Study on High-dimensional Extension of Pick’s Theorem

Zhu Guangyuan
(Beijing 101 Middle School, China)
Tutor: Liu Jianyu

Abstract: Due to existence of Reeve tetrahedron counterexample, Pick’s theorem cannot be simply high-dimensional extended unconditionally. In this paper, n-dimension extension of Pick’s theorem is proposed under sufficient conditions including cube, 2^{n-1}-prism etc., and necessary conditions of extension are discussed as well. Formula obtained from n-dimensional extension of Pick’s theorem maintains the simple form of Pick’s theorem, and there is no convex restriction under sufficient conditions. Meanwhile, high-dimensional extension of Euler’s formula is obtained through the special case of cube. N-dimensional simple connected closed region in Euclidean space can be approximated through unit cube to further establish connection between lattice geometry and general Euclidean space geometry.

Key words: Pick’s theorem; lattice point; high-dimensional space; n-edge; cube combination

1 Preface: Pick’s Theorem and 3-dimensional Counterexample

In a plane rectangular coordinate system, points of which the x-coordinate and y-coordinate are both integers are called lattice points (also called integral points). Obviously the definition above can be extended to rectangular coordinate systems of any dimension.

If all vertexes of a polygon are lattice points, then the polygon is referred to as lattice polygon, of which the area satisfies the following formula:

Formula 1-1: S=C+B/2-1

S is the area of polygon, C is quantity of lattice points in the polygon, and B is quantity of lattice points on the edges of polygon.

This formula is called Pick’s theorem since it was proposed by Pick. It is an interesting theorem. For a simple polygon of which the vertexes are lattice points, Pick’s theorem describes the relation between its area S and quantity of lattice points in it and on its edge, not restricted by polygon convexity.

In the reference [1][2] and [3], proving of Pick’s theorem was provided. Author of reference [2] might have confused Pick and the French mathematician Picard (the question is raised in the editor’s note at end of the book), therefore he introduced

Pick' theorem in the 1st chapter of the book of which the theme shall be Picard Theorem. However, it is still a rather good article of Pick's theorem; especially it has provided simple proving of the theorem, which will be quoted in the section. However, different method was adopted for counting lattice points. In future, the counting method will be adopted for completing counting of lattice points of high-dimensional geometry.

Adopt the following method for counting lattice points and internal points on the edges and in the area of lattice polygon:

Step 1, counting vertexes of polygon.

Step 2, counting internal points on edges. Plus quantity of internal points on edges and vertexes of the polygon, then we get quantity of lattice points on the edges.

Step 3, counting internal points in the area. Plus quantity of internal points in the area and lattice points on the edges, and then we get quantity of lattice points in the area of polygon.

The method of counting is convenient for counting internal points and lattice points on the edges and in the area of high-dimensional geometry.

Unless otherwise stipulated, all geometries mentioned here are lattice geometries. For example, triangle means lattice triangle, 3-dimensional cube means lattice 3-dimensional cube, and cuboid means lattice cuboid etc.

Lemma 1-1: Pick' theorem is valid for rectangle with edges parallel to coordinate axis.

Proving:

Given a rectangle with coordinates of vertexes A(0,0), B(m,0), C(m,n), D(0,n).

Internal point: (m-1)(n-1)

Edge points: internal points on edges+ vertexes = 2(m-1+n-1)+4=2m+2n

Lemma 1-2: Pick's theorem is valid for right-angled triangle with right-angle edges parallel to coordinate axis.

Proving:

In lemma 1-1, coordinates of vertexes of triangle ABC are A(0,0),B(m,0),C(m,n).

Given quantity of points on AC edge is l, then

Internal points: C=1/2[(m-1)(n-1)-(l-2)]

Edge points: B=m+n+l-1

Area: S=mn/2

Satisfying Pick's theorem S=C+B/2-1.

Lemma 1-3: Pick's theorem is valid for arbitrary triangle.

Proving:

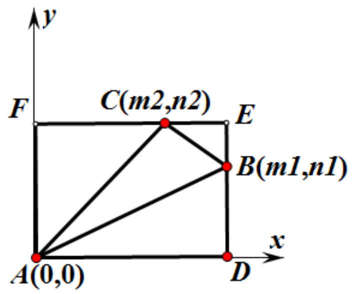


Fig.1-1 Arbitrary Triangle Embedded in Rectangle

Triangle ABC is shown as fig.1-1. Use T_1,T_2 and T_3 for triangles BEC, CFA and ADB respectively, then known from lemma 1-2, triangle T_i satisfies Pick's theorem:

$S(T_i)=C(T_i)+B(T_i)/2-1(i=1,2,3)$

Known from lemma 1-1, rectangle ADEF satisfies Pick's theorem:

$S_{ADEF}=C(R)+B(R)/2-1$

Then for triangle ABC:

$$\begin{aligned} S(T) &= S_{ADEF}-S(T_1)-S(T_2)-S(T_3) \\ &= C(R)-C(T_1)-C(T_2)-C(T_3)+1/2[B(R)-B(T_1)-B(T_2)-B(T_3)]+2 \\ &=[C(T)+B(T)-3]+1/2[-B(T)]+2 \\ &= C(T)+ B(T)/2-1 \end{aligned}$$

Theorem 1-1: Pick's theorem is valid for arbitrary polygon (including concave polygon).

Proving:

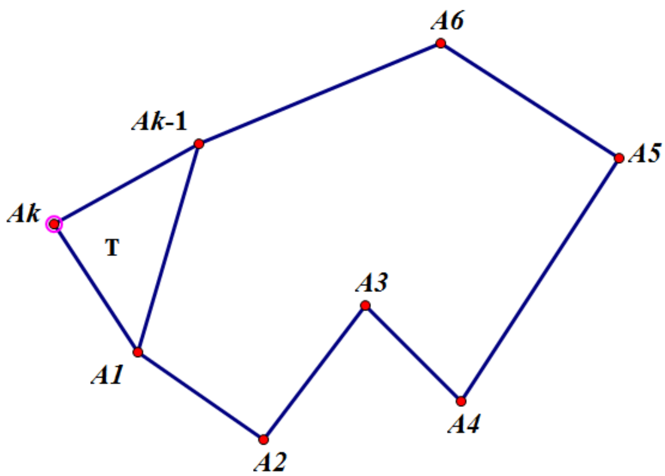


Fig.1-2 Arbitrary Polygon

Given $A_1A_2…A_n$ is a lattice polygon shown as fig.1-2. Use S_n for its area, C_n for quantity of lattice points in it and B_n for quantity of lattice points on edges. Prove $S_n=C_n+1/2B_n-1$. Adopt mathematical induction. When n=3, known from lemma 1-3 that the proposition is valid. Assuming when n=k-1, the proposition is valid, then when n=k, since there is at least one internal angle in the polygon, as shown in fig.1-2, we can assume the angle as A_k, which is less than π. Connect A_1 and A_{k-1} to get:

Mark triangle $A_1 A_k A_{k-1}$ as T, then:

$S_k=S_{k-1}+S(T)=[C_{k-1}+C(T)]+1/2[B_k-1+B(T)]-2$

Assume quantity of lattice points on A_1A_{k-1} is 1, then:

$S_k =(C_k-l+2)+1/2(B_k +2l-2)-2= C_k +1/2\ B_k -1$

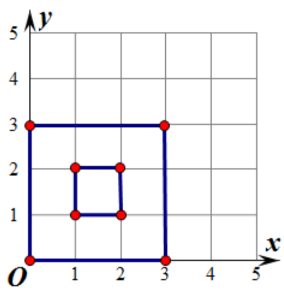
i.e. when n=k, the proposition is valid.

Extension 1-1: Lattice regular triangle does not exist.

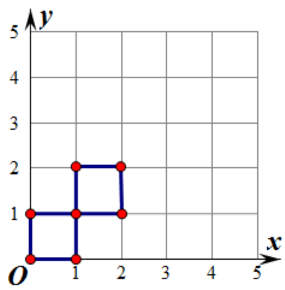
Proving:

Given triangle ABC is lattice regular triangle. Assuming rectangle ADEF is lattice rectangle with edges parallel to coordinate axis embedded with triangle ABC (refer to fig.1-1). It is known from Pick's theorem that the area of triangle ABC is a rational number. However, square of edge a of triangle ABC is integer (Pythagorean Theorem), then area of triangle ABC S=$\frac{\sqrt{3}}{2}$ a^2 is irrational. It is contradictory.

It is shown in the following text that in a 3-dimensional space, lattice regular triangle exists. Moreover, extension 1-1 can be adopted for proving that some 3-dimensional geometry does not exist.



(1)



(2)

Fig.1-3 2-dimensional Non-simple Connected Lattice Geometry

Pick's theorem is applicable for concave polygon, but is not valid for non-simple connected space. See fig.1-3. It has not been found in articles concerning Pick's theorem. Since the following n-dimensional unit cube combination satisfying high-dimensional Pick's theorem is of simple connectivity restriction as well, the application scope of Pick's theorem is clearly defined here.

Combination of two polygons through single point mode is defined as informal combination, as shown in fig. 1-3(2). It is clear that under this condition, it is even impossible to confirm whether there is internal point in the combination.

