N10

Abstract

The purpose of this paper is to find a good way approximate e.In our research ,we get some formulas to approximate e by **L'Hospital's rule**.

This study has a open vast vistas.We get a kind of functions to approximate e by **Talyor's theorem**.By these functions we can get some numbers which can replace e in approximate calculation.

Key words.e,approximate

A Study of the Approximation of e

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1 Lemma

- **L'Hospital's rule**

  Functions f,F are differentiable on an open interval I except possibly at a point a contained in I,if

  - $\lim\limits_{x\to a} f(x)=0, \lim\limits_{x\to a} F(x)=0$;
  - $F'(x)\neq 0$ for all x in I with $x\neq a$;
  - $\lim\limits_{x\to a}\frac{f'(x)}{F'(x)}$ exits(or is ∞),

  then $\lim\limits_{x\to a}\frac{f(x)}{F(x)}=\lim\limits_{x\to a}\frac{f'(x)}{F'(x)}$

  Proof. WLOG,we assume $f(a)=F(a)=0$,then by **Cauchy mean value theorem**[1],we get $\exists\ \xi\in(a,x)$ or(x,a),s.t.

  $$\frac{f(x)}{F(x)}=\frac{f(x)-f(a)}{F(x)-F(a)}=\frac{f'(\xi)}{F'(\xi)}$$

  so

  $$\lim_{x\to a}\frac{f(x)}{F(x)}=\lim_{x\to a}\frac{f'(\xi)}{F'(\xi)}=\lim_{x\to a}\frac{f'(x)}{F'(x)}$$

  □

- **Talyor's theorem**

  If function $f(x)$ is $(n+1)^{th}$ differentiable on (a,b),then $\forall x\in(a,b)$,we have

  $$f(x)=\sum_{k=0}^{n}f^{(k)}(x_0)\frac{(x-x_0)^k}{k!}+R_n(x) \text{ and } R_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}.$$

  Proof. Let

  $$P_n(x)=\sum_{k=0}^{n}f^{(k)}(x_0)\frac{(x-x_0)^k}{k!},$$

  we just to prove

  $$R_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}.$$

  When $n=0$,by **Cauchy mean value theorem**,we know the proposition is true;if the proposition is true for n,now we prove that it is true for $n+1$.

  By inductive assumption,$R_n(x)$ is $(n+1)^{th}$ differentiable on (a,b) and

  $$R_n^{(k)}(x_0)=0(k\in[0,n]\cap\mathbb{Z}).$$

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[1]The proof of **Cauchy mean value theorem** is presented in paper[1]

So by **Cauchy mean value theorem**,we get

$$\frac{R_n^{(k)}(\xi_k)}{A_{n+1}^k(\xi_k-x_0)^{k+1}}=\frac{R_n^{(k)}(\xi_k)-R_n^{(k)}(x_0)}{A_{n+1}^k[(\xi_k-x_0)^{k+1}-0]}=\frac{R_n^{(k+1)}(\xi_{k+1})}{A_{n+1}^{k+1}(\xi_{k+1}-x_0)^k}$$

$(\xi_0=x,\xi_{k+1}\in(\xi_k,x_0)$ or $(x_0,\xi_k),\xi_n=\xi)$,so

$$\frac{R_n(x)}{(x-x_0)^{n+1}}=\frac{R_n^{(n+1)}(\xi)}{(n+1)!}=\frac{f^{(n+1)}(\xi)}{(n+1)!}\Rightarrow R_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}.$$

□

By **Talyor's theorem**,we have

$$\ln(1+x)=\sum_{k=1}^{n}\frac{(-1)^{k+1}x^k}{k}+O(x^{n+1}).$$

- $\lim\limits_{x\to0^+}\frac{x-\ln(1+x)}{x^2}=\frac{1}{2}$.

  Proof. $\lim\limits_{x\to0^+}\frac{x-\ln(1+x)}{x^2}\overset{\textbf{L'Hospital's rule}}{=}\lim\limits_{x\to0^+}\frac{1-\frac{1}{1+x}}{2x}=\frac{1}{2}$. □

- [2] For $x\in\mathbb{R}_+$,$(1+\frac{1}{x})^x<e<(1+\frac{1}{x})^{x+1}$,and we have $\lim\limits_{x\to+\infty}(1+\frac{1}{x})^x=\lim\limits_{x\to+\infty}(1+\frac{1}{x})^{x+1}=e$.

