An Asymptotic Formula in Number Theory

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Abstract

Let $\mathrm{r}(n)$ denote the arithmetic function whose Dirichlet series is

$$\frac{\zeta^2(2s-2)}{\zeta^2(4s-4)}\prod_p(1+p(p^s-1)^{-1}).$$

We obtain the asymptotic formula

$$\sum_{n\le x}\mathrm{r}(n) = \frac{225}{2\pi^4}x^2\prod_p\left(1-\frac{1}{p(p+1)}\right)+\frac{2\zeta(\frac{1}{2})}{\pi^2}x^{\frac{3}{2}}\prod_p\left(1-\frac{1}{p^{\frac{3}{2}}(p+p^{\frac{1}{2}}+1)}\right) \quad (1)$$
$$+\mathcal{O}(x^{1.417+\epsilon(2x)}\log x),$$

by applying Perron's formula to the Dirichlet series of $\mathrm{r}(n)$ where $\epsilon(x)=\dfrac{1+o(1)}{\log(\log x)}$.

Contents

1 Introduction

Let rad(n) denote the radical of an integer n, which is the product of the distinct prime numbers dividing n, or equivalently,

$$\mathrm{rad}(n) = \prod_{\substack{p|n \\ p \text{ prime}}} p.$$

Assume rad$(1) = 1$, so that rad(n) is multiplicative.
The best estimation for $\sum_{n\leq x} \mathrm{rad}(n)$ is

$$\sum_{n\leq x} \mathrm{rad}(n) = \frac{x^2}{2} \prod_p \left(1 - \frac{1}{p(p+1)}\right) + \mathcal{O}(x^{\frac{3}{2}}),$$

obtained by E.Cohen in his articles [6] and[7]. Alternative derivations for Cohen's result are available in [5] and Tenenbaum's book [2]

However, in this thesis, we obtain asymptotics for the sum of the arithmetic function r(n) which is closely related to rad(n). The Dirichlet series of r(n) satisfies

$$\begin{aligned} R(s) &= \frac{\zeta^2(2s-2)}{\zeta^2(4s-4)} \sum_{n\geq 1} \frac{\mathrm{rad}(n)}{n^s} \\ &= \left[\prod_p \left(1 + \frac{1}{p^{2s-2}}\right)\right]^2 \prod_p \left(1 + \frac{p}{p^s - 1}\right). \end{aligned} \tag{2}$$

If we define

$$q(n) = \begin{cases} 1 & \text{if } n \text{ is a square number,} \\ 0 & \text{if } n \text{ is not a square number,} \end{cases} \tag{3}$$

we can express r(n) as the Dirichlet convolution rad $* q(n)|\mu(\sqrt{n})|n * q(n)|\mu(\sqrt{n})|n$, or equivalently

$$\mathrm{r}(n) = \sum_{def=n} \mathrm{rad}(d)q(e)q(f)|\mu(\sqrt{e})\mu(\sqrt{f})|ef,$$

where d, e and f are divisors of n.
Applying Perron's formula to the Dirichlet series of r(n), we can have estimates that involve more main terms and smaller error term. Further research beyond this thesis may be conducted in the future to recover estimates of $\sum_{n\leq x} \mathrm{rad}(n)$ from $\sum_{n\leq x} \mathrm{r}(n)$.

2 Outline of Proof

Let $R(s)$ denote the Dirichlet series of r(n), which is

$$\begin{aligned} R(s) &= \sum_{n\geq 1} \frac{\mathrm{r}(n)}{n^s} \\ &= \frac{\zeta^2(2s-2)}{\zeta^2(4s-4)} \prod_p \left(1 + \frac{p}{p^s-1}\right) \\ &= \frac{\zeta(s)\zeta(s-1)\zeta(2s-2)}{\zeta(4s-4)} \prod_p (1 - \frac{1}{p^{4s-4}} - \frac{1}{p^{3s-2}} - \frac{1}{p^s} + \frac{1}{p^{2s-1}} + \frac{1}{p^{4s-3}}). \end{aligned} \tag{4}$$

Our aim is to extract more information about $\sum_{n\leq x} \mathrm{r}(n)$ by applying Perron's formula to

$R(s)$.

