The Rationality of the Vertices of a Regular Simplex or Semiregular Simplex in $\mathbb{R}^n$ or L^P-space over $\mathbb{R}^2$

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Abstract: When the vertices of the undersurface of a regular simplex or semiregular simplex in $\mathbb{R}^n(n \geq 2)$ or L^P-space over $\mathbb{R}^2$ are all rational, whether the remaining vertex is rational is discussed in the article.
Key words: $\mathbb{R}^n$ space; L^P-space over $\mathbb{R}^2$; (semi)regular simplex; rational point

0 Introduction

Our teacher once led our discussion of similar problems about $\mathbb{R}^2$ square and $\mathbb{R}^3$ cube in a certain Descartes coordinate system of Euclidean space, which led to very different results. Due to this, one of the authors has studied the following problem: *If the vertices of a facet of $\mathbb{R}^n$ cube are all rational, whether the vertices of the opposite facet are rational.* The research led to the following results: *When n is even, the vertices mentioned above must be rational; and when n is odd, they are rational iff the length of the cube's edges is rational*[1]. On the other hand, we can not only extend our problem to other values of dimension, but also to other sorts of geometric figures. One of our classmates has once studied regular tetrahedron and regular octahedron in $\mathbb{R}^3$, which led to a same result that *If the vertices of one of its face are rational, then other vertices are rational iff the length of the edges are rational times of $\sqrt{2}$.*

This time, following the suggestion from our instructor, we generalize the regular tetrahedron to other values of dimension, which is regular simplex. Then we extend further to semiregular simplex, and studied it.

We claim that, in the following part of the paper, without other specification, our discussion is based on a certain given Descartes coordinate system. Firstly, we give the definition of regular simplex and semiregular simplex:

Definition 0.1 We define that, a $\mathbb{R}^n(n \geq 1)$ *regular simplex* is a geometric figure that is derived by connecting $n+1$ points in $\mathbb{R}^k(k \geq n)$ space, that the distances between any two of these points are constant. The distance mentioned above is called the *edge length* of the regular simplex, denoted by *l*. A geometric figure that is derived by connecting any certain *n* points out of $n+1$ mentioned above is then called the *undersurface* of this regular simplex, while the remaining vertex is called the *apex* of this regular simplex.

Definition 0.2 We define that, a $\mathbb{R}^n(n \geq 2)$ *semiregular simplex* is a geometric figure that is derived by connecting the vertices of a $\mathbb{R}^{n-1}$ regular simplex with a certain point that is not on the minimal hyperplane that contains the regular simplex mentioned above, that the distance from that certain point to the different vertices of the regular simplex is constant. We call the $\mathbb{R}^{n-1}$ regular simplex just mentioned the *undersurface* of this semiregular simplex, which its edge length is called the *edge length* of this semiregular simplex, also denoted by *l*. And we call that certain point the *apex* of this semiregular simplex.

We then give the definition of rational points and rational vectors:

Definition 0.3 We define that, a *rational point* in $\mathbb{R}^n$ is a point that all of its *n* coordinates are rational. Similarly, a *rational vector* in $\mathbb{R}^n$ is a vector given all of its *n* components rational.

Our proof strategy is similar to the strategy used by the author and the classmate mentioned above. Particularly, we find a certain rational point on the undersurface of the given regular simplex or semiregular simplex, which we have assumed that the vertices of this undersurface are all rational. Then we set this point as the origin, without loss of generality. It is followed that the vectors pointed from the origin to the vertices of the undersurface are all rational. Based on these vectors, we are able to construct a rational vector that is collinear with the vector pointed from the origin to the apex of this regular or semiregular simplex, that we can compute the ratio of the modulus of the two vectors, which then leads to the rationality of the apex.

We proved that:

Theorem 0.4 (Main Theorem 1) *Given a* $\mathbb{R}^n(n \geq 2)$ *semiregular simplex in* $\mathbb{R}^n$ *space, if its undersurface's vertices are all rational, then, when n is even, the apex is rational iff* $lh\sqrt{2n}$ *is rational; when n is odd, the apex is rational iff* $h\sqrt{n}$ *is rational.*

The notation *h* used here denotes the altitude of this semiregular simplex, which can be understood as the length of the line segment through the apex and perpendicular to the undersurface, but that is not a suitable definition. For that, we use an equivalent definition, see **Definition 3.2.1** in ***Chapter 3***.

Theorem 0.5 (Main Theorem 2) *Given a* $\mathbb{R}^n(n \geq 2)$ *regular simplex in* $\mathbb{R}^n$ *space, if its undersurface's vertices are all rational, then, when n is even, the apex is rational iff* $\sqrt{n+1}$ *is rational; when n is odd, the apex is rational iff* $l\sqrt{2n+2}$ *is rational.*

On the other hand, we'll also show that:

Theorem 0.6 (Main Theorem 3) *Given a* $\mathbb{R}^n(n \geq 2)$ *semiregular or regular simplex in* $\mathbb{R}^n$ *space, if its undersurface's vertices are all rational, then the apex is rational iff its volume is rational.*

We not only generalize our problem in respect of values of dimension, but also in respect of the definition of the distance. We studied the situation in a L^P-space over $\mathbb{R}^2$. Firstly we claim that, similar to the situation in $\mathbb{R}^n$. We set up a certain Descartes coordinate system in the given L^P-space over $\mathbb{R}^2$. Then similarly we may define *rational point*, *regular simplex* and *edge length of regular simplex*, which we won't give details here.

Euler, in 1769, conjectured, when $n > 1$, if:

$$\sum_{i=1}^{n} a_i^k = b^k$$

Where $a_i(\forall i \in \{1,2,\dots,n\})$ and b are all nontrivial integers, then $n \geq k$[2]. The conjecture was shown to be false by L. J. Lander and T. R. Parkin in 1966[3]. But then they, together with J. L. Selfridge, based on the Euler's conjecture just mentioned, conjectured in 1967 that:

Conjecture 0.7[4] *When* $k > 3$, *k is an integer, and:*

$$\sum_{i=1}^{n} a_i^k = \sum_{j=1}^{m} b_j^k$$

Where m and n are not both equal to 1, $a_i \neq b_j(\forall i \in \{1,2,\dots,n\}, j \in \{1,2,\dots,m\})$ *and are all* *positive integers, then* $m+n \geq k$.

This conjecture still remains open. Here, we'll just use a version of that conjecture which is weaker and also still remaining open:

Conjecture 0.8 *When* $k \in [5,+\infty) \cap \mathbb{N}$ *and:*

$$x^k + y^k = z^k + w^k$$

Then $\{x,y\} \neq \{z,w\}$ *and x, y, z, w all being positive integers could not be both satisfied.*

Then, we introduce our result:

Theorem 0.9 (Main Theorem 4) *Given a regular simplex in* $L^P(P \in [5,+\infty) \cap \mathbb{N})$*-space over* $\mathbb{R}^2$*, if two of its vertices are both rational, and that* **Conjecture 0.8** *holds, then the last vertex cannot be rational.*

For other values of P, we also obtain some results. We introduce the results in L^1 and L^∞ space first. We define:

Definition 0.10 Given two different points in L^1-space over $\mathbb{R}^2$ $A(x,y)$ and $B(z,w)$, if $|z-x| = |w-y|$, then we call *A*, *B* a pair of *standard translation points*.

Definition 0.11 Given two different points in L^∞-space over $\mathbb{R}^2$ $A(x,y)$ and $B(z,w)$, if $x = z$ or $y = w$, then we call *A*, *B* a pair of *standard translation points*.

We can then give our results as following:

Theorem 0.12 (Main Theorem 5) *Given a regular simplex in* L^1 or L^∞*-space over* $\mathbb{R}^2$*, if two of its vertices are both rational and not a pair of standard translation points, then the last vertex must be rational.*

For the reason why we exclude the situation of standard translation points, see **Theorem 6.1.1** and **Theorem 6.1.2**. For the remaining values of *P*, confined to the ability of the authors, our results are still inchoate, see ***Section 2*** of ***Chapter 6***.

1 The Definition and Properties of Vector Product in $\mathbb{R}^n$

In this chapter, the definition and properties of vector product in $\mathbb{R}^n$ will be given, in order to construct the basic tool for the discussion of $\mathbb{R}^n$ regular and semiregular simplex followed.

Definition 1.1[5] Assuming there are $n-1$ vectors in $\mathbb{R}^n (n \geq 2)$, whose coordinates under an orthonormal basis $\boldsymbol{e_1}$, $\boldsymbol{e_2}$, …, $\boldsymbol{e_n}$ are $\boldsymbol{a_1}(k_{11}, k_{12}, \dots, k_{1n})$, $\boldsymbol{a_2}(k_{21}, k_{22}, \dots, k_{2n})$, …, $\boldsymbol{a_{(n-1)}}\left(k_{(n-1)1}, k_{(n-1)2}, \dots, k_{(n-1)n}\right)$, then their *vector product* is defined as following:

$$\boldsymbol{a_1} \times \boldsymbol{a_2} \times \dots \times \boldsymbol{a_{(n-1)}} \overset{\text{def}}{=} \begin{vmatrix} e_1 & e_2 & \cdots & e_{(n-1)} & e_n \\ k_{11} & k_{12} & \cdots & k_{1(n-1)} & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2(n-1)} & k_{2n} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ k_{(n-1)1} & k_{(n-1)2} & \cdots & k_{(n-1)(n-1)} & k_{(n-1)n} \end{vmatrix}$$

Obviously, the well-known vector product in $\mathbb{R}^3$ is the special case of the vector product we just defined with $n = 3$.

Theorem 1.2 $\left(\boldsymbol{a_1} \times \boldsymbol{a_2} \times \dots \times \boldsymbol{a_{(n-1)}}\right) \cdot \boldsymbol{a_i} = 0 (\forall i \in \{1,2,\dots,n-1\})$.