2 First strengthening

Now we try to strengthen the conclusion to

$$(1+\frac{1}{x})^x<e<(1+\frac{1}{x})^{x+\frac{1}{2}}$$

Proof. Let

$$f(x)\equiv\frac{1}{\ln(1+\frac{1}{x})}-x\equiv g(\frac{1}{x}),$$

Then

$$g(x)=\frac{1}{\ln(1+x)}-\frac{1}{x}\text{ and }e=(1+\frac{1}{x})^{x+f(x)},$$

We have

$$g'(x)=\frac{(\ln(1+x)+\frac{x}{\sqrt{1+x}})(\ln(1+x)-\frac{x}{\sqrt{1+x}})}{x^2\ln^2(1+x)}.$$

Let

$$h(x)\equiv\ln(1+x)-\frac{x}{\sqrt{1+x}},$$

$\because$

$$h'(x)=-\frac{(\sqrt{1+x}-1)^2}{2(1+x)^{\frac{3}{2}}}<0,h(0)=0,$$

$\therefore$ when $x>0,g'(x)<0\Rightarrow g(x)$ is decreasing on $\mathbb{R}_+\Rightarrow f(x)$ is increasing on $\mathbb{R}_+$.
We have $\lim\limits_{x\to0^+}f(x)=0$ and $\lim\limits_{x\to+\infty}f(x)=\lim\limits_{x\to0^+}g(x)=\lim\limits_{x\to0^+}\frac{x-\ln(1+x)}{x\ln(1+x)}\overset{\textbf{L'Hospital's rule}}{=}\lim\limits_{x\to0^+}\frac{\frac{x}{1+x}}{\ln(1+x)+\frac{x}{1+x}}$
$=\lim\limits_{x\to0^+}\frac{x}{x+(1+x)\ln(1+x)}\overset{\textbf{L'Hospital's rule}}{=}\lim\limits_{x\to0^+}\frac{1}{2+\ln(1+x)}=\frac{1}{2}$.
So

$$f(x)\in(0,\frac{1}{2})\Rightarrow(1+\frac{1}{x})^x<e<(1+\frac{1}{x})^{x+\frac{1}{2}}.$$

By the way,when $x\to\infty$,$(1+\frac{1}{x})^{x+\frac{1}{2}}$ approximate e better than $(1+\frac{1}{x})^x$. □

[2] The proof of this conclusion is presented in paper[1]

3 Second strengthening

Now we compare the approximation to e between $(1+\frac{1}{x})^x$ and $(1+\frac{1}{x})^{x+\frac{1}{2}}$.

