Study on a Type of Geometric Iterations

Zhang Hengjia

Dalian Jinzhou Senior High School, Liaoning Province

Instructor: Li Jie

Abstract

Iteration is one of the long historic puzzles in Geometry, and the iteration in triangle is the fundamental question to be solved. In this work, some iterative questions which have not been studied before were proposed and explored. Firstly, applying coordinate method, the positions of the limiting points were successfully obtained, which facilitate solving the questions. The questions then further extended to two aspects for further study: 1) Extension of the medians to vertical lines, and arbitrary bisector lines. 2) Extension of the triangle to polygons. During the process of the study, coordinate method and vector method with its linear iterative principle are proposed to solve the iteration problems. It was found that both coordinate and vector methods can reach good results. It was found that the key to solve the iteration problems by coordinate method is to find the iteration points via recursive formula, while it is crucial to directly find out the positions of the iterative limitation points based on the rules of no change on the linear combination. The coordinate method is a kind of pervasive method but with complicated calculations, while the vector method is simple but rather limited to only some special cases. Furthermore, the properties of the interesting spiral lines, which were obtained by grouping the iteration points, were also subject to investigation.

Key words: iteration, coordinate method, vector method, triangle, polygon

Introduction

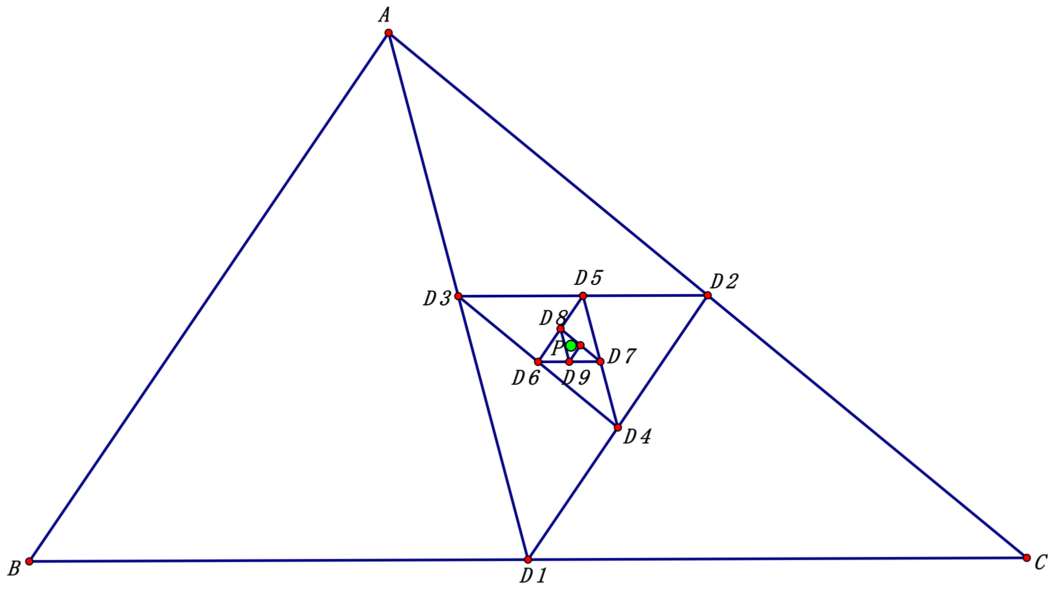
Iteration is one of the long historical puzzles in Geometry, and the iteration in triangle serves as the most fundamental questions. Iteration is a process that repeating same or similar things with the aim of approaching a desired goal, target or result. Each repetition of the process is also called iteration and the result of one iteration is used as the starting point for the next iteration. [1] Combination of Geometry and Algebra calculations is usually required to solve the iteration problems. Study on the iteration of the triangle itself is quite common [2], but iteration study on some particular elements of the triangle is rare.

In this work, preliminary work has been performed on the iteration of the elements in triangle and polygons. Firstly, starting from the question on the iteration of the medians in triangle, the position of the limiting points was obtained by coordinate method. The special characters of these limiting points have been investigated. The questions then further extended to two aspects for further study: 1) extension of the median to vertical lines and arbitrary bisector lines, and 2) extension of the triangle to polygons. It was found that both coordinate and vector methods can well resolve the problems. The key to solve the iteration problems by coordinate method is to find the iteration points via recursive formula followed by getting the position of the iterative limiting points by taking the limit of the coordinate data. And based on the rules of no change on the linear combination, the key to solve the iteration problem by vector method is to directly find out the positions of the iterative limitation points by undetermined coefficient method. The coordinate method is a kind of pervasive method but with complicated calculations. The more edges of the target geometry, the more complicated the calculation is. The vector method is simple but it is rather limited to some special cases, because a prerequisite of linear relationship is required for calculation.

Furthermore, in the study of the iteration of the equal lines in triangle, it was found that the area of the triangle could be divided into the parts with the same ratio by a group of spiral lines obtained by grouping the iteration points. In the study of the iteration problems in polygons, superposition of the gravity centers between each iterative limiting point and the polygons was found.

1. Iterations of the medians in triangle

Problem 1: Passing a vertex point A to make the median $AD_1$of $\triangle ABC$; then passing point D_1make the D_1D_2median of $\triangle ACD_1$; passing D_2 to make D_2D_3 median of $\triangle AD_1D_2\ldots$ passing point D_n to makeD_nD_{n+1}median of triangle $\triangle D_{n-2}D_{n-1}D_n$ …… etc. Is there a point P to meet the condition of $\lim_{n\to\infty} D_n = P$? If there is, then what is the position of the point P in the triangle?



1.1 Proof of existence of P point.

As can be seen by the iterative process, the lengths of the medianss in iterations $\{|D_{4n}D_{4n+1}|\}$, $\{|D_{4n+1}D_{4n+2}|\}$, $\{|D_{4n+2}D_{4n+3}|\}$, $\{|D_{4n+3}D_{4n+4}|\}$ are decreasing in the ratio of $\frac{1}{4}$ and converge to 0, which implies that the lengths of the three sides of the triangles tend to be 0. Accordingly, the iterated triangles converge to a single point. Because $\Delta D_{n-2}D_{n-1}D_n \supset \Delta D_{n-1}D_nD_{n+1}$ $(\forall n \geq 3)$, so there is only one point P to satisfy $\bigcap_{n=3}^{\infty} \Delta D_{n-2}D_{n-1}D_n = \{P\}$, therefore D_n converges at a point P.

Meanwhile, it can also deduce that P is inside triangle of the N^{th} iteration.

1.2 Answer for problem 1

Establish coordinate system by taking $\overrightarrow{D_1C}$ and $\overrightarrow{D_1A}$ as axis x and y, respectively.

Given that $D_1(0, 0)$, $x_C = 1$,

$x_1 = x_{D_1} = 0, x_n = x_{D_{4n-4}} = x_{D_{4n-3}}$

The followings equations can be established based on the given parameters:

$$x_1 + 3x_2 = 4x_3$$
$$x_2 + 3x_3 = 4x_4$$
$$\ldots\ldots$$

$$x_n + 3x_{n+1} = 4x_{n+2}(1)$$

Namely $-4(x_{n+2} - x_{n+1}) = x_{n+1} - x_n$

Multiplication gives $x_{n+1} - x_n = \frac{1}{4} * (-\frac{1}{4})^{n-1}$

Then sum of the above formula gives $x_n = \frac{1}{4} \times \frac{1-\left(-\frac{1}{4}\right)^{n-1}}{1+\frac{1}{4}} = \frac{1}{5} - \frac{1}{5}(\frac{-1}{4})^{n-1}$

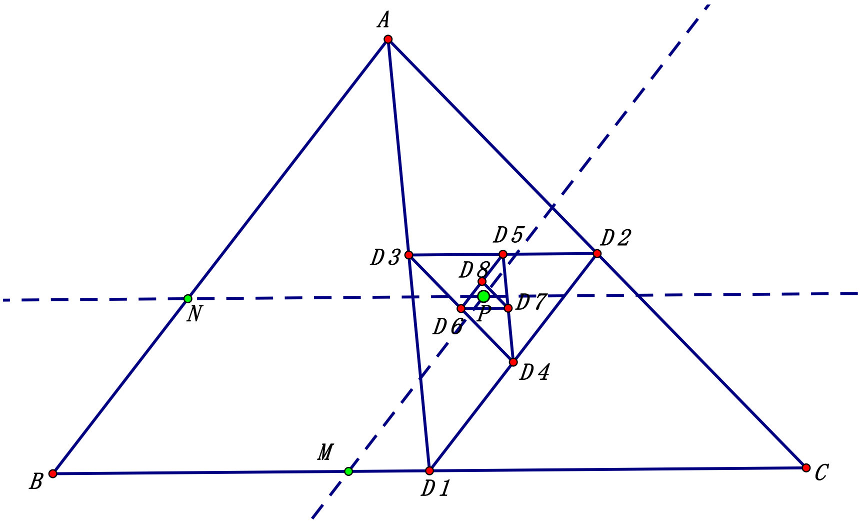
Then $\lim_{n\to\infty} x_n = \frac{1}{5} - \frac{1}{5} \times 0 = \frac{1}{5}$

Similarly, $\lim_{n\to\infty} y_n = \frac{2}{5} y_A$

Answer to problem 1: If we plot parallel lines by passing through point P to three sides of the triangle respectively, then the ratios of the lengths of parallel lines and the lengths of three sides are in the ratios of 2:3,2:3,1:4, respectively, which fix the position of point P in the triangle.

