On Concurrent Lines Related to Miquel Points

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Abstract

Starting from an AIME problem, the Miquel circles created by two Miquel Points are studied in this paper, and then we come to a conclusion that no matter how the figure changes, three radical axes of the two corresponding Miquel circles are always concurrent. We explore from the shallower to the deeper, and establish our proof from the specialized cases to the general cases, using the properties of radical axes and Miquel Points as well as complex numbers. Furthermore, we study the cases with three sets of Miquel Points and Miquel circles and discover three collinear points and handle it with the help of computer. We also generalize the theorem into high-dimensional cases, finding out that the corresponding theorem is still true in some cases. In addition, still in other cases the theorem isn't true, but with some restrictions some beautiful properties can be derived.

Key words: Plane geometry, Miquel Point, Radical axis, Concurrent lines

Catalogue

1 Introduction

We find a problem in 2010 American Invitational Mathematics Examination (AIME). As is shown in **Figure 1**, in ΔABC, $AC=13$, $BC=14$, and $AB=15$. Points M and D lie on $\overline{AC}$ with $AM=MC$, $\angle ABD=\angle BDC$. Points N and E lie on $\overline{AB}$ with $AN=NB$, $\angle ACE=\angle ECB$. Let P be the point, other than A, of intersection of the circumcircles of ΔAMN and ΔADE. Ray AP meets BC at Q. The ratio BQ/CQ can be written in the form m/n, where m and n are relatively prime positive integers. Find $m-n$. [1]This is Problem 15 of 2010 AIME II,

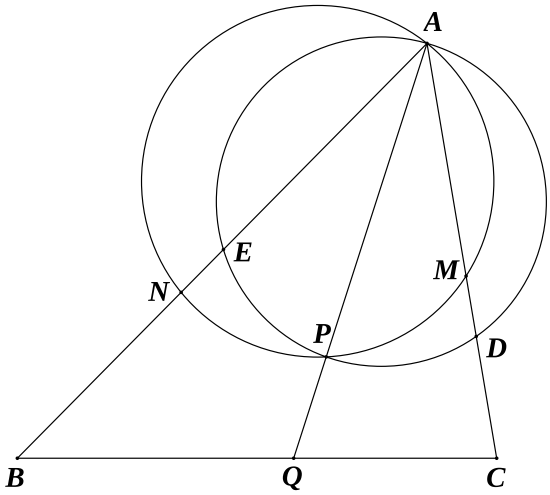

Figure 1 2010 AIME Problem 15

which is not easy to solve. According to the answer provided by the MAA, the similarity of triangles and area-method can be used to solve this problem, which are, however, too complicated to use.

After doing some further researches over this problem, we found the proportion formula has kind of symmetry. Therefore, we made a guess and used Sketchpad to delve. We discovered that AQ together with other two lines initiating from vertices B,C have some fantastic properties. Then we wrote this article to explore the problem.

2 Notations and Some Properties

In this section, we introduce some notations that we use in this article.
The following definition are all with respect to Euclidean Plane $\mathbb{R}^2$. Although using Projective Plane $\mathbb{P}^2$ sometimes brings convenience when talking about concurrent lines, the circumcircle, which plays a main role in the following theorems, cannot be well defined in Projective Plane. Therefore, we decide to base our theorem on the Euclidean Plane.

We first introduce some notations and definitions in plane geometry.

Notation 1[2] Let A and B be two different points on the plane, then we use $\overline{AB}$ to represent the segment starting from one point another, and we use AB to denote the line passing A and B. When we use $\overline{AB}$ to represent the distance from A to B, we consider it directed, which means after choosing a positive direction, the distance goes with the direction measures positive and that goes against the direction measures negative. Therefore, $\overline{AB} = -\overline{BA}$.

Notation 2[2] Let A, B, C be three different points on the plane, then we use $\angle ABC$ to represent the directed angles, which means angles measured in the counter-clockwise direction is positive, and angles measured in the clockwise direction is negative.

Definition 1[3] On a plane, the **power** of a point P with respect to a circle ω of center O and radius r is defined by $\rho(P) = \left|\overline{OP}\right|^2 - r^2$.