Proof $\left(\boldsymbol{a_1} \times \boldsymbol{a_2} \times \dots \times \boldsymbol{a_{(n-1)}}\right) \cdot \boldsymbol{a_i}$

$$= \begin{vmatrix} e_1 & e_2 & \cdots & e_{(n-1)} & e_n \\ k_{11} & k_{12} & \cdots & k_{1(n-1)} & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2(n-1)} & k_{2n} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ k_{(n-1)1} & k_{(n-1)2} & \cdots & k_{(n-1)(n-1)} & k_{(n-1)n} \end{vmatrix} \cdot \boldsymbol{a_i}$$

$$= \begin{vmatrix} k_{i1} & k_{i2} & \cdots & k_{i(n-1)} & k_{in} \\ k_{11} & k_{12} & \cdots & k_{1(n-1)} & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2(n-1)} & k_{2n} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ k_{(n-1)1} & k_{(n-1)2} & \cdots & k_{(n-1)(n-1)} & k_{(n-1)n} \end{vmatrix} = 0. \qquad \blacksquare$$

A direct corollary follows:

Corollary 1.3 $\boldsymbol{a_1} \times \boldsymbol{a_2} \times \ldots \times \boldsymbol{a_{(n-1)}}$ *is orthogonal to all* $\boldsymbol{a_i} (\forall i \in \{1,2,\ldots,n-1\})$.

For the discussion following, it is necessary for us to define:

Definition 1.4 We define that, a parallelogram is called a $\mathbb{R}^2$ *parallelotope*. Then, defining by induction, $\mathbb{R}^{n+1}(n \geq 2)$ *parallelotope* is a subset of the entire $\mathbb{R}^k(k \geq n+1)$ space derived from translating a $\mathbb{R}^n$ parallelotope along a vector which is not on the minimal hyperplane that contains the above-mentioned $\mathbb{R}^n$ parallelotope.

On the other hand, it is well-known that the determinant has the following properties:

1. Multi-linearity:

$$D(\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, a\boldsymbol{x_i} + \boldsymbol{b}, \ldots, \boldsymbol{x_{(n-1)}}, \boldsymbol{x_n})$$
$$= aD(\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_i}, \ldots, \boldsymbol{x_{(n-1)}}, \boldsymbol{x_n}) + D(\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{b}, \ldots, \boldsymbol{x_{(n-1)}}, \boldsymbol{x_n})$$

2. Anticommutativity:

$$D(\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_i}, \ldots, \boldsymbol{x_j}, \ldots, \boldsymbol{x_{(n-1)}}, \boldsymbol{x_n}) = 0(\boldsymbol{x_i} = \boldsymbol{x_j})$$

3. Reducibility:

$$D(\boldsymbol{e_1}, \boldsymbol{e_2}, \ldots, \boldsymbol{e_{(n-1)}}, \boldsymbol{e_n}) = 1$$

Actually, only the determinant has these three properties, see **Reference [6]**.

According to **Definition 1.4**, we observe that $\mathbb{R}^2$ parallelotope is actually parallelogram, while $\mathbb{R}^3$ parallelotope is actually parallelepiped. The area of the former: $S(\boldsymbol{a}, \boldsymbol{b})$ and the volume of the latter: $V(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ are all equipped with the three above-mentioned properties (Strictly speaking, their area or volume should be the absolute value of a certain function equipped with these properties), thus when calculating the volume of $\mathbb{R}^n(n \geq 2)$ parallelotope, it is not difficult to prove that it also has these three properties. Moreover, we know that these three properties define a unique function, which leads to the following result:

Theorem 1.5 *Assuming a* $\mathbb{R}^n(n \geq 2)$ *parallelotope in* $\mathbb{R}^n$ *having* $\boldsymbol{x_1}$, $\boldsymbol{x_2}$, ..., $\boldsymbol{x_{(n-1)}}$, $\boldsymbol{x_n}$ *as edges, then its volume can be computed as following:*

$$V(\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_{(n-1)}}, \boldsymbol{x_n}) = |D(\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_{(n-1)}}, \boldsymbol{x_n})|.$$

As for $\mathbb{R}^m$ parallelotope in $\mathbb{R}^n(n > m)$, we may construct a *m*-dimensional Descartes coordinate system on the minimal hyperplane that contains it, then using the method **Theorem 1.5** gives to compute its volume. On the other hand, the volume mentioned here inherits the properties of area in $\mathbb{R}^2$ and volume in $\mathbb{R}^3$, so when there is a vector among $\boldsymbol{x_1}$, $\boldsymbol{x_2}$, …, $\boldsymbol{x_{(n-1)}}$, $\boldsymbol{x_n}$ that is orthogonal to all other $n-1$ vectors, e.g. $\boldsymbol{x_n}$, then:

$$V(\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_{(n-1)}}, \boldsymbol{x_n}) = \|\boldsymbol{x_n}\| \cdot V(\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_{(n-1)}}).$$

Generally, when the included angle between $\boldsymbol{x_n}$ and the hyperplane defined by $\boldsymbol{x_1}$, $\boldsymbol{x_2}$, …, $\boldsymbol{x_{(n-1)}}$ is θ (For the specific definition of this included angle, see **Definition 2.2** in ***Chapter 2***), then:

$$V(\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_{(n-1)}}, \boldsymbol{x_n}) = \|\boldsymbol{x_n}\| \cdot \sin\theta \cdot V(\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_{(n-1)}}).$$

Theorem 1.6[5] $\|\boldsymbol{a_1} \times \boldsymbol{a_2} \times \dots \times \boldsymbol{a_{(n-1)}}\| = V(\boldsymbol{a_1}, \boldsymbol{a_2}, \dots, \boldsymbol{a_{(n-1)}})$.

Proof Assuming the cofactor of $\boldsymbol{e_i} (\forall i \in \{1,2,\dots,n\})$ in:

$$\begin{vmatrix} e_1 & e_2 & \cdots & e_{(n-1)} & e_n \\ k_{11} & k_{12} & \cdots & k_{1(n-1)} & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2(n-1)} & k_{2n} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ k_{(n-1)1} & k_{(n-1)2} & \cdots & k_{(n-1)(n-1)} & k_{(n-1)n} \end{vmatrix}$$

is A_i, also assuming that $\boldsymbol{a} = \boldsymbol{a_1} \times \boldsymbol{a_2} \times \dots \times \boldsymbol{a_{(n-1)}}$, then:

$$V(\boldsymbol{a}, \boldsymbol{a_1}, \boldsymbol{a_2}, \dots, \boldsymbol{a_{(n-1)}})$$

$$= \left\| \begin{matrix} A_1 & A_2 & \cdots & A_{n-1} & A_n \\ k_{11} & k_{12} & \cdots & k_{1(n-1)} & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2(n-1)} & k_{2n} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ k_{(n-1)1} & k_{(n-1)2} & \cdots & k_{(n-1)(n-1)} & k_{(n-1)n} \end{matrix} \right\|$$

$$= A_1^2 + A_2^2 + \cdots + A_{(n-1)}^2 + A_n^2$$

On the other hand:

$$V(\boldsymbol{a}, \boldsymbol{a_1}, \boldsymbol{a_2}, \dots, \boldsymbol{a_{(n-1)}})$$

$$= \|\boldsymbol{a}\| \cdot V(\boldsymbol{a_1}, \boldsymbol{a_2}, \dots, \boldsymbol{a_{(n-1)}})$$

$$= V(\boldsymbol{a_1}, \boldsymbol{a_2}, \dots, \boldsymbol{a_{(n-1)}}) \cdot \sqrt{A_1^2 + A_2^2 + \cdots + A_{(n-1)}^2 + A_n^2}$$

Thus:

$$V(\boldsymbol{a_1}, \boldsymbol{a_2}, \dots, \boldsymbol{a_{(n-1)}}) = \sqrt{A_1^2 + A_2^2 + \cdots + A_{(n-1)}^2 + A_n^2} = \|\boldsymbol{a}\|. \qquad \blacksquare$$

The theorem gives us a new way to calculate the modulus of the vector product, which will be used later.

2 The Definition of Projection Vector and Some Related Topics

In the discussion following **Theorem 1.5**, we have mentioned the included angle between a vector and a hyperplane, which we need to give strict definition and discuss some related topic. Since that, we give the following definition first:

Definition 2.1 Assuming there is a vector $\boldsymbol{p}$ and a linear hyperplane $\mathbf{G}$, the *projection vector* of $\boldsymbol{p}$ on $\mathbf{G}$, which is $\boldsymbol{p'}$, has following properties:

1. $\boldsymbol{p'} \in \mathbf{G}$;
2. $\exists \boldsymbol{q}$, such that $\boldsymbol{p'} + \boldsymbol{q} = \boldsymbol{p}$ and $\boldsymbol{q} \in \mathbf{G}^{\perp}$.

We observe that, the projection of a vector in $\mathbb{R}^3$ onto a plane consists with **Definition 2.1**. And since there is only a unique way to decompose a vector, the projector vector must be unique.

Then we can define:

Definition 2.2 Assuming there is a vector $\boldsymbol{p}$ and a linear hyperplane $\mathbf{G}$, the *included angle* θ between $\boldsymbol{p}$ and $\mathbf{G}$ is the included angle between $\boldsymbol{p}$ and its projection vector on $\mathbf{G}$.

According to this definition, we can know that for any vector $\boldsymbol{p}$ and any linear hyperplane $\mathbf{G}$, $0 \le \theta \le \frac{\pi}{2}$ holds.

Here, we'll discuss a related problem, firstly we define that:

Definition 2.3 Assuming there are $m(1 \le m \le n)$ vectors in $\mathbb{R}^n$: $\boldsymbol{a_1}$, $\boldsymbol{a_2}$, …, $\boldsymbol{a_m}$, which satisfy $\|\boldsymbol{a_1}\| = \|\boldsymbol{a_2}\| = \cdots = \|\boldsymbol{a_m}\| = u(u \ne 0)$ and $\boldsymbol{a_i} \cdot \boldsymbol{a_j} = \text{const.}(\forall i,j \in \{1,2,\dots,m\}, i \ne j)$. Then we define that, the *generalized bisecting vector* of these *m* vectors is a vector $\boldsymbol{a}$ which satisfies $\boldsymbol{a} \cdot \boldsymbol{a_i} = \text{const.}(\forall i \in \{1,2,\dots,m\})$.