1.3 Expression formula of point P.

Make PM // AB and PN // BC by passing point P, it is possible to get the expression formula of point P by using parallelogram BMPN.



P $\left(\frac{2x_A+2x_C+x_B}{5}, \frac{2y_A+2y_C+y_B}{5}\right)$

1.4 In view of other points.

1.4.1 Different P in same triangle

The model can be readily viewed from different points:

① Take A as starting point, turn towards AC——>P_1

② Take A as starting point, turn towards AB——>P_2

③ Take B as starting point, turn towards AB——>P_3

④ Take B as starting point, turn towards BC——>P_4

⑤ Take C as starting point, turn towards BC——>P_5

⑥ Take C as starting point, turn towards AC——>P_6

Obviously, $P_1 = P_6, P_2 = P_3, P_4 = P_5$
We assume$P_1 = P_6 = B', P_2 = P_3 = C', P_4 = P_5 = A'$
Then we can get the following formula;

$B'(\frac{2x_A+2x_C+x_B}{5}, \frac{2y_A+2y_C+y_B}{5})$

$A'(\frac{2x_B+2x_C+x_A}{5}, \frac{2y_B+2y_C+y_A}{5})$

$C'(\frac{2x_A+2x_B+x_C}{5}, \frac{2y_A+2y_B+y_C}{5})$

1.4.2 Properties of the triangle A′B′C′

The triangle A′B′C′has the following properties:

1） The triangle A′B′C′would have the same gravity center as that of the triangle ABC.

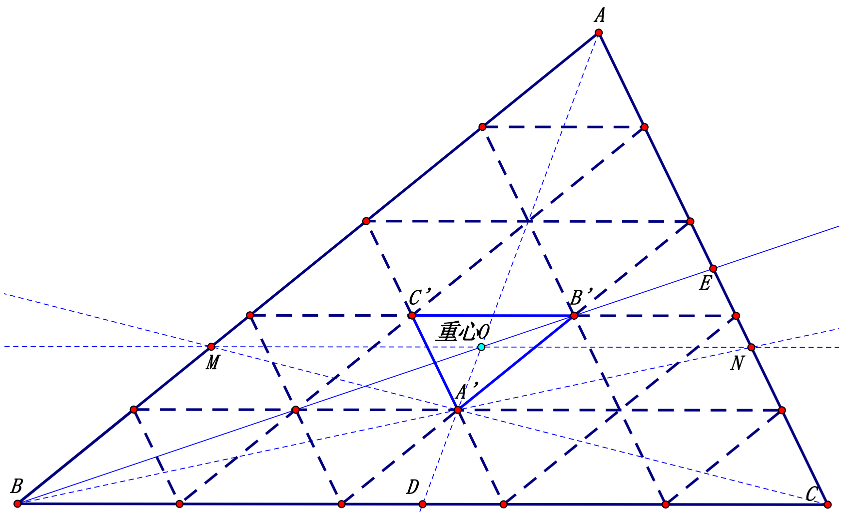
2) The gravity centers of triangle O, B′, B shares the median of the AC side with OB: O B′=5:1. If AC midpoint is taken as E, then BOB′E forms harmonic progression of points. (Similar proposition is true for the other two points).

3) Triangle A′B′C′ is homothetic to triangle ABC on the gravity center O with the homothetic ratio of 1:5.

Proof: 1） $\frac{x_{A'}+x_{C'}+x_{B'}}{3}=\frac{1}{3}\sum_{cyc}\frac{2x_A+2x_C+x_B}{5}=\frac{\sum_{cyc} x_A}{3}$

Similarly, $\frac{\sum_{cyc} y_{A'}}{3} = \frac{\sum_{cyc} y_A}{3}$

The above 2） and 3） can be seen from the following figure.



It can be seen that BO: B′O: B′E=10: 2: 3

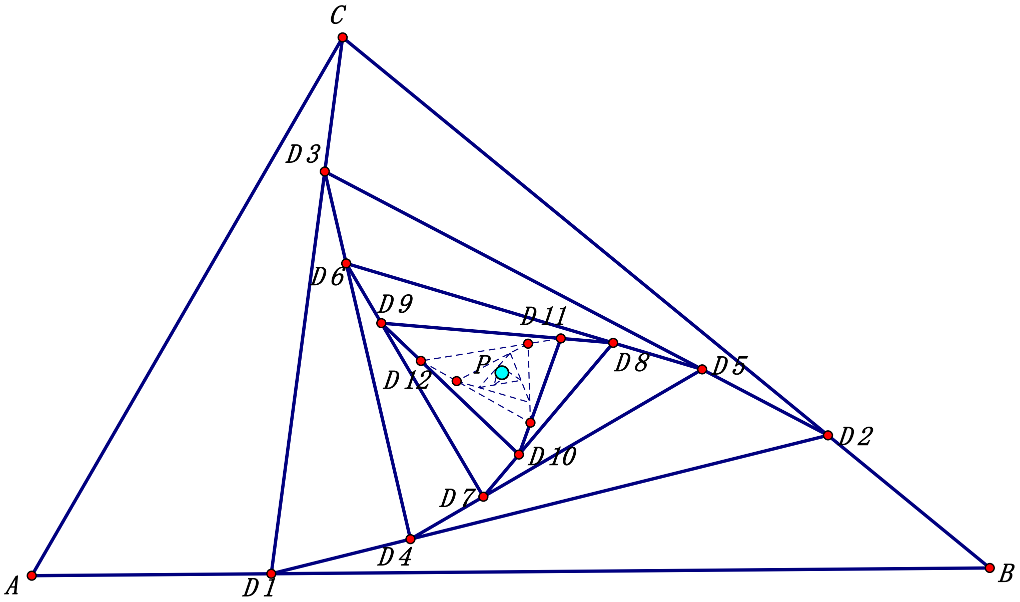
In fact, link CA’ and BA’ and extrapolate it to intersect with line AB and AC at points M and N, respectively. Then AOA’D is a diagonal of complete quadrangle AMNA’BC, and AOA’D is harmonic progression of points.

While $\frac{OA'}{OA} = \frac{OB'}{OB} = \frac{OC'}{OC} = \frac{1}{5}$ and A′B′ // AB, B′C′ // BC, A′C′ // AC.

That is, triangle A'B'C' is homothetic to triangle ABC on the gravity center O with the homothetic ratio of 1:5.

2. Arbitrary bisector iteration in triangle

Problem 2: Make point D_1 on AB, so that $AD_1:AB = \varphi \in (0,1)$, and connect the points C and D_1; locate a point D_2 on BC, so that $BD_2:BC = \varphi$, and connect the points D_1 and D_2; locate a point D_3 on CD_1, so that $CD_3:CD_1 = \varphi$, and connect the points D_2 and D_3…… locate a point D_n on $D_{n-3}D_{n-2}$, so that $D_{n-3}D_n$: $D_{n-3}D_{n-2} = \varphi$, and connect the points D_{n-1} and D_n…… etc. Is there a point P that meets the condition of $\lim_{n\to\infty} D_n = P$? If there is, then what is the position of the point P in the triangle?



2.1 Proof of the existence of the limit point in problem 2

In triangle $D_{n-2}D_{n-1}D_n$, by Stewart Theorem, we have

$$|D_nD_{n+1}|^2 = (1-\varphi)|D_{n-2}D_n|^2 + \varphi|D_{n-1}D_n|^2 + \varphi(\varphi-1)|D_{n-2}D_{n-1}|^2.$$

By Cosine Theorem, we have

$$|D_{n-2}D_n|^2 = (1-\varphi)^2|D_{n-3}D_{n-2}|^2$$

Where $|D_nD_{n+1}| < max\{|D_{n-2}D_n|, |D_{n-1}D_n|\}$.

Thus, $|D_nD_{n+1}|$ is decreasing. Since $|D_nD_{n+1}| > 0$, By Monotone Bounded Principle, the limit $\lim_{n\to\infty}|D_nD_{n+1}|$ exists.

Let $\lim_{n\to\infty}|D_nD_{n+1}| = t$.

Then $t^2 = (1-\varphi)^3t^2 + \varphi t^2 + \varphi(\varphi-1)t^2$

Factorization to give:

$$t^2(1-\varphi)(3-\varphi)\varphi = 0.$$

Since $1-\varphi \neq 0$ and $(1-\varphi)(3-\varphi)\varphi \neq 0$,

We have $\lim_{n\to\infty}|D_nD_{n+1}| = t = 0.$

The circumference of the triangle $\Delta D_{n-5}D_{n-3}D_{n-4}$ converges to 0 and $D_n \in \triangle D_{n-5}D_{n-3}D_{n-4}$.

Because $\Delta D_{n-2}D_{n-1}D_n \supset \Delta D_{n-1}D_nD_{n+1}$ $(\forall n \geq 3)$, so there is only one point P to satisfy $\bigcap_{n=3}^{\infty} \Delta D_{n-2}D_{n-1}D_n = \{P\}$, therefore D_n converges at a point P.