Property 1 (Power of a Point Theorem) Given a circle ω and a point P, draw a line l through P and intersect ω at two points A, B, and then on the power of P with respect to ω we have

$$\rho(P) = \overline{PA} \cdot \overline{PB}.$$

Definition 2[3] The locus of a point having equal power with regard to two given non-concentric circles is called the **radical axes** of these two circles.

Property 2[3] Radical axis of two circles is always a certain line perpendicular to their line of centers. In particular, if the circles intersect, the radical axis is the line through their points of intersections. If the circles are tangent, it is the common tangent of two circles.

In order to complete the proof of our theorem, we also introduce some notations on vectors and complex numbers here. In the following part of the article, we use boldface letter (such as $\mathbf{a}$) to represent a vector.

Notation 3 Let z be a complex number, then we denote $\operatorname{Re}\{z\} = \operatorname{Re} z$ as the real part of z and $\operatorname{Im}\{z\} = \operatorname{Im} z$ as the imaginary part of z. $|z|$ is used to represent the length of z and $\arg z$ is used to represent the principle value of the argument of z. But because this article mainly deal with points and lines, i doesn't stand for imaginary units but index without special announcement.

Then we introduce some knowledge from analytic geometry and linear algebra, which is used in the **Section 7**.

Notation 4 Let $\mathbf{A}$ be a matrix, then we denote the transpose of $\mathbf{A}$ as d $\mathbf{A}'$.

Notation 5 Let d be a positive integer representing the number of dimensions. Let the point of Euclidean space $\mathbb{R}^d$ be represented by column vectors $\mathbf{x} = (x_1, \cdots, x_d)'$ having the Euclidean norm $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$, where the scalar product (also known as inner product) is defined by

$$\langle \mathbf{y}, \mathbf{z} \rangle = \mathbf{y}'\mathbf{z} = y_1 z_1 + \cdots + y_d z_d$$

for $\mathbf{y} = (y_1, \cdots, y_d)'$ and $\mathbf{z} = (z_1, \cdots, z_d)'$.

Notation 6 Let $\mathbf{A}$ be a square matrix, then we use $\det \mathbf{A}$ to represent its determinant and use $\operatorname{rank}(\mathbf{A})$ to represent its rank. We use $\mathbf{A}^*$ to represent cofactor matrix of $\mathbf{A}$.

Definition 3[4] Let $\mathbf{A}$ be a square matrix, and $\mathbf{A}$ is called **skew-symmetric** iff

$$\mathbf{A}' = -\mathbf{A}.$$

In cases of high dimensions beyond 2, we have a similar definition about the power of a sphere and the radical (hyper)plane.

Definition 4 The **power** of a point $\mathbf{x}_1$ with respect to a $(d-1)$-sphere $S = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x} - \mathbf{x}_0\| = r\}$ of radius r and center $\mathbf{x}_0$ is defined by

$$\rho(P) = \|\mathbf{x}_1 - \mathbf{x}_0\|^2 - r^2.$$

Definition 5 The locus of a point having equal power to two given non-concentric $(d-1)$-spheres is called the **radical (hyper)plane** of these two spheres.

Property 3 Similarly, radical (hyper)plane of two spheres is always a certain (hyper)plane perpendicular to their line of centers, which can be proved by both analytic geometric method as well as some other methods.

Then we introduce the notation of the sign of permutation.

Notation 7 Given n as a positive integer, let σ be a permutation of $(1, 2, \cdots, n)$, we denote $\operatorname{sgn}(\sigma)$ be the sign of permutation.

3 Lemmas

Some basic theorem in circles and triangles are used in this article.

Lemma 1 (Law of Sines) Let a,b,c be the length of the opposite side of vertices A,B,C in ΔABC, and then

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R,$$

where R is the radius of the circumcircle of ΔABC.

Lemma 2 (Theorem of Euler Line) Let G,O,H be the centroid, the circumcenter and the orthocenter of ΔABC, respectively, and then G,O,H is on the same line called **Euler Line** with respect to ΔABC.

The trigonometric form of Ceva's Theorem and its converse theorem are employed in this paper. For it is well known, we don't provide its proof.