$$
\begin{aligned}
& \lim_{x\to+\infty} \frac{e-(1+\frac{1}{x})^x}{x[(1+\frac{1}{x})^{x+\frac{1}{2}}-e]} \\
= \quad & \lim_{x\to 0^+} \frac{x[e-(1+x)^{\frac{1}{x}}]}{(1+x)^{\frac{1}{x}+\frac{1}{2}}-e} \\
\overset{\textbf{L'Hospital's rule}}{=} \quad & \lim_{x\to 0^+} \frac{e-(1+x)^{\frac{1}{x}}-(1+x)^{\frac{1}{x}-1}+\frac{\ln(1+x)}{x}(1+x)^{\frac{1}{x}}}{\frac{1}{2}(1+x)^{\frac{1}{x}-\frac{1}{2}}+\frac{1}{x}(1+x)^{\frac{1}{x}-\frac{1}{2}}-\frac{\ln(1+x)}{x^2}(1+x)^{\frac{1}{x}-\frac{1}{2}}} \\
\overset{\textbf{L'Hospital's rule}}{=} \quad & \lim_{x\to 0^+} (1+x)^{-\frac{1}{2}} \frac{1+\frac{2x(1+x)\ln(1+x)-x^2-(1+x)^2\ln^2(1+x)}{x^3}}{-\frac{1}{4}+\frac{x^2(1+x)\ln(1+x)+(1+x)^2\ln^2(1+x)-x^2(1+2x)}{x^4}} \\
= \quad & \frac{1+\lim\limits_{x\to 0^+}\frac{2x(1+x)\ln(1+x)-x^2-(1+x)^2\ln^2(1+x)}{x^3}}{-\frac{1}{4}+\lim\limits_{x\to 0^+}\frac{x^2(1+x)\ln(1+x)+(1+x)^2\ln^2(1+x)-x^2(1+2x)}{x^4}} \\
= \quad & \frac{1-\lim\limits_{x\to 0^+}\frac{1}{x}(\frac{x-(1+x)\ln(1+x)}{x^2})^2}{-\frac{1}{4}+\lim\limits_{x\to 0^+}[\frac{\ln(1+x)}{x}+(\frac{\ln(1+x)}{x})^2+2\frac{\ln(1+x)-x}{x^2}\frac{\ln(1+x)+x}{x}+\frac{x^2\ln(1+x)+\ln^2(1+x)-x^2}{x^4}]} \\
\overset{\textbf{L'Hospital's rule}}{=} \quad & \frac{1}{-\frac{1}{4}+\lim\limits_{x\to 0^+}\frac{x^2\ln(1+x)+\ln^2(1+x)-x^2}{x^4}} \\
& \lim_{x\to 0^+} \frac{x^2\ln(1+x)+\ln^2(1+x)-x^2}{x^4} \\
\overset{\textbf{Taylor's formula}}{=} \quad & \lim_{x\to 0^+} \frac{(x-\frac{x^2}{2}+\frac{x^3}{3}+O(x^4))^2-x^2+x^2(x-\frac{x^2}{2}+O(x^3))}{x^4} \\
= \quad & \lim_{x\to 0^+} \frac{\frac{5}{12}x^4+O(x^5)}{x^4} \\
= \quad & \frac{5}{12}
\end{aligned}
$$

So

$$\lim_{x\to+\infty} \frac{e-(1+\frac{1}{x})^x}{x[(1+\frac{1}{x})^{x+\frac{1}{2}}-e]} = \frac{1}{-\frac{1}{4}+\frac{5}{12}} = 6.$$

So we can use

$$\frac{6x(1+\frac{1}{x})^{x+\frac{1}{2}}+(1+\frac{1}{x})^x}{6x+1}$$

to approximate e.

Now we prove when $x \in \mathbb{R}+$,$e > \frac{6x(1+\frac{1}{x})^{x+\frac{1}{2}}+(1+\frac{1}{x})^x}{6x+1}$.

Proof. Let

$$f(x) = \frac{6x(1+\frac{1}{x})^{x+\frac{1}{2}}+(1+\frac{1}{x})^x}{6x+1}.$$

$\because \lim\limits_{x\to 0^+} f(x) = e$,
$\therefore$ we just need to prove $f(x)$ is increasing.
Let

$$g(x) \equiv f(\frac{1}{x}) \Rightarrow g(x) = (1+x)^{\frac{1}{x}} \frac{6\sqrt{1+x}+x}{6+x}.$$

$$f(x)\ \textit{is increasing} \iff g(x)\ \textit{is decreasing} \Leftarrow g'(x)<0$$
$$\iff \frac{(1+x)^{\frac{1}{x}}}{(x+6)^2}\{\frac{(6+x)(6\sqrt{1+x}+x)}{x^2}[\frac{x}{1+x}-\ln(1+x)]+\frac{12+6\sqrt{1+x}-3x}{\sqrt{1+x}}\}<0$$
$$\iff \frac{(6+x)(6\sqrt{1+x}+x)}{x^2}[\frac{x}{1+x}-\ln(1+x)]+\frac{12+6\sqrt{1+x}-3x}{\sqrt{1+x}}<0$$
$$\iff \sqrt{1+x}(6+x)(6\sqrt{1+x}+x)[\ln(1+x)-\frac{x}{1+x}]+x^2(3x-12-6\sqrt{1+x})>0 \tag{1}$$