2.2 Answer for problem 2:

Taking D_1 as the origin point, and D_1B and D_1C as axis x and y to establish coordinate system,

Let x(D_1)=0, x(B)=1, m=$\frac{1}{\varphi}$ (geometric meaning of m is the mth equal diversion point)

According to definite proportion formula, we can get x(D_2)=$\frac{m-1}{m}$,x(D_3)=0,x(D_4)= $\frac{m-1}{m^2}$

Then we assume a(n)=x(D_{n+1}), a(1)=$\frac{m-1}{m}$,a(2)=0, a(3)= $\frac{m-1}{m^2}$

And $a(n+3) = \frac{m-1}{m}a(n) + \frac{1}{m}a(n+1)$

Corresponding characteristic equation and characteristic roots are $x^3 = \frac{m-1}{m} + \frac{1}{m}x$

$$\{1, \frac{-m-\sqrt{4m-3m^2}}{2m}, \frac{-m+\sqrt{4m-3m^2}}{2m}\}$$

Let $a(n) = A \times (\frac{-m-\sqrt{4m-3m^2}}{2m})^n + B \times (\frac{-m+\sqrt{4m-3m^2}}{2m})^n + C$

After putting into the former parameters, it can be worked out that: $C = \frac{-1+m}{-1+3m}$

Characteristic roots：$x_{2,3} = \frac{-m\pm\sqrt{4m-3m^2}}{2m} = -\frac{1}{2} \pm \sqrt{\frac{1}{m} - \frac{3}{4}}$

① Where m$\in (1, \frac{4}{3}]$

Because:$\frac{1}{m} - \frac{3}{4} \in [0, \frac{1}{4})$

Then:$x_2 \in \left(-1, -\frac{1}{2}\right]$, $x_3 \in [-\frac{1}{2}, 0)$

So:$\lim_{n\to+\infty} {x_2}^n = \lim_{n\to+\infty} {x_3}^n = 0$

Namely:$\lim_{n\to+\infty} a(n) = C = \frac{-1+m}{-1+3m}$

② Where $m \in \left(\frac{4}{3}, +\infty\right)$

$$x_{2,3} = -\frac{1}{2} \pm \sqrt{\frac{1}{m} - \frac{3}{4}} = \pm\sqrt{\frac{3}{4} - \frac{1}{m}}\,i - \frac{1}{2}$$

Because: $|x_{2,3}| = \sqrt{1 - \frac{1}{m}} \in \left(\frac{1}{4}, 1\right)$

So: $\lim_{n\to+\infty} {x_2}^n = \lim_{n\to+\infty} {x_3}^n = \lim_{n\to+\infty} \left(\sqrt{1-\frac{1}{m}}\right)^n = 0$

Namely: $\lim_{n\to+\infty} a(n) = C = \frac{-1+m}{-1+3m}$

2.3 Further discussion on the problem 2

2.3.1 The extreme case

If we make $\varphi \to 0^+$ in problem 2, namely $m \to +\infty$

We can get: $\lim_{n\to+\infty} \frac{-1+m}{-1+3m} = \frac{1}{3}$

Namely, point P tends to reach the gravity center of triangle ABC.

What we take here is the repeated limit

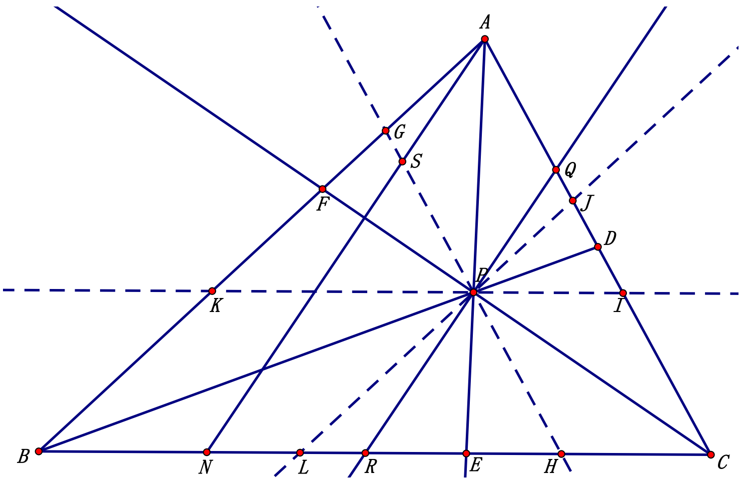
$$\lim_{m\to+\infty} \lim_{n\to+\infty} a_n$$

Rather than its double limit

$$\lim_{\substack{n\to+\infty \\ m\to+\infty}} a(m,n)$$

(Its repeated limit $\lim_{n\to\infty} \lim_{m\to\infty} a_n$ does not exist; this will be further discussed in the following sections)

2.3.2 Expression of the position of point P



It is known that $\frac{BC}{BN} = m$, following the above solution, we can get:

$$\frac{AG}{AB} = \frac{CH}{CB} = \frac{-1+m}{-1+3m}$$

And the following equations can be derived on BC: $\frac{BR}{CR}=\frac{m+1}{2m-2}$

Use Menelaus Law on triangle BCD

$\frac{CR}{BR}\cdot\frac{BP}{DP}\cdot\frac{DQ}{CQ}=\frac{2m-2}{m+1}\cdot\frac{2m}{m-1}\cdot\frac{DQ}{CQ}=1$

Followed by applying the proportion relation on AC, the above formula can be solved to reach: $\frac{AD}{CD}=1$

Namely, D is the midpoint of AC, and because $\frac{BP}{DP}=\frac{2m}{m-1}$

$$\overrightarrow{BP}=\frac{m}{3m-1}\overrightarrow{BA}+\frac{m}{3m-1}\overrightarrow{BC}$$

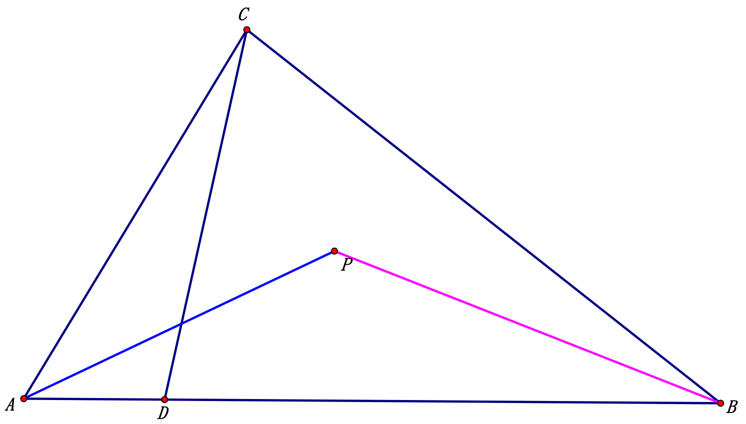
Then the coordinate of point P :

$$P\left(\frac{(m-1)x_B+mx_A+mx_C}{3m-1},\frac{(m-1)y_B+my_A+my_C}{3m-1}\right)$$

2.4 An alternative idea to the solutions of problem 2

2.4.1 Answer for problem 2

It can be found out that the position of P can be determined by linear combination of a group of sides of triangle. And in any cases of iteration, this linear relation remains unchanged. So the following very simple solution can be acquired:



Assuming that $\overrightarrow{AP}=\alpha\overrightarrow{AC}+\beta\overrightarrow{AB}$

Then on the basis of the above mentioned assumption, we can get: $\overrightarrow{BP}=\alpha\overrightarrow{BD}+\beta\overrightarrow{BC}$

Taking point A as the origin point, planimetric rectangular coordinate system can be established.
Assuming A（0, 0）, D（k, 0）, B（a，0），C（b，c），P（m，n）

Then: $\begin{pmatrix}m\\n\end{pmatrix}=\alpha\begin{pmatrix}b\\c\end{pmatrix}+\beta\begin{pmatrix}a\\0\end{pmatrix}$

And $\begin{pmatrix}m-a\\n\end{pmatrix}=\alpha\begin{pmatrix}k-a\\0\end{pmatrix}+\beta\begin{pmatrix}b-a\\c\end{pmatrix}$

Taking α and β as unknown numbers, calculate the above formula, we can get:

$$\binom{\alpha}{\beta} = \begin{pmatrix} \frac{a}{3a-k} \\ \frac{a}{3a-k} \end{pmatrix}$$

To solve the problem 2, k=$\frac{a}{m}$ is brought into the above formula, then: $\alpha = \beta = \frac{m}{3m-1}$, which is completely identical with our previous result.

2.4.2Extension of the vector method

We can obtain a more general conclusion than the vector method in problem 2.

Linear Iteration Principle in triangles:

In the triangle$D_1D_2D_3$, the point D_n is uniquely determined by

$\overrightarrow{D_{n-3}D_n} = \alpha\overrightarrow{D_{n-3}D_{n-2}} + \beta\overrightarrow{D_{n-3}D_{n-1}}$.

If there exists unique limit $\lim_{n\to\infty} D_n = P$, then for $\forall n \in \mathbb{N}_+$, $\overrightarrow{D_nP} = A \cdot \overrightarrow{D_nD_{n+1}} + B \cdot \overrightarrow{D_nD_{n+2}}$, where $A = \frac{1-\beta}{3-\alpha-2\beta}$, $B = \frac{1}{3-\alpha-2\beta}$.

Proof by Mathematical Induction method:

Firstly, for n=1, divide D_n to three sequences $\{D_{3k}\}$, $\{D_{3k+1}\}$, $\{D_{3k+2}\}$ by congruence module 3.