A direct result from the definition:

Theorem 2.4 *Assuming there are $m(m \ge 1)$ vectors: $\boldsymbol{a_1}$, $\boldsymbol{a_2}$, …, $\boldsymbol{a_m}$, and other $n(n \ge 1)$ vectors: $\boldsymbol{b_1}$, $\boldsymbol{b_2}$, …, $\boldsymbol{b_n}$, satisfying $\|\boldsymbol{a_1}\| = \|\boldsymbol{a_2}\| = \cdots = \|\boldsymbol{a_m}\| = \|\boldsymbol{b_1}\| = \|\boldsymbol{b_2}\| = \cdots = \|\boldsymbol{b_n}\| = u(u \ne 0)$ and $\boldsymbol{a_i} \cdot \boldsymbol{a_j} = \boldsymbol{a_k} \cdot \boldsymbol{b_r} = \boldsymbol{b_s} \cdot \boldsymbol{b_t} = \text{const.}(\forall i,j,k \in \{1,2,\dots,m\}, r,s,t \in \{1,2,\dots,n\}, i \ne j, s \ne t)$, also $m+n$ is no greater than the dimension of the entire space. If there is a vector $\boldsymbol{p}$ which is a generalized bisecting vector of $\boldsymbol{a_1}$, $\boldsymbol{a_2}$, …, $\boldsymbol{a_m}$, also of $\boldsymbol{a_m}$, $\boldsymbol{b_1}$, $\boldsymbol{b_2}$, …, $\boldsymbol{b_n}$, then $\boldsymbol{p}$ is a generalized bisecting vector of $\boldsymbol{a_1}$, $\boldsymbol{a_2}$, …, $\boldsymbol{a_m}$, $\boldsymbol{b_1}$, $\boldsymbol{b_2}$, …, $\boldsymbol{b_n}$.*

Theorem 2.5 *Assuming there are $m(1 \le m < n)$ linearly independent vectors in $\mathbb{R}^n$: $\boldsymbol{a_1}$, $\boldsymbol{a_2}$, …, $\boldsymbol{a_m}$, satisfying $\|\boldsymbol{a_1}\| = \|\boldsymbol{a_2}\| = \cdots = \|\boldsymbol{a_m}\| = u$ and $\boldsymbol{a_i} \cdot \boldsymbol{a_j} = v(\forall i,j \in \{1,2,\dots,m\}, i \ne j)$. Now assume the hyperplane defined by them is $\mathbf{G}$, and one of their generalized bisecting vectors is $\boldsymbol{a}$, then the projection vector of $\boldsymbol{a}$ on $\mathbf{G}$ which is $\boldsymbol{a'}$ is collinear with $\boldsymbol{a_1} + \boldsymbol{a_2} + \cdots + \boldsymbol{a_m}$.*

Proof Assuming $p = \frac{k}{(m-1)v+u^2}\left(k = \boldsymbol{a} \cdot \boldsymbol{a_i}(\forall i \in \{1,2,\dots,m\})\right)$, consider the following vector $\boldsymbol{r} = \boldsymbol{a} - p(\boldsymbol{a_1} + \boldsymbol{a_2} + \cdots + \boldsymbol{a_m})$. Firstly:

$$\begin{aligned}
&\boldsymbol{r} \cdot \boldsymbol{a_i} \\
&= \boldsymbol{a} \cdot \boldsymbol{a_i} - p(\boldsymbol{a_1} \cdot \boldsymbol{a_i} + \boldsymbol{a_2} \cdot \boldsymbol{a_i} + \cdots + \boldsymbol{a_m} \cdot \boldsymbol{a_i}) \\
&= k - \frac{k}{(m-1)v+u^2} \cdot [(m-1)v+u^2] \\
&= 0
\end{aligned}$$

On the other hand, since $\boldsymbol{a_1}$, $\boldsymbol{a_2}$, …, $\boldsymbol{a_m}$ are linearly independent, $\boldsymbol{r} \in \mathbf{G}^{\perp}$. Moreover, $p(\boldsymbol{a_1} + \boldsymbol{a_2} + \cdots + \boldsymbol{a_m}) \in \mathbf{G}$ and $p(\boldsymbol{a_1} + \boldsymbol{a_2} + \cdots + \boldsymbol{a_m}) + \boldsymbol{r} = \boldsymbol{a}$, so $p(\boldsymbol{a_1} + \boldsymbol{a_2} + \cdots + \boldsymbol{a_m})$ is a projection vector of $\boldsymbol{a}$ on $\mathbf{G}$. By the uniqueness of projection vector, $\boldsymbol{a'} = p(\boldsymbol{a_1} + \boldsymbol{a_2} + \cdots + \boldsymbol{a_m})$.■

For the discussion latter, the following theorem is needed:

Theorem 2.6 *Given vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{a'}$, $\boldsymbol{b'}$, $\boldsymbol{r}$, such that $\boldsymbol{a} + \boldsymbol{a'} = \boldsymbol{b} + \boldsymbol{b'} = \boldsymbol{r}$, $\|\boldsymbol{a}\| = \|\boldsymbol{b}\|$ and $\|\boldsymbol{a'}\| = \|\boldsymbol{b'}\|$, then $\boldsymbol{r}$ is a generalized bisecting vector of $\boldsymbol{a}$, $\boldsymbol{b}$.*

Proof We observe that $\boldsymbol{r} \cdot \boldsymbol{a} = \boldsymbol{a}^2 + \boldsymbol{a} \cdot \boldsymbol{a'}$. Also, $\boldsymbol{r}^2 = \boldsymbol{a}^2 + 2\boldsymbol{a} \cdot \boldsymbol{a'} + \boldsymbol{a'}^2 = \boldsymbol{b}^2 + 2\boldsymbol{b} \cdot \boldsymbol{b'} + \boldsymbol{b'}^2$, together with $\|\boldsymbol{a}\| = \|\boldsymbol{b}\|$ and $\|\boldsymbol{a'}\| = \|\boldsymbol{b'}\|$, we can obtain that:

$$\boldsymbol{a}\cdot\boldsymbol{a}'=\boldsymbol{b}\cdot\boldsymbol{b}'$$

Also, $\boldsymbol{r}\cdot\boldsymbol{b}=\boldsymbol{b}^2+\boldsymbol{b}\cdot\boldsymbol{b}'$, thus $\boldsymbol{r}\cdot\boldsymbol{a}=\boldsymbol{r}\cdot\boldsymbol{b}$, so $\boldsymbol{r}$ is a generalized bisecting vector of $\boldsymbol{a},\boldsymbol{b}$. ■

3 The Properties of $\mathbb{R}^{\boldsymbol{n}}$ Regular Simplex and Semiregular Simplex

From this chapter, since $\mathbb{R}$ regular simplex is just a line segment, we'll just discuss $\mathbb{R}^n(n\geq 2)$ regular simplex and semiregular simplex. Also we'll just discuss the regular simplex and semiregular simplex which the minimal hyperplane containing its undersurface is linear, since other kinds of regular simplex and semiregular simplex can become the former kind by translation. By the way, directly from the definition, we can observe that $\mathbb{R}^n$ regular simplex is actually a special case of $\mathbb{R}^n$ semiregular simplex.

(1) Qualitative Properties

Definition 3.1.1 We define that, for a $\mathbb{R}^k(k\leq n)$ regular simplex in $\mathbb{R}^n$, its *center* is a point on the minimal hyperplane that contains the regular simplex, whose distance to the different vertices of the regular simplex is constant.

Theorem 3.1.2 *Assuming $\mathbb{R}^{n-1}$ regular simplex exists, and equipped with centers, then $\mathbb{R}^n$ semiregular simplex and regular simplex both exist.*

Proof We can construct a $\mathbb{R}^n$ semiregular simplex as following:

Construct a $\mathbb{R}^{n-1}$ regular simplex, and mark one of its centers. Then we are able to find a point not on the minimal hyperplane that contains the semiregular simplex which will let the vector pointing from the above-mentioned center to this point belongs to the orthogonal complement of the above-mentioned hyperplane. Then, directly from the definition of the center, we know that connecting this point with all the vertices of the $\mathbb{R}^{n-1}$ regular simplex mentioned above derives a $\mathbb{R}^n$ semiregular simplex.

Similarly, we can construct a $\mathbb{R}^n$ regular simplex. ■

Theorem 3.1.3 *Assuming $\mathbb{R}^n$ semiregular simplex exists, and its undersurface has a unique center, then the line segment through it and the apex is perpendicular to the undersurface.*

Proof When $n=2$, the theorem holds obviously.

When $n>2$, suppose the vector $\boldsymbol{r}$ is a vector pointing from the unique center of the undersurface to the apex, also suppose $\boldsymbol{a_i}(\forall i\in\{1,2,\dots,n\})$ are vectors pointing from the center to the various vertices of the undersurface. Then assume $\boldsymbol{a_i'}=\boldsymbol{r}-\boldsymbol{a_i}(\forall i\in\{1,2,\dots,n\})$. Clearly, $\boldsymbol{a_i'}$ are vectors pointing from the vertices of the undersurface to apex. Then, derived from the definition of semiregular simplex and center of regular simplex, we obtain that:

$$\|\boldsymbol{a_i}\|=\text{const.}\,(\forall i\in\{1,2,\dots,n\})$$
$$\|\boldsymbol{a_i'}\|=\text{const.}\,(\forall i\in\{1,2,\dots,n\})$$

Since that, together with **Theorem 2.6**, given any pair of $\boldsymbol{a_i}$ and $\boldsymbol{a_j}(i,j\in\{1,2,\dots,n\},i\neq j)$, $\boldsymbol{r}$ is a generalized bisecting vector of them. On the other hand, $\forall\boldsymbol{a_i}$, $\boldsymbol{a_j}$ and $\boldsymbol{a_k}(i,j,k\in\{1,2,\dots,n\},|\{i,j,k\}|=3)$, $\boldsymbol{a_i}-\boldsymbol{a_j}$ and $\boldsymbol{a_i}-\boldsymbol{a_k}$ both correspond to an edge of the undersurface, by the definition of the regular simplex, we obtain that:

$$\|\boldsymbol{a_i}-\boldsymbol{a_j}\|=\|\boldsymbol{a_i}-\boldsymbol{a_k}\|$$

Using **Theorem 2.6** again, we obtain that $\boldsymbol{a_i}$ is a generalized bisecting vector of $\boldsymbol{a_j}$ and $\boldsymbol{a_k}$. Considering that the choice of $\boldsymbol{a_i}$, $\boldsymbol{a_j}$ and $\boldsymbol{a_k}$ is arbitrary, we get:

$$\boldsymbol{a_i}\cdot\boldsymbol{a_j}=\text{const.}\,(\forall i,j\in\{1,2,\dots,n\},i\neq j)$$