Lemma 3[5] (Ceva's Theorem) If P_1,P_2,P_3 are chosen on the lines of sides of $\Delta A_1A_2A_3$, then line A_1P_1,A_2P_2,A_3P_3 are concurrent iff

$$\frac{\sin\angle P_1A_1A_2}{\sin\angle P_1A_1A_3}\cdot\frac{\sin\angle P_2A_2A_3}{\sin\angle P_2A_2A_1}\cdot\frac{\sin\angle P_3A_3A_1}{\sin\angle P_3A_3A_2}=-1.$$

Miquel's Theorem is another important theorem related to the problem that we study, and we have some corollaries about this theorem.

Lemma 4[6] (Miquel's Theorem) As is shown in **Figure 2**, given an arbitrary triangle $\Delta A_1A_2A_3$, P_1,P_2,P_3 are on sides $\overline{A_2A_3},\overline{A_3A_1},\overline{A_1A_2}$ respectively, then the circumcenters of $\Delta A_1P_2P_3,\Delta A_2P_3P_1,\Delta A_3P_1P_2$ will meet at a point P, which is called the Miquel Point for the triad $P_1P_2P_3$ with respect to $\Delta A_1A_2A_3$.**Proof** Let the point P

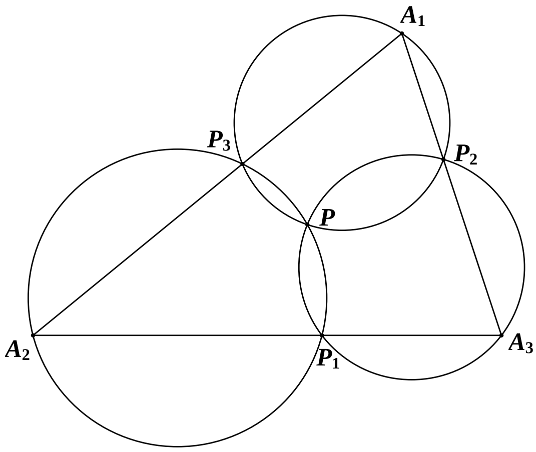

Figure 2 Miquel's Theorem

be of intersection of two of the circles $\Delta A_1P_2P_3$ and $\Delta A_2P_3P_1$, which lie in the triangle, distinct to P_3. Then at once $\angle A_3P_1P=\angle A_2P_3P=\angle A_1P_2P$, which shows that

P, P_1, P_2, A_3 are concyclic. □

Remark Although the original **Lemma 1** is about P_1, P_2, P_3 which are on the sides of $\Delta A_1A_2A_3$, but the circumcircles still meet at the same point as long as P_1, P_2, P_3 are on the lines of the sides, even if P is out of the triangle. Thus, the corollary is based on A_2A_3, A_3A_1, A_1A_2 instead of $\overline{A_2A_3}, \overline{A_3A_1}, \overline{A_1A_2}$.

Corollary 1 As is shown in **Figure 2**, given a triangle $\Delta A_1A_2A_3$, two points P_1, P_2 lie on A_3A_1, A_2A_3 respectively. Let P_3 be a point on the plane. If circumcircles of $\Delta A_1P_2P_3, \Delta A_2P_3P_1, \Delta A_3P_1P_2$ meet at the same point P distinct to P_3, then P_3 lies on A_1A_2. That is, P is exactly the Miquel Point of the triad $P_1P_2P_3$ with respect to $\Delta A_1A_2A_3$.

Proof Obviously $\angle A_3P_1P = \angle A_2P_3P$, $\angle PP_2A_3 = \angle PP_3A_1$. Note that $\angle A_3P_1P + \angle PP_2A_3 = \pi$, and we obtain $\angle A_2P_3P + \angle PP_3A_1 = \pi$, which means A_1, P_3, A_2 are on the same line. As is shown in **Lemma 4**, P is the Miquel Point of $P_1P_2P_3$ with respect to $\Delta A_1A_2A_3$. □

Corollary 2 As is shown in **Figure 3**, given an arbitrary triangle $\Delta A_1A_2A_3$, P_1, P_2, P_3 are on A_2A_3, A_3A_1, A_1A_2 respectively, with P being the Miquel Point for the traid $P_1P_2P_3$ with respect to $\Delta A_1A_2A_3$. If P_1, P_2, P_3 are the projection of vertices A_1, A_2, A_3 on the edges A_2A_3, A_3A_1, A_1A_2 respectively, then P is the orthocenter of $\Delta A_1A_2A_3$.