First, we use an effective form of Perron's formula to derive

$$\sum_{n\leq x} \mathrm{r}(n) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} R(s)x^s\frac{ds}{s} + \mathcal{O}(x^{1+\epsilon(2x)}) + \mathcal{O}(\frac{x^{2+1+\epsilon(2x)}\log x}{T}). \tag{5}$$

Using some suitable contour, we can then apply the residue theorem to obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-iT}^{c+iT} R(s)\frac{x^s}{s}ds =& \frac{225}{2\pi^4}x^2\prod_p\left(1-\frac{1}{p(p+1)}\right) + \frac{2\zeta(\frac{1}{2})}{\pi^2}x^{\frac{3}{2}}\prod_p\left(1-\frac{1}{p^{\frac{3}{2}}(p+p^{\frac{1}{2}}+1)}\right)\\ &+\frac{1}{2\pi i}\left(-\int_{d-iT}^{c-iT}+\int_{d+iT}^{c+iT}+\int_{d-iT}^{d+iT}\right)R(s)\frac{x^s}{s}ds. \end{aligned}\tag{6}$$

The remaining work is to estimate the integral on the right hand side of (6).
Combining some results on the Riemann Zeta function $\zeta(s)$, we can get

$$\left(-\int_{d-iT}^{c-iT}+\int_{d+iT}^{c+iT}+\int_{d-iT}^{d+iT}\right)R(s)\frac{x^s}{s} = \mathcal{O}(\frac{x^{1.51}}{T^{0.84}}) + \mathcal{O}(\frac{x^2}{T}) + \mathcal{O}(x^{1.3}T^{0.2}\sqrt{\log T}). \tag{7}$$

Choosing $T = x^{0.583}$ and putting the estimates together into (5), we can finally obtain the asymptotic formula

$$\begin{aligned}\sum_{n\leq x}\mathrm{r}(n) =& \frac{225}{2\pi^4}x^2\prod_p\left(1-\frac{1}{p(p+1)}\right) + \frac{2\zeta(\frac{1}{2})}{\pi^2}x^{\frac{3}{2}}\prod_p\left(1-\frac{1}{p^{\frac{3}{2}}(p+p^{\frac{1}{2}}+1)}\right)\\ &+\mathcal{O}(x^{1.417+\epsilon(2x)}\log x).\end{aligned}\tag{8}$$

3 Manipulations

Throughout the following sections, we follow the convention $s = \sigma + it$, where σ and t denote the real part ($\Re$) and imaginary part ($\Im$) of s, respectively.

3.1 Effective Perron's Formula

Let $F(s) = \sum_{n=1}^{\infty}\frac{a(n)}{n^s}$ be a Dirichlet series with finite abscissa of absolute convergence σ_α, with $a(n)$ being an arbitrary arithmetic function.
Suppose that there exists some real number $\alpha \geq 0$ such that, for $\sigma > \sigma_a$,

$$\sum_{n\leq 1}\frac{|a(n)|}{n^s} = \mathcal{O}((\sigma-\sigma_a)^{-\alpha}),$$

and there exists a non-decreasing function $B(x)$ satisfying $|a(n)| < B(n)$.
Then for $x \geq 2, T \geq 2, \Re(s) = \sigma \leq \sigma_a, c = \sigma_a - \sigma + \frac{1}{\log x}$, we have

$$\sum_{n\leq x}\frac{a(n)}{n^s} = \frac{1}{2\pi i}\int_{c-iT}^{c+iT}F(s+w)x^w\frac{dw}{w} + \mathcal{O}\left(x^{\sigma_a-\sigma}\frac{(\log x)^\alpha}{T} + \frac{B(2x)}{x^\sigma}\left(1+x\frac{\log x}{T}\right)\right).$$

For the proof, see Tenenbaum's book [2].