Conclusion 1-1: Pick's theorem is invalid for non-simple connected area, as shown in Fig. 1-3(1).

Conclusion 1-2: Pick's theorem is invalid for informal combination, as shown in Fig. 1-3(2).

Unless otherwise stipulated, geometries mentioned in the following text are simple-connected normal combined geometries.

There is counterexample of Pick's theorem in 3-dimensional space. Given Reeve tetrahedron (0,0,0),(1,0,0),(0,1,0),(1,1,k), k is arbitrary positive integer. It is not difficult to prove that for arbitrary k, there is no integral point in the tetrahedron and on its edges except for vertexes. Therefore, volume might be different with the same quantity of integral points.

Counterexample 1-1: Volume of Reeve tetrahedron is not related to quantity of lattice points on edges.

It is a fatal blow for high-dimensional extension of Pick's theorem. Arbitrary nature of k decides that there is geometry existing in high-dimensional space and its volume is not depending on quantity of lattice points on edges. Pick's theorem cannot be high-dimensional extended unconditionally. There comes the following question:

Sufficient conditions for validity of Pick's theorem and the formula under sufficient conditions.

Sufficient and necessary conditions for validity of Pick's theorem.

Pick's theorem cannot be high-dimensional extended simply and unconditionally, while Ehrhart polynomial can be an indirect extension [4]. Condition of Ehrhart polynomial is convex polyhedron, which itself is given through quantity of integral points. The highest coefficient corresponds to volume, second highest coefficient corresponds to boundary volume, and the constant term corresponds to Euler number. However, Ehrhart polynomial is of convexity condition and not of the simple form of Pick's theorem. Moreover, the coefficient is difficult for calculation. At present, only highest coefficient, second-highest coefficient and constant term can be obtained [5] [6].

In this paper, n-dimensional extension, such as cube , 2^{n-1}-prism etc., of Pick's theorem is done, and simple form of formula 1-1 satisfying Pick's theorem is proposed, which is suitable for concave extension and capable of giving out coefficient of formula under n-dimensional condition. It is proven that simple connected closed region of arbitrary n-dimensional Euclidean space can be approximated through unit cube combination.

2 3-dimensional Extension of Pick's Theorem

2.1 3-dimensional Cube and Cuboid

Firstly consider the situation that edges of cube are parallel to coordinate axis. Firstly, due to existence of counterexample 1-1, study of 3-dimensional polyhedron generally adopts cube and cuboid with edges parallel to coordinate axis of the complement edges of the polyhedron; Secondly, cube with edges parallel to coordinate axis is of 2-dimensional extension of simple form of Pick's theorem, while counterexample exists in prism which does not satisfy the condition; Thirdly, arbitrary 3-dimensional simple connected closed geometry can be approximated through combination of unit cubes of which the edges are parallel to coordinate axis, and unit cube with edges parallel to coordinate axis satisfies 2-dimensional simple extension of Pick's theorem.

Agreement: Cubes and cuboids mentioned in the section (section 2.1) are cubes and cuboids with edges parallel to coordinate axis.

There are 8 vertexes, 12 edges and 6 faces of 3-dimensional cube. Given a cube with edge length of l, then:

Internal points: $C=(l-1)^3$

Plus internal points on 12 edges and 8 vertexes to obtain the points on edges.

Edge points: $B=12(l-1)+8=12l-4$

Plus internal points on 6 faces and points on 12 edges to obtain the points on faces.

Face points: $A=6(l-1)^2+12l-4=6l^2+2$

Through calculation, it is obtained $C+A/2-B/4-1=l^3$, equal to $V= l^3$.

Formula 2-1: $V=C+A/2-B/4-1$, i.e. 3-dimensional extension of Pick's theorem.

Geometry satisfying formula 2-1 is referred to as geometry satisfying 3-dimensional Pick's theorem.

Theorem 2-1: 3-dimensional cube satisfies 3-dimensional Pick's theorem.

Given edges of cuboid are l_1, l_2 and l_3, then:

Internal points: $C=(l_1-1)(l_2-1)(l_3-1)= l_1 l_2 l_3-(l_1 l_2+ l_2 l_3+ l_3 l_1)+(l_1+ l_2+ l_3)-1$

Edge points: $B=4(l_1-1)+ 4(l_2-1)+ 4(l_3-1)+8=4(l_1+ l_2+ l_3)-4$

Face points: $A=2(l_1-1)(l_2-1)+2(l_2-1)(l_3-1)+2(l_3-1)(l_1-1)+ 4(l_1+ l_2+ l_3)-4$
$=2(l_1 l_2+ l_2 l_3+ l_3 l_1)+2$

Through calculation it is obtained: $C+A/2-B/4-1=l_1l_2l_3$, equal to volume $V=l_1l_2l_3$, obtaining:

Theorem 2-2: 3-dimensional cuboid satisfies 3-dimensional Pick's theorem.

2.2 Other Geometries (Pyramid, Prism) in 3-dimensional Space

In this section (section 2.2), other geometries (pyramid and prism) in 3-dimensional space are discussed. Pyramid and prism are defined as bottom face is parallel to coordinate plane (edges of bottom face are not restricted to be parallel to coordinate axis) and height is parallel to coordinate axis. It is complying with general condition of pyramid and prism as well. Pyramid and prism with height not parallel to bottom face are generally called oblique pyramid and oblique prism.

Given a pyramid with bottom face of grid 2x2 and height h.

Internal points: C= h-1

Face points: A=internal points of bottom face + edge points of bottom face + vertexes= 1+8+1=10

Edge points: B= edge points of bottom face + vertexes= 8+1=9

Through calculation, it is obtained C+A/2-B/4-1=$h+\frac{3}{4}$, not equal to V=$\frac{4}{3}h$, obtaining:

Counterexample 2-1: pyramid does not satisfies V=C+A/2-B/4-1.

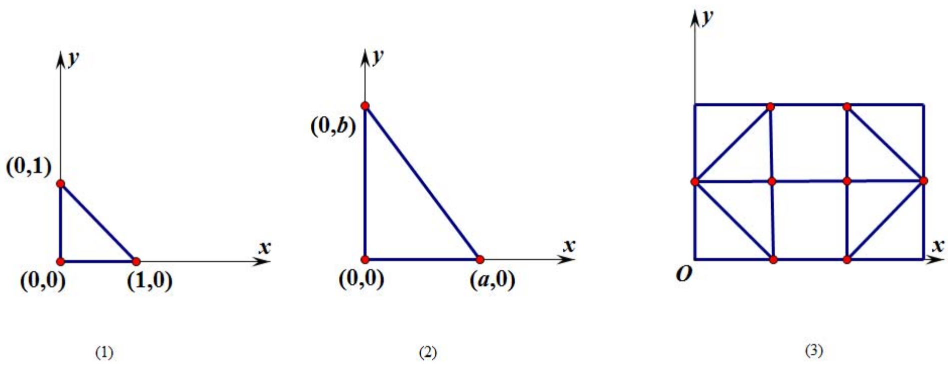


Fig.2-1 Bottom Face of Triangular and Hexagonal Prism

Given a prism with height h and bottom face of unit isosceles right triangle with vertexes (0,0),(1,0),(0,1), as shown in fig. 2-1 (1).

h=1:C=0,B=6,A=6, C+A/2-B/4-1=1/2 equal to V=1/2.

h=2:C=0,B=9,A=9, C+A/2-B/4-1=5/4 not equal to V=1.

h=1 is a special case of coincidence (value of volume of prism is exactly equal to that of bottom area), obtaining:

Counterexample 2-2: Isosceles right triangular prism with unit right angle edges does not satisfy V=C+A/2-B/4-1 when h>1.

Actually, for a right triangular prism with vertexes (0,0),(a,0),(0,b), as shown in fig. 2-1(2) and height h:

Area of triangle: $S=\frac{ab}{2}$

Edge points: B=(a-1)+(b-1)+3=a+b+1

It is known from Pick's theorem S=C+B/2-1, quantity of internal points of triangle is $C=\frac{ab}{2}-\frac{a+b+1}{2}+1$

Obtaining that for right triangular prism:

Internal points: $C^{*}=(\frac{ab}{2}-\frac{a+b+1}{2}+1)(h-1)$

Edge points: $B^{*}=2(a-1+b-1)+3(h-1)+6$

Face points: $A^{*}=2C+(a+b+1)(h+1)$. The previous method of counting is not adopted here. For prism, it is more convenient to adopt the method of adding twice of internal points of bottom face with edge points and then multiplying with (h+1).

Through calculation it is obtained $C^{*}+A^{*}/2-B^{*}/4-1=(\frac{ab}{2}+\frac{1}{4})h-\frac{1}{4}$ not equal to $V=\frac{abh}{2}$

It is a special case of coincidence when h=1, which is called trivial. Obtaining:

Counterexample 2-3: Non-trivial prism generated from arbitrary right triangle with right angle edges parallel to coordinate axis does not satisfy V=C+A/2-B/4-1.

For a hexagon shown in fig.2-1(3), C=2,B=6, through Pick's theorem S=C+B/2-1=4, obtaining the area is 4.

Given a prism with bottom face of hexagon and height of 5.