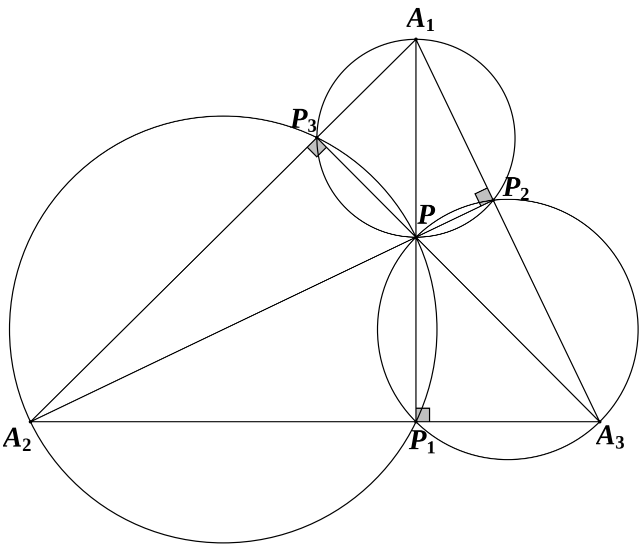

Figure 3 P is the Miquel Point and the orthocenter as well

Proof As is shown in **Figure 3**, from the characteristic of triangle, we derive that A_1, P_2, P_3, H are concyclic, where H is orthocenter of $\Delta A_1A_2A_3$. Then we know that P must be on the circumcircle of the quadrilateral $A_1P_2P_3H$. Similarly, P is on the circumcircle of the quadrilateral $A_2P_3P_1H$ and $A_3P_1P_2H$. Thus, we learn that P is the same point as point H, the orthocenter of $\Delta A_1A_2A_3$. □

Corollary 3 As is shown in **Figure 4**, given an arbitrary triangle $\Delta A_1A_2A_3$, P_1, P_2, P_3 are on A_2A_3, A_3A_1, A_1A_2 and P is the Miquel Point for the traid $P_1P_2P_3$ with respect to $\Delta A_1A_2A_3$. If P_1, P_2, P_3 are midpoints of the edges $\overline{A_2A_3}, \overline{A_3A_1}, \overline{A_1A_2}$ respectively, then P is the circumcenter of $\Delta A_1A_2A_3$.

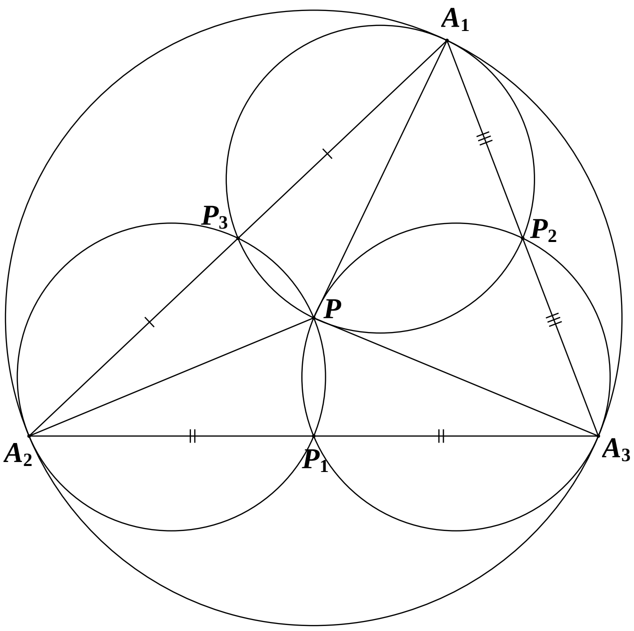

Figure 4 P is the Miquel Point and the circumcenter as well

Proof. From characteristics of triangle, A_1, P_2, P_3, O lie on the same circle, where O is the circumcenter of $\Delta A_1A_2A_3$. And then P must be on the circumcircles of the quadrilaterals $A_1P_2P_3O$, $A_2P_3P_1O$ and $A_3P_1P_2O$. Thus, we obtain that P is the circumcenter of $\Delta A_1A_2A_3$. □

