > x$ (Left side of the medial axis), and it is true.

$f(n)=\dfrac{A_{n,n+1}}{(2n+1)!}$ is decreasing.

3.4 Comparison between several triangles

Pascal's triangle	Stirling's triangle of the second kind
1	1
1 1	1 1
1 2 1	1 3 1
1 3 3 1	1 7 6 1
1 4 6 4 1	1 15 25 10 1
1 5 10 10 5 1	1 31 90 65 15 1
......	......

Pascal's triangle **Stirling's triangle of the second kind**

1
1 1
2 3 1
6 11 6 1
24 50 35 10 1
120 274 225 85 15 1
......

Stirling's triangle of the first kind

1
1 1
1 4 1
1 11 11 1
1 26 66 26 1
1 57 302 302 57 1
......

Tian Ji's triangle

Graph 7 (Comparison between several triangles)

3.4.1 Stirling numbers of the second kind

Definition If set A 's subsets $\left\{A_1, A_2, ..., A_k\right\}$ satisfies:

(1) $A_i \neq \varnothing$; (2)To any $i \neq j$, $A_i \cup A_j = \varnothing$; (3) $A_1 \cup A_2 \cup ... \cup A_k = A$.

Then we call $\left\{A_1, A_2, ..., A_k\right\}$ is a partition of set A. The number of ways to part a set of n elements into k nonempty subsets are denoted by $S_{n,k}$. The $S_{n,k}$ is called **the Stirling numbers of the second kind. [2]**

Then we consider the recurrence of Stirling numbers of the second kind:

$$S_{n,k} = S_{n-1,k-1} + k \cdot S_{n-1,k}$$

How does it come?

（1）If a single element set $\left\{a_1\right\}$ belongs to one of the partition, then the number of partition of this kind is $S_{n-1,k-1}$.

（2）If a single element set $\left\{a_1\right\}$ doesn't belong to one of the partition, then it belongs to one of the k subsets of $\complement_A \left\{a_1\right\}$, then the number of partition of this kind is $k \cdot S_{n-1,k}$.

We just add them together to reach the recurrence.

3.4.2 Stirling numbers of the first kind

Definition The number of ways of setting i different elements on j circles is denoted by $s_{i,j}$. It is called **the Stirling numbers of the first kind. [2]**

Then we consider the recurrence of Stirling numbers of the first kind:

$$s_{i,j} = s_{i-1,j-1} - (i-1)s_{i-1,j}$$

How does it come?

（1）If the $i-1$ elements set $j-1$ circles, then the last element set a circle alone. There are $s_{i-1,j-1}$ ways.

（2）If the $i-1$ elements set j circles, then just put the last element on the left (or right) side of any element. There are $(i-1)\cdot s_{i-1,j}$ ways.

We just add them together to reach the recurrence.

3.4.3 The relation between several triangles

First, we think of the way we get them:

Consider the Tian Ji's triangle:

(1) If H_{2n+1} accidentally races with H_{2n+2} , then we add a "lose" to Tian Ji

(2) If H_{2n+1} changes its order with a weaker horse H_k (it should belong to Tian Ji) and H_k wins in the "n horse racing", then we add a "lose" to Tian Ji, while if H_k loses, we add a "win" to Tian Ji.

So, it will not be surprising why the recurrence of those triangles are so similar, they are all composed by two situations.

Then, we think of the recurrence of them:

Pascal triangle: $P_{i,j} = P_{i-1,j} + P_{i,j-1}$ Stirling Number of sec. kind: $S_{n,k} = S_{n-1,k-1} + k\cdot S_{n-1,k}$

Stirling Number of the first kind: $S_{n,k} = S_{n-1,k-1} + k\cdot S_{n-1,k}$

Tian Ji's triangle: $A_{i,j} = (i+1)\cdot A_{i,j-1} + j\cdot A_{i-1,j}$

Actually, no matter which it is, Pascal's triangle or Stirling's triangle, or Tian Ji's triangle, their recurrence has a form like this:

$$K_{i,j} = f(i,j)\cdot K_{i,j-1} + g(i,j)\cdot K_{i-1,j} \text{ or } K_{i,j} = f(i,j)\cdot K_{i-1,j-1} + g(i,j)\cdot K_{i-1,j}$$

We just use different $(1,1)$, $(1,h(i))$, $(1,h(j))$, $(m(i),n(j))$ to replace $(f(i,j),g(i,j))$, so that we can get different recurrences. And different recurrences can get different triangles. So we can say that

Multiple Tian Ji racing horses is an actual combinatorial explanation of the recurrence

$$A_{i,j}=(i+1)\cdot A_{i,j-1}+j\cdot A_{i-1,j}$$

Last, we just give some small theorems between them:

Theorem 3 $P_{i,j}\le S_{i,j}\le A_{i,j}$.