By **Theorem 2.4**, we know that $\boldsymbol{r}$ is a generalized bisecting vector of $\boldsymbol{a_1}$, $\boldsymbol{a_2}$, …, $\boldsymbol{a_n}$. Also, we know that $\boldsymbol{a_1}$, $\boldsymbol{a_2}$, …, $\boldsymbol{a_n}$ are linearly dependent, but if any $n-1$ of them are still linearly dependent, then the center of the undersurface of this semiregular simplex must be on the hyperplane with a dimension less than $n-1$ defines by those $n-1$ vectors, becoming the center of the regular simplex on the above-mentioned hyperplane. By the fact that the moduli of $\boldsymbol{a_1}$, $\boldsymbol{a_2}$, …, $\boldsymbol{a_n}$ are the same, and the included angles between any pair of them are also the same, the only one vector not in those $n-1$ must also be on the hyperplane, which, according to the condition given, the definition of the center and the definition of the regular simplex, the vertex corresponding to the vector must be the same as the center mentioned above, letting the vector become $\mathbf{0}$, thus makes no sense. Hence any $n-1$ of $\boldsymbol{a_1}$, $\boldsymbol{a_2}$, …, $\boldsymbol{a_n}$ must be linearly independent. Consequently, the hyperplane defined by $\boldsymbol{a_1}$, $\boldsymbol{a_2}$, …, $\boldsymbol{a_{n-1}}$ and the hyperplane defined by $\boldsymbol{a_2}$, $\boldsymbol{a_3}$, …, $\boldsymbol{a_n}$ are both the minimal hyperplane that contains the undersurface, denoted by **G**. By the definition of the generalized bisecting vector, $\boldsymbol{r}$ is a generalized bisecting vector of $\boldsymbol{a_1}$, $\boldsymbol{a_2}$, …, $\boldsymbol{a_{n-1}}$, also the generalized bisecting vector of $\boldsymbol{a_2}$, $\boldsymbol{a_3}$, …, $\boldsymbol{a_n}$. Then according to **Theorem 2.5**, the projection vector of $\boldsymbol{r}$ on **G**, denoted by $\boldsymbol{r'}$, is collinear with both $\boldsymbol{a_1}+\boldsymbol{a_2}+\cdots+\boldsymbol{a_{n-1}}$ and $\boldsymbol{a_2}+\boldsymbol{a_3}+\cdots+\boldsymbol{a_n}$.

Here we discuss whether or not $\boldsymbol{a_1}+\boldsymbol{a_2}+\cdots+\boldsymbol{a_{n-1}}$ and $\boldsymbol{a_2}+\boldsymbol{a_3}+\cdots+\boldsymbol{a_n}$ are linearly dependent. If they are, then we may assume:

$$\boldsymbol{a_1}+\boldsymbol{a_2}+\cdots+\boldsymbol{a_{n-1}}=\beta(\boldsymbol{a_2}+\boldsymbol{a_3}+\cdots+\boldsymbol{a_n})$$

Considering we can actually exchange the two sides of the equality, we get $\beta=\pm1$. If $\beta=1$, then:

$$\boldsymbol{a_1}+\boldsymbol{a_2}+\cdots+\boldsymbol{a_{n-1}}=\boldsymbol{a_2}+\boldsymbol{a_3}+\cdots+\boldsymbol{a_n}$$

Which leads to $\boldsymbol{a_1}=\boldsymbol{a_n}$, a contradictory result. Otherwise, if $\beta=-1$, then:

$$\boldsymbol{a_1}+\boldsymbol{a_2}+\cdots+\boldsymbol{a_{n-1}}=-\boldsymbol{a_2}-\boldsymbol{a_3}-\cdots-\boldsymbol{a_n}$$

Thus:

$$\boldsymbol{a_1}+\boldsymbol{a_n}+2(\boldsymbol{a_2}+\boldsymbol{a_3}+\cdots+\boldsymbol{a_{n-1}})=\mathbf{0}$$

We observe that the choice of $\boldsymbol{a_1}$ and $\boldsymbol{a_n}$ is arbitrary, thus:

$$-\boldsymbol{a_i}-\boldsymbol{a_j}+2\sum_{k=1}^{n}\boldsymbol{a_k}=\mathbf{0}(\forall i,j\in\{1,2,\dots,n\},i\neq j)$$

Adding all the equalities with this form, we get:

$$\boldsymbol{a_1}+\boldsymbol{a_2}+\cdots+\boldsymbol{a_{n-1}}+\boldsymbol{a_n}=\mathbf{0}$$

So:

$$\boldsymbol{a_i}+\boldsymbol{a_j}=\mathbf{0}(\forall i,j\in\{1,2,\dots,n\},i\neq j)$$

Which is contradictory considering three vectors all negating each other. In summary, $\boldsymbol{a_1}+\boldsymbol{a_2}+\cdots+\boldsymbol{a_{n-1}}$ must be linearly independent with $\boldsymbol{a_2}+\boldsymbol{a_3}+\cdots+\boldsymbol{a_n}$, which leads to $\boldsymbol{r'}=\mathbf{0}$, then $\boldsymbol{r}\in\mathbf{G}^{\perp}$. ■

Theorem 3.1.4 *If $\mathbb{R}^{n-1}$ regular simplex and $\mathbb{R}^n$ regular simplex exists, and $\mathbb{R}^{n-1}$ regular simplex has a unique center, then $\mathbb{R}^n$ regular simplex has a unique center.*

Proof According to **Theorem 3.13**, we know that the line segment through the center of the undersurface of this $\mathbb{R}^n$ regular simplex and the apex is perpendicular to the undersurface. Together with the definition of the center, connecting the possible center of this $\mathbb{R}^n$ regular simplex with the vertices of the undersurface must derive a $\mathbb{R}^n$ semiregular simplex. Thus the line segment through this possible center and the center of the undersurface must also be perpendicular to the undersurface, which leads to the conclusion that the apex of this $\mathbb{R}^n$ regular simplex, this possible center, and the center of the undersurface must be collinear.

We now calculate the position of this possible center. Assume the distance between the apex and the center of the undersurface is u and the ratio of the distance from the center of the undersurface to the vertices of the undersurface to *l* is *a*, which is unique according to the condition given. Also suppose the distance between this possible center and the apex is *p*. Then we obtain:

$$\begin{cases} a^2l^2 + u^2 = l^2 \\ a^2l^2 + (u-p)^2 = p^2 \end{cases}$$

Under suitable restrictions, the equation group has and only has one solution:

$$p = \frac{l}{2\sqrt{1-a^2}}$$

Thus $u-p$, the distance between the possible center and the center of the undersurface is unique. This shows that $\mathbb{R}^n$ regular simplex has a unique center. ■

We observe that, using **Theorem 3.1.2** together with **Theorem 3.1.4**, and using the fact that $\mathbb{R}$ regular simplex exists with unique center, i.e. the midpoint of the line segment, we can get the following result:

Theorem 3.1.5 $\forall n \geq 2$, $\mathbb{R}^n$ *regular simplex and semiregular simplex both exist, while* $\mathbb{R}^n$ *regular simplex has a unique center.*

Thus **Theorem 3.1.3** can be rewritten as:

Corollary 3.1.6 *The line segment through the center of the undersurface of a* $\mathbb{R}^n$ *semiregular simplex and the apex of it is perpendicular to the undersurface.*

(2) Quantitative Properties

Because of the fact pointed out by **Corollary 3.1.6**, the line segment through the apex of a $\mathbb{R}^n$ semiregular simplex and the center of its undersurface fits the usual definition of the altitude of a geometric figure. Thus it's proper for us to define:

Definition 3.2.1 We call the line segment through the apex of a $\mathbb{R}^n$ semiregular simplex and the center of its undersurface its *altitude*, we also use the term *altitude* to represent the length of the above-mentioned line segment, denoted by *h*.

Obviously, two $\mathbb{R}^n$ semiregular simplexes are congruent if their edge lengths and altitudes are equal respectively.

For the discussion following, this definition will be needed:

Definition 3.2.2 We call any vector that points from the center of a $\mathbb{R}^n$ regular simplex to a vertex of it the *center vector* of this regular simplex, while its modulus is called the *radius* of this regular simplex, denoted by *r*. We also call the ratio of *r* to the edge length the regular simplex's *r-l ratio*, denoted by *a*.

We'll discuss the relation between *a* and the value of dimension *n*, firstly.

Theorem 3.2.3 *There is a definite relation between a and n:* $a_n = \frac{\sqrt{n}}{\sqrt{2(n+1)}}$.

Proof Considering a $\mathbb{R}^{n-1}$ regular simplex, whose *r-l* ratio is *a* in the proof of **Theorem 3.1.4**, we now denote its *r-l* ratio by a_{n-1}. And consider a $\mathbb{R}^n$ regular simplex whose edge length is the same as the edge length of the $\mathbb{R}^{n-1}$ regular simplex mentioned above, thus its radius is *p* in the proof of **Theorem 3.1.4**. Its *r-l* ratio is:

$$a_n = \frac{p}{l} = \frac{1}{2\sqrt{1-a_{n-1}^2}}$$

Observing that $a_2 = \frac{1}{\sqrt{3}}$, we can get:

$$a_n = \frac{\sqrt{n}}{\sqrt{2(n+1)}}$$ ■

We now discuss the included angle between the center vectors of $\mathbb{R}^n$ regular simplex.

Theorem 3.2.4 *There is a definite relation between the cosine of the included angle between the center vectors of a* $\mathbb{R}^n$ *regular simplex,* $\cos\theta$*, and the value of dimension n:* $\cos\theta_n = -\frac{1}{n}$.

Proof Consider the undersurface of this regular simplex, which is a $\mathbb{R}^{n-1}$ regular simplex. Then assume two of its center vectors are $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$. Also, suppose the vector pointing from the apex of the above-mentioned $\mathbb{R}^n$ regular simplex to its undersurface's center is $\boldsymbol{h}$.