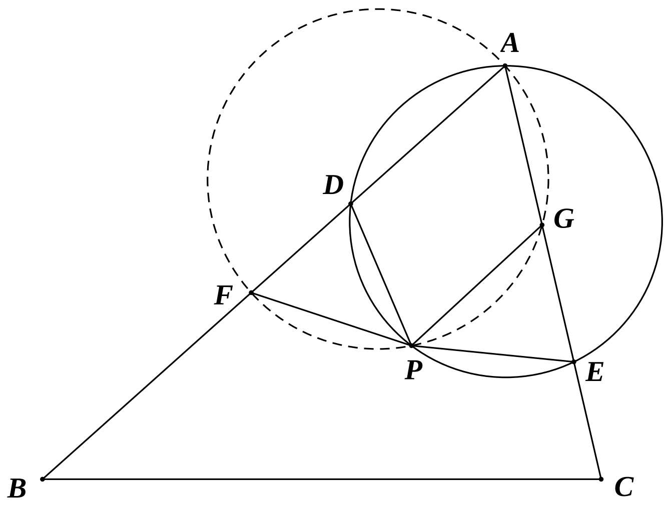

Figure 5 Two similar triangles

Lemma 5 As is shown in **Figure 5**, given ΔABC, points D and F lie on $\overline{AB}$

and other two points G and E lie on $\overline{AC}$. Assume circumcircle of ΔAFG and ΔADE meet at point P, which is distinct to A, then $\Delta PDF \sim \Delta PEG$.

Proof Because of concyclic, $\angle PGE = \angle PFD$, $\angle FDP = \angle GEP$. Then we obtain that $\Delta PDF \sim \Delta PEG$. □

Then we introduce a series of theorems in the analytical geometry and linear algebra. Let d be a positive integer for the number of dimensions in the following texts.

Lemma 6 A $(d-1)$-sphere S passing the points $\mathbf{x}_0, \cdots, \mathbf{x}_d \in \mathbb{R}^d$, which are not on a (hyper)plane, then S have the equation

$$\begin{vmatrix} \|\mathbf{x}_0\|^2 & \mathbf{x}_0' & 1 \\ \vdots & \vdots & \vdots \\ \|\mathbf{x}_d\|^2 & \mathbf{x}_d' & 1 \\ \|\mathbf{x}\|^2 & \mathbf{x}' & 1 \end{vmatrix} = 0.$$

Lemma 7 Given a $(d-1)$-sphere S with the equation

$$f(\mathbf{x}) = \|\mathbf{x}\|^2 + \sum_{k=1}^{d} B_k x_k + C = 0,$$

where $B_1, B_2, \cdots, B_d, C$ are all reals. For a point $\mathbf{x}_1 \in \mathbb{R}^d$, the power of $\mathbf{x}_1$ with respect to S is equal to $f(\mathbf{x}_1)$.

Lemma 8 Given two $(d-1)$-spheres S, T with equations

$$f(\mathbf{x}) = \|\mathbf{x}\|^2 + \sum_{k=1}^{d} B_k x_k + C = 0, g(\mathbf{x}) = \|\mathbf{x}\|^2 + \sum_{k=1}^{d} B_k' x_k + C' = 0,$$

respectively, where $B_1, B_2, \cdots, B_d, C, B_1', B_2', \cdots, B_d', C'$ are all reals, then the radical (hyper)plane of S, T has the equation

$$f(\mathbf{x}) - g(\mathbf{x}) = \sum_{k=1}^{d} (B_k - B_k') x_k + (C - C') = 0.$$

And that radical (hyper)plane is a (hyper)plane can be directly derived.

Lemma 9 Given $d+1$ (hyper)planes $u_0, u_1, \cdots, u_d$ and for $j \in \{0, \cdots, d\}$ u_j has the equation that

$$f_j(\mathbf{x}) = \sum_{k=1}^{d} A_{jk} x_k + A_{j(d+1)} = 0,$$

where $A_{j1}, \cdots, A_{jd}, A_{j(d+1)}$ are all reals, and then $u_0, u_1, \cdots, u_d$ are concurrent (or parallel) iff

$$\begin{vmatrix} A_{01} & A_{02} & \cdots & A_{0(d+1)} \\ A_{11} & A_{12} & \cdots & A_{1(d+1)} \\ \vdots & \vdots & \ddots & \vdots \\ A_{d1} & A_{d2} & \cdots & A_{d(d+1)} \end{vmatrix} = 0.$$