Since $R(s)$ has a simple pole at $s = 2$, to get $\sum_{n \le x} \mathrm{r}(n)$, we can apply the theorem to $R(s)$ with $s = 0, \sigma = 0, \alpha = 1, c = 2 + \frac{1}{\log x}, B(x) = x^{1+\epsilon(2x)}$ and we have

$$\sum_{n \le x} \mathrm{r}(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} R(w) x^w \frac{dw}{w} + \mathcal{O}(x^{1+\epsilon(2x)}) + \mathcal{O}(\frac{x^{2+\epsilon(2x)} \log x}{T}).$$

More specifically, to determine $B(x)$, we begin by

$$R(s) < \prod_p (1 + e\frac{p}{p^s} + e\frac{p^2}{p^{2s}} + e\frac{p^3}{p^{3s}} + e\frac{p^4}{p^{4s}} + \cdots),$$

where $e = 2.718\cdots$ is the base of the natural logarithm. Thus we have

$$r(n) < e^{\omega(n)} n,$$

where $\omega(n)$ is the arithmetic function that counts the number of distinct primes dividing n. Using estimates for $\omega(n)$ from Tenebaum's book [2], we can get

$$\omega(n) < \epsilon(n) \log n = \frac{\log n}{\log(\log n)} \left(1 + o(1)\right),$$

where $\epsilon(n) = \dfrac{1 + o(1)}{\log(\log n)}$, so that we have

$$B(2x) = (2x)^{1+\epsilon(2x)} = \mathcal{O}(x^{1+\epsilon(2x)})$$

here.

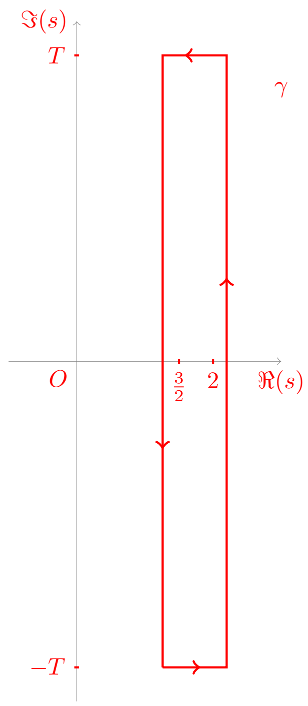

Figure 1: Contour γ

3.2 Evaluation of Integral

To evaluate the integral, we can use a rectangular contour γ with four corners at $c-iT$, $c+iT$, $d+iT$ and $d-iT$, as shown in Figure 1, where $d=1.3$ and $c=2+\frac{1}{\log x}$.
To be convenient, we'll use s instead of w in the integral.
By residue theorem,

$$\frac{1}{2\pi i}\oint_{\gamma} R(s)\frac{x^s}{s}ds = \frac{x^2}{2}\text{Res}[R(s),2] + \frac{2x^{\frac{3}{2}}}{3}\text{Res}[R(s),\frac{3}{2}].$$

For

$$R(s) = \frac{\zeta(s)\zeta(s-1)\zeta(2s-2)}{\zeta(4s-4)}\prod_p(1-\frac{1}{p^{4s-4}}-\frac{1}{p^{3s-2}}-\frac{1}{p^s}+\frac{1}{p^{2s-1}}+\frac{1}{p^{4s-3}}), \tag{9}$$