Solving the equation group in the proof of **Theorem 3.1.4**, we get:

$$\|\boldsymbol{h}\| = \sqrt{1 - a_{n-1}^2} \cdot l$$

On the other hand, the radius of this $\mathbb{R}^n$ regular simplex is:

$$r = a_n l$$

Hence, the distance from the center of this regular simplex to the center of this regular simplex's undersurface is:

$$\begin{aligned}
\|\boldsymbol{h}\| - r &= \left(\sqrt{1 - a_{n-1}^2} - a_n\right) \cdot l \\
&= \left(\sqrt{1 - a_{n-1}^2} - \frac{1}{2\sqrt{1 - a_{n-1}^2}}\right) \cdot l \\
&= \frac{1 - 2a_{n-1}^2}{2\sqrt{1 - a_{n-1}^2}} \cdot l \\
&= \frac{1 - 2a_{n-1}^2}{2 - 2a_{n-1}^2} \sqrt{1 - a_{n-1}^2} \cdot l \\
&= \frac{1 - 2a_{n-1}^2}{2 - 2a_{n-1}^2} \cdot \|\boldsymbol{h}\|
\end{aligned}$$

We observe that $a_{n-1}^2 = \frac{n-1}{2n}$, thus:

$$\frac{1 - 2a_{n-1}^2}{2 - 2a_{n-1}^2} = \frac{1}{n+1}$$

Therefore, the vector pointing from the center of the regular simplex to the center of its undersurface is:

$$\frac{1}{n+1}\boldsymbol{h}$$

We thus get two center vectors of this $\mathbb{R}^n$ regular simplex:

$$\boldsymbol{a_1} + \frac{1}{n+1}\boldsymbol{h} \text{ and } \boldsymbol{a_2} + \frac{1}{n+1}\boldsymbol{h}$$

Since we know that given any two center vectors of a $\mathbb{R}^n$ regular simplex, the included angle between them is independent of the choice of the two vectors, we can compute the included angle between only one pair of center vectors, e.g. the ones that are mentioned above. On the other hand, we also know that:

$$\boldsymbol{a_1} \cdot \boldsymbol{h} = 0$$
$$\boldsymbol{a_2} \cdot \boldsymbol{h} = 0$$

$$\boldsymbol{h}^2 = (1 - a_{n-1}^2)l^2 = \frac{1 - a_{n-1}^2}{a_{n-1}^2}\boldsymbol{a_1}^2 = \frac{n+1}{n-1}\boldsymbol{a_1}^2$$

We compute:

$$\begin{aligned}\left(\boldsymbol{a_1} + \frac{1}{n+1}\boldsymbol{h}\right) \cdot \left(\boldsymbol{a_2} + \frac{1}{n+1}\boldsymbol{h}\right) &= \boldsymbol{a_1}^2 \cdot \cos\theta_{n-1} + \frac{\boldsymbol{h}^2}{(n+1)^2} \\ &= \boldsymbol{a_1}^2 \cdot \left(\cos\theta_{n-1} + \frac{1}{n^2-1}\right)\end{aligned}$$

On the other hand:

$$\begin{aligned}\left\|\boldsymbol{a_1} + \frac{1}{n+1}\boldsymbol{h}\right\| \cdot \left\|\boldsymbol{a_2} + \frac{1}{n+1}\boldsymbol{h}\right\| &= \left(\boldsymbol{a_1} + \frac{1}{n+1}\boldsymbol{h}\right)^2 \\ &= \boldsymbol{a_1}^2 + \frac{\boldsymbol{h}^2}{(n+1)^2} \\ &= \boldsymbol{a_1}^2 \cdot \frac{n^2}{n^2-1}\end{aligned}$$

As a result:

$$\cos\theta_n = \frac{\left(\boldsymbol{a_1} + \frac{1}{n+1}\boldsymbol{h}\right) \cdot \left(\boldsymbol{a_2} + \frac{1}{n+1}\boldsymbol{h}\right)}{\left\|\boldsymbol{a_1} + \frac{1}{n+1}\boldsymbol{h}\right\| \cdot \left\|\boldsymbol{a_2} + \frac{1}{n+1}\boldsymbol{h}\right\|} = \frac{\left(\cos\theta_{n-1} + \frac{1}{n^2-1}\right)}{\frac{n^2}{n^2-1}} = \frac{(n^2-1)\cos\theta_{n-1} + 1}{n^2}$$

Considering $\cos\theta_2 = -\frac{1}{2}$, we get:

$$\cos\theta_n = -\frac{1}{n}$$ ■

Next up, we'll discuss based on the second equation given after **Theorem 1.5**, and we'll regard a line defined by only one vector or a common plane defined by only two vectors as a hyperplane.

Theorem 3.2.5 *Assuming a $\mathbb{R}^n$ regular simplex, and choosing $m (m < n)$ of its center vectors, suppose the hyperplane defined by these m vectors is* **G**. *Then choose a vector different from the m vectors mentioned above,* $\boldsymbol{p}$. *The sine of the included angle between* $\boldsymbol{p}$ *and* **G** *is* $\sqrt{\frac{(n+1)(n-m)}{n(n-m+1)}}$.

Proof Within the proof of **Theorem 3.1.3**, we have pointed out that if we assume the center vectors of this $\mathbb{R}^n$ regular simplex are $\boldsymbol{a_1}$, $\boldsymbol{a_2}$, …, $\boldsymbol{a_{n+1}}$, we'll get:

$$\|\boldsymbol{a_i}\| = \text{const.} (\forall i \in \{1,2,\dots,n+1\})$$
$$\boldsymbol{a_i} \cdot \boldsymbol{a_j} = \text{const.} (\forall i,j \in \{1,2,\dots,n+1\}, i \neq j)$$

Here, we suppose $\|\boldsymbol{a_i}\| = v (\forall i \in \{1,2,\dots,n+1\})$ and $\boldsymbol{a_i} \cdot \boldsymbol{a_j} = u (\forall i,j \in \{1,2,\dots,n+1\}, i \neq j)$. We observe that, $\forall \boldsymbol{a_i} (i \in \{1,2,\dots,n+1\})$, and choosing several center vectors of this $\mathbb{R}^n$ regular simplex different from $\boldsymbol{a_i}$, then $\boldsymbol{a_i}$ is a generalized bisecting vector of them. Hence, by **Definition 2.2** and **Theorem 2.5**, and assuming the *m* vectors mentioned in the statement of the theorem are $\boldsymbol{a_1}$, $\boldsymbol{a_2}$, …, $\boldsymbol{a_m}$, $\boldsymbol{r} = \boldsymbol{a_1} + \boldsymbol{a_2} + \cdots + \boldsymbol{a_m}$, then the included angle between $\boldsymbol{p}$ and **G**, θ, satisfy the following equation:

$$\cos\theta = |\cos(\widehat{\boldsymbol{p},\boldsymbol{r}})|$$

We calculate:

$$\boldsymbol{p} \cdot \boldsymbol{r} = \boldsymbol{a_1} \cdot \boldsymbol{p} + \boldsymbol{a_2} \cdot \boldsymbol{p} + \cdots + \boldsymbol{a_m} \cdot \boldsymbol{p} = m \cdot u$$
$$\boldsymbol{r}^2 = \boldsymbol{a_1}^2 + \boldsymbol{a_2}^2 + \cdots + \boldsymbol{a_m}^2 + \sum_{i,j \in \{1,2,\dots,m\}, i \neq j} \boldsymbol{a_i} \cdot \boldsymbol{a_j}$$

$= mv^2 + (m^2 - m)u$
$= m(v^2 + m \cdot u - u)$

Consequently:

$$\cos(\widehat{\boldsymbol{p}, \boldsymbol{r}}) = \frac{\boldsymbol{p} \cdot \boldsymbol{r}}{\|\boldsymbol{p}\| \cdot \|\boldsymbol{r}\|} = \frac{m \cdot u}{v \cdot \sqrt{m(v^2 + m \cdot u - u)}} = \frac{\sqrt{m} \cdot u}{v \cdot \sqrt{v^2 + m \cdot u - u}}$$

We observe that:

$$v = a_n l = \frac{\sqrt{n} \cdot l}{\sqrt{2(n+1)}}$$

$$u = v^2 \cdot \cos\theta_n = -\frac{l^2}{2n+2}$$

So:

$$\cos(\widehat{\boldsymbol{p}, \boldsymbol{r}}) = -\sqrt{\frac{m}{n(n-m+1)}}$$

Thus:

$$\sin\theta = \sqrt{\frac{(n+1)(n-m)}{n(n-m+1)}} \qquad \blacksquare$$

We can now prove the following important theorem:

Theorem 3.2.6 *Assuming a $\mathbb{R}^n$ regular simplex, the volume of a parallelotope having any n of the regular simplex's center vectors as edges is* $\frac{l^n}{2^{\frac{n}{2}} \cdot \sqrt{n+1}}$.

Proof When we calculate this volume, we are actually calculating the product of all of its edge lengths, together with all the $\sin\theta$, which is the sine of the included angle between an edge and the hyperplane defined by several other edges. There are $n-1$ included angles with this form, whose sine values are the ones we get in **Theorem 3.2.5**, setting *m* from 1 to $n-1$. Based on this point of view, also using the results from **Theorem 3.2.3** and **Theorem 3.2.5**, we can get the value of the desired volume:

$$\left(\sqrt{\frac{n}{2n+2}} \cdot l\right)^n \cdot \prod_{m=1}^{n-1} \sqrt{\frac{(n+1)(n-m)}{n(n-m+1)}}$$

We compute:

$$\left(\sqrt{\frac{n}{2n+2}} \cdot l\right)^n \cdot \prod_{m=1}^{n-1} \sqrt{\frac{(n+1)(n-m)}{n(n-m+1)}} = \frac{l^n}{2^{\frac{n}{2}} \cdot \sqrt{n+1}} \qquad \blacksquare$$

4 The Proof of Main Theorem 1, 2 and 3

Theorem 4.1 *Given a $\mathbb{R}^n$ regular simplex in $\mathbb{R}^k (k \geq n)$ with all the vertices rational, then its center is rational.*

Proof When $n = 2$, $\mathbb{R}^n$ regular simplex is actually a equilateral triangle, thus the theorem holds obviously when $n = 2$.