Lemma 10 Given $d+1$ (hyper)planes $u_0, u_1, \cdots, u_d$ and for $j \in \{0, \cdots, d\}$ u_j has the equation that

$$f_j(\mathbf{x}) = \sum_{k=1}^{d} A_{jk} x_k + A_{j(d+1)} = 0,$$

where $A_{j1}, \cdots, A_{jd}, A_{j(d+1)}$ are all reals, then $u_0, u_1, \cdots, u_d$ meet at one line (or parallel) iff

$$\operatorname{rank}\begin{pmatrix} A_{01} & A_{02} & \cdots & A_{0(d+1)} \\ A_{11} & A_{12} & \cdots & A_{1(d+1)} \\ \vdots & \vdots & \ddots & \vdots \\ A_{d1} & A_{d2} & \cdots & A_{d(d+1)} \end{pmatrix} \le d-1.$$

Lemma 11 and **Lemma 12** are about some properties of skew-symmetric determinants, which we introduce in order to handle some skew-symmetric matrix in the **Section 7**.

Lemma 11[3] Let $\mathbf{A}$ be a $n \times n$ skew-symmetric matrix that

$$\mathbf{A} = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1(n-1)} & a_{1n} \\ -a_{12} & 0 & \cdots & a_{2(n-2)} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{1(n-1)} & -a_{2(n-1)} & \cdots & 0 & a_{(n-1)n} \\ -a_{1n} & -a_{2n} & \cdots & -a_{(n-1)n} & 0 \end{pmatrix},$$

then there exists a polynomial of a_{ij}, $1 \le i < j \le n, i, j \in \{1, 2, \cdots, n\}$, called the Pfaffian of $\mathbf{A}$ or $\operatorname{pf} \mathbf{A}$, such that

$$\det \mathbf{A} = (\operatorname{pf} \mathbf{A})^2.$$

And if $2 \mid n$, then $\operatorname{pf} \mathbf{A}$ can be explicitly represented as

$$\operatorname{pf} \mathbf{A} = \frac{1}{2^{\frac{n}{2}} \left(\frac{n}{2}\right)!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{\frac{n}{2}} a_{\sigma(2i-1)\sigma(2i)},$$

where S_n is a set including all the permutations of $(1, 2, \cdots, n)$.

Lemma 12[3] Let $\mathbf{A}$ be a skew-symmetric matrix, then we have

$$2 \mid \operatorname{rank}(\mathbf{A}).$$

4 Specialized Cases

We use the same method which is used to handle the problem in introduction to make other four circles produced by other two vertices. To sum up, we have six circles now. In this part, we take two of these three triads, including midpoints of sides, intersections of angular bisectors and the opposite sides, vertices' projection on the opposite sides, to explore. We surprisingly find some astonishing properties, but we have no idea how to prove it for the very first time. These specialized cases enlighten us to prove the general cases, and are presented below without proof.

Case 1 As is shown in **Figure 6**, given ΔABC, D, E, F are the midpoints of $\overline{BC}, \overline{CA}, \overline{AB}$ respectively and G, H, I are the intersections of angular bisectors and BC, CA, AB respectively. Let the radical axis of the circumcircles of ΔAIH and ΔAEF be l_1, the radical axis of the circumcircles of ΔBDF and ΔBGI be l_2 and the radical axis of the circumcircles of ΔCDE and ΔCGH be l_3, and in this case l_1, l_2, l_3 are concurrent (or parallel).