Sum the iterated formulas in three sequences to give,

$$\overrightarrow{D_1D_{3n+1}} = \alpha\sum_{k=1}^{n}\overrightarrow{D_{3k-2}D_{3k-1}} + \beta\sum_{k=1}^{n}\overrightarrow{D_{3k-2}D_{3k}}$$

$$\overrightarrow{D_2D_{3n+2}} = \alpha\sum_{k=1}^{n}\overrightarrow{D_{3k-1}D_{3k}} + \beta\sum_{k=1}^{n}\overrightarrow{D_{3k-1}D_{3k+1}}$$

$$\overrightarrow{D_3D_{3n+3}} = \alpha\sum_{k=1}^{n}\overrightarrow{D_{3k}D_{3k+1}} + \beta\sum_{k=1}^{n}\overrightarrow{D_{3k}D_{3k+2}}$$

As $n \to +\infty$, we get

$$\overrightarrow{D_1P} = \alpha\sum_{k=1}^{\infty}\overrightarrow{D_{3k-2}D_{3k-1}} + \beta\sum_{k=1}^{\infty}\overrightarrow{D_{3k-2}D_{3k}}$$

$$\overrightarrow{D_2P} = \alpha\sum_{k=1}^{\infty}\overrightarrow{D_{3k-1}D_{3k}} + \beta\sum_{k=1}^{\infty}\overrightarrow{D_{3k-1}D_{3k+1}}$$

$$\overrightarrow{D_3P} = \alpha\sum_{k=1}^{\infty}\overrightarrow{D_{3k}D_{3k+1}} + \beta\sum_{k=1}^{\infty}\overrightarrow{D_{3k}D_{3k+2}}$$

Furthermore,

$$(1-\alpha-\beta)\overrightarrow{D_1P}+(1-\beta)\overrightarrow{D_2P}+\overrightarrow{D_3P}$$

$$=(1-\alpha-\beta)\left(\alpha\sum_{k=1}^{\infty}\overrightarrow{D_{3k-2}D_{3k-1}}+\beta\sum_{k=1}^{\infty}\overrightarrow{D_{3k-2}D_{3k}}\right)$$

$$+(1-\beta)\left(\alpha\sum_{k=1}^{\infty}\overrightarrow{D_{3k-1}D_{3k}}+\beta\sum_{k=1}^{\infty}\overrightarrow{D_{3k-1}D_{3k+1}}\right)+\alpha\sum_{k=1}^{\infty}\overrightarrow{D_{3k}D_{3k+1}}$$

$$+\beta\sum_{k=1}^{\infty}\overrightarrow{D_{3k}D_{3k+2}}$$

$$=\alpha\sum_{u=1}^{3}\sum_{k=1}^{\infty}\overrightarrow{D_{3k}D_{3k+u}}+\beta\sum_{u=1}^{3}\sum_{k=1}^{\infty}\overrightarrow{D_{3k}D_{3k+2u}}$$

$$-\left(\alpha^2\sum_{k=1}^{\infty}\overrightarrow{D_{3k-2}D_{3k-1}}+2\alpha\beta\sum_{k=1}^{\infty}\overrightarrow{D_{3k-2}D_{3k}}+\beta^2\sum_{k=1}^{\infty}\overrightarrow{D_{3k-1}D_{3k}}\right)$$

$$=-\left(\alpha\left(\alpha\sum_{k=1}^{\infty}\overrightarrow{D_{3k-2}D_{3k-1}}+\beta\sum_{k=1}^{\infty}\overrightarrow{D_{3k-2}D_{3k}}\right)+\beta\left(\alpha\sum_{k=1}^{\infty}\overrightarrow{D_{3k-2}D_{3k}}+\beta\sum_{k=1}^{\infty}\overrightarrow{D_{3k-1}D_{3k}}\right)\right)$$

$$=-\alpha\sum_{k=1}^{\infty}\overrightarrow{D_{3k-2}D_{3k+1}}-\beta\sum_{k=1}^{\infty}\overrightarrow{D_{3k-3}D_{3k}}=\vec{0}$$

$$\Leftrightarrow\overrightarrow{D_1P}=\frac{1-\beta}{3-\alpha-2\beta}\overrightarrow{D_1D_2}+\frac{1}{3-\alpha-2\beta}\overrightarrow{D_1D_3}$$

Thus, when n=1, the conclusion holds right.

Now suppose that the conclusion also holds right when n=k($k\in\mathbb{N}_+$), i.e. $\overrightarrow{D_kP}=A\cdot\overrightarrow{D_kD_{k+1}}+B\cdot\overrightarrow{D_kD_{k+2}}$.

For n=k+1,

$$\overrightarrow{D_{k+1}P}-\left(A\cdot\overrightarrow{D_{k+1}D_{k+2}}+B\cdot\overrightarrow{D_{k+1}D_{k+3}}\right)$$

$$=\overrightarrow{D_{k+1}D_k}+\overrightarrow{D_kP}-A\cdot\left(\overrightarrow{D_{k+1}D_k}+\overrightarrow{D_kD_{k+2}}\right)-B\cdot\left(\overrightarrow{D_{k+1}D_k}+\overrightarrow{D_kD_{k+3}}\right)$$

$$=\overrightarrow{D_{k+1}D_k}+\overrightarrow{D_kP}-A\cdot\overrightarrow{D_{k+1}D_k}-A\cdot\overrightarrow{D_kD_{k+2}}-B\cdot\overrightarrow{D_{k+1}D_k}-B\cdot\left(\alpha\overrightarrow{D_nD_{n+1}}+\beta\overrightarrow{D_nD_{n+2}}\right)$$

$$=\overrightarrow{D_kD_{k+1}}\cdot(2A+B-1-B\beta)+\overrightarrow{D_nD_{n+2}}\cdot(B-A-B\beta)$$

$$=\vec{0},$$

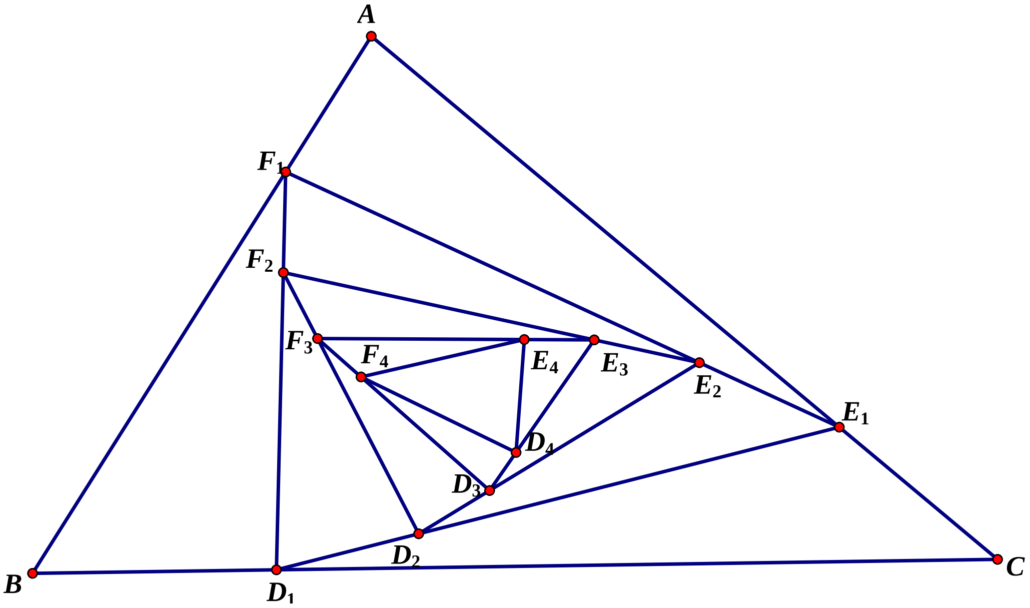
which means that the conclusion is also correct.

Therefore, for $\forall n \in \mathbb{N}_+$, $\overrightarrow{D_nP} = \frac{1-\beta}{3-\alpha-2\beta} \cdot \overrightarrow{D_nD_{n+1}} + \frac{1}{3-\alpha-2\beta} \cdot \overrightarrow{D_nD_{n+2}}$.

Specifically, regarding to problem 2, it suffices to take $\alpha = \varphi$, $\beta = 0$.

2.4.3 Applications of the vector method in the iterations of figures

It is worth to note that the vector method applied here is not limited to the problems of the iterations of triangle. For example, it also applies to the following problem 3.



Problem 3: In triangle ABC, take $BD_1:BC=CE_1:CA=AF_1:AB=k$, then perform iteration in the following sequences: $D_nD_{n+1} : D_nE_n = E_nE_{n+1} : E_nF_n = F_nF_{n+1} : F_nD_n = k$. Let $\lim_{n\to\infty} D_n = \lim_{n\to\infty} E_n = \lim_{n\to\infty} F_n = P$. How to find the position of point P?

Answer: Let B be the origin point. Denote C(c, 0), A(a, b), $\overrightarrow{BP} = \alpha\overrightarrow{BA} + \beta\overrightarrow{BC}$.

According to the Linear Iteration Principle, we have $\overrightarrow{D_1P} = \alpha\overrightarrow{D_1F_1} + \beta\overrightarrow{D_1E_1}$.