We observe that, in the proof of **Theorem 3.2.4**, we give an expression of some of the center vectors:

$$\boldsymbol{a_i} + \frac{1}{n+1}\boldsymbol{h}(i \in \{1,2,\dots,n\})$$

That is, a center vector of a $\mathbb{R}^n$ regular simplex can be obtained from adding $\frac{1}{n+1}$ times of the vector pointing from the apex to the center of the undersurface to a center vector of the undersurface. We now consider a $\mathbb{R}^s(s<n)$ regular simplex as an element of the above-mentioned $\mathbb{R}^n$ regular simplex. If the center of it is rational, and the vertices of it are all rational, too, then all of its center vectors are rational. Next, consider a $\mathbb{R}^{s+1}$ regular simplex as an element of the $\mathbb{R}^n$ regular simplex but has that $\mathbb{R}^s$ regular simplex as the undersurface. Its apex is rational, while the center of its undersurface has also been assumed as rational. Thus the vector pointing from its apex to the center of the undersurface is rational. And using the fact that $\frac{1}{n+1}$ is rational, we conclude that $\boldsymbol{a_i} + \frac{1}{n+1}\boldsymbol{h}(\forall i \in \{1,2,\dots,n\})$ is rational, which leads to the result that the center of this $\mathbb{R}^{s+1}$ regular simplex is rational.

Notice that when $s=2$, we have pointed out that its center is rational. Thus using the result we just obtain to deduct step by step, we get that the center of the $\mathbb{R}^n$ regular simplex is rational, which is the desired result. ■

In **Main Theorem 1**, **2** and **3**, we have assumed that the vertices of the undersurface of a semiregular simplex or regular simplex are all rational, so by **Theorem 4.1**, the center of the undersurface is rational, thus we can assume this center is the origin without loss of generality, which gives a specific way of translating the semiregular simplex or regular simplex so that the minimal hyperplane that contains this simplex's undersurface is linear, which can be used to meet the requirement given in the beginning of ***Chapter 3***. In the following proofs of **Main Theorem 1**, **2** and **3**, we'll all the way assume the center of the undersurface of the semiregular simplex or regular simplex is the origin.

Theorem 4.2 (Main Theorem 1) *Given a $\mathbb{R}^n(n \geq 2)$ semiregular simplex in $\mathbb{R}^n$ space, if its undersurface's vertices are all rational, then, when n is even, the apex is rational iff $lh\sqrt{2n}$ is rational; when n is odd, the apex is rational iff $h\sqrt{n}$ is rational.*

Proof We choose $n-1$ of the center vectors of the undersurface of this semiregular simplex, which are both rational. Assume their vector product is ***p***. According to the definition of the vector product and the properties the determinant has, ***p*** is rational.

Furthermore, by **Corollary 1.3** and **Corollary 3.1.6**, ***p*** is collinear with the vector ***u*** which points from the center of the undersurface to the apex. Then, by **Theorem 1.6** and **Theorem 3.2.6**, we can obtain that:

$$||\boldsymbol{p}|| = \frac{l^{n-1}}{2^{\frac{n-1}{2}} \cdot \sqrt{n}}$$

On the other hand, $||\boldsymbol{u}|| = h$, which leads to the ratio between these two values:

$$\frac{||\boldsymbol{p}||}{||\boldsymbol{u}||} = \frac{l^{n-1}}{2^{\frac{n-1}{2}} \cdot h\sqrt{n}}$$

We observe that l^2 is rational. Then, when *n* is even, we could know that $\frac{||\boldsymbol{p}||}{||\boldsymbol{u}||}$ is rational iff

$lh\sqrt{2n}$ is rational; when *n* is odd, $\frac{\|\boldsymbol{p}\|}{\|\boldsymbol{u}\|}$ is rational iff $h\sqrt{n}$ is rational.

On the other hand, according to the discussion we just made, the apex of the $\mathbb{R}^n$ semiregular simplex is rational iff $\frac{\|\boldsymbol{p}\|}{\|\boldsymbol{u}\|}$ is rational, which leads to the desired result. ■

Theorem 4.3 (Main Theorem 2) *Given a* $\mathbb{R}^n(n \geq 2)$ *regular simplex in* $\mathbb{R}^n$ *space, if its undersurface's vertices are all rational, then, when n is even, the apex is rational iff* $\sqrt{n+1}$ *is rational; when n is odd, the apex is rational iff* $l\sqrt{2n+2}$ *is rational.*

Proof Notice that this theorem is actually a special case of **Main Theorem 1**. We observe that, for a $\mathbb{R}^n$ regular simplex:

$$h = \sqrt{1 - a_{n-1}^2} \cdot l$$

According to **Theorem 3.2.3**:

$$a_{n-1}^2 = \frac{n-1}{2n}$$

Thus:

$$h = \sqrt{\frac{n+1}{2n}} \cdot l$$

Using this fact in **Main Theorem 1** leads to the desired result. ■

Theorem 4.4 *The volume of a* $\mathbb{R}^n$ *semiregular simplex is* $\frac{l^{n-1}h\sqrt{n}}{n!\cdot 2^{\frac{n-1}{2}}}$.

Proof Firstly, we suppose the volume (specifically, the $n-1$ dimensional volume) of the undersurface of this semiregular simplex is *S*, then obviously the volume of the semiregular simplex is:

$$V = \int_0^h \left(\frac{x}{h}\right)^{n-1} S\, dx = \frac{1}{n} Sh$$

Now we move on to calculating *S*. Notice that the undersurface is a $\mathbb{R}^{n-1}$ regular simplex which is a special case of semiregular simplex. And we know that, for a $\mathbb{R}^n$ regular simplex:

$$h = \sqrt{\frac{n+1}{2n}} \cdot l$$

So:

$$S = l \cdot \prod_{k=2}^{n-1} \frac{1}{k} \cdot \sqrt{\frac{k+1}{2k}} \cdot l$$

By calculation:

$$S = \frac{l^{n-1}\sqrt{n}}{(n-1)! \cdot 2^{\frac{n-1}{2}}}$$

Thus:

$$V = \frac{l^{n-1}h\sqrt{n}}{n! \cdot 2^{\frac{n-1}{2}}}$$ ■

Theorem 0.6 (Main Theorem 3) *Given a* $\mathbb{R}^n(n \geq 2)$ *semiregular or regular simplex in* $\mathbb{R}^n$ *space, if its undersurface's vertices are all rational, then the apex is rational iff its volume is rational.*

Proof Since regular simplex is a special case of semiregular simplex, we just need to give a proof for $\mathbb{R}^n$ semiregular simplex.

According to **Theorem 4.4**, the volume of this semiregular simplex is:

$$\frac{l^{n-1}h\sqrt{n}}{n! \cdot 2^{\frac{n-1}{2}}}$$

Where we notice that l^2 and $n!$ are both rational, then, when n is even, we can obtain that it's rational iff $lh\sqrt{2n}$ is rational; when n is odd, it's rational iff $h\sqrt{n}$ is rational. Together with **Main Theorem 1**, this leads to the desired result. ■

In fact, in a paper by one of the authors which was submitted recently, we have proved a stronger result, that is **Main Theorem 3** holds for all $\mathbb{R}^n$ right pyramid, which we won't give details here[7].

5 The Discussion of Regular Simplex in $L^P(P \in [5, +\infty) \cap \mathbb{N})$-space over $\mathbb{R}^2$

Theorem 5.1 *In* $L^P(1 < P < \infty)$*-space over* $\mathbb{R}^2$*, a circle has and only has two intersection points with a circle of the same radius whose center is on the former circle.*

Proof In order to prove the theorem, a lemma is needed:

Theorem 5.1 – Lemma 1 *On a Euclidean plane, for a closed and strictly convex figure with continuous boundary, any arc of it cannot be translated from another arc of it.*

Proof of the Lemma Suppose this sort of situation can happen, and let these two arcs be $\widehat{AB}$ and $\widehat{CD}$ respectively. By strictly convexity, they cannot be straight line segment. We firstly consider the situation when *A*, *B*, *C* and *D* are collinear, just like the **Fig. 1** below. It makes no sense, obviously.

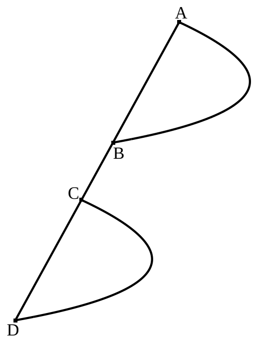

Fig. 1 The situation when *A*, *B*, *C* and *D* are collinear

When these four points are not collinear, we get a parallelogram *ABDC*, whose inner points all belong to the convex figure. The area between $\widehat{AB}$ and *AB*, and the area between $\widehat{CD}$ and *CD* also belong to the figure, which will lead to the result that for at least one of the arcs, there will be points belong to the figure in both sides of it, which is contradictory. ■

The lemma pointed out a fact, that on a Euclidean plane, for two closed and strictly convex figures which do not coincide and with continuous boundary. If one can become another by translation, then any pair of the arcs of them could not coincide.

Theorem 5.1 – Lemma 2 *On a Euclidean plane, for two closed and strictly convex figures* *which do not coincide and with continuous boundaries, if one can become another by translation, then their boundaries has at most two intersection points.*

Proof of the Lemma Suppose there are more than two intersection points, consider two intersection points which have one and only one intersection point between them (They must exist by the condition given and the fact pointed out after **Lemma 1**). Let them be *A* and *B*, while the unique intersection point between them be *C*. We now consider the arc $\widehat{AB}$ of one of the figures, while it's easy to see that the area between $\widehat{AB}$ and *AB* belongs to both of the two figures, which leads to the result that *C* is on *AB*.