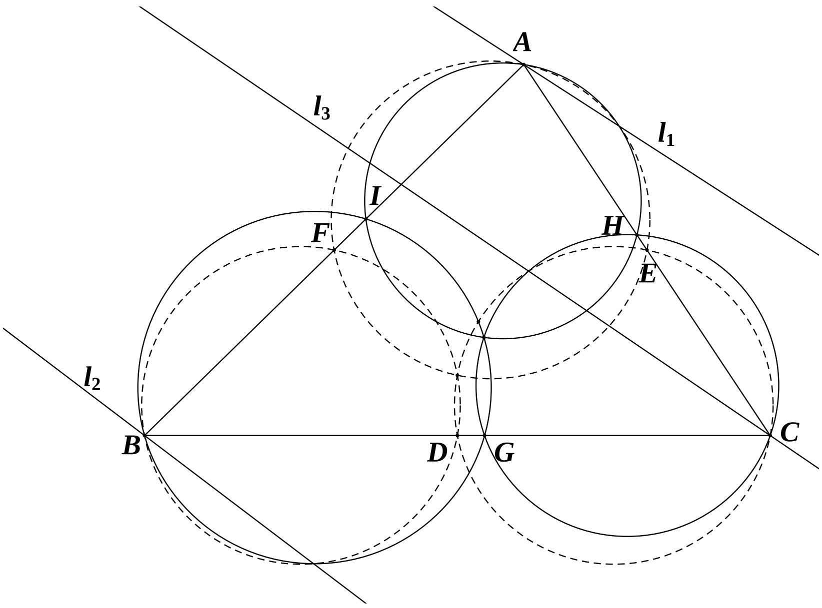

Figure 6 The figure of **Case 1**

(Note: The three radical axes indeed meet at a point, which is too far to be shown.)

Case 2 As is shown in **Figure 7**, given ΔABC, D, E, F are the intersections of angular bisectors and BC, CA, AB respectively and G, H, I are the projections of vertices A, B, C onto BC, CA, AB respectively. Let the radical axis of the circumcircles of ΔAIH and ΔAEF be l_1, the radical axis of the circumcircles of ΔBDF and ΔBGI be l_2 and the radical axis of the circumcircles ΔCDE and ΔCGH be l_3, and in this case l_1, l_2, l_3 are concurrent (or parallel).

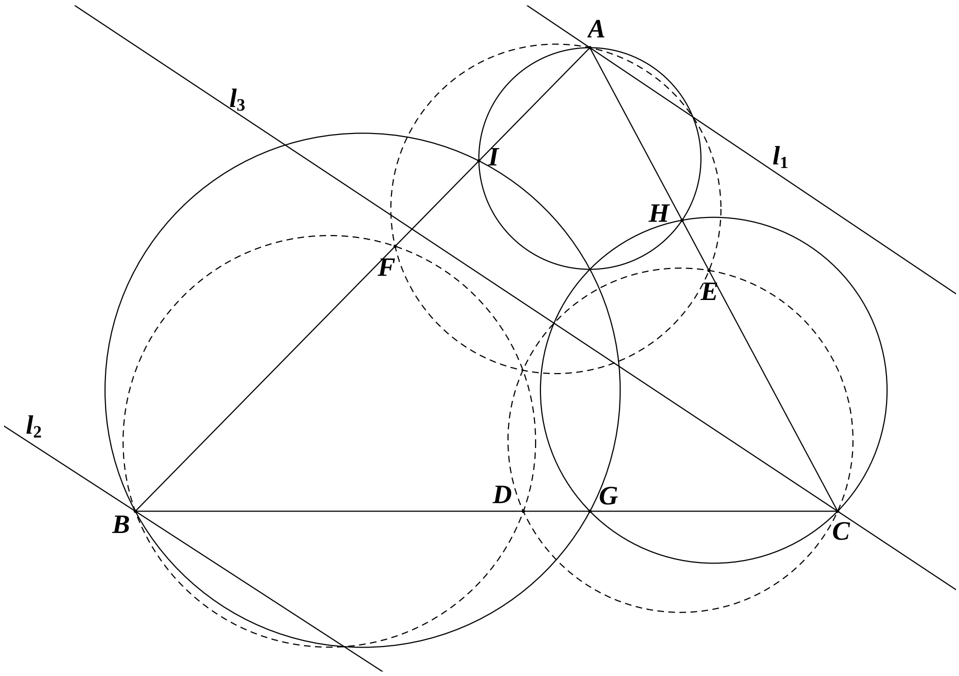

Figure 7 The figure of **Case 2**

(Note: The three radical axes indeed meet at a point, which is too far to be shown.)