Internal points: $C^{*}=2x(5-1)=8$

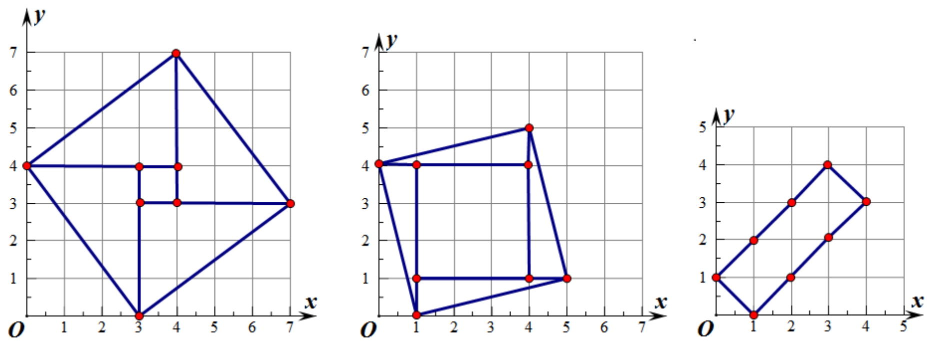
Edge points: $B^{*}=6x(5+1)=36$

Face points: $A^{*}=2+2+6x\ (5+1)=40$

Through calculation it is obtained $C^{*}+A^{*}/2-B^{*}/4-1=18$, not equal to V=20

Actually, equation is not valid if the height is arbitrary positive integer.

Counterexample 2-4: There is hexagon prism not satisfying V=C+A/2-B/4-1.



(1) (2) (3)

Fig.2-2 Bottom Face of Square and Rectangular Prism

(1) As shown in fig.2-2(1), for a square (3,0),(0,4),(4,7),(7,3), S=5x5=25,B=4. Through Pick's theorem S=25=C+B/2-1, it is obtained C=24.

Given a prism with bottom face of this square and height of 5. The prism is a cube with edge length of 5.

Internal points: C^*=24x(5-1) =96

Edge points: B^*=4x(5+1)=24

Face points: A^*=4x(5+1)=24

Through calculation, it is obtained that $C^*+A^*/2-B^*/4-1=125$, equal to V=25x5=125.

Actually, equation is valid if the height is arbitrary positive integer.

Conclusion 2-1: There is square prism (bottom edge not parallel to coordinate axis) satisfying 3-dimensional Pick's theorem.

Conclusion 2-2: There is cube (bottom edge not parallel to coordinate axis) satisfying 3-dimensional Pick's theorem.

(2) As shown in fig.2-2(2), for a square (1,0),(0,4),(4,5),(5,1), $S=\sqrt{17}\sqrt{17}=17$,B=4.

Through Pick's theorem S=17=C+B/2-1, it is obtained C=16.

Given a prism with bottom face of this square and height of 5. The prism is a cube with edge length of 5.

Internal points: C^*=16x(5-1) =64

Edge points: B^*=4x(5+1)=24

Face points: A^*=16+16+4x(5+1)=56

Through calculation, it is obtained that $C^*+A^*/2-B^*/4-1=85$, equal to V=17x5=85.

Actually, equation is valid if the height is arbitrary positive integer.

Conclusion 2-3: Validity of conclusion 2-1 is not restricted to that the edge length is integer.

(3)As shown in fig.2-2(3), for a rectangle (1,0),(0,1),(3,4),(4,3), C=3,B=8. Through Pick's theorem S=C+B/2-1=6, it is obtained that area is 6.

Given a prism with bottom face of this rectangle and height of 5.

Internal points: $C^*=3x(5-1)=12$

Edge points: $B^*=4x(4+0+4)=32$

Face points: $A^*=3+3+8x(5+1)=54$

Through calculation, it is obtained that $C^*+A^*/2-B^*/4-1=30$, equal to $V=6x5=30$.

Actually, equation is valid if the height is arbitrary positive integer.

Conclusion 2-4: There is rectangular prism (bottom edge not parallel to coordinate axis) satisfying 3-dimensional Pick's theorem.

Combing conclusion 2-1, conclusion 2-2, conclusion 2-3 and conclusion 2-4, it is obtained:

Theorem 2-3: There are square and rectangular prisms with edges of bottom face not parallel to coordinate axis satisfying 3-dimensional Pick's theorem.

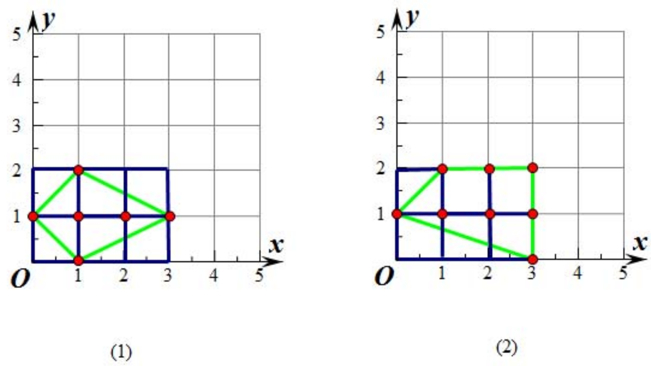


Fig.2-3 Bottom Face of Quadrangular Prism

(1) Given a quadrangular prism without internal points on edges, as shown in fig.2-3(1), with height of h, then:

Edge points: $B=4$

Through Pick's theorem $S=C+B/2-1$, internal points of quadrangle is $C=S-1$.

Obtaining that for quadrangular prism:

Internal points: $C^*=(S-1)(h-1)$

Edge points: $B^*=4(h+1)$

Face points: $A^*=2(S-1)+4(h+1)$

Through calculation, it is obtained $C+A/2-B/4-1=Sh$, equal to $V=Sh$

Conclusion 2-5: Arbitrary quadrangular prism without internal points on edges satisfies 3-dimensional Pick's theory.

(2) Given a quadrangular prism with internal points on edges, as shown in fig.2-3(2), $C=2,B=6$. Through Pick's theorem $S=C+B/2-1=4$, it is obtained that area is 4.

Given a prism with bottom face of the quadrangle and height of 5.

Internal points: $C^*=2x(5-1)=8$

Edge points: B^*=4x(5+1)+2x2=28

Face points: A^*=2+2+6x6=40

Through calculation, it is obtained $C^*+A^*/2-B^*/4-1=20$, equal to V=4x5=20

Actually, the equation is valid for quadrangle with internal point on any side, of which the height is arbitrary positive integer.

Conclusion 2-6: Arbitrary quadrangular prism with internal point on any side satisfies 3-dimensional Pick's theorem.

Combing conclusion 2-5 and conclusion 2-6, it is obtained that:

Theorem 2-4: Arbitrary irregular quadrangular prism satisfying 3-dimensional Pick's theory.

At last, given arbitrary N-sides polygon to find out a necessary and sufficient condition for N-sides prism satisfying 3-dimensional Pick's theorem.

Given prism with arbitrary N-sides polygon C=c,B=b, and height =h. From Pick's theorem S=c+b/2-1.

Through calculation:

C+A/2-B/4-1=ch+Bh/2-Nh/4+N/4-1

V= Sh=ch+Bh/2-h

Necessary and sufficient condition for satisfying Pick's theorem is that the above-mentioned two equations equal to each other. Through simplification, it is obtained (h-1)(N-4)=0.

Validness of the equation requires N=4 or h=1, obtaining:

Theorem 2-5: Necessary and sufficient condition for non-trivial N-sides prism satisfying 3-dimensional Pick's Theorem is N=4.

Counterexamples 2-1, 2-2, 2-3 and 2-4 show that there are pyramid and prism in 3-dimensional space not satisfying V=C+A/2-B/4-1. The problem exists in n-dimensional space higher than 3-dimension as well. It means that general geometry in high-dimensional space cannot be simply high-dimensional extended unconditionally to obtain the simple formula of Pick's theorem in 2-dimensional space.

Conclusions 2-1, 2-2, 2-3 and 2-4 give out that square and rectangular prisms of which edge of bottom face are not parallel to coordinate axis satisfy 3-dimensional Pick's theorem. It means that cube and cuboid parallel to coordinate axis are not necessary conditions for satisfying 3-dimensional Pick's theorem.

Is there any square or rectangular prism of which all edges are not parallel to coordinate axis satisfying 3-dimensional Pick's theorem? In the following text, a cuboid will be structured and verified that whether it satisfies 3-dimensional Pick's theorem.

2.3 3-dimensional Cuboid with All Edges not Parallel to Coordinate Axis

Since geometry in lattice space is restricted by integral point, many geometries existing in general Euclidean space do not exist in lattice space. For example, if lattice right triangle rigging-angle cone (bottom face is right triangle and the other 3 faces are isosceles triangles, refer to right triangle rigging-angle cone $A_2D_1C_2$-D_2 in fig.2-4) parallel to coordinate plane was existing, it would be rather convenient to structure 3-dimensional cube with all edges not parallel to coordinate axis. However, it is deducted from extension 1-1 that plane lattice right triangle is not existing.

Conclusion 2-7: Right triangular pyramid with bottom face parallel to coordinate plane does not exist.

Conclusion 2-8: Right triangular prism with bottom face parallel to coordinate plane does not exist.