Then, we get:

$$\begin{cases} \alpha = \frac{1}{3} \\ \beta = \frac{1}{3} \end{cases}$$

The result implies that the point P is the gravity center of the triangle ABC.

Furthermore, we can obtain a similar conclusion as **Triangles Linear Iteration Principle**:

In the triangle$A_1B_1C_1$, the iterative triangle$A_nB_nC_n$ is uniquely determined by

$\overrightarrow{A_{n-1}A_n} = \alpha_1\overrightarrow{A_{n-1}B_{n-1}} + \beta_1\overrightarrow{A_{n-1}C_{n-1}}$, $\overrightarrow{B_{n-1}B_n} = \alpha_2\overrightarrow{B_{n-1}C_{n-1}} + \beta_2\overrightarrow{B_{n-1}A_{n-1}}$, $\overrightarrow{C_{n-1}C_n} =$

$\alpha_3\overrightarrow{C_{n-1}A_{n-1}} + \beta_3\overrightarrow{C_{n-1}B_{n-1}}$. If there exists unique limit $\lim_{n\to\infty} A_nB_nC_n = P$, then for $\forall n \in \mathbb{N}_+$,

$a \cdot \overrightarrow{A_nP} + b \cdot \overrightarrow{B_nP} + c \cdot \overrightarrow{C_nP} = \vec{0}$

where $a = \frac{\alpha_2\alpha_3+\alpha_3\beta_2+\beta_2\beta_3}{p}$, $b = \frac{\alpha_1\alpha_3+\alpha_1\beta_3+\beta_1\beta_3}{p}$, $c = \frac{\alpha_2\alpha_1+\alpha_2\beta_1+\beta_2\beta_1}{p}$,

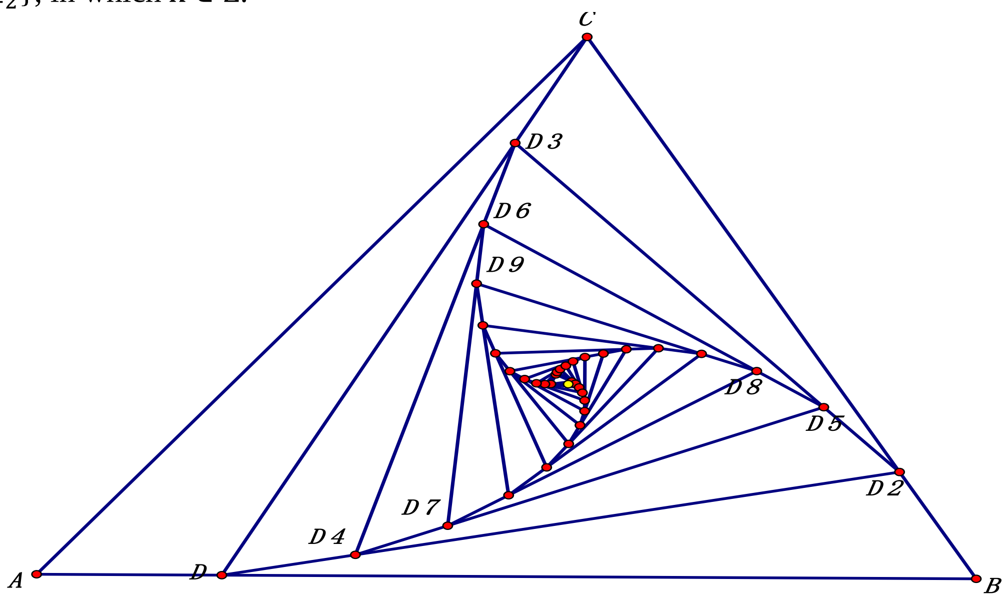
$p=\alpha_2\alpha_3 + \alpha_3\beta_2 + \beta_2\beta_3 + \alpha_1\alpha_3 + \alpha_1\beta_3 + \beta_1\beta_3 + \alpha_2\alpha_1 + \alpha_2\beta_1 + \beta_2\beta_1$

Specifically, when $\alpha_1 = \alpha_2 = \alpha_3$, $\beta_1 = \beta_2 = \beta_3$,P is the gravity center of the triangle.

2.5 Study on the trajectory of points

2.5.1 The equation of the curve

When we draw the following graph (a proportion that is limit iteration was performed), we can intuitively see the distribution of the points are divided into three groups $\{D_{3k}\}$, $\{D_{3k+1}\}$ and $\{D_{3k+2}\}$, in which $k \in Z$.



It can be seen that all points in the three groups gradually approach point P. However, as the points are discrete, it is impossible to form a continuous curve.

If we assume that $m \to +\infty$, then

$$\lim_{n\to\infty} a(n) = \frac{1}{3}2^{-1-n}\left(2^{1+n} + \left(-1 - i\sqrt{3}\right)\left(-1 + i\sqrt{3}\right)^n + i\left(-1 - i\sqrt{3}\right)^n\left(i + \sqrt{3}\right)\right)$$

$$= \frac{1}{3}(1 - \mathrm{Cos}[\frac{2n\pi}{3}] + \sqrt{3}\mathrm{Sin}[\frac{2n\pi}{3}])$$

(a(n) in the above formula is the a(n) assumed in “answer for problem 2”, which was used for the measurement of component of $\overrightarrow{AP}$ on $\overrightarrow{AB}$).

We found that repeated limit $\lim_{n\to\infty}\lim_{m\to\infty} a_n$ does not exist. In other words, if we take the limit directly on $a(n)$, it will cause focusing of all points on three points A, B and C, therefore, we cannot take limit like the two previous problems. But it is sure that there must be one curve that may enable all points in one of the three groups mentioned above to fall on it. But searching of such kind of curve is challenging, because taking limit directly as we discussed before is impossible.

In fact, we can also acquire n=g(x, m) from x=f (n, m), then bring into y=u(n, m) to acquire y=w(x, m), then assume $m \to +\infty$ to acquire equation meeting x and y. It is impossible to acquire analytic solution, so this method cannot be applied. On the other hand, we can consider polar coordinate system, but it was found that $\tan\theta = 0$ when limit is taken in calculation results.

Furthermore, we have compared this spiral with other known spirals (such as Archimedes spiral and equiangular spiral), but these spirals does not meet the properties of the spiral examined in this work.

In summary of the above mentioned, we cannot acquire accurate equation of the spiral yet, and cannot determine the specific type of the spiral.

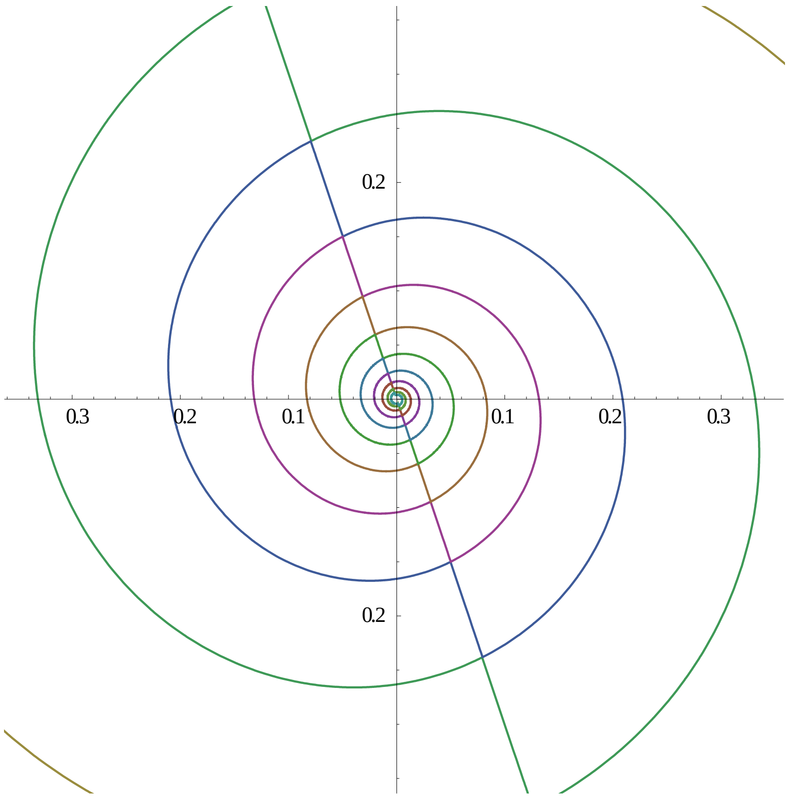
However, it is not too hard to solve a special condition. When the triangle is an isosceles right triangle, and the median iteration is from the right angle vertex， there is a spiral equation passing all the iterative points:

(Take the limiting point P as the pole， the line parallel to the hypotenuse as the polar axis.)

$$\rho = \sqrt{\frac{8^{1-\frac{4\left(\tan^{-1}\left(3-\frac{10}{3+\tan\theta}\right)+k\pi\right)}{9\pi}}}{5}}, \quad k \in \mathbb{Z}, \quad \vartheta \in \left[k\pi + \tan^{-1}(3), k\pi + \frac{\pi}{2} + \tan^{-1}\left(\frac{1}{3}\right)\right]$$

Obviously, the equation above is a combination of a series of equations, but the whole curve is continuous and smooth. To simplify the equation, it can be written as:

$$3 - \frac{10}{3+\tan\theta} = \tan\left(\frac{9}{4}\pi \log_8\left(\frac{5}{8}\rho^2\right)\right)$$



In the case of n bisector, the subject is the same. for the oblique coordinate system, the results can also be a similar formula.