Let $\sqrt{1+x}=a$,then we have $a>1$,$x=a^2-1$.
Use a substitute x,we get

$$(1)\iff (2\ln a+\frac{1}{a^2}-1)a(a^2+6a-1)(a^2+5)+(a^2-1)^2(3a^2-6a-15)>0$$
$$\iff h(a)\equiv \ln a+\frac{(a^2-1)(3a^5+5a^4-24a^3-10a^2-15a+5)}{2a^2(a^2+6a-1)(a^2+5)}>0.$$

$\because h(1)=0$,so we just need to prove

$$h'(x)>0$$
$$\iff \frac{2a(a+1)^2(a-1)}{4a^4(a^2+6a-1)^2(a^2+5)^2}(3a^8+35a^7+85a^6+471a^5+195a^4-795a^3+1395a^2-575a+50)>0$$
$$\iff 3a^8+35a^7+85a^6+471a^5+195a^4-795a^3+1395a^2-575a+50>0$$
$$\iff 3a^8+35a^7+56a^6+625a^2+50+29a^3(a^3-1)+471a^3(a^2-1)+195a^2(a-1)^2+575a(a-1)>0$$

Now it is obviously. □

4 Third strengthening

Now we find the best constant λ which makes $\frac{(6x+\lambda)(1+\frac{1}{x})^{x+\frac{1}{2}}+(1+\frac{1}{x})^x}{6x+1+\lambda}$ close to e.
Let
$$\lambda(x)=\frac{(6x+1)e-(1+\frac{1}{x})^x-6x(1+\frac{1}{x})^{x+\frac{1}{2}}}{(1+\frac{1}{x})^{x+\frac{1}{2}}-e},$$
then we have
$$\frac{(6x+\lambda(x))(1+\frac{1}{x})^{x+\frac{1}{2}}+(1+\frac{1}{x})^x}{6x+1+\lambda(x)}=e.$$

$$\lim_{x\to+\infty}\frac{(6x+1)e-(1+\frac{1}{x})^x-6x(1+\frac{1}{x})^{x+\frac{1}{2}}}{(1+\frac{1}{x})^{x+\frac{1}{2}}-e}$$
$$=\lim_{x\to 0^+}\frac{e(6+x)-(1+x)^{\frac{1}{x}+\frac{1}{2}}-x(1+x)^{\frac{1}{x}}}{x((1+x)^{\frac{1}{x}+\frac{1}{2}}-e)}$$
$$\overset{\textbf{L'Hospital's rule}}{=}\lim_{x\to 0^+}\frac{e-(1+x)^{\frac{1}{x}}+(\frac{\ln(1+x)}{x}-\frac{1}{1+x})(1+x)^{\frac{1}{x}}+3[\frac{2}{x}(\frac{\ln(1+x)}{x}-\frac{1}{1+x})-\frac{1}{x+1}](1+x)^{\frac{1}{x}+\frac{1}{2}}}{\frac{x}{2(1+x)}(1+x)^{\frac{1}{x}+\frac{1}{2}}+[(1+x)^{\frac{1}{x}+\frac{1}{2}}-e]-(\frac{\ln(1+x)}{x}-\frac{1}{1+x})(1+x)^{\frac{1}{x}+\frac{1}{2}}} \tag{2}$$

$$\because \lim_{x\to 0^+}[\frac{2}{x}(\frac{\ln(1+x)}{x}-\frac{1}{1+x})-\frac{1}{x+1}]=\lim_{x\to 0^+}\frac{2(1+x)\ln(1+x)-x^2-2x}{x^2(1+x)}\overset{\textbf{L'Hospital's rule}}{=}\lim_{x\to 0^+}\frac{2\ln(1+x)-2x}{2x}=0,$$

$\therefore$ the numerator and denominator of (2) both are approximate to 0 when $x \to 0^+$,

$$(2) \overset{\textbf{L'Hospital's rule}}{=} \lim_{x\to 0^+} \frac{1}{\sqrt{1+x}} \cdot \frac{\sqrt{1+x}(\frac{3}{2} + \frac{12x+6}{x^2} - \frac{6(1+x)\ln(1+x)}{x^2} - \frac{6(1+x)^2\ln^2(1+x)}{x^4}) + 1 - \frac{1}{x} + \frac{2(1+x)\ln(1+x)}{x^2} - \frac{(1+x)^2\ln^2(1+x)}{x^3}}{\frac{3x^2+4x+4}{4x} - \frac{(x+2)(x+1)}{x^2}\ln(1+x) + \frac{(1+x)^2}{x^3}\ln^2(1+x)}$$