Conclusion 2-7 shows that lattice right triangle rigging-angle cone parallel to coordinate plane does not exist. Therefore, lattice geometry shall be structured carefully and verified strictly.

It is not easy to structure a 3-dimensional cuboid with all edges not parallel to coordinate axis (i.e. all faces not parallel to coordinate planes) without aid of computer; even such cuboid is structured successfully with aid of computer; it is still difficult to understand its geometric intuition. In the following text, a 3-dimensional cuboid with all edges not parallel to coordinate axis will be structured manually for verifying whether the cuboid satisfies 3-dimensional Pick's theorem.

First of all, a cube $A_1B_1C_1D_1$-$A_2B_2C_2D_2$ is structured in the first quadrant from coordinate origin with edge length of 3. D_2 superposes coordinate origin O, shown in fig.2-4.

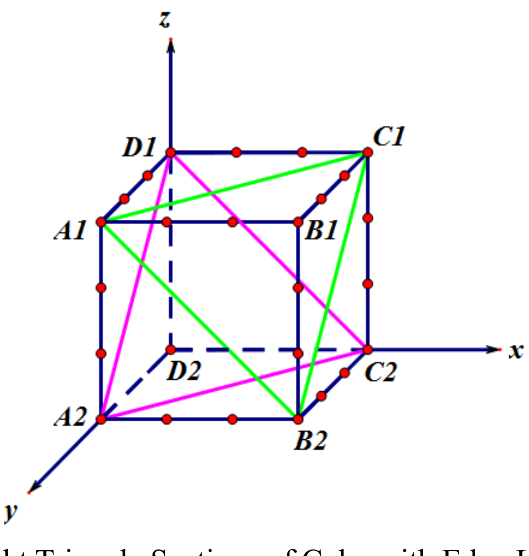


Fig.2-4 Right Triangle Sections of Cube with Edge Length of 3

Through solid geometry and space analytic geometry, it is known that plane of triangle $A_2D_1C_2$ is parallel to that of $C_1B_2A_1$, normal direction is $1\mathbf{i}+1\mathbf{j}+1\mathbf{k}$, and main diagonal vector of cube is $\mathbf{OB}_1$. Crossover points of triangle $A_2D_1C_2$, $C_1B_2A_1$ and segment OB trisects the segment OB. Distance between plane of triangle $A_2D_1C_2$ and that of $C_1B_2A_1$ is $\sqrt{3}$.

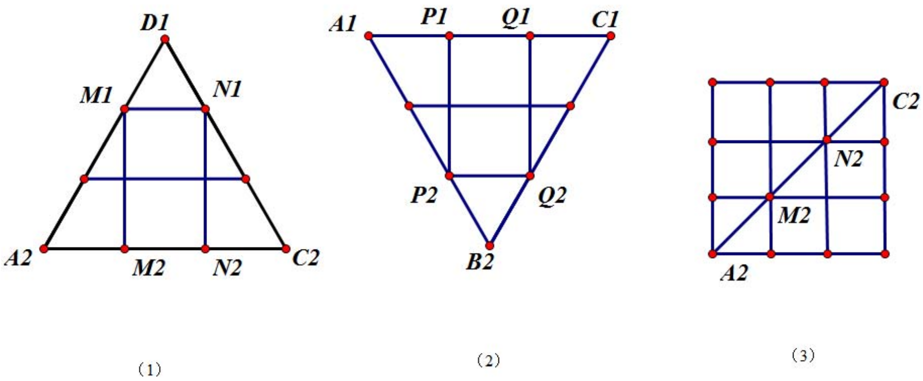


Fig.2-5 Right Triangle Section Normal View and Bottom Face Normal View of Cube

Fig.2-5 is normal views of triangle $C_1B_2A_1$. M_1 and N_1 are trisection points of triangle $A_2D_1C_2$. M_2 and N_2 are respectively foot points of M_1 and N_1 on edge A_2C_2. P_2 and Q_2 are trisection points of triangle $C_1B_2A_1$. P_1 and Q_1 are respectively foot points of P_2 and Q_2 on edge A_1C_1. Coordinates of all points are in Tab. 2-1 as follows:

$A_2(0,3,0)$	$D_1(0,0,3)$	C_2 (3,0,0)	
$C_1(3,0,3)$	$B_2(3,3,0)$	$A_1(0,3,3)$	
$M_1(0,1,2)$	$N_1(1,0,2)$	$M_2(1,2,0)$	$N_2(2,1,0)$
$P_1(1,2,3)$	$Q_1(2,1,3)$	$P_2(2,3,1)$	$Q_2(3,2,1)$

Table 2-1 Table of Vertexes of Cuboid

Prove that P_1M_1, P_2M_2, Q_1N_1 and Q_2N_2 are perpendicular to plane of triangle $C_1B_2A_1$. Take P_1M_1 as example.

Method 1: Calculate distance of P_1M_1. Distance of P_1M_1 is $\sqrt{3}$, equal to distance between plane of triangle $A_2D_1C_2$ and that of $C_1B_2A_1$.

Method 2: Calculate vector of M_1P_1. Vector of $\mathbf{M_1P_1}$ is $1\mathbf{i}+1\mathbf{j}+1\mathbf{k}$, equal to normal direction vector of plane of triangle $C_1B_2A_1$.

Therefore, $P_1M_1N_1Q_1$- $P_2M_2N_2Q_2$ is a 3-dimensional cuboid with all edges not parallel to coordinate axis, as shown in fig.2-6.

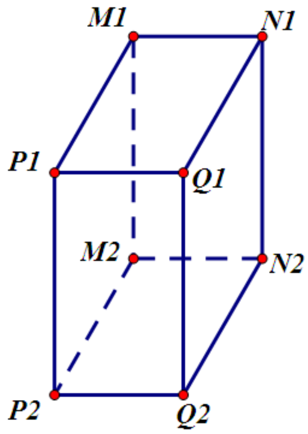


Fig.2-6 Right Triangle Section Normal View of Cuboid

Edge length of cuboid $P_1M_1N_1Q_1$- P_2M_2 N_2Q_2 are respectively $P_1P_2=\sqrt{6}$, $P_1Q_1=\sqrt{2}$, $P_1M_1=\sqrt{3}$.

Internal points: C=2

(1,1,2),(2,2,1)

Face points: A= 14

(P_1,M_1,N_1,Q_1,P_2,M_2,N_2,Q_2 and (1,1,1),(2,2,2),(1,2,1),(1,2,2),
(2,1,1),(2,1,2))

Edge points: B = 8(P_1,M_1,N_1,Q_1,P_2,M_2,N_2,Q_2)

Through calculation, it is obtained C+A/2-B/4-1=6, equal to $V=\sqrt{6}\ \sqrt{2}\ \sqrt{3}=6$, obtaining:

Theorem 2-5: There is 3-dimensional cuboid with all edges not parallel to coordinate axis satisfying 3-dimensional Pick's theorem.

2.4 Combination of 3-dimensional Cubes (Cube Block)

Agreement: Cube and cuboid mentioned in the section (section 2.4) are cube and cuboid with edges parallel to coordinate axis.

The following agreements are made for discussing combination of cubes:

Agreement 1: Two geometries are combined on lattice point to ensure that combination of geometries is lattice geometry.

Agreement 2: Combination of geometries is simple connected.

Agreement 3: Geometries are combined on face (neither point nor line). Combination satisfying the condition is defined as formal combination.

Formal combination defined in agreement 3 extends the formal combination defined in the preface. The definition will be further extended in high-dimensional space.

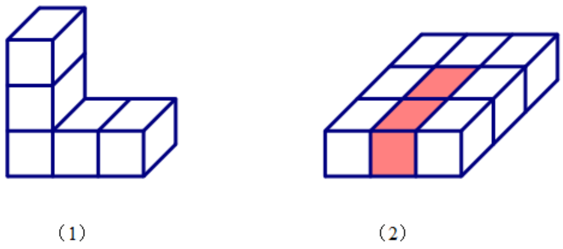


Fig.2-7 3-dimensional Unit Cube Pipe Combination

Pipe combination of 3-dimensional unit cubes is defined as a series of unit cubes, of which the first cube has only one face connected to the post order cube, middle cubes have only one front face connected to the front order cube and only one back face connected to the post order cube, and the tail cube has only one face connected to the front order cube, as shown in 2-7.

Given pipe combination of r pieces of unit cubes.

Internal points: C=0

Face points: A=4(r+1)

Edge points: B= 4(r+1)

Through calculation, it is obtained C+A/2-B/4-1=r, equal to V=r, obtaining:

Theorem 2-6: Pipe combination of arbitrary unit cubes satisfies 3-dimensional Pick's theorem.

Combination of cubes is not necessary to satisfy convexity condition. Therefore, under condition of combination of cubes, 3-dimensional Pick's theorem is not confined to convexity. It is a feature not possessed by results of many 3-dimensional lattice questions and even other 3-dimensional geometric questions.

Given a (3k)x(3k)x(3k) cube composed of 3x3x3 cube. With the method of structuring 3-dimensional Hilbert curve given by reference [7], an ergodic (3k)x(3k)x(3k) cube pipe combination composed of 3x3x3 cubes can be structured, as shown in fig.2-8.