using

$$\lim_{s\to 1}(s-1)\zeta(s) = 1,$$

we can get

$$\begin{aligned}
\text{Res}[R(s),2] &= \lim_{s\to 2}(s-2)R(s)\\
&= \lim_{s\to 2}\frac{\zeta^2(2s-2)}{\zeta^2(4s-4)}[(s-1)-1]\zeta(s-1)\zeta(s)\prod_p(1-\frac{1}{p^{2s-2}}-\frac{1}{p^s}+\frac{1}{p^{2s-1}})\\
&= \frac{\zeta^2(2)}{\zeta^2(4)}\zeta(2)\prod_p\left(1-\frac{2p-1}{p^3}\right)\\
&= \frac{\zeta^2(2)}{\zeta^2(4)}\prod_p\left(\frac{1-\frac{2p-1}{p^3}}{1-\frac{1}{p^2}}\right)\\
&= \frac{\zeta^2(2)}{\zeta^2(4)}\prod_p\left(1-\frac{1}{p(p+1)}\right),
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
\text{Res}[R(s),\frac{3}{2}] &= \lim_{s\to\frac{3}{2}}(s-\frac{3}{2})R(s)\\
&= \lim_{s\to\frac{3}{2}}\frac{1}{2}[(2s-2)-1]\zeta(2s-2)\frac{\zeta(s)\zeta(s-1)}{\zeta(4s-4)}\\
&\quad \prod_p(1-\frac{1}{p^{4s-4}}-\frac{1}{p^{3s-2}}-\frac{1}{p^s}+\frac{1}{p^{2s-1}}+\frac{1}{p^{4s-3}})\\
&= \frac{1}{2}\frac{\zeta(\frac{3}{2})\zeta(\frac{1}{2})}{\zeta(2)}\prod_p\left(1-\frac{p^{\frac{3}{2}}+p^{\frac{1}{2}}-1}{p^3}\right)\\
&= \frac{1}{2}\frac{\zeta(\frac{1}{2})}{\zeta(2)}\prod_p\left(\frac{1-p^{-\frac{3}{2}}-p^{-\frac{5}{2}}+p^{-3}}{1-p^{-\frac{3}{2}}}\right)\\
&= \frac{1}{2}\frac{\zeta(\frac{1}{2})}{\zeta(2)}\prod_p\left(1-\frac{1}{p^{\frac{3}{2}}(p+p^{\frac{1}{2}}+1)}\right).
\end{aligned} \tag{11}$$

Putting the residue back into the formula, we have obtained

$$\begin{aligned}\sum_{n\le x}\mathrm{r}(n) &= \frac{1}{2\pi i}\int_{c-iT}^{c+iT} R(s)\frac{x^s}{s}ds + \mathcal{O}(x^{1+\epsilon(2x)}) + \mathcal{O}(\frac{x^{2+\epsilon(2x)}\log x}{T}) \\ &= \frac{225}{2\pi^4}x^2\prod_p\left(1-\frac{1}{p(p+1)}\right) + \frac{2\zeta(\frac{1}{2})}{\pi^2}x^{\frac{3}{2}}\prod_p\left(1-\frac{1}{p^{\frac{3}{2}}(p+p^{\frac{1}{2}}+1)}\right) \\ &\quad + \frac{1}{2\pi i}\left(-\int_{d-iT}^{c-iT}+\int_{d+iT}^{c+iT}+\int_{d-iT}^{d+iT}\right)R(s)\frac{x^s}{s}ds \\ &\quad + \mathcal{O}(x^{1+\epsilon(2x)}) + \mathcal{O}(\frac{x^{2+\epsilon(2x)}\log x}{T}). \end{aligned} \tag{12}$$

It remains to estimate the integral on the three sides other than $\Re(s) = c$.

3.3 Estimates for Bounded Factors in $R(s)$

We choose $d = 1.3$. In the rectangle contour, we have uniformly that,

$$|\zeta(s)| < \zeta(\sigma) < \zeta(1.3),$$

Let $P(s)$ denote the product

$$\prod_p\left(1-\frac{1}{p^{4s-4}}-\frac{1}{p^{3s-2}}-\frac{1}{p^s}+\frac{1}{p^{2s-1}}+\frac{1}{p^{4s-3}}\right).$$

Since

$$\sum_p\left(-\frac{1}{p^{4s-4}}-\frac{1}{p^{3s-2}}-\frac{1}{p^s}+\frac{1}{p^{2s-1}}+\frac{1}{p^{4s-3}}\right)$$

converges absolutely for $\Re(s) \ge d$.
we have

$$P(s) = \mathcal{O}(1)$$

in the rectangle.