Theorem 4 For a Stirling Number of the second kind: $S_{i,j}$, there is $S_{n,2}=2^{n-1}-1$(It is popular and its prood is easy to be made); for $A_{i,j}$ in Tian Ji's triangle, there is

$$A_{n,2}=2^{n+1}-(n+2)$$

Proof $A_{1,2}=2^{1+1}-(1+2)=1$, it satisfies the above equality.

Consider k satisfying it, which is to say $A_{k,2}=2^{k+1}-(k+2)$, then

$$A_{k+1,2}=2A_{k,2}+k+1=2^{k+2}-(k+3)$$

So $k+1$ satisfies it. Theorem 5 is true.

3.5 Inversion of Tian Ji's triangle

Many triangles have **inversion** formulas. For instance, Pascal's triangle has a inversion formula as:

$$f(n)=\sum_{k=0}^{n}P_{n,k}\cdot g(k)\Leftrightarrow g(n)=\sum_{k=0}^{n}(-1)^{n-k}P_{n,k}\cdot f(k)$$

Among, $P_{n,k}$ and $(-1)^{n-k}P_{n,k}$ are two **reciprocal kernels. [4]**

Actually, when we inverse a progression in the form of

$$f(n)=\sum_{k=0}^{n}a_{n,k}\cdot g(k)\Leftrightarrow g(n)=\sum_{k=0}^{n}b_{n,k}\cdot f(k)$$

, we are just admitting that there are several triangles, satisfying:

$$\begin{bmatrix} f(1)\\ f(2)\\ \vdots\\ f(x)\end{bmatrix}=\begin{bmatrix} a_{11} & 0 & \cdots & 0\\ a_{21} & a_{22} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & a_{n3} & a_{nn}\end{bmatrix}\begin{bmatrix} g(1)\\ g(2)\\ \vdots\\ g(x)\end{bmatrix}=\begin{bmatrix} a_{11} & 0 & \cdots & 0\\ a_{21} & a_{22} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & a_{n3} & a_{nn}\end{bmatrix}\begin{bmatrix} b_{11} & 0 & \cdots & 0\\ b_{21} & b_{22} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ b_{n1} & b_{n2} & b_{n3} & b_{nn}\end{bmatrix}\begin{bmatrix} f(1)\\ f(2)\\ \vdots\\ f(x)\end{bmatrix}$$

so for $A=[a_{mn}]$ and $B=[b_{mn}]$, we can find $AB=I$. It is true in all inversions.

And, for Stirling triangle of two kinds, we have (Here, $s_{n,k}$ is singed, thus have some negative items.)

$$f(n)=\sum_{k=0}^{n}s_{n,k}\cdot g(k)\Leftrightarrow g(n)=\sum_{k=0}^{n}S_{n,k}\cdot f(k)$$

So it is not surprise why we treat $s_{n,k}$ and $S_{n,k}$ as 'twins', as well as $P_{n,k}$ and its signed one.

Then we consider the inversion of Tian Ji's triangle. Since $AB=I$, we can easily know that what we are to do is to find a matrix B which satisfies $AB=I$. (Among, A stand for Tian Ji's triangle.)

$$\begin{bmatrix}1 & & & & \\ 1 & 1 & & 0 & \\ 1 & 4 & 1 & & \\ 1 & 11 & 11 & 1 & \\ 1 & 26 & 66 & 26 & 1\end{bmatrix}\cdot\begin{bmatrix}b_{11} & & 0\\ \vdots & \ddots & \\ b_{n1} & \cdots & b_{nn}\end{bmatrix}=\begin{bmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{bmatrix}\left(\begin{bmatrix}1 & & & & \\ 1 & 1 & & 0 & \\ 1 & 4 & 1 & & \\ 1 & 11 & 11 & 1 & \\ 1 & 26 & 66 & 26 & 1\end{bmatrix}=\begin{bmatrix}A_{0,1} & & & & \\ A_{0,2} & A_{1,1} & & 0 & \\ A_{0,3} & A_{1,2} & A_{2,1} & & \\ \vdots & \vdots & \vdots & \ddots & \\ A_{0,n} & A_{1,n} & A_{2,n} & \cdots & A_{n,n}\end{bmatrix}\right)$$

So, we can set some equations:

(1) Multiply the first column in matrix A with every row in matrix B: $\sum_{k=1}^{n}b_{k,n}=0$;

(2) Multiply the n^{th} column in matrix A with the n^{th} row in matrix B: $b_{k,k}=0=1$;

(3) Multiply every $i(i\le n)^{\text{th}}$ column in matrix A with the n^{th} row in matrix B:

$$\sum_{k=1}^{n}A_{k-i,i}\cdot b_{k,n}=0$$

When we try every $i(i\le n)$, we can get n equations, and we also have n unknowns. We can get the twin triangle of Tian Ji's triangle. We denote the twin triangle of Tian Ji's triangle by a_{mn}.Thus,

$$f(n)=\sum_{k=0}^{n}a_{n,k}\cdot g(k)\Leftrightarrow g(n)=\sum_{k=0}^{n}A_{n,k}\cdot f(k)$$

and below is the **Inversion of Tian Ji's triangle**.

$$
\begin{array}{cccccc}
 & & 1 & & & \\
 & -1 & & 1 & & \\
 & 3 & 4 & & 1 & \\
-23 & & 33 & -11 & & 1 \\
425 & -620 & 220 & & -26 & & 1 \\
30771 & -44605 & 15560 & -1784 & 57 & 1 \\
 & & \cdots\cdots & & &
\end{array}
$$

Graph8 (Inversion of Tian Ji's triangle)

4 Tian Ji's Winning Rate

4.1 Tian Ji's Winning Rate in odd situations

We set a table to observe the inverse number of Tian Ji's winning rate $d(n)=\dfrac{(2n+1)!}{\sum\limits_{i>j,i+j=2n+1} A_{i,j}}$.

Horses	3	5	7	9	11	13	15	17	19
d	6	4.44	3.84	3.51	3.30	3.13	3.04	2.95	2.92
Horses	21	23	25	27	29	31	33	35	37
d	2.86	2.81	2.76	2.74	2.70	2.70	2.68	2.66	2.64
Horses	39	……			87	89	91	93	95
d	2.62	……			2.335	2.340	2.336	2.332	2.328

Graph 9 (Inverse Number of Tian Ji Winning Rate)

Statement $d(n)=\dfrac{(2n+1)!}{\sum\limits_{i>j,i+j=2n+1} A_{i,j}}$ is decreasing. (which is to say Tian Ji's winning rate is increasing)

Proof Take note of $\sum\limits_{i>j,i+j=2n+1} A_{i,j}+\sum\limits_{i<j-1,i+j=2n+1} A_{i,j}+\sum\limits_{i=j-1=n,i+j=2n+1} A_{i,j}=(2n+1)!$, and every item in $\sum\limits_{i>j,i+j=2n+1} A_{i,j}$ corresponds with an item in $\sum\limits_{i<j-1,i+j=2n+1} A_{i,j}$, which is supported by Theorem 1. So

$$\sum_{i>j,i+j=2n+1} A_{i,j}=\sum_{i<j-1,i+j=2n+1} A_{i,j}$$

And $f(n)=\lim\limits_{n\to\infty}\dfrac{A_{n,n+1}}{(2n+1)!}$ is decreasing, so $d(n)$ is decreasing. Tian Ji's chance is increasing.

Corollary $d(n)$ has a limit.

Proof It is because $d(n)$ is decreasing , and has a lower bound 2, so we can have a definition of the limit.

Definition We define the limit of $d\left(n\right)=\dfrac{\left(2n+1\right)!}{\sum\limits_{i>j,i+j=2n+1} A_{i,j}}$, $\lim\limits_{n\to\infty}\dfrac{\left(2n+1\right)!}{\sum\limits_{i>j,i+j=2n+1} A_{i,j}}$ as Tian Ji Winning Number.

4.2 Tian Ji Intervals

In the research above, we just study the Tian Ji winning rate in odd horses situations, and we find that Tian Ji winning rate is increasing. Then I calculate the Tian Ji winning rate in even situations, as there are draw games, I just sum the results Tian Ji exactly win together, and don't sum draw games in. The data is on the following.

Horses	3	4	5	6	7	8	9	10	11
Rate	0.1667	0.0416	0.2252	0.0805	0.2604	0.1126	0.2850	0.1390	0.303

Table 10 (Tian Ji Winning Rate in all situations)

So I find that when we calculate the even games, we can find that the property of monotone don't exist anymore. Maybe because we we haven't handled the 'draw game' well, so it might not be fair to Tian Ji, as it is not fair for Tian Ji to give up all his draw games. So we multiple the numbers of draw games with a number h, actually it changes when the number of horses changes, so we define it as a function $h\left(2n\right)$. To keep the winning rate in the property of monotone, we can get an interval for $h\left(2n\right)$ to vary in. We denote the interval (a set) by $H\left(2n\right)$ I give the intervals on the following table.