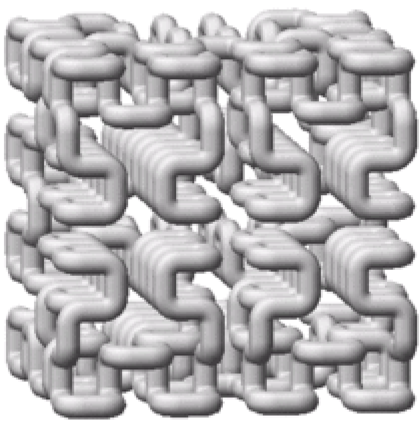


Fig.2-8 Scheme of 3-dimensional Hilbert Curve

For convenience of high-dimensional extension, another proving of theorem 2-6 is given out here, and method of mathematical induction is adopted.

When r=2, it can be proved with direct calculation. Assume that the theorem is valid when r=k. Then for r=k+1:

Given that pipe of k piece of unit cubes and unit cube satisfying 3-dimensional Pick's theorem satisfy respectively:

$V_1=C_1+A_1/2-B_1/4-1$

$V_2=C_2+A_2/2-B_2/4-1$

Volume of cube pipe is V, internal point is C, and point on faces is B.

Internal points: $C= C_1+ C_2$

Face points: $A= A_1+A_2-4$

Edge points: $B= B_1+ B_2-4$

Satisfying V=V1+V2= C+A/2-B/4-1.

It is directly obtained from theorem 2-6 that:

Conclusion 2-1: Cuboid composed of r unit cubes satisfies 3-dimensional Pick's theorem.

Combine the above-mentioned cuboid on r×1 face. Given two cuboids satisfying 3-dimensional Pick's theorem satisfy respectively:

$V_1=C_1+A_1/2-B_1/4-1$

$V_2=C_2+A_2/2-B_2/4-1$

Volume of cube pipe is V, internal point is C, and points on faces is B.

Internal points: $C= C_1+ C_2$

Face points: $A= A_1+A_2-2(r-1)-4$

Edge points: $B= B_1+ B_2-4(r-1)-4$

Satisfying $V=V_1+V_2= C+A/2-B/4-1$.

Satisfying $V=V_1+V_2= C+A/2-B/4-1$.

Combining mathematical induction and above calculation, it is obtained that:

Conclusion 2-2: r×s×1 cuboid composed of r×s pieces of unit cubes satisfies 3-dimensional Pick's theorem.

Combine the above-mentioned cuboid on r×s face. Given two cuboids satisfying 3-dimensional Pick's theorem satisfy respectively:

$V_1=C_1+A_1/2-B_1/4-1$

$V_2=C_2+A_2/2-B_2/4-1$

Volume of cube pipe is V, internal point is C, and point on faces is B.

Internal points: $C= C_1+ C_2 +(r-1)(s-1)$

Face points: $A= A_1+A_2-2(r-1)(s-1)-2(r-1)-2(2-1)-4$

Edge points: $B= B_1+ B_2-4(r-1)-4(s-1)-4$

Satisfying $V=V_1+V_2= C+A/2-B/4-1$.

Combining mathematical induction and above calculation, it is obtained that:

Theorem 2-7: r×s×t cuboid composed of r×s×t pieces of unit cubes satisfies

3-dimensional Pick's theorem (i.e. 3-dimensional cuboid satisfies 3-dimensional Pick's theorem).

Theorem 2-7 can be used as another proof for theorem 2-2.

The key of extending theorem 2-7 is that internal points of cuboid generated on combination face are exactly half of the sum of internal points on the two combined faces; internal points on the edge integrating combination faces are exactly half of the sum of internal points on the two edges integrated, and meanwhile coefficient of 3-dimensional Pick's theorem satisfies the condition. The meaning of geometry with positive face point coefficient and negative edge point coefficient is that face points are not part of internal points and increment of internal points offsets half of sum of face points decrement; while edge points are part of face points, then decrement of face points offsets half of sum of edge points decrement. It ensures that Pick's theorem satisfies additivity of volume. It is known that additivity is a basic property of volume; therefore, sign of coefficient of 3-dimensional Pick's theorem expresses additivity of volume. In the following text, it shows that n-dimensional Pick's theorem satisfies the additivity of volume as well.

3 N-dimensional Extension of Pick's Theorem

The section (section 3.1-3.5) will be focused on n-dimensional extension of Pick's theorem under condition that edges are parallel to coordinate axis.

Agreement: Cube and cuboid mention in this section (section 3.1-3.5) are cube and cuboid with edge parallel to coordinate axis.

3.1 Edge Calculation of High-dimensional Cube

For high-dimensional cube extension of Pick's theorem, the first question to be solved is quantity of vertexes, edges and faces of high-dimensional cube. For example, quantity of lattice points on edges and faces of 4-dimensional cube?

(1) Given a 3-dimensional unit cube in first quadrant of 3-dimensional Cartesian coordinate system.

① Quantity of vertexes. Every vertex has three coordinates and every coordinate takes two values 0 or 1. $2^3=8$.

② Quantity of edges. Every point on the edge has 3 coordinates, among which 2 are fixed and 1 is flexible. The fixed coordinates are of two options: 0 or 1. For example, for edge on Z-axis, x=0,y=0 is fixed and z is moving between [0,1]. There are 2 fixed among the 3 coordinates, and then there will be $C_3^2=3$ options. And there are 2 options (0 or 1) for places fixing the fixed coordinates. The total number of options is the quantity of edges of 3-dimensional cube. Therefore, edges of 3-dimensional cube are $2^2C_3^2 = 12$.

③ Quantity of faces. Every point on the face has 3 coordinates, among which 1 is fixed and 2 are flexible. The fixed coordinate is of two options: 0 or 1. There are 1 fixed among the 3 coordinates, and then there will be $C_3^1 = 3$ options. And there are 2 options (0 or 1) for places fixing the fixed coordinate. The total number of options is the quantity of faces of 3-dimensional cube. Therefore, faces of 3-dimensional cube are $2C_3^1 = 6$.

(2) Continue to discuss 4-dimension. Given a 4-dimensional unit cube in first quadrant of 4-dimensional Cartesian coordinate system.

①Quantity of vertexes. Every vertex has 4 coordinates and every coordinate takes two values 0 or 1. 2^4=8.

②Quantity of edges. Every point on the edge of 4-dimensional cube has 4 coordinates, among which 3 are fixed and 1 is flexible. The fixed coordinates are of two options: 0 or 1. There are 3 fixed among the 4 coordinates, then there will be $C_4^3 = 4$ options. And there are 2 options (0 or 1) for places fixing the fixed coordinates. The total number of options is the quantity of edges of 4-dimensional cube. Therefore, edges of 4-dimensional cube are $2^3C_4^3 = 32$.

③Quantity of faces. Every point on the face has 4 coordinates, among which 2 are fixed and 2 are flexible. The fixed coordinate is of two options: 0 or 1. There are 2 fixed among the 4 coordinates, then there will be $C_4^2 = 6$ options. And there are 2 options (0 or 1) for places fixing the fixed coordinate. The total number of options is the quantity of faces of 4-dimensional cube. Therefore, faces of 4-dimensional cube are $2^2C_4^2 = 24$.

④Quantity of hyperplanes. There are hyperplanes existing in 4-dimensional cube. Every point on the hyperplane of 4-dimensional cube has 4 coordinates, among which 1 is fixed and 3 are flexible. The fixed coordinate is of two options: 0 or 1. There is 1 fixed among the 4 coordinates, then there will be $C_4^1 = 4$ options. And there are 2 options (0 or 1) for places fixing the fixed coordinate. The total number of options is the quantity of hyperplanes of 4-dimensional cube. Therefore, hyperplanes of 4-dimensional cube are $2\ C_4^1 = 8$.

(3) Extended to n-dimension.

①Quantity of vertexes. Every vertex has n coordinates and every coordinate takes two values 0 or 1. 2^n.

②Quantity of edges. Every point on the edge of n-dimensional cube has n coordinates, among which n-1 are fixed and 1 is flexible. The fixed coordinates are of two options: 0 or 1. There are n-1 fixed among the n coordinates, and then there will be C_n^{n-1} options. And there are 2 options (0 or 1) for places fixing the fixed coordinates. The total number of options is the quantity of edges of n-dimensional cube. Therefore, edges of n-dimensional cube are $2^{n-1}C_n^{n-1}$.

③Quantity of faces. Every point on the face of n-dimensional cube has n coordinates, among which n-2 are fixed and 2 are flexible. The fixed coordinate is of two options: 0 or 1. There are n-2 fixed among the n coordinates, then there will be C_n^{n-2} options. And there are 2 options (0 or 1) for places fixing the fixed coordinate. The total number of options is the quantity of faces of n-dimensional cube. Therefore, faces of n-dimensional cube are $2^{n-2}C_n^{n-2}$.

④Quantity of hyperplanes. There are m-dimensional hyperplanes existing in high-dimensional cube (2<m<n). Every point on the m-dimensional hyperplane of n-dimensional cube has n coordinates, among which n-m is fixed and m are flexible. The fixed coordinate is of two options: 0 or 1. There is n-m fixed among the n coordinates, and then there will be $C_n^{\mathrm{n-m}}$ options. And there are 2 options (0 or 1) for places fixing the fixed coordinate. The total number of options is the quantity of m-dimensional hyperplanes of n-dimensional cube. Therefore, m-dimensional hyperplanes of n-dimensional cube are $2^{n-m}C_{\mathrm{n}}^{n-m}$.