$$= \frac{\lim\limits_{x\to 0^+}[\sqrt{1+x}(\frac{3}{2} + \frac{12x+6}{x^2} - \frac{6(1+x)\ln(1+x)}{x^2} - \frac{6(1+x)^2\ln^2(1+x)}{x^4}) + 1 - \frac{1}{x} + \frac{2(1+x)\ln(1+x)}{x^2} - \frac{(1+x)^2\ln^2(1+x)}{x^3}]}{\lim\limits_{x\to 0^+} \frac{3x^2+4x+4}{4x} - \frac{(x+2)(x+1)}{x^2}\ln(1+x) + \frac{(1+x)^2}{x^3}\ln^2(1+x)}$$

$$= \frac{(3)}{(4)}$$

$$(3) = \lim_{x\to 0^+} \sqrt{1+x}(-\frac{19}{2x} + \frac{12x+6}{x^3} - \frac{6\ln(1+x)}{x^3} - \frac{6(2x+1)\ln^2(1+x)}{x^5}) + \lim_{x\to 0^+} \sqrt{1+x}(\frac{12}{x} - \frac{6x\ln(1+x)}{x^3} - \frac{6x^2}{x^5}) + \lim_{x\to 0^+}(\frac{1}{x} - \frac{\sqrt{1+x}}{x}) + \lim_{x\to 0^+}(-\frac{1}{x^2} + \frac{2(1+x)\ln(1+x)}{x^3} - \frac{(1+x)^2\ln^2(1+x)}{x^4})$$

$$= (5) + (6) + (7) + (8).$$

$$(5) = \lim_{x\to 0^+} \frac{-19x^4 + 12x^2(2x+1) - 12x^2\ln(1+x) - 12(2x+1)\ln^2(1+x)}{2x^5}$$

$$\overset{\textbf{Taylor's formula}}{=} \lim_{x\to 0^+} \frac{-19x^4 + 24x^3 + 12x^2 - 12x^2(x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4)) - 12(2x+1)(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + O(x^5))^2}{2x^5}$$

$$= \lim_{x\to 0^+} \frac{-13x^4 + 12x^3 + 12x^2 - 4x^5 + O(x^6) - (2x+1)(12x^2 - 12x^3 + 11x^4 - 10x^5 + O(x^6))}{2x^5}$$

$$= \lim_{x\to 0^+} \frac{-13x^4 + 12x^3 + 12x^2 - 4x^5 + O(x^6) - (12x^2 + 12x^3 - 13x^4 + 12x^5 + O(x^6)}{2x^5}$$

$$= \lim_{x\to 0^+} \frac{-16x^5 + O(x^6)}{2x^5}$$

$$= -8$$

$$(6) = 6\lim_{x\to 0^+} \frac{x - \ln(1+x)}{x^2} \cdot \frac{2x + \ln(1+x)}{x}$$

$$= 9$$

$$(7) = \lim_{x\to 0^+} \frac{1 - \sqrt{1+x}}{x} = \lim_{x\to 0^+} \frac{-x}{x(1+\sqrt{1+x})}$$

$$= -\frac{1}{2}$$

$$(8) = -\lim_{x\to 0^+} \frac{[x - (1+x)\ln(1+x)]^2}{x^4}$$

$$= -(\lim_{x\to 0^+} \frac{x - (1+x)\ln(1+x)}{x^2})^2$$

$$\overset{\textbf{L'Hospital's rule}}{=} -(\lim_{x\to 0^+} \frac{1 - \ln(1+x) - 1}{2x})^2$$

$$= -\frac{1}{4}$$

$$\text{So } (3) = -8 + 9 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

$$(4) = \lim_{x\to 0^+}(\frac{3}{4} - \frac{x^2\ln(1+x)}{x^3} + \frac{x^2\ln^2(1+x)}{x^4}) + \lim_{x\to 0^+} \frac{x - \ln(1+x)}{x^2} \cdot \frac{x^2 + x - (2x+1)\ln(1+x)}{x^2}$$

$$= \frac{3}{4} + \frac{1}{2}\lim_{x\to 0^+} \frac{x^2 + x - (2x+1)\ln(1+x)}{x^2}$$

$$= \frac{3}{4} + \frac{1}{2}\lim_{x\to 0^+}(\frac{x^2 - 2x\ln(1+x)}{x^2} + \frac{x - \ln(1+x)}{x^2})$$

$$= \frac{1}{2}$$

So

$$\lambda = \lim_{x \to 0^+} \lambda(x) = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}.$$