This formula is obtained by changing the horizontal and vertical coordinates into triangular form, with trigonometric identities and inverse trigonometric functions n is eliminated. But for the more general case, polar coordinate method is much more complicated.
It is not hard not find that for the general equation we can get two center symmetrical spirals, which we call conjugate spirals.
Though it is not the spiral we want, it is still meaningful as a spiral passing all iterative points.
(The figure is the formula obtained when k take 0-12)

2.5.2 Properties of the spiral

In summary of the above mentioned, we cannot acquire accurate equation of the spiral yet, and cannot determine the specific type of the spiral. But it is possible to get several properties of the broken line which is close to the spiral as follows:

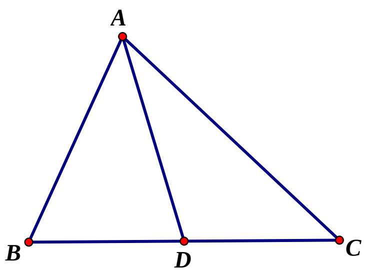
(1) Where the iterative triangle is acute triangle, for each group of points, with the increase of n, decrease of $|D_nP|$.

This can be proved with the fact that the result of the partial derivative of n for $|D_nP|$ is more than 0. However, a simple proof method is proposed which applies the conclusion in “alternative idea” and the induction principle.
According to the following lemma:

The median of acute triangle is longer than half of its segment.

Proving of the lemma:

As in the following figure, AD in the acute triangle ABC bisects BC, what shall be proved is AD>CD only.

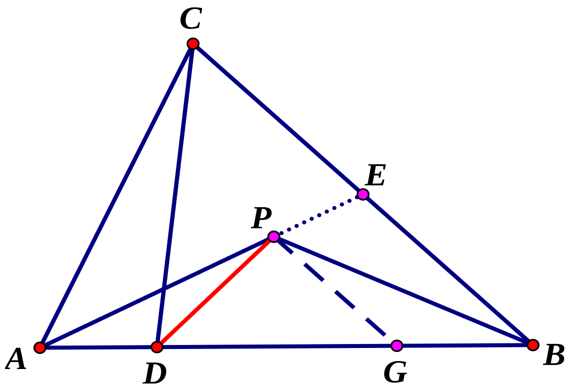


From pappus theorem $AD^2 = \frac{AB^2+BC^2}{2} - CD^2$

The triangle ABC is acute triangle $\Leftrightarrow AB^2 + BC^2 > AC^2$

Bring into the above formula to get: $AD^2 = \frac{AB^2+BC^2}{2} - CD^2 > \frac{4CD^2}{2} - CD^2 = CD^2$

That is: AD>CD



Proof of the original problem:
As shown in the figure, triangle ABC and triangle BCD are acute triangles.
P is a limit point, A isD_n, D isD_{n+1}, according to the raised problem, what shall be proved is AP>DP only in the figure.
∵P∈Δ_{BCD}, P is on AE, which is serve as the median of the triangle constantly.
∴DP＜max {AP, PG}＜AP
By using the induction principle, the original proposition is proved to be true.
It is noteworthy that the condition in property I is sufficient but not necessary.

(2) Spiral cuts the original triangle into three parts, set their areas as S_1, S_2 and S_3, respectively. The original triangle area is set as S.

Then values of $\frac{S_1}{S}, \frac{S_2}{S}$ and $\frac{S_3}{S}$ are only related to m (unrelated to the shape of triangle)
And S_1, S_2 and S_3 are of geometric progression (this is unrelated to either m nor shape of triangle)

As for the polygonal line$\{D_{3n+t}\}$, we firstly consider the area enclosed by broken lines where t=1 and t=2.

$$S_2 = S_{BDD_2} + \sum_{i=1}^{\infty} S_{D_{3i-1}D_{3i+1}D_{3i+2}}$$

In which $S_{BDD_2} = (1-\phi)\phi \cdot S, S_{D_{3i-1}D_{3i+1}D_{3i+2}} = (1-\phi)^{3i+1}\phi \cdot S$

$$\therefore S_2 = \sum_{i=0}^{\infty}(1-\phi)^{3i+1}\phi \cdot S$$

$$\text{And} \because \phi \in (0,1)$$

$$\therefore \text{the series converge at } \frac{(1-\phi)\phi \cdot S}{1-(1-\phi)^3} = \frac{(\phi-1)S}{\phi^2+3\phi-3}$$

Similarly, we can get: $S_1 = \frac{S}{-\phi^2-3\phi+3}, S_3 = \frac{(\phi-1)^2 S}{-\phi^2-3\phi+3}$

Obviously,

$$\frac{S_1}{S} = \frac{1}{-\phi^2-3\phi+3}, \frac{S_2}{S} = \frac{(\phi-1)}{\phi^2+3\phi-3}, \frac{S_3}{S} = \frac{(\phi-1)^2 S}{-\phi^2-3\phi+3}$$

And $S_1 \cdot S_3 = {S_2}^2$

That is, S1, S2 and S3 form geometric relationship.

At the extreme situation, the effect of polygonal line is identical with the spiral, and it can be concluded that: spiral will divide triangle into three equivalent parts.

And at more ordinary situation, if we let increment between spiral and polygonal line equivalent, the above mentioned conclusion is still true. (This property provides another feature for the spiral.)

(3) About the length of the broken lines
Similar to (2), we can take limit from polygonal line, but the function we get is so complicated that it is difficult for us to determine convergence and divergence of its series.

If we assume $|D_{p-1}D_p| = L_P$

Then Stewart law gives recursion $L_{n+3}{}^2 = \phi L_{n+2}{}^2 + (1-\phi)^3 L_n{}^2 - \phi(1-\phi)L_{n+1}{}^2$
It is possible to determineL_n, but we could not get the process the series so far, therefore, the length of the spiral needs to be further studied (see specific value as per attachment).

(As verified,$\sum L_P{}^2$ converges at $\frac{(1-\phi)^2 L_1{}^2+(1-\phi)L_2{}^2+L_3{}^2}{(1-\phi)^2\phi}$, but $\sum L_P{}^2 < \sum L_P$)

However, it can be proved that the length of the spiral passing all iterative points is convergent.

$$I_k = \int_{-\tan^{-1}(3)}^{\pi-\tan^{-1}(3)} \sqrt{\rho^2 + \left(\frac{\partial\rho}{\partial\theta}\right)^2}\, d\theta = \frac{\left(\sqrt{2^{\frac{1}{3}(5-4k)}} - \sqrt{2^{\frac{1}{3}-\frac{4k}{3}}}\right)\sqrt{5(9\pi^2 + 4\mathrm{Log}[2]^2)}}{\mathrm{Log}[32]}$$

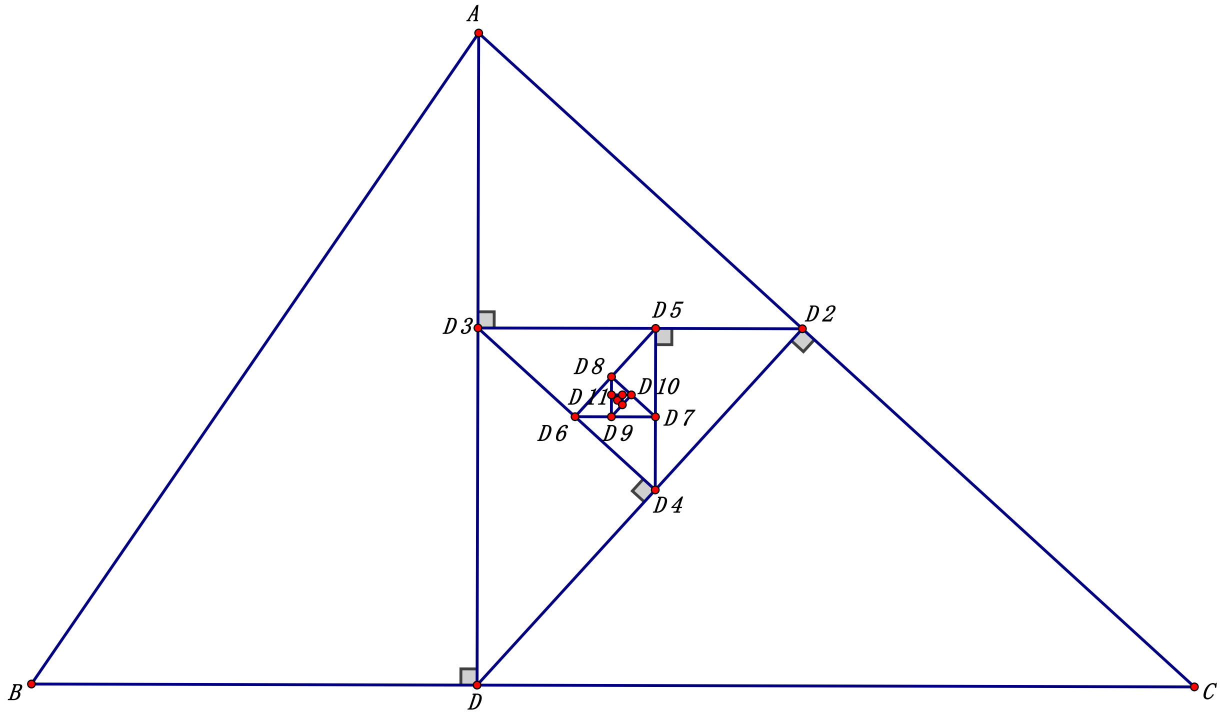
$$I = \sum_{k=1}^{\infty} \frac{\left(\sqrt{2^{\frac{1}{3}(5-4k)}} - \sqrt{2^{\frac{1}{3}-\frac{4k}{3}}}\right)\sqrt{5(9\pi^2 + 4\mathrm{Log}[2]^2)}}{\mathrm{Log}[32]} = \frac{2^{1/6}\sqrt{5(9\pi^2 + 4\mathrm{Log}[2]^2)}}{\mathrm{Log}[32]}$$