Since hyperplanes of different dimensions are existing in high-dimensional space, point, edge, face and hyperplane are defined as edge, vertex 0-edge, edge 1-edge, face 2-edge and m-dimensional hyperplane m-edge. Quantity of edges in 3 to 5 dimension cube is given out in table 3-1.

	0-edge	1-edge	2-edge	3-edge	4-edge
3-dimensional cube	8	12	6		
4-dimensional cube	16	32	24	8	
5-dimensional cube	32	80	80	40	10

Table 3-1 Quantity of Edges in 3 to 5-dimensional Cube

3.2 4-dimensional Cube

It is known from section 3.1 that in 4-dimensional cube, there are 16 0-edges, 32 1-edges,24 2-edges and 8 3-edges.

Given a 4-dimensional cube with edge length of 1, then:

Internal point: $C=(l-1)^4$

1-edge point: $B_1=32(l-1)+16=32l-16$

2-edge point: $B_2=24(l-1)^2+32l-16=24l^2-16l+8$

3-edge point: $B_3=8(l-1)^3+24l^2-16l+8=8l^3+8l$

Through calculation it is obtained that: $C+B_3/2-B_2/4-B_1/8-1=l^4$, equal to $V= l^4$,

Formula 3-1: $V= C+B_3/2-B_2/4-B_1/8-1$, i.e. 4-dimensional extension of Pick's theorem.

Geometry satisfying formula 3-1 is called a geometry satisfying 4-dimensional Pick's theorem.

Theorem 3-1: 4-dimensional cube satisfies 4-dimensional Pick's theorem.

Turn to 4-dimensional Euler's formula:

In 3-dimensional Euler's formula V-E+F=2, V is quantity of vertexes, E is quantity of edges and F is quantity of faces.

From table 3-1, 4-dimensional Euler's formula can be obtained.

Mark quantity of m-edge as N_m, then $N_0-N_1+N_2-N_3=16-32+24-8=0$, as:

Formula 3-2: $N_0-N_1+N_2-N_3=0$

N_i is quantity of i-edge point.

Formula 3-2 is 4-dimensional extension of Euler's formula.

3.3 N-dimensional Cube

n-dimensional cube extension of Pick's theorem is realized in the following text. First of all, a lemma concerning combinatorial identity is proved.

Lemma 3-1: $\sum_{i=m}^{n-1}(-1)^{i-m}C_i^{i-m}C_n^{n-i}=-(-1)^{n-m}C_n^{n-m}$

Proving:

Step 1, prove $\sum_{i=m}^{n-1}(-1)^{i-m}\dfrac{1}{(i-m)!(n-i)!}=-(-1)^{n-m}\dfrac{1}{(n-m)!}$

①When n-m is odd:

$$\sum_{i=m}^{n-1}(-1)^{i-m}\frac{1}{(i-m)!(n-i)!}$$

$$=(-1)^{m-m}\frac{1}{(m-m)!(n-m)!}+\sum_{i=m+1}^{n-1}(-1)^{i-m}\frac{1}{(i-m)!(n-i)!}$$

First term = $-(-1)^{n-m}\dfrac{1}{(n-m)!}$

Attention shall be paid to the second term that term k (i=m+k) in the formula, $(-1)^{k}\dfrac{1}{k!(n-m-k)!}$, are equal to the value of term n-m-k (i=n-k), $(-1)^{n-m-k}\dfrac{1}{(n-m-k)!k!}$ but with opposite sign, and their sum is 0. (First term and last term, second term and second last term … When n-m is odd, n-m-1 is even, making them matching in pairs). Therefore, the second term is 0.

$\sum_{i=m}^{n-1}(-1)^{i-m}\dfrac{1}{(i-m)!(n-i)!}=-(-1)^{n-m}\dfrac{1}{(n-m)!}$ is valid.

② When n-m is even:

$$\sum_{i=m}^{n-1}(-1)^{i-m}\frac{1}{(i-m)!(n-i)!}$$

$$=\sum_{i=m}^{n}(-1)^{i-m}\frac{1}{(i-m)!(n-i)!}-(-1)^{n-m}\frac{1}{(n-m)!}$$

Attention shall be paid to the first term that term k (i=m+k-1) in the formula, $(-1)^{k-1}\dfrac{1}{(k-1)!(n-m-k+1)!}$, are equal to the value of term n-m-k+1 (i=n-k+1), $(-1)^{n-m-k-1}\dfrac{1}{(n-m-k+1)!(k-1)!}$ but with opposite sign, and their sum is 0. Therefore, the first term is 0.

$\sum_{i=m}^{n-1}(-1)^{i-m}\dfrac{1}{(i-m)!(n-i)!}=-(-1)^{n-m}\dfrac{1}{(n-m)!}$ is valid.

Step 2, prove $\sum_{i=m}^{n-1}(-1)^{i-m}C_{i}^{i-m}C_{n}^{n-i}=-(-1)^{n-m}C_{n}^{n-m}$

$$\sum_{i=m}^{n-1}(-1)^{i-m}C_{i}^{i-m}C_{n}^{n-i}$$

$$=\sum_{i=m}^{n-1}(-1)^{i-m}\frac{i!}{m!(m-1)!}\frac{n!}{i!(n-i)!}$$

$$=\sum_{i=m}^{n-1}(-1)^{i-m}\frac{1}{(i-m)!(n-i)!}\frac{n!}{m!}$$

$=-(-1)^{n-m}\frac{1}{(n-m)!}\frac{n!}{m!}$

(In step 1, $\sum_{i=m}^{n-1}(-1)^{i-m}\frac{1}{(i-m)!(n-i)!}=-(-1)^{n-m}\frac{1}{(n-m)!}$ has been proved.)

$=-(-1)^{n-m}\frac{n!}{m!(n-m)!}$

$=-(-1)^{n-m}C_n^{n-m}$

Theorem 3-2: Volume of n-dimensional cube V $=C+B_{n-1}/2-B_{n-2}/4-\ldots\ldots B_1/2^{n-1}-1$, i.e. n-dimensional extension of Pick's theorem (n-dimensional satisfies n-dimensional Pick's theorem).

Proving:

For n-dimensional cube with edge length of 1

Internal points in cube $C=(l-1)^n$

Plus internal points on $2^{n-1}C_n^{n-1}$ edges with 2^n vertexes to get points on edges:

Points on 1-edge $B_1=J_1=2^{n-1}C_n^{n-1}(l-1)+2^n$

Plus $2^{n-2}C_n^{n-2}$ internal points on faces with points on $2^{n-1}C_n^{n-1}$ edges to get points on faces:

Points on 2-edge $B_2=J_2+B_1=2^{n-2}C_n^{n-2}(l-1)^2+B_1$

...

Points on m-edge $B_m=J_m+B_{m-1}=2^{n-m}C_n^{n-m}(l-1)^m+B_{m-1}$

…

Points on n-1 edge $B_{n-1}=J_{n-1}+B_{n-2}=2C_n^1(l-1)^{n-1}+B_{n-2}$

Firstly calculate $B_{n-1}/2-B_{n-2}/4-\ldots-B_1/2^{n-1}$:

$B_{n-1}/2-B_{n-2}/4-\ldots B_1/2^{n-1}$

$=(J_{n-1}+B_{n-2})/2-(J_{n-2}+B_{n-3})/4\ldots-(J_m+B_{m-1})/2^{n-m}\ldots-J_1/2^{n-1}$

$=J_{n-1}/2+J_{n-2}/4\ldots+J_m/2^{n-m}\ldots+J_1/2^{n-1}$

$=[2C_n^1(l-1)^{n-1}]/2+[2^2C_n^2(l-1)^{n-2}]/4\ldots$

$+[2^{n-m}C_n^{n-m}(l-1)^m]/2^m\ldots+[2^{n-1}C_n^{n-1}(l-1)+2^n]/2^{n-1}$

$=[2C_n^1\ (l-1)^{n-1}]/2+[2^2C_n^2\,(l-1)^{n-2}]/4\ldots$

$+[2^{n-m}C_n^{n-m}\ (l-1)^m]/2^m\ldots+[2^{n-1}C_n^{n-1}\ (l-1)\ ]/2^{n-1}+2$

$=C_n^1\ (l-1)^{n-1}+C_n^2\,(l-1)^{n-2}\ldots+C_n^m\,(l-1)^m\ldots+C_n^{n-1}\ (l-1)+2$

For $(l-1)^{n-1}$, $(l-1)^{n-2}$, … , $(l-1)$, expand coefficient of term $l^{n-1},\ldots,l^1,l^0$ of summation formula.