In summary,we can use

$$\frac{(6x+\frac{1}{2})(1+\frac{1}{x})^{x+\frac{1}{2}} + (1+\frac{1}{x})^x}{6x+\frac{3}{2}}$$

to approximate e.

By the way,is there exist a strict inequality between e and $\frac{(6x+\frac{1}{2})(1+\frac{1}{x})^{x+\frac{1}{2}}+(1+\frac{1}{x})^x}{6x+\frac{3}{2}} \equiv f(x)$?

The verification indicates that $f(1) < e < f(0.1)$,actually,the root of the equation $f(x)$ is in the interval $(0.4919825571, 0.4919825575)$.So the strict inequality is complex,we won't discuss it below.

If $x \to +\infty$,$f(x)$ will approximate e well;if x approaching the root before,$f(x)$ will approximate e well,too. In special,let $x = \frac{1}{2}$,we get

$$f(\frac{1}{2}) = \frac{21+\sqrt{3}}{9} \approx 2.71823351,$$

the deviation between $f(\frac{1}{2})$ and e is less than $5 \cdot 10^{-5}$,we can use it in some approximate calculation.

5 Table

Now we show the table about the approximation between e and the four function before(round it to four significant digits).

	1	10	100
$e-(1+\frac{1}{x})^x$	$7.182 \cdot 10^{-1}$	$1.245 \cdot 10^{-1}$	$1.347 \cdot 10^{-2}$
$(1+\frac{1}{x})^{x+\frac{1}{2}} - e$	$1.101 \cdot 10^{-1}$	$2.058 \cdot 10^{-3}$	$2.243 \cdot 10^{-5}$
$e - \frac{6x(1+\frac{1}{x})^{x+\frac{1}{2}}+(1+\frac{1}{x})^x}{6x+1}$	$8.201 \cdot 10^{-3}$	$1.716 \cdot 10^{-5}$	$1.870 \cdot 10^{-8}$
$e - \frac{(6x+\frac{1}{2})(1+\frac{1}{x})^{x+\frac{1}{2}}+(1+\frac{1}{x})^x}{6x+\frac{3}{2}}$	$3.117 \cdot 10^{-4}$	$2.837 \cdot 10^{-7}$	$3.668 \cdot 10^{-11}$

6 Vista

6.1 Generalization

Let's consider the formula

$$\frac{(1+\frac{1}{x})^x + (6x+\sum\limits_{k=0}^{n}\frac{a_k}{x^k})(1+\frac{1}{x})^{x+\frac{1}{2}}}{6x+\sum\limits_{k=0}^{n}\frac{a_k}{x^k}+1},$$

which a_k is the best constant to let formula approximate e.
Let

$$f(x) \equiv \frac{e-(1+\frac{1}{x})^x}{\frac{1}{x}^{x+\frac{1}{2}} - e} - 6x,$$

then we have

$$e = \frac{(1+\frac{1}{x})^x + (6x+f(x))(1+\frac{1}{x})^{x+\frac{1}{2}}}{6x+f(x)+1},$$

by reciprocal substitution,we have

$$f(\frac{1}{x}) = \frac{e-(1+x)^{\frac{1}{x}}}{(1+x)^{\frac{1}{x}+\frac{1}{2}} - e} - \frac{6}{x} \equiv g(x).$$