Using Euler Product, we can show that $|\frac{1}{\zeta(s)}| < \frac{\zeta(\sigma)}{\zeta(2\sigma)}$, so that we have

$$\frac{1}{\zeta(4s-4)} = \mathcal{O}(1)$$

in the rectangle.

3.4 Modified Lindölef's Theorem

Let Ω be a half-strip in the complex plane

$$\Omega = \{s \in \mathbb{C} | \sigma_1 \le \Re(s) = \sigma \le \sigma_2 \text{ and } \Im(s) = t \ge t_0 > 0\} \subsetneq \mathbb{C}.$$

Suppose that f is holomorphic on Ω. If p, q are such constants that $|f(s)| = \mathcal{O}(t^p)$ on $\Re(s) = \sigma_1$ and $|f(s)| = \mathcal{O}(t^q)$ on $\Re(s) = \sigma_2$, and if there is a constant A such that $\frac{|f(\sigma+it)|}{t^A}$ is bounded on Ω, then

$$|f(\sigma+it)| = \mathcal{O}(t^{k(\sigma)})$$

throughout Ω, where

$$k(\sigma) = \frac{q-p}{\sigma_2 - \sigma_1}(\sigma - \sigma_1) + p$$

is the affine function which is p at σ_1 and q at σ_2. For proof, see Edwards' monograph [1].

Also needed from Titchmarsh's monograph [3] and Tenenbaum's book [2] is the conclusion that for $t \geq 1$, we have

$$\zeta(s) = \mathcal{O}\left(t^{\kappa(\sigma)} \log t\right),$$

where

$$\kappa(\sigma) \leq \begin{cases} \frac{1}{3}(1-\sigma) & \text{for } \frac{1}{2} < \sigma \leq 1, \\ \frac{1}{6}(3-4\sigma) & \text{for } 0 \leq \sigma \leq \frac{1}{2}. \end{cases} \tag{13}$$

3.5 Integrals on the Horizontal Sides

To estimate $\zeta(s-1)\zeta(2s-2)$ in the whole rectangle, we can apply the Lindölef's theorem separately to $\zeta(s-1)$ and $\zeta(2s-2)$.

First we partition the rectangle into two sets, with their real parts satisfying $1.3 \leq \sigma \leq 1.51$ and $1.51 < \sigma \leq c$, respectively. Then we can use estimate (13) to derive that, for $t \geq 1$ on $\sigma = 1.3$,

$$\zeta(s-1)\zeta(2s-2) = \mathcal{O}(t^{0.44} \log^2 t),$$

on $\sigma = 1.51$,

$$\zeta(s-1)\zeta(2s-2) = \mathcal{O}(t^{0.16} \log t),$$

and on $\sigma = c$,

$$\zeta(s-1)\zeta(2s-2) = \mathcal{O}(1).$$

Since $\zeta(s-1)\zeta(2s-2)$ is bounded by polynomial of t on the strip

$$\Omega = \{s \in \mathbb{C} | 1.3 \leq \Re(s) = \sigma \leq c \text{ and } \Im(s) = t \geq t_0 > 0\}$$

(see Ford's thesis [4]), we can apply modified Lindölef's theorem to obtain bound for $\sigma(s-1)\sigma(2s-2)$ in the two strips $1.3 \leq \sigma \leq 1.51$ and $1.51 < \sigma \leq c$ separately.
In $1.3 \leq \Re(s) = \sigma \leq 1.51$,

$$\sigma(s-1)\sigma(2s-2) = \mathcal{O}(t^{k_1(\sigma)}),$$

where $k_1(\sigma)$ is the affine function that reaches 0.44 on $\sigma = 1.3$ and 0.16 on $\sigma = 1.51$.
Similarly, in $1.51 \leq \Re(s) = \sigma \leq c$,

$$\sigma(s-1)\sigma(2s-2) = \mathcal{O}(t^{k_2(\sigma)}),$$

where $k_2(\sigma)$ is the affine function that reaches 0 on $\sigma = c$ and 0.16 on $\sigma = 1.51$.