Now consider the arc $\widehat{AC}$ and arc $\widehat{CB}$ of the above-mentioned one of two figures, where the area between $\widehat{AC}$ and *AB* belongs to that figure. Similar result holds for the arc $\widehat{CB}$. Thus, the two arcs are in the same side of *AB*, just like the following **Fig. 2**, which is contradictory with the convexity of the figure:

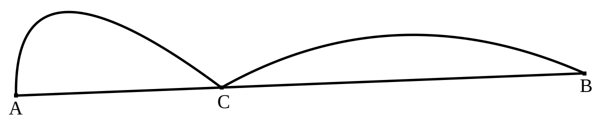

Fig. 2 The two arcs in the same side of *AB*

This gives the desired result. ■

We notice that, for the given two L^P circles, any one of them can become another by translation. Also, they are closed, strictly convex, with continuous boundaries, and don't coincide. We prove their strictly convexity as following:

Theorem 5.1 – Lemma 3 *In* $L^P(1 < P < \infty)$*-space over* $\mathbb{R}^2$, *circles are strictly convex.*

Proof of the Lemma We observe that, the center and the radius of this L^P circle do not affect the result. Thus we'll assume that this is the unit circle. The circle has two lines of symmetry, which are *x*-axis and *y*-axis. As a result, we just need to prove the circle's boundary in quadrant I is strictly concave.

Inside the quadrant I, we can denote the circle's boundary by the following equation:

$$y = \sqrt[P]{1 - x^P}$$

So:

$$\frac{d^2y}{dx^2} = (1 - P)y^{1-2P}x^{P-2}$$

Notice that in quadrant I, both *x* and *y* are positive, and $P > 1$, hence:

$$\frac{d^2y}{dx^2} < 0$$

This leads to the desired result. ■

So, according to **Lemma 2**, there are at most two intersection points between the two L^P circles. On the other hand, any one of the two circles has a part inside another circle, e.g. the center of another circle, O_1; also a part outside another circle, e.g. the antipodal point of O_1. Then by the continuity of the two circles, there are at least two intersection points between their boundaries, which proves the desired result. ■

Theorem 5.2 *Given a regular simplex in* $L^P(1 < P < \infty)$*-space over* $\mathbb{R}^2$, *there cannot be a point whose distances to the different vertices of the regular simplex all equal to the edge length of the regular simplex.*

Proof We let *O*, *A*, *B* denote the three vertices of the regular simplex, and set *O* as the origin without loss of generality. We construct a L^P circle whose center is O and radius is $\left\|\overrightarrow{OA}\right\|_P$, denoted by $\odot O$. Notice that A, B are both on $\odot O$. Furthermore, we construct another L^P circle that has A as center and the same radius as $\odot O$, denoted by $\odot A$. Then B is also on this circle. By **Theorem 5.1**, there are and only are two intersection points between the boundaries of two circles, while one of them is B. Suppose the point satisfying the requirement in the statement of this theorem does exist, denoted by C, then C can only be that another intersection point which is unique.

Theorem 5.2 – Lemma 1 *B is not on OA.*

Proof of the Lemma We know that there are only two intersection points between OA and the boundary of $\odot O$, which are A and the antipodal point of A. Both of them do not satisfy $\left\|\overrightarrow{AB}\right\|_P = \left\|\overrightarrow{OA}\right\|_P$, which proved the lemma. ■

We divide the entire plane into two parts by OA. By **Lemma 1**, B is in one of the two parts, which we'll call the *right half*, while the another part will be called the *left half*. On the other hand, OA also divides $\odot A$ (as well as the boundary of $\odot A$) into two parts obviously.

Theorem 5.2 – Lemma 2 *C is in the left half.*

Proof of the Lemma We already know that OA divides the boundary of $\odot A$ into two parts. We denote the one that is in the right half by $\widehat{OBA}$, while the one that is in the left half by $\widehat{OA}$. Now considering a point N traveling from A to O along $\widehat{OBA}$, in this process, as the boundary of $\odot A$ is continuous and smooth, $\left\|\overrightarrow{ON}\right\|_P$ must vary continuously. Similarly, if N travels along $\widehat{OA}$, $\left\|\overrightarrow{ON}\right\|_P$ must also vary continuously, which ensures that there must be two points on $\widehat{OA}$ and $\widehat{OBA}$ respectively, denoted by M, satisfying $\left\|\overrightarrow{OM}\right\|_P = \left\|\overrightarrow{OA}\right\|_P$. We also know that there can only be at most two M's, that the one on $\widehat{OBA}$ is B, so the one on $\widehat{OA}$ must be C. ■

We now construct a L^P circle centered at C, with radius equals to $\left\|\overrightarrow{OA}\right\|_P$, denoted by $\odot C$. Then all of O, A, B must be on it. Again by **Theorem 5.1**, we know that the intersection points between it and $\odot O$ must only be A and B. Since that, the one of the two arcs (without the endpoints) having A and B as endpoints must be in $\odot O$ totally, another must be outside $\odot O$ totally. Let the latter one be denoted by $\widehat{AB}$.

Theorem 5.2 – Lemma 3 *There cannot be a point on $\widehat{AB}$ and also in the left half.*

Proof of the Lemma Assuming there is a point like this, we denote it by Q. By **Theorem 5.1 – Lemma 3**, $\odot C$ is strictly convex. So that the area between $\widehat{OBQ}$ and OQ belongs to $\odot C$, which has a part in $\odot O$, thus makes some parts of $\odot C$ in both sides of $\widehat{OAQ}$, respectively. This is contradictory. ■

Considering the antipodal point E of B on $\odot C$, by the strictly convexity of $\odot O$, E must be outside $\odot O$ (That's because, if not so, then C is on BE but also on the boundary of $\odot O$, which is contradictory with the strictly convexity of $\odot O$), thus E is on $\widehat{AB}$. Notice that B is in the right half, C is in the left half, which makes E in the left half, contradictory with **Lemma 3**, thus proves the desired result. ■

These two theorems show that our definition for the regular simplex in $L^P(1 < P < \infty)$-space over $\mathbb{R}^2$ is proper, that it can have three vertices, and a 'regular simplex' with four vertices does not exist.

Theorem 5.3 *If* **Conjecture 0.8** *holds, then there cannot be integers x, y, z, w which at least one of them is nontrivial that satisfy the following equation:*

$$|x|^k+|y|^k=|z|^k+|w|^k=|x-z|^k+|y-w|^k$$

Where $k\in[5,+\infty)\cap\mathbb{N}$.

Proof We discuss, when *x*, *y*, *z*, *w* are all nontrivial:

If $x=z$ and $y=w$, then *x*, *y*, *z*, *w* will all be trivial, a contradiction.

If $x=z$ and $y=-w$, then $|x|^k+|y|^k=|2y|^k$, which is impossible due to Fermat's Last Theorem.

Similarly, it's impossible that $x=-z$ and $y=w$.

If $x=-z$ and $y=-w$, then $|x|^k+|y|^k=2^k(|x|^k+|y|^k)$, which is impossible.

If $x=w$ and $y=z$, then $|x|^k+|y|^k=2|x-y|^k$, contradictory with **Conjecture 0.8**.

If the situation which is $x=w$ and $y=-z$ holds or the situation which is $x=-w$ and $y=z$ holds, then $|x|^k+|y|^k=|x+y|^k+|x-y|^k$, contradictory with **Conjecture 0.8**.

If $x=-w$ and $y=-z$, then $|x|^k+|y|^k=2|x+y|^k$, also contradictory with **Conjecture 0.8**.

If other cases hold, then it will directly contradict **Conjecture 0.8**.

When there is one and only one 0 among *x*, *y*, *z*, *w*, it will directly contradict Fermat's Last Theorem, thus it's impossible.

When there are two and only two 0 among *x*, *y*, *z*, *w*:

If $x=z=0$, then it's impossible that $y=w$; but if $y=-w$, then $|y|^k=2^k|y|^k$, not possible; other cases are not possible, either.

Similarly, it's impossible that $y=w=0$.

If $x=w=0$, then one of $y=z$ and $y=-z$ must hold, whether which one holds, we can get that $|y|^k=2|y|^k$, not possible.

Similarly, it's impossible that $y=z=0$.

Other cases are impossible, obviously. ■

A direct corollary:

Corollary 5.4 *If* **Conjecture 0.8** *holds, then there cannot be rational numbers x, y, z, w which at least one of them is nontrivial that satisfy the following equation:*

$$|x|^k+|y|^k=|z|^k+|w|^k=|x-z|^k+|y-w|^k$$

Where $k\in[5,+\infty)\cap\mathbb{N}$.

Theorem 5.5 (Main Theorem 4) *Given a regular simplex in* $L^P(P\in[5,+\infty)\cap\mathbb{N})$*-space over* $\mathbb{R}^2$*, if two of its vertices are both rational, and that* **Conjecture 0.8** *holds, then the last vertex cannot be rational.*

Proof According to the definition, this regular simplex has three vertices. Without loss of generality, we assume that one of the rational vertices mentioned in the statement of the theorem is the origin. Let another rational vertex be $A(x,y)$, and the remaining vertex be $B(z,w)$. Obviously, at least one of *x*, *y*, *z*, *w* is nontrivial, and:

$$\left\|\overrightarrow{OA}\right\|_P=\left\|\overrightarrow{OB}\right\|_P=\left\|\overrightarrow{AB}\right\|_P$$

Thus:

$$|x|^P+|y|^P=|z|^P+|w|^P=|x-z|^P+|y-w|^P$$

According to **Corollary 5.4**, *x*, *y*, *z*, *w* cannot be all rational when **Conjecture 0.8** holds, but *A* is rational, thus *B* cannot be rational. ■

6 Regular Simplex of Other Cases in L^P-space over $\mathbb{R}^2$

(1) $P = 1$ or $P = \infty$

In this section, we study the regular simplex in L^1 or L^∞-space over $\mathbb{R}^2$. We want to answer that when two of its vertices are rational, whether the last vertex is rational. But we need to exclude a situation, which is that the two known vertices are a pair of standard translation points, for the following reason:

Theorem 6.1.1 *For a pair of standard translation points O, A in L^1-space over $\mathbb{R}^2$, there are uncountably infinitely many points B with uncertain rationality, such that:*

$$\left\|\overrightarrow{OA}\right\|_1 = \left\|\overrightarrow{OB}\right\|_1 = \left\|\overrightarrow{AB}\right\|_1$$

Proof The statement of the theorem is actually equivalent with the following statement: *There are uncountably infinitely many intersection points between the boundary of a circle centered at O, with a radius of $\left\|\overrightarrow{OA}\right\|_1$ and the boundary of a circle centered at A with equal radius with uncertain rationality, when O and A are a pair of standard translation points in L^1-space over $\mathbb{R}^2$.* The latter statement can be shown by the following figure intuitively:

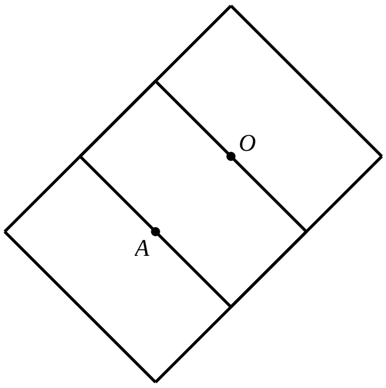

Fig. 3 *O* and *A* are a pair of standard translation points

We let *O* be the origin without loss of generality, and discuss only the situation when *A* is in quadrant III, since the situation of other cases can be proved similarly.