Coefficient of constant term is:

$$\sum_{i=1}^{n-1}(-1)^i C_i^i C_n^{n-i}+2=\begin{cases}2 & \text{n is odd}\\ 0 & \text{n is even}\end{cases}$$

Coefficient of one term is:

$$\sum_{i=1}^{n-1}(-1)^{i-1}C_i^{i-1}C_n^{n-i}=-(-1)^{n-1}C_n^{n-1}\ \text{(Lemma 3-1)}$$

…

Coefficient of m term is:

$$\sum_{i=m}^{n-1}(-1)^{i-m}C_i^{i-m}C_n^{n-i}=-(-1)^{n-m}C_n^{n-m}\ \text{(Lemma 3-1)}$$

…

Coefficient of n-1 term is:

$$(-1)^0 C_{n-1}^0 C_n^1=n$$

Then

$B_{n-1}/2-B_{n-2}/4-\ldots B_1/2^{n-1}$

$$=nl^{n-1}\ldots-(-1)^{n-m}\ C_n^{n-m}\ \ l^m\ldots-(-1)^{n-1}C_n^{n-1}\ \ l+\begin{cases}2 & \text{n is odd}\\ 0 & \text{n is even}\end{cases}\qquad(1)$$

Expand internal points $C=(l-1)^n$ to get:

$$C=\sum_{i=0}^{n}(-1)^i C_n^i l^{n-i}$$

$$=l^n-n\,l^{n-1}\ldots+(-1)^{n-m}\ C_n^{n-m}\ \ l^m\ldots+(-1)^{n-1}C_n^{n-1}\ \ l+(-1)^n\qquad(2)$$

Mind that $V=l^n$, $(-1)^n+\begin{cases}2 & \text{n is odd}\\ 0 & \text{n is even}\end{cases}=1$

(1)+(2): $C+B_{n-1}/2-B_{n-2}/4-\ldots B_1/2^{n-1}=V+1$

Obtaining: $V=C+B_{n-1}/2-B_{n-2}/4-\ldots\ldots-B_1/2^{n-1}-1$

Turn to n-dimensional Euler's formula:

Quantity of m-edge is $2^m C_n^m$. It is marked $N_m=2^m C_n^m$, calculating

$N_0-N_1+N_2-\ldots+(-1)^{n-1}N_{n-1}$

$=2^n-2^{n-1}C_n^{n-1}+\ldots+(-1)^m\ 2^m C_n^m+\ldots+(-1)^{n-1}\ 2C_n^1$

$=(2-1)^n-(-1)^n$

$=1-(-1)^{n-1}$

$=\begin{cases}2 & \text{n is odd}\\ 0 & \text{n is even}\end{cases}$

Obtaining:

Formula 3-3: $N_0-N_1+N_2-\ldots+(-1)^{n-1}N_{n-1}=1-(-1)^{n-1}=\begin{cases}2 & \text{n is odd}\\ 0 & \text{n is even}\end{cases}$

N_i is quantity of i-edge points.

Formula 3-3 is n-dimensional extension of Euler's formula.

3.4 Combination of n-dimensional Cubes

The following text will be focused on discussing combination of n-dimensional cubes. Following agreements are made:

Agreement 1: Two geometries are combined on lattice point to ensure that combination of geometries is lattice geometry.

Agreement 2: Combination of geometries is simple connected.

Agreement 3: Geometries are combined on (n-1)-edge. Combination satisfying the condition is defined as formal combination.

Formal combination defined in agreement 3 extends the formal combination defined in the preface and section 2.4. and is suitable for arbitrary combination of n-dimensional cubes.

Imitating 3-dimensional cube combination, n-dimensional cube combination is structured.

Given that two unit cubes satisfying n-dimensional Pick's theorem satisfy respectively:

$V_1= C_1+B_{1,n-1}/2-B_{1,n-2}/4-\ldots\ldots B_{1,1}/2^{n-1}-1$ (1)

$V_2= C_2+B_{2,n-1}/2-B_{2,n-2}/4-\ldots\ldots B_{2,1}/2^{n-1}-1$ (2)

Internal points of n-dimensional cube generated on (n-1)-edge of combination of two n-dimensional cube is exactly half of the sum of internal points on two (n-1)-edges. Mark the generated internal points is s, then:

Internal points: $C= C_1+ C_2 +s$

Face points: $B_{n-1}= B_{1,n-1}+ B_{2,n-1}-2s$

For arbitrary $1<m<n-1$, internal points on (m-1)-edge combined by m-edges is exactly half of the sum of internal points on the two (m-1)-edges combined. Given that internal points on m-edge of integrated combination faces is t, and coefficient of n-dimensional Pick's theorem exactly satisfies the condition, then:

m-edge: $B_m = B_{1,m} + B_{2,m} - t - 2^{n-1}$

(m-1)-edge: $B_{m-1} = B_{1,m-1} + B_{2,m-1} - 2t - 2^{n-1}$

(1)+(2):$V=V_1+V_2=C+B_{n-1}/2-B_{n-2}/4-\ldots\ldots B_1/2^{n-1}-1$. Obtaining:

Theorem 3-3: Combination of two n-dimensional unit cubes satisfies n-dimensional Pick' s theorem.

Pipe combination of n-dimensional unit cubes is defined as a series of unit cubes, of which the first cube has only one (n-1)-edge connected to the sequential cube, middle cubes have only one front (n-1)-edge connected to the front order cube and only one back (n-1)-edge connected to the post order cube, and the tail cube has only one (n-1)-edge connected to the front order cube.

Structure with recurrence a n-dimensional cube pipe composed of r pieces of n-dimensional cubes, satisfying agreement 1, 2 and 3. Carry out argumentation in proving of theorem 2-6 on the combination face to get:

Theorem 3-4: Pipe combination of arbitrary n-dimensional unit cubes satisfies n-dimensional Pick's theorem.

Given a $(3k)^n$ cube composed of 3^n cubes. With the method of structuring 3-dimensional Hilbert curve given by reference [7], an ergodic $(3k)^n$ cube pipe combination composed of 3^n cubes can be structured.

In the same way, extend theorem 2-7 to n-dimension following the method of theorem 2-7 and argumentation of theorem 3-3 to get:

Theorem 3-4: n-dimensional cuboid satisfies n-dimensional Pick's theorem.

3.5 Corresponding Theorem

In following text, it will be proved that all simple connected closed geometries in n-dimensional Euclidean space can be approximated through combination of n-dimensional unit cubes with edges parallel to coordinate axis, and n-dimensional unit cubes with edges parallel to coordinate axis satisfy simple extension same to that of 2-dimensional Pick's theorem, to further establish connection between lattice geometry and geometry in general Euclidean space. It shows that even being restricted to combination of n-dimensional unit cubes with edges parallel to coordinate axis, it still corresponds to simple connected closed region of n-dimensional Euclidean space.

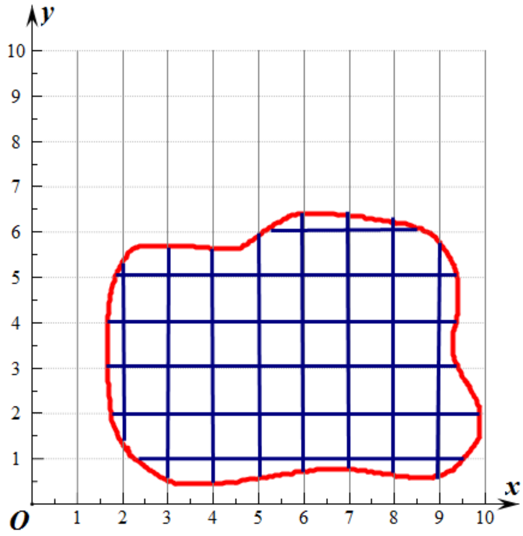


Fig.3-1 Approximation of 2-dimensional Combination Square to 2-dimensional Simple Connected Closed Region

Given a general Euclidean space, it is known from multivariate calculus [8] that plane simple connected closed region D can be approximated through grids of plane rectangular coordinate system, as shown in fig.3-1, which means that for arbitrary ε>0, there is δ>0 making grids of plane rectangular coordinate system satisfy that when edge length of grids is less than δ, difference between area of D and that of grids in D is less than ε. Space simple connected closed region D can be approximated through grids of space rectangular coordinate system, meaning that for arbitrary ε>0, there is δ>0 making grids of space rectangular coordinate system satisfy that when edge length of grids is less than δ, difference between volume of D and that of grids in D is less than ε. Obviously, the result is not related to dimension (dimension only influences value of δ with given ε, but not its existence), and it is obtained that n-dimensional space simple connected closed region D can be approximated through grids of n-dimensional space rectangular coordinate system, which means that for arbitrary ε>0, there is δ>0 making grids of n-dimensional space rectangular coordinate system satisfy that when edge length of grids is less than δ, difference between volume of D and that of grids in D is less than ε. It is known that grids in D are combination of n-dimensional unit cube. Take length less than δ as unit length to get:

Theorem 3-5 (corresponding theorem): Proper unit length u can be appointed to make simple connected closed region in n-dimensional Euclidean space has approximation of combination of n-dimensional unit cubes, and n-dimensional unit cube satisfies n-dimensional Pick's theorem.

Geometric meaning of corresponding theorem is that, arbitrary simple connected closed geometry can be approximated through combination of n-dimensional unit cubes with edges parallel to coordinate axis, and the n-dimensional unit cube satisfies Pick's theorem. Multiply volume of combination of lattice unit cubes V with u^n to get ε approximation of geometry volume. In this way, it established connection between combination of n-dimensional unit cubes in lattice space and n-dimensional Euclidean space simple connected closed region in general Euclidean space.