Using the results in Section 3.3, we can get

$$\left|\int_{d+iT}^{c+iT} R(s)\frac{x^s}{s} ds\right| = \frac{1}{T}\mathcal{O}\left(\int_{d+iT}^{c+iT} |\zeta(s-1)\zeta(2s-2)x^s||ds|\right).$$

Using the estimation for$\zeta(s-1)\zeta(2s-2)$ obtained above, we have

$$\begin{aligned}\int_{d+iT}^{c+iT} |\zeta(s-1)\zeta(2s-2)x^s||ds| &\leq \int_{1.3}^{1.51} |\zeta(\sigma-1+it)\zeta(2\sigma-2+2it)|x^\sigma d\sigma \\ &+ \int_{1.51}^{c} |\zeta(\sigma-1+it)\zeta(2\sigma-2+2it)|x^\sigma d\sigma \\ &= \mathcal{O}(x^{1.51}T^{0.44}\log T) + \mathcal{O}(x^2), \end{aligned} \tag{14}$$

For $|\int_{d-iT}^{c-iT} R(s)\frac{x^s}{s}ds|$, using the identity $\zeta(\bar{s}) = \overline{\zeta(s)}$ we can get the same estimate.
In conclusion,

$$\left(\int_{d+iT}^{c+iT} + \int_{d-iT}^{c-iT}\right)\zeta(s-1)\zeta(2s-2)x^s ds = \mathcal{O}(\frac{x^{1.51}\log T}{T^{0.56}}) + \mathcal{O}(\frac{x^2}{T}) \tag{15}$$

3.6 Integral on the Vertical Side

To estimate the integral on the vertical side, we need estimate for mean value of $|\zeta(s)|^2$ in the rectangle.
For $\frac{1}{2} < \sigma < 1$, we have

$$\int_1^T |\zeta(\sigma+it)|^2 dt = \mathcal{O}(T). \tag{16}$$

For proof, see the monographs [3] and [1].

First, we separate the mean value part.

$$\begin{aligned}\int_d^{d+iT} R(s)\frac{x^s}{s}ds &= \int_d^{d+i} R(s)\frac{x^s}{s} + \int_{d+i}^{d+iT} R(s)\frac{x^s}{s} \\ &= \mathcal{O}(x^{1.3}) + x^{1.3}\mathcal{O}(\int_1^T |\frac{\zeta(s-1)\zeta(2s-2)}{s}|dt), \end{aligned} \tag{17}$$

and then

$$\begin{aligned}\int_1^T |\zeta(s-1)\zeta(2s-2)||\frac{dt}{s}| &< \int_1^T |\zeta(s-1)\zeta(2s-2)||\frac{dt}{t}| \\ &\leq \sqrt{\left(\int_1^T |\zeta(0.3+it)|^2\frac{dt}{t}\right)\left(\int_1^T |\zeta(0.6+2it)|^2\frac{dt}{t}\right)}, \end{aligned} \tag{18}$$

where we use the Cauchy-Schwarz inequality in the second line.

Let $f(T) = \int_1^T |\zeta(0.6+2it)|^2 dt$. Then we can use equation (16) to obtain

$$f(T) = \frac{1}{2}\int_1^T |\zeta(0.6+2it)|^2 d(2t) = \mathcal{O}(T),$$

so we have

$$\begin{aligned}\int_1^T |\zeta(0.6+2it)|^2\frac{dt}{t} &= \frac{f(t)}{t}\bigg|_1^T + \int_1^T \frac{f(t)}{t^2}dt \\ &= \mathcal{O}(1) + \mathcal{O}(\log T). \end{aligned} \tag{19}$$