3. Further Extension

Further extension of the iteration problems has been also performed and the results can be summarized as follows:

3.1 Extension on perpendicular line direction

Problem 4: For a triangle ABC, passing a vertex point A to make the perpendicular line $AD_1$of $\triangle ABC$, then passing pointD_1, make the perpendicular line $D_1D_2$of $\triangle ACD_1$,……passing point D_n make the perpendicular line $D_nD_{n+1}$of$\triangle D_{n-2}D_{n-1}D_n$ ……etc. If there is a point P that meets the condition of $\lim_{n\to\infty} D_n = P$? If there is, what is the position of the point P in the triangle?



(The proof of the existence of point P is similar to that in Problem 2, which is not shown here)
Establish Cartesian coordinate system by taking D as the origin point,

Let X(C)=a，X(A)=h，$\angle C=\alpha$，$k=\sin^2\alpha$，$DD_2=d$

$$k=\frac{AD_1\cdot\sin\alpha}{\frac{AD_1}{\sin\alpha}}=\frac{AD_2}{AC}=\frac{AD_3}{AD}=\frac{D_2D_4}{D_1D_2}=\frac{D_2D_5}{D_3D_2}=\ldots\ldots$$

$$=\frac{D_{2m}D_{2m+2}}{D_{2m-1}D_{2m}}=\frac{D_{2m}D_{2m+3}}{D_{2m}D_{2m+1}}\ (m\epsilon\mathbb{N}_+)$$

Let $b_n=Y(D_{2n})\ (n\epsilon\mathbb{N}_+), b_1=(1-k)h, b_2=(1-k)^2h,$

We get: $b_{2m+1}=k\cdot b_{2m-1}+(1-k)b_{2m}\ \ b_{2m}=k\cdot b_{2m-3}+(1-k)b_{2m-1}$

If we assume $\text{sub}-\text{series}\ \{c_n\}:\ c_n=b_{2n-1}$ of $\{b_n\}$

Then we get $c_1=(1-k)h, c_2=k(1-k)h+(1-k)^3h$

And $c_{n+1}=b_{2n+1}=k\cdot b_{2n-1}+(1-k)b_{2n}$

$$=k\cdot b_{2n-1}+(1-k)[k\cdot b_{2n-3}+(1-k)b_{2n-1}]$$

$$=k(1-k)c_{n-2}+(k^2-k+1)c_{n-1}$$

As per difference equation, it can be worked out that:

$$c_n=-\frac{h(-1+k)(-1+((-1+k)k)^n)}{-1-k+k^2}$$

And $\because k=\sin^2\alpha\in(0,1), (-1+k)k\in\left(-\frac{1}{4},0\right)$

$$\therefore \lim_{n\to\infty}c_n=\lim_{n\to\infty}\frac{h(1-k)(-1+((-1+k)k)^n)}{-1-k+k^2}=\frac{h(1-k)(-1+\lim_{n\to\infty}[((-1+k)k)^n])}{-1-k+k^2}$$

$$=\frac{h(1-k)}{1+k-k^2}$$

In iterated triangleAD_1D_2, $\angle C=\frac{\pi}{2}-\alpha$ and $k'=\cos^2\alpha=1-k.$

Then the distance from point P to AC

$$d_{P-AC}=\frac{d(1-k')}{1+k'-k'^2}=\frac{dk}{1+k-k^2},$$

And consequently, the projection of P to $\overrightarrow{DC}$ is

$$\left(\left(1-\frac{d_{P-AC}}{d}\right)-\frac{1-k}{1+k-k^2}\right)a=\frac{k(1-k)a}{1+k-k^2}.$$

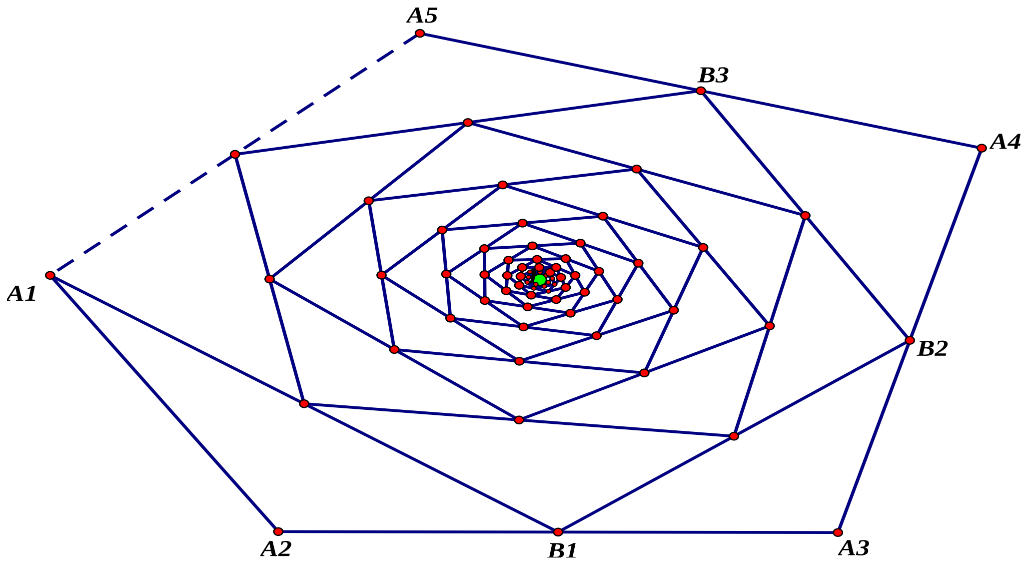
In summary, we can get the position of point P:

$$\overrightarrow{DP}=\frac{k-k^2}{1+k-k^2}\overrightarrow{DC}+\frac{1-k}{1+k-k^2}\overrightarrow{DA}, \text{In which } k=\sin^2\angle C$$

It is easy to see that if directly applied vector method is not possible, it is because of the condition unable to meet the premise, the linear relationship. But we can separate the iteration to use the vector method. The conclusion is the same as the above.

3.2 Extension in convex polygon

Problem5: Take one point A_1 of convex polygon $A_1A_2 \dots A_M$ and connect it with midpoint B_1 of line A_2A_3 , then connect point B_1 with midpoint B_2 of line A_3A_4 ……connect point B_{M-1} with midpoint B_M of line A_1B_1……(repeat sequentially)……If there is a point P that meets the condition of $\lim_{n\to\infty} D_n = P$? If there is, what is the position of the point P in the triangle? (Similarly, it is allowed to generalize the above contents in polygon)



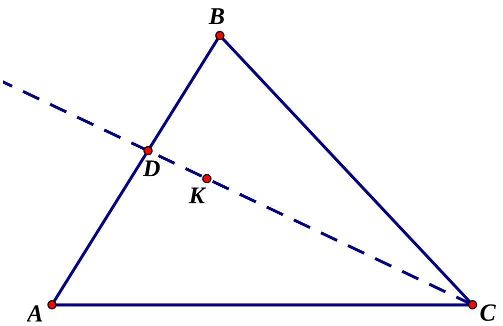
If the above coordinate method can still be adopted for solving the problems of polygon, then n-polygon requires resolving N^{th} algebraic equation. However, similar to the method for solving the Problem 2, we can also utilize iterative principle and undetermined coefficients to work out the coordinates of the limit points.

3.2.1 Lemma:

For the point K of a convex polygon $A_1A_2 \dots A_n$, there is a group of non-negative real numbers $\{\alpha_1, \alpha_2, \dots\dots, \alpha_n\}$, where$\sum_{k=1}^{n} \alpha_k = 1$, then $\sum_{k=1}^{n} \alpha_k \overrightarrow{KA_k} = \vec{0}$.