In combination of unit cubes, cube, cuboid and pipe of unit cubes combined by unit cubes with different levels of edges all satisfy Pick' theorem.

4 Discussion on n-dimensional Geometry with Edges not Parallel to Coordinate Axis

4.1 Existence of n-dimensional Geometry with Edges not Parallel to Coordinate Axis Satisfying Pick's Theorem

Given a prism with bottom face of parallelogram, of which vertexes are (0,0),(1,1),(2,1),(1,0), and height of 1. The geometry is a 3-dimensional hexahedron, which is known from conclusion 2-5 satisfies 3-dimensional Pick's theorem. 8 vertexes of the geometry are:

(0,0,0),(1,1,0),(2,1,0),(1,0,0),
(0,0,1),(1,1,1),(2,1,1),(1,0,1).

Given a 4-dimensional geometry:

(0,0,0,0),(1,1,0,0),(2,1,0,0),(1,0,0,0),
(0,0,1,0),(1,1,1,0),(2,1,1,0),(1,0,1,0),
(0,0,0,h),(1,1,0,h),(2,1,0,h),(1,0,0,h),
(0,0,1,h),(1,1,1,h),(2,1,1,h),(1,0,1,h).

Consider 3-dimensional hexahedron as 3-dimensional hyperplane (or 3-edge), then 4-dimensional geometry can be considered as a 4-dimensional prism with bottom face of the hyperplane and height of h. It is used as high-dimensional extension of concept of prism.

The 4-dimensional geometry has:

Internal points: C=0

1-edge points: $B_1=8(h+1)$

2-edge points: $B_2=8(h+1)$

3-edge points: $B_3=8(h+1)$

Through calculation it is obtained: $C+B_3/2-B_2/4-B_1/8-1=h$, equal to volume V=h

Conclusion 4-1: There is 4-dimensional non-cuboid satisfying 4-dimensional Pick's theorem (there is 4-dimensional geometry with edges not parallel to coordinate axis satisfying 4-dimensional Pick's theorem).

Given above-mentioned 4-dimensional geometry with height h=1, and it can be considered as 4-dimensional hyperplane (or 4-edge), which can be further used for structuring 5-dimensional prism with bottom face of this hyperplane and height of h satisfying 5-dimensional Pick's theorem. In this way of recurrence, n-dimensional geometry with edges not parallel to coordinate axis satisfying n-dimensional Pick's theorem can be structured, obtaining:

Theorem 4-1: There is n-dimensional non-cuboid satisfying n-dimensional Pick's theorem (there is n -dimensional geometry with edges not parallel to coordinate axis satisfying n -dimensional Pick's theorem).

At last, given arbitrary (n-1)-dimensional hyperplane with N vertexes to find out a necessary and sufficient condition for n-dimensional N-prism satisfying n-dimensional Pick's theorem.

Given prism with arbitrary (n-1)-dimensional hyperplane $C=c, B_1=b_1, B_2=b_2, \ldots\ldots B_m=b_m, \ldots\ldots$, $B_{n-2}=b_{n-2}$, and height =h. From Pick's theorem $S=c+b_{n-2}/2-b_{n-2}/4-\ldots\ldots-b_1/2^{n-2}-1$.

Points on 1-edge: $B_1=2b_1+N(h-1)$

Points on 2-edge: $B_1+2(b_2-b_1)+(b_1-N)(h-1)$

……

Points on m-edge: $B_m=B_{m-1}+2(b_m-b_{m-1})+(b_{m-1}-b_{m-2})(h-1)$

……

Points on (n-1)-edge: $B_{n-1}=B_n-2+2c+(b_n-2-b_{n-3})(h-1)$

Through calculation:

$C+B_{n-1}/2-B_{n-2}/4-\ldots\ldots-B_1/2^{n-1}-1$

$=ch+b_{n-2}h/2-b_{n-3}h/4-\ldots\ldots-b_1h/2^{n-2}-(Nh-N)/2^{n-1}-1$

$V=Sh=(c+b_{n-2}/2-b_{n-3}/4-\ldots\ldots-b_1/2^{n-2}-1)h$

Necessary and sufficient condition for satisfying Pick's theorem is that the above-mentioned two equations equal to each other. Through simplification, it is obtained $(h-1)(N/2^{n-1}-1)=0$.

Validness of the equation requires $N=2^{n-1}$ or h=1, obtaining:

Theorem 4-2: Necessary and sufficient condition for non-trivial n-dimensional N-prism satisfying n-dimensional Pick's Theorem is $N=2^{n-1}$.

4.2 Computation Formula of Internal Points on 1-Edge of n-dimensional Space

It is known from section 2.3 that it is extremely difficult to calculate quantity of lattice points of geometry in n-dimensional space. It is obstacle for studying n-dimensional lattice geometry as well. However, general formula of calculating internal points of 1-edge of n-dimensional space can be obtained. Then:

Theorem 4-3: For two arbitrary points $A(a_1,a_2,\ldots,a_n)$, $B(b_1,b_2,\ldots,b_n)$ in n-dimensional space, internal points on segment AB $I_{AB}=(b_1-a_1, b_2-a_2,\ldots, b_n-$

a_n)-1=d-1, in which d=(b_1- a_1, b_2- a_2,…, b_n - a_n,) is greatest common divisor of b_1- a_1, b_2- a_2,…, b_n - a_n.

Proving:

Given parameter equation of line AB as:

$$\begin{cases} x_1 = a_1 + \dfrac{b_1 - a_1}{d} t \\ x_2 = a_2 + \dfrac{b_2 - a_2}{d} t \\ \cdots\cdots \\ x_n = a_n + \dfrac{b_n - a_n}{d} t \end{cases}$$

If t is integer, then $x_1,x_2,\ldots,x_n$ are integers;

If $x_1,x_2,\ldots,x_n$ are integers, then t must be rational number. Given $t=\frac{p}{q}$, $\frac{p}{q}$ is fraction in lowest term.

Since x_i, a_i are integers, then $q \mid \frac{b_i - a_i}{d} p$ (i=1,2,…,n).

Since $\frac{p}{q}$ is fraction in lowest term, then $q \mid \frac{b_i - a_i}{d}$.

Since $(\frac{b_1 - a_1}{d}, \frac{b_2 - a_2}{d}, \ldots, \frac{b_n - a_n}{d})=1$, then |q|=1, therefor t must be integer.

On line segment AB, $a_i \le x_i \le b_i$, then $0 \le t \le d$.

Therefore, internal point on segment AB is d-1. In special case, if d=1, there is no internal point on segment AB.

5 Conclusion and Conjecture

Due to existence of Reeve tetrahedron counterexample, Pick's theorem cannot be simply high-dimensional extended unconditionally. In this paper, n-dimension extension of Pick's theorem is proposed under sufficient conditions including cube, 2^{n-1}-prism etc., and necessary conditions of extension are discussed as well. Formula obtained from n-dimensional extension of Pick's theorem maintains the simple form of Pick's theorem, and there is no convex restriction under sufficient conditions. Meanwhile, high-dimensional extension of Euler's formula is obtained through the special case of cube. N-dimensional simple connected closed region in Euclidean space can be approximated through unit cube to further establish connection between lattice geometry and general Euclidean space geometry.

Theorem 2-3 and 2-4 show that there are square and rectangular prisms with edges of bottom face not parallel to coordinate axis (i.e. cube and cuboid with edges of bottom face not parallel to coordinate axis) and 3-dimensional cuboid with all edges not parallel to coordinate axis satisfying 3-dimensional Pick's theorem. It clues that for sufficient conditions of satisfying n-dimensional Pick's theorem, parallelism between edge and coordinate axis might not be the nature, while orthogonality of edges contained by the geometry is the nature, which induces the following conjecture:

Conjecture 5-1: Except for trivial condition, cuboid (not restricted by edge parallel to coordinate axis) is sufficient condition for satisfying n-dimensional Pick's theorem.

Section 3.3 and 3.4 are equal to give out proving for the conjecture with condition that edges are parallel to coordinate axis.

It is known from theorem 2-4, 2-5 and 4-1 that there is irregular quadrangular prism, 3-dimensional cuboid with all edges not parallel to coordinate axis and n-dimensional geometry with edges not parallel to coordinate axis satisfying Pick's theorem. Therefore, cuboid with edges parallel to coordinate axis cannot be considered as necessary condition for high-dimensional extension of Pick's theorem, and condition of conjecture 5-1 cannot be used as necessary condition either. Therefore, geometry satisfying n-dimensional Pick's theorem is a larger category than those determined by sufficient conditions discussed here.

Study of sufficiency and necessity under general situation involves calculation of quantity of lattice points on n-edge of space, which exceeds scope of elementary method. It is hoped that the study can be carried out with advanced mathematical tools, like algebraic geometry etc.

At last, the paper is ended with comparison between Pick's theorem and high-dimensional extended n-dimensional Pick's theorem.

Mark B_{n-1} as B:

S=C+B/2-1

$$V=C+B/2-\sum_{i=2}^{n-1}\frac{1}{2^i}B_{n-i}-1$$

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