Using the functional equation for $\zeta(s)$ and the complex Stirling formula for $\Gamma(s)$, we can get, for $0 < \sigma < 1$,

$$\zeta(s) = \mathcal{O}(t^{\frac{1}{2}-\sigma}|\zeta(1-s)|),$$

Using this estimate, we can get

$$\begin{aligned}\int_1^T |\zeta(0.3+it)|^2 \frac{dt}{t} &= \int_1^T |\zeta(0.3-it)|^2 \frac{dt}{t} \\ &= \mathcal{O}(\int_1^T t^{0.4}|\zeta(0.7+it)|^2 \frac{dt}{t}) \\ &= \mathcal{O}(\int_1^T |\zeta(0.7+it)|^2 \frac{dt}{t^{0.6}}). \end{aligned} \tag{20}$$

Using methods similar to (10), we can get

$$\int_1^T |\zeta(0.3+it)|^2 \frac{dt}{t} = \mathcal{O}(T^{0.4}).$$

Combining these results into (18), we can get

$$\int_1^T |\frac{\zeta(s-1)\zeta(2s-2)}{s}|dt = \mathcal{O}(T^{0.2}\sqrt{\log T}).$$

Similarly, we have

$$\int_{-T}^1 |\frac{\zeta(s-1)\zeta(2s-2)}{s}|dt = \mathcal{O}(T^{0.2}\sqrt{\log T}).$$

In conclusion,

$$\int_{d-iT}^{d+iT} R(s)\frac{x^s}{s}ds = \mathcal{O}(x^{1.3}T^{0.2}\sqrt{\log T}). \tag{21}$$

3.7 Conclusion

Combining the estimates in the previous sections we can get, for x big enough,

$$\begin{aligned}\sum_{n\le x} \mathrm{r}(n) = \frac{225}{2\pi^4}x^2 \prod_p \left(1-\frac{1}{p(p+1)}\right) + \frac{2\zeta(\frac{1}{2})}{\pi^2}x^{\frac{3}{2}}\prod_p\left(1-\frac{1}{p^{\frac{3}{2}}(p+p^{\frac{1}{2}}+1)}\right) \\ + \mathcal{O}(x^{1+\epsilon(2x)}) + \mathcal{O}(\frac{x^{2+\epsilon(2x)}\log x}{T}) + \mathcal{O}(\frac{x^{1.51}\log T}{T^{0.56}}) \\ + \mathcal{O}(\frac{x^2}{T}) + \mathcal{O}(x^{1.3}T^{0.2}\sqrt{\log T}). \end{aligned} \tag{22}$$

Choosing $T = x^{0.583}$, we can get

$$\begin{aligned}\sum_{n\le x} \mathrm{r}(n) =& \frac{225}{2\pi^4}x^2 \prod_p \left(1-\frac{1}{p(p+1)}\right) + \frac{2\zeta(\frac{1}{2})}{\pi^2}x^{\frac{3}{2}}\prod_p\left(1-\frac{1}{p^{\frac{3}{2}}(p+p^{\frac{1}{2}}+1)}\right) \\ &+ \mathcal{O}(x^{1.417+\epsilon(2x)}\log x), \end{aligned} \tag{23}$$

where $\epsilon(x) = \dfrac{1+o(1)}{\log(\log x)}$. For $\epsilon(x)$, detailed calculations show that we have $1.417+\epsilon(2x) < 1.5$ when $x < e^{92}$ and $x > e^{e^{21.8}}$.

4 Some Further Thoughts

Recently, the author was considering asymptotics for the sum $\sum_{n\le x}\frac{1}{\mathrm{rad}(n)}$. Using results from Tenebaum's book [2], we can prove the limit

$$\lim_{x\to+\infty}\frac{\sum_{n\le x}\frac{1}{\mathrm{rad}(n)}}{x} = 0.$$

Moreover, generalizing the results from [9], we can show that the sum $\sum_{n\le x}\frac{1}{\mathrm{rad}(n)}$ grows faster than $C_A(\log x)^A$ for any $A>0$ where C_A is a positive constant depending on A. However, the author has not derived any aymptotics now.

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