Since that, the coordinates of *A* satisfy:

$$A(-x, -x)$$

Where $x > 0$. Choose any *B* satisfying:

$$-2x \le x_B \le -x \text{ and } y_B = x_B + 2x$$

We can check that, *B* satisfies $\left\|\overrightarrow{OA}\right\|_1 = \left\|\overrightarrow{OB}\right\|_1 = \left\|\overrightarrow{AB}\right\|_1 = 2x$. Obviously there are uncountably infinitely many *B*'s with uncertain rationality. ■

Similarly:

Theorem 6.1.2 *For a pair of standard translation points O, A in L^∞-space over $\mathbb{R}^2$, there are uncountably infinitely many points B with uncertain rationality, such that:*

$$\left\|\overrightarrow{OA}\right\|_\infty = \left\|\overrightarrow{OB}\right\|_\infty = \left\|\overrightarrow{AB}\right\|_\infty$$

Its proof is similar to the proof of **Theorem 6.1.1**, which we won't give details here. But we'll give the following figure to show this theorem intuitively:

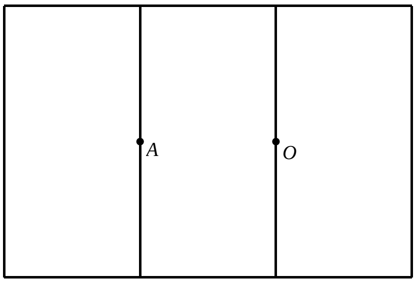

Fig. 4 O and A are a pair of standard translation points

According to the previous two theorems, we can see that, if the known two vertices are a pair of standard translation points, even if we know their exact positions, we won't be able to determine the last vertex's rationality, thus we cannot discuss further in this situation. On the other hand, whether L^1 or L^∞ circle, we know that in fact it's a square. Thus, two L^1 or L^∞ circles with equal radius, if one's center is on another's boundary, and the two centers are not a pair of standard translation points, then there are obviously two and only two intersection points between the two circles' boundaries.

Theorem 0.12 (Main Theorem 5) *Given a regular simplex in L^1 or L^∞-space over $\mathbb{R}^2$, if two of its vertices are both rational and not a pair of standard translation points, then the last vertex must be rational.*

Proof Here we only give a proof of the situation in L^1-space over $\mathbb{R}^2$, the situation in L^∞-space over $\mathbb{R}^2$ can be proved similarly.

Let the two known vertices be denoted by O and A, and assume O is the origin without loss of generality. Let the coordinates of A be (x, y), where we assume A is in quadrant I and $x > y$. Other cases can be proved similarly which we won't give details here.

We consider $B\left(\frac{x+y}{2}, -\frac{x+y}{2}\right)$, where we can check that $\left\|\overrightarrow{OA}\right\|_1 = \left\|\overrightarrow{OB}\right\|_1 = \left\|\overrightarrow{AB}\right\|_1 = x + y$.

We also consider $C\left(\frac{x-y}{2}, \frac{x+3y}{2}\right)$, similarly, we can check that $\left\|\overrightarrow{OA}\right\|_1 = \left\|\overrightarrow{OC}\right\|_1 = \left\|\overrightarrow{AC}\right\|_1 = x + y$.

We already know that there are two and only two proper points, thus only B and C satisfy our requirements. B and C are rational, obviously, which gives the desired result. ■

(2) Other Values of *P*

For other values of P, confined to the ability of the authors, we'll just introduce some inchoate results.

For $P = 3$ and $P = 4$, we notice that, **Theorem 5.1** and **5.2** still hold, which means that when two of the vertices of a regular simplex in L^3 or L^4-space over $\mathbb{R}^2$ are fixed, the last vertex can only has two possible positions. And, a 'regular simplex' with four vertices in L^3 or L^4-space over $\mathbb{R}^2$ does not exist.

In the proof of **Theorem 5.5**, we have pointed out that, whether or not the last vertex of the regular simplex in L^P-space over $\mathbb{R}^2$ can be rational depends on when given rational numbers x, y which at least one of them is nontrivial, whether or not the equation $|x|^P + |y|^P = |z|^P + |w|^P = |x - z|^P + |y - w|^P$ of z, w has a rational solution. When $P \in [5, +\infty) \cap \mathbb{N}$, under the assumption that **Conjecture 0.8** holds, we have proved that there cannot be such a rational solution. But, When $k = 3$ or $k = 4$ (That is, $P = 3$ or $P = 4$), **Conjecture 0.8** does not hold, which we can provide the following counterexamples[1]:

$$2^3 + 16^3 = 9^3 + 15^3$$
$$7^4 + 239^4 = 157^4 + 227^4$$

For more counterexamples of the situation when $k = 4$, see **Reference [4]** and **[8]**.

However, these counterexamples don't necessarily mean that there are some rational solutions of the equation mentioned above. We know that, when $k = 2$, there are lots of equations that are in the form of $a^k + b^k = c^k + d^k$ $(a, b, c, d \in \mathbb{N}^+)$, but a regular simplex in L^2-space over $\mathbb{R}^2$ cannot has three rational vertices, which is the common situation of the regular triangle in normal $\mathbb{R}^2$. Hence, we still need to directly analyze the equation mentioned above.

We use computers to search for solution directly. Of course, it's impossible to search for rational solutions even in a very small interval, but for this equation, it is easy to show that the existence of rational solutions is equivalent with the existence of integer solutions, thus we can search for just the integer solutions. In a specific interval, this sort of search can be completed in a finite amount of time. This is our result (In the search, we have used Wolfram Mathematica®) :

Theorem 6.2.1 *When $k = 3$ or $k = 4$, there can't be integers x, y, z, w in the interval $[-20000, 20000]$ which at least one of them is nontrivial satisfying the following equation:*

$$|x|^k + |y|^k = |z|^k + |w|^k = |x - z|^k + |y - w|^k$$

Of course, this can't solve the problem, but it's enough to let us conjecture that:

Conjecture 6.2.2 *When $k = 3$ or $k = 4$, there can't be integers x, y, z, w which at least one of them is nontrivial satisfying the following equation:*

$$|x|^k + |y|^k = |z|^k + |w|^k = |x - z|^k + |y - w|^k$$

A direct corollary of this conjecture is that for a regular simplex in L^3 or L^4-space over $\mathbb{R}^2$, if two of its vertices are rational, the last vertex cannot be rational.

For nonintegers $P > 1$, we didn't get any valuable result, but we can point out that, **Theorem 5.1** and **5.2** still hold. Since the last vertex can only has two possible positions, then the method to determine its rationality should exist.

When $P < 1$, we know that, the function $f(\boldsymbol{a}) = \left(|\boldsymbol{a}_x|^P + |\boldsymbol{a}_y|^P\right)^{\frac{1}{P}}$ is no longer a norm. However, L^P-space over $\mathbb{R}^2$ still exists, though the circle is no longer convex, making **Theorem 5.1** no longer hold. Here, we use the situation of $P = \frac{1}{2}$ as an example. Given two vertices of a regular simplex in $L^{\frac{1}{2}}$-space over $\mathbb{R}^2$, the last vertex can at most have six possible positions, as the following figure shows:

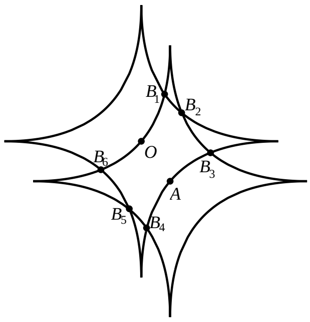

Fig. 5 Given *O*, *A*, there are six possible positions of *B*

[1] These two counterexamples are provided by Prof. FENG Rong-quan from the Peking University.

Of course, in some situations, there can only be four or two possible positions of *B*, which depends on the specific positions of *O* and *A*.

We didn't get any valuable result for $P < 1$, either. Actually, considering the variety of the possibilities of the last vertex, it is doubtful whether the method to determine its rationality exists.

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Reference

[1] GAO Wei-chen. *n* 维空间中 *n* 维立方体顶点的有理性 (The Rationality of the Vertices of an *n*-cube in *n*-dimensional Space[1])[J]. 数学通报 (Bulletin of Mathematics), 2015, 54[9]: 52-54

[2] Dickson L. History of the Theory of Numbers, Vol 2[M]. Chelsea: Dover Publications, 2005. 648

[3] L. J. Lander, T. R. Parkin. Counterexample to Euler's Conjecture on Sums of Like Powers[J]. Bulletin of the American Mathematical Society, 1966, [72]: 1079

[4] L. J. Lander, T. R. Parkin, J. L. Selfridge. A Survey of Equal Sums of Like Powers[J]. Mathematics of Computation, 1967, 99[21]: 446-459

[5] XIA Pan-qiu. The External Product in Higher Euclidean Space[J]. College Mathematics, 2011, 27[4]: 159-164

[6] ZHANG Xian-ke. 高等代数学 (Advanced Algebra)[M]. Beijing: Tsinghua University Press, 2004. 43-46

[7] GAO Wei-chen. $\mathbb{R}^n$中正棱锥顶点的有理性 (The Rationality of the Vertices of a Right Pyramid in $\mathbb{R}^n$)[J], Submitted

[8] L. J. Lander, T. R. Parkin. Equal Sums of Biquadrates[J]. Mathematics of Computation, 1966, 95[20]: 450-451

[1] This English name, as well as the following ones in brackets, was translated into English by the authors of this paper because of the lack of official translation.