Proof of lemma:

If n=3



As shown in the above figure, $\exists m, n \in R^+, \overrightarrow{KD} = m\overrightarrow{KA} + n\overrightarrow{KB}$

And ∵D,K,C are collinear,

$\therefore \exists p, q \in R^+, \vec{0} = p\overrightarrow{KD} + q\overrightarrow{KC} = p\left(m\overrightarrow{KA} + n\overrightarrow{KB}\right) + q\overrightarrow{KC} = pm\overrightarrow{KA} + pn\overrightarrow{KB} + q\overrightarrow{KC}$

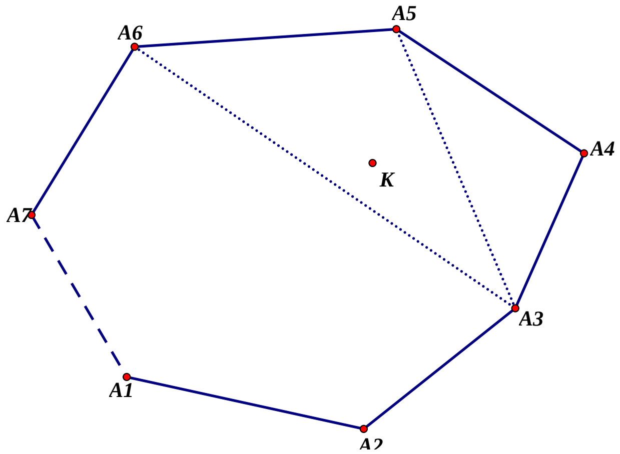
Where $pm + pn + q = 1$

In the case that the K is on triangle side, it is still true.

If n>3, it is surely that, K is inside at least one triangle which takes three vertexes on the polygon , it is known from the above that:

$$\exists a, b, c \in R^+, a\overrightarrow{KB} + b\overrightarrow{KB} + c\overrightarrow{KC} = \vec{0}, \text{and } a + b + c = 1$$

Take the coefficient of the other vectors as 0, then a group of non-negative real numbers $\{\alpha_1, \ \alpha_2, \ \ldots\ldots, \ \alpha_n\}$ can be obtained, where $\sum_{k=1}^{n} \alpha_k = 1$, then $\sum_{k=1}^{n} \alpha_k \overrightarrow{KA_k} = \vec{0}$.



In fact, similar to the discussions of vector method in triangles, one can prove the invariance of the coefficients in above Lemma under iterations. Thus, we could call it **Linear Iteration Principle In Polygon.**

3.2.2 Answer of original problem:

Take O（0，0）as the original point to establish the coordinate system,

And suppose A_n coordinate is（p_n, q_n）,P(X,Y)，$B_n(r_n, t_n)$

From the lemma, we suppose

$$\overrightarrow{OP} = \sum_{m=1}^{M} \alpha_m \overrightarrow{OA_m}$$

From the induction principle, there is

$$\overrightarrow{OP} = \alpha_1\overrightarrow{OB_1} + \alpha_2\overrightarrow{OA_3} + \cdots + \alpha_{M-1}\overrightarrow{OA_n} + \alpha_M\overrightarrow{OA_1}$$

$$\overrightarrow{OP} = \alpha_1\overrightarrow{OB_2} + \alpha_2\overrightarrow{OA_4} + \cdots + \alpha_{M-2}\overrightarrow{OA_n} + \alpha_{M-1}\overrightarrow{OA_1} + \alpha_M\overrightarrow{OB_1}$$

......

$$\overrightarrow{OP} = \alpha_1\overrightarrow{OB_{M-2}} + \alpha_2\overrightarrow{OA_M} + \alpha_3\overrightarrow{OA_1} + \alpha_4\overrightarrow{OB_1} + \cdots + \alpha_M\overrightarrow{OB_{M-3}}$$

Then, there is equation set:

$$\begin{bmatrix} p_1 & \cdots & p_n \\ \vdots & \ddots & \vdots \\ r_{M-2} & \cdots & r_{M-3} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \cdots \\ \alpha_M \end{bmatrix} = \begin{bmatrix} X \\ \cdots \\ X \end{bmatrix}$$

$$\begin{bmatrix} q_1 & \cdots & q_n \\ \vdots & \ddots & \vdots \\ t_{M-2} & \cdots & t_{M-3} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \ldots \\ \alpha_M \end{bmatrix} = \begin{bmatrix} Y \\ \ldots \\ Y \end{bmatrix}$$

Combines $\sum_{k=1}^{n} \alpha_k = 1$

By Cramer's Rule $\{\alpha_1, \alpha_2, \ldots\ldots, \alpha_n\}$ can be worked out, then point P position can be established.

In fact, we could obtain more equations. However, it is unnecessary to list all equations for reasons as follows. For $p, n \in \mathbb{N}_+$, we have

$$\sum_{p=t}^{t+M-1} \alpha_{p-t+1} \overrightarrow{OB_{nM+p}} = \sum_{p=t}^{t+M-1} \alpha_{p-t+1} \cdot \frac{\overrightarrow{OB_{(n-1)M+p}} + \overrightarrow{OB_{(n-1)M+p+1}}}{2}$$

$$= \frac{1}{2} \sum_{p=t}^{t+M-1} \alpha_{p-t+1} \cdot \overrightarrow{OB_{(n-1)M+p}} + \frac{1}{2} \sum_{p=t}^{t+M-1} \alpha_{p-t+1} \cdot \overrightarrow{OB_{(n-1)M+p+1}} = \frac{X}{2} + \frac{X}{2} = X$$

Then, by the Inductive Principle, the remaining equations are linearly dependent on the equations listed.

As for more general conditions, taking m points of equal parts only changes the coordinate of B_n. Because the limited numbers of points for coordinates is available in the method mentioned above, its process for the solution of the problem is easy.

3.2.3 A derived conclusion

Wonderful results for polygon could be derived.

From analysis of problem 1, it is known that the limit point positions are different if the initial iteration is started from the different fixing points. Suppose that it is P_n, the graph formed with P_n would have the same gravity center with the original polygon.

Proof of the above assumption:

It is known that $\sum_{k=1}^{M} \alpha_k \overrightarrow{PA_k} = \vec{0}$

As for different P superposition, there is

$$\sum_{r=1}^{M} \sum_{k=1}^{M} \alpha_k \overrightarrow{P_r A_{k+r-1 \bmod (M)}} = \sum_{k=1}^{n} \left(\alpha_k \sum_{r=1}^{M} \overrightarrow{P_r A_{k+r-1 \bmod (M)}} \right) = \vec{0}$$

$$\because \sum_{r=1}^{M} \overrightarrow{P_r A_{k+r-1 \bmod (M)}} = \sum_{r=1}^{M} \overrightarrow{P_r A_{k+r-1 \bmod (M)}} + \sum_{r=1}^{M} \overrightarrow{A_{k+r-1 \bmod (M)} A_{k+r+1-1 \bmod (M)}}$$

$$= \sum_{r=1}^{M} \overrightarrow{P_r A_{k+r+1-1 \bmod (M)}} = \cdots = \sum_{r=1}^{M} \overrightarrow{P_r A_r}$$

And $\because \alpha_k$ is not always 0

$$\therefore \sum_{r=1}^{M} \overrightarrow{P_r A_{k+r-1 \bmod (M)}} = \sum_{r=1}^{M} \overrightarrow{P_r A_r} = \vec{0}$$

Suppose the gravity center of the original polygon is O, and the gravity center of the graph formed

withP_n is O' [3]
Then

$$\sum_{k=1}^{M} \overrightarrow{OA_k} = \vec{0}$$

$$\sum_{k=1}^{M} \overrightarrow{O'P_k} = \vec{0}$$

Take the difference to give

$$\vec{0} = \sum_{k=1}^{M} \overrightarrow{OA_k} - \sum_{k=1}^{M} \overrightarrow{O'P_k} = \sum_{k=1}^{M} \left(\overrightarrow{P_kA_k} + \overrightarrow{OO'}\right) = M \cdot \overrightarrow{OO'}$$

So O and O' are coincident.

The derived conclusion: the graph formed with P_n has the same gravity center with the original polygon.

3.3 Extension to angular bisector direction

For a triangle ABC, passing a vertex A, make $\triangle ABC$ angular bisector AD_1;then passing D_1, make $\triangle ACD_1$ angular bisector D_1D_2; passing D_2, make $\triangle AD_1D_2$ angular bisector D_2D_3…… passing point D_n, make $\triangle D_{n-2}D_{n-1}D_n$ angular bisector D_nD_{n+1}……If there is a point P hat meets the condition of $\lim_{n\to\infty} D_n = P$? If there is, what is the position of the point P in the triangle?

(Not solved yet)

It can be concluded at present that: similar to the above contents, firstly we establish coordinate system by taking D_1 as the original point, and let DC=1, $x_n = x(D_n)$
Then

$$x_n = \frac{x_{n-3}a_{n-2} + x_{n-2}b_{n-1}.}{a_{n-2} + b_{n-1}}$$

In which a_n and b_n are determined as per the following recursion: (In which preceding finite term of $\{a_n\}$ and $\{b_n\}$ can be resolved)

$$a_n = a_{n-1} \cdot \sqrt{a_{n-2}b_{n-1}. \cdot \left(1 - \frac{1}{(a_{n-2} + b_{n-1})^2}\right)}$$

$$b_n = \frac{a_{n-2}a_{n-1}.}{a_{n-2} + b_{n-1}}$$

The major difficulty of this problem is that the iterated formulas are not linear. So we cannot use the vector method and complexity of the nonlinear iterated formulas is too large.

Acknowledgement

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Reference

[1] Fractal in Wikepedia, https://en.wikipedia.org/wiki/Fractal.

[2] Chang Gengzhe, P. J. Davis, Iterative Processes in Elementary Geometry, JOURNAL OF CHANGSHA UNIVERSITY, 1998, 12(4),19-28.

[3] Center of mass in Wikipedia, https://en.wikipedia.org/wiki/Center_of_mass#Center_of_gravity.