$2n$	4	6	8	10	12
$H\left(2n\right)$	$\left(\frac{3}{11},\frac{2}{5}\right)$	$\left(\frac{52}{151},\frac{3}{7}\right)$	$\left(\frac{5955}{15619},\frac{4}{9}\right)$	$\left(0.4046,\frac{5}{11}\right)$	$\left(0.4193,\frac{6}{13}\right)$

Table 11 (Tian Ji Interval)

Among, Tian Ji's winning rate is calculated by the equality

$$p\left(2n\right)=\frac{h\cdot A_{n,n}+\left(\frac{2n!}{2}-A_{n,n}\right)}{2n!}$$

It can be easily proved that both sides of the interval are increasing, as we can prove that $\dfrac{A_{n,n}}{2n!}$ is decreasing as we prove $\dfrac{A_{n,n+1}}{\left(2n+1\right)!}$ is decreasing. So we can get some property of the interval and the inexistence of the desired number h satisfying all the n varying from 3 to infinitely large.

Theorem 5 As $0.4193 > \frac{2}{5}$, so there isn't a number h, making

$$p(2n-1)=\frac{1}{d\left(n-1\right)}<\frac{h\cdot A_{n,n}+\left(\frac{2n!}{2}-A_{n,n}\right)}{2n!}<\frac{1}{d\left(n\right)}=p(2n+1)$$

satisfies all the $n\in N$. It is to say the sum aggregate of those intervals $H\left(i\right)$ is an empty set.

Theorem 6 We can also find that there is a coincidence that all the right side of the intervals $H\left(i\right)$ can be represented by the form of $\frac{n}{2n+1}$.

Proof The right side of the interval $H\left(2n\right)$, which can be denoted by h, satisfies:

$$\frac{h\cdot A_{n,n}+\left(\frac{2n!}{2}-A_{n,n}\right)}{2n!}=p\left(2n+1\right)=\frac{\frac{\left(2n+1\right)!-A_{n,n+1}}{2}}{\left(2n+1\right)!}$$

$$\Leftrightarrow\left(2n+1\right)\left(2h\cdot A_{n,n}+2n!-2A_{n,n}\right)=\left(2n+1\right)!-A_{n,n+1}$$
$$\Rightarrow\left(2-2h\right)A_{n,n}=A_{n,n+1}$$
$$\Rightarrow h=\frac{n}{2n+1}.$$

5 The philosophy significance of Tian Ji's chance of winning

We increase the $2n+1$, just to increase the distance between two team, as every horse you add to Tian Ji's team is less faster than that of the King's. So when I first did the research, I guessed that Tian Ji's chance was decreasing until an unbelievable limit, which is very low. However, after I have known that it is actually increasing, I must think about it. Why would it be so?

Actually under many branches of philosophy, there is a concept of **contradiction[3]**, which stands for *the unity of opposites*. We just watch Tian Ji's winning and the King's winning as opposites, then what is the result? Contradiction tells us that conflicting unites the opposites from departing them. Maths also tells that Tian Ji's chance of winning is increasing while the King's is decreasing. Conflicting also unites them from departing them.

Then what is the eventual consequence of the unity? It is a **new balance[3]**, just like Ying and Yang(the two extremes in ancient Chinese culture)will reach a 'medium state', which is to say Tian Ji's chance of winning will reach a peculiar value when the number of horses is approaching infinitely large. Maths also tells us that there is a limit.

The coincidence between Tian Ji and the King, as well as the connections between philosophy and Maths, needs more exploration.

6 Generating Function with double arguments

For the recurrence of progression $\{A_{i,j}\}$

$$A_{i,j} = (i+1)\cdot A_{i,j-1} + j\cdot A_{i-1,j} \qquad (**)$$

If we suppose the generating function of double index progression $\{A_{i,j}\}$ is $f(x,y) = \sum_{i,j=0}^{\infty} A_{i,j}\cdot x^i y^j$.