After expand this formula at $x=0$ by **Talyor's formula**,we get

$$g(x)=\sum_{k=0}^{n}a'_k x^k+O(x^{n+1}),$$

we also have

$$\sum_{k=0}^{+\infty}\frac{a_k}{x^k}=f(x)\Rightarrow\sum_{k=0}^{+\infty}a_k x^k=g(x)\Rightarrow\forall k\in\mathbb{N},a_k=a'_k$$

so we just need to expand $g(x)$ to get the approximant:

$$g(x)=\frac{1}{2}+\frac{1}{10}x-\frac{27}{160}x^2+\frac{18233}{100800}x^3-\frac{5179}{28800}x^4+\frac{352741}{2016000}x^5-\frac{10908403}{64512000}x^6+\frac{218799997903}{1341204480000}x^7-\frac{28155913529}{178827264000}x^8$$
$$+\frac{530510007035063}{348713164800000}x^9-\frac{13037704174051}{88562073600000}x^{10}+O(x^{11}).$$

So $a_0=\frac{1}{2}$(this data agree well with our computing result),$a_1=\frac{1}{10},a_2=-\frac{27}{160}\cdots$

The comparison between the formula and e is complex,we won't discuss it here.

6.2 Change templet

All above is the consideration about the weighted average of $(1+\frac{1}{x})^x$ and $(1+\frac{1}{x})^{x+\frac{1}{2}}$,we can consider the better one.

First we consider

$$(1+\frac{1}{x})^{x+\sum\limits_{k=0}^{n}\frac{a_k}{x^k}}$$

(a_k is the best constant which makes the formula approximate e).

Let

$$f(x)\equiv\frac{1}{\ln(1+x)}-\frac{1}{x},$$

then we have

$$e=(1+\frac{1}{x})^{x+f(x)}.$$

Similarly,after expand $f(x)$ at $x=0$ by **Talyor's formula**,we get

$$f(x)=\frac{1}{2}-\frac{1}{12}x+\frac{1}{24}x^2-\frac{19}{720}x^3+\frac{3}{160}x^4-\frac{863}{60480}x^5+\frac{275}{24192}x^6-\frac{33953}{3628800}x^7+\frac{8183}{1036800}x^8-\frac{3250433}{47900600}x^9+\frac{4671}{788480}x^{10}+O(x^{11}).$$

So $a_0=\frac{1}{2},a_2=-\frac{1}{12},a_3=\frac{1}{24}\cdots$ we also have the recurrence relation:

$$a_0=\frac{1}{2},a_{n+1}=\frac{(-1)^{n+1}}{n+3}+\sum_{k=0}^{n}\frac{(-1)^n}{n+2-k}a_k.$$

Second we consider the weighted average.Consider

$$\lim_{x\to+\infty}\frac{(1+\frac{1}{x})^{x+\frac{1}{2}}-e}{x(e-(1+\frac{1}{x})^{x+\frac{1}{2}-\frac{1}{12x}})}=\lim_{x\to0^+}\frac{x[(1+x)^{\frac{1}{x}+\frac{1}{2}}-e]}{(e-(1+x)^{\frac{1}{x}+\frac{1}{2}-\frac{x}{12}})},$$

by calculate,we know this limit equal to 2,

so we can use

$$\frac{2x(1+\frac{1}{x})^{x+\frac{1}{2}-\frac{1}{12x}}+(1+\frac{1}{x})^{x+\frac{1}{2}}}{2x+1}$$

to approximate e.

Similarly we can use

$$\frac{(2x+\frac{4}{15})(1+\frac{1}{x})^{x+\frac{1}{2}-\frac{1}{12x}}+(1+\frac{1}{x})^{x+\frac{1}{2}}}{2x+\frac{19}{15}}$$

to approximate e.

Let's show the result by table(round it to four significant digits):

	1	10	100
$\frac{2x(1+\frac{1}{x})^{x+\frac{1}{2}-\frac{1}{12x}}+(1+\frac{1}{x})^{x+\frac{1}{2}}}{2x+1}-e$	$4.314\cdot10^{-4}$	$1.283\cdot10^{-6}$	$1.484\cdot10^{-10}$
$e-\frac{(2x+\frac{4}{15})(1+\frac{1}{x})^{x+\frac{1}{2}-\frac{1}{12x}}+(1+\frac{1}{x})^{x+\frac{1}{2}}}{2x+\frac{19}{15}}$	$5.972\cdot10^{-6}$	$5.889\cdot10^{-9}$	$7.860\cdot10^{-14}$

7 Reference

[1]高等数学.上册/同济大学数学系编。–7版。–北京：高等教育出版社。