Then we can get a result in the form of functional equation:

Theorem 6 $f(x,y) = \sum_{i,j=0}^{\infty} A_{i,j}\cdot x^i y^j$ satisfies

$$\frac{\partial}{\partial x}\left(f(x,y)\cdot x\cdot y\right) + \left[\frac{\partial}{\partial y}\left(f(x,y)\cdot x\right)\right]\cdot y + x^2 y + 4xy^2 + y^3 = f(x,y)$$

Corollary

$$x\cdot y\cdot f_x^{'}(x,y) + x\cdot f_y^{'}(x,y) + x^2 y + 4xy^2 + y^3 = (1-y)\cdot f(x,y)$$

Explanation As it is a Generating Function with double arguments which has something to do with derivative, and it is rarely discovered in references of Combinatory Theory, or references of Functions, I just work out the final result in the above, and find it hard to move any further.

7 Tian Ji Distribution

7.1 The Graph of Tian Ji Distribution

In the proof of Theorem 2, the relation of $A_{n-1,n}$ and $A_{n,n-1}$ is the key to success, and it relates to the data in the whole row. So we just take out an odd row, and observe it.

We consider the situation of 15, and observe the 15th row:

1 32752 13824089 848090912 15041229521 102776998928 311387598411 447538817472 311387598411 102776998928 15041229521 848090912 13824089 32752 1

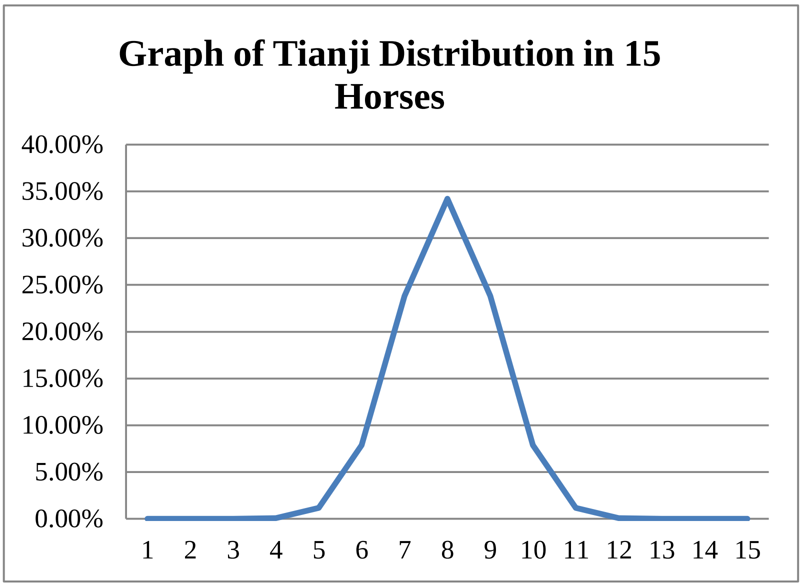
Then we set the probability of every result $\left(\frac{A_{i,n-i}}{n!}\right)$ in a row. (correct to several former digits)

$$\frac{1}{15!}, 2.504\times10^{-8}, 1.057\times10^{-5}, 6.485\times10^{-4}, 0.0115, 0.0786, 0.2381,$$

$$0.3422, 0.2381, 0.0786, 0.0115, 6.485\times10^{-4}, 1.057\times10^{-5}, 2.504\times10^{-8}, \frac{1}{15!}$$

Definition We call the distribution like this as **Tian Ji Distribution**.

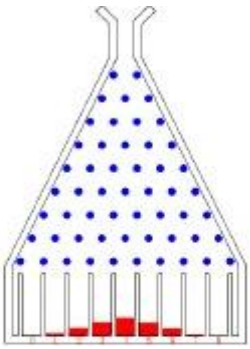
We set a graph of Tian Ji Distribution in 15 horses:



Graph 12 (Graph of Tian Ji Distribution in 15 Horses)

7.2 Baffle model of Tian Ji Distribution

For the Binomial Distribution of Pascal triangle, we can find the baffle model like this: In the baffle model, the ball is put above, after it meets a small stick, it turns to two sides of equal probability. After it rolls into a row, it will get a distribution of probability, and it is called **Binomial Distribution**. If we multiply the distribution data with 2^k (k is the number of row), it comes the data of Pascal triangle.



Graph 13 (Baffle model of Tian Ji Distribution)

To Tian Ji Distribution, we also have the similar baffle model. When a ball meets a small stick, it will not simply turns to two sides of equal probability, but have different probability: To the stick below the $A_{i,j}$, it has a probability of $\frac{i+1}{1+i+j}$ to turn left, a probability of $\frac{j}{1+i+j}$ to turn right. Similarly, if we multiply the distribution data with $n!$, it comes the data of Tian Ji's triangle.

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