Case 3 As is shown in **Figure 8**, given ΔABC, D, E, F are the midpoints of $\overline{BC}, \overline{CA}, \overline{AB}$ respectively and G, H, I, are the projections of vertices A, B, C onto BC, CA, AB respectively. Let the radical axis of the circumcircles of ΔAIH and ΔAEF be l_1, the radical axis of the circumcircles of ΔBDF and ΔBGI be l_2 and the radical axis of the circumcircles of ΔCDE and ΔCGH be l_3, and in this case

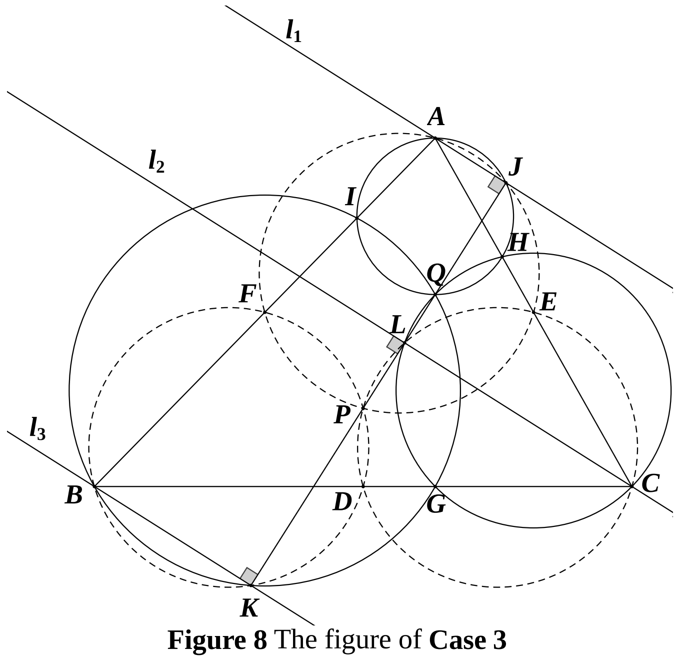

Figure 8 The figure of **Case 3**

(Note: The three radical axes are indeed parallel.)

l_1, l_2, l_3 are parallel.

However, we find **Case 3** is not very complicated, so we manage to prove it.

Proof We denote the point J as the intersection of the circumcircles of ΔAIH and ΔAEF different from A, the point K as the intersection of the circumcircles of ΔBDF and ΔBGI different from B and the point L as the intersection of the circumcircles of ΔCDE and ΔCGH different from C.

From **Corollary 2**, we know that Q is the orthocenter of ΔABC. And from **Corollary 3**, we know that P is the circumcenter of ΔABC. Therefore according to **Lemma 2**, PQ is the Euler line of ΔABC and $AI \perp IQ, AF \perp FP$. Then considering the circles, we have $AJ \perp JQ, AJ \perp JP$, and therefore we obtain that J, Q, P are collinear.

In a similar way, we can prove that K, P, Q and P, L, Q are also collinear. Therefore, P, Q, J, K, L are on the same line, so PQ is perpendicular to l_1, l_2, l_3, and that the l_1, l_2, l_3 are parallel can be derived. □

Therefore, as a byproduct, this theorem comes out.

Theorem 1 In **Case 3**, we denote the point J as the intersection of the circumcircles of ΔAIH and ΔAEF different from A, the point K as the intersection of the circumcircles of ΔBDF and ΔBGI different from B and the point L as the intersection of the circumcircles of ΔCDE and ΔCGH different from C. Let P, Q respectively be the Miquel Point for triads DEF, GHI with respect to ΔABC, and then P, Q, J, K, L are on the same line, which is the Euler line of ΔABC, perpendicular to l_1, l_2, l_3.

5 General Cases

After working on special cases, we then analyze the general ones. We set two points on each sides, and use the same way used in **Section 4** to construct circles and radical axes. We surprisingly find that even though the points are not specialized, the three constructed radical axes are still concurrent (or parallel). Thus, we make a conjecture and finally prove the theorem below.

Theorem 2 As is shown in **Figure 9**, given arbitrary $\Delta A_1A_2A_3$, points P_1,P_2,P_3 are on A_2A_3,A_3A_1,A_1A_2 respectively and Q_1,Q_2,Q_3 are also on A_2A_3,A_3A_1,A_1A_2 respectively. Let l_k be the radical axis of the circumcircle of $\Delta A_kP_{k+1}P_{k+2}$ and that of $\Delta A_kQ_{k+1}Q_{k+2}$ for $k\in\{1,2,3\}$ (for notation convenience, we regard $P_4=P_1,P_5=P_2,Q_4=Q_1,Q_5=Q_2$), and then l_1,l_2,l_3 are concurrent (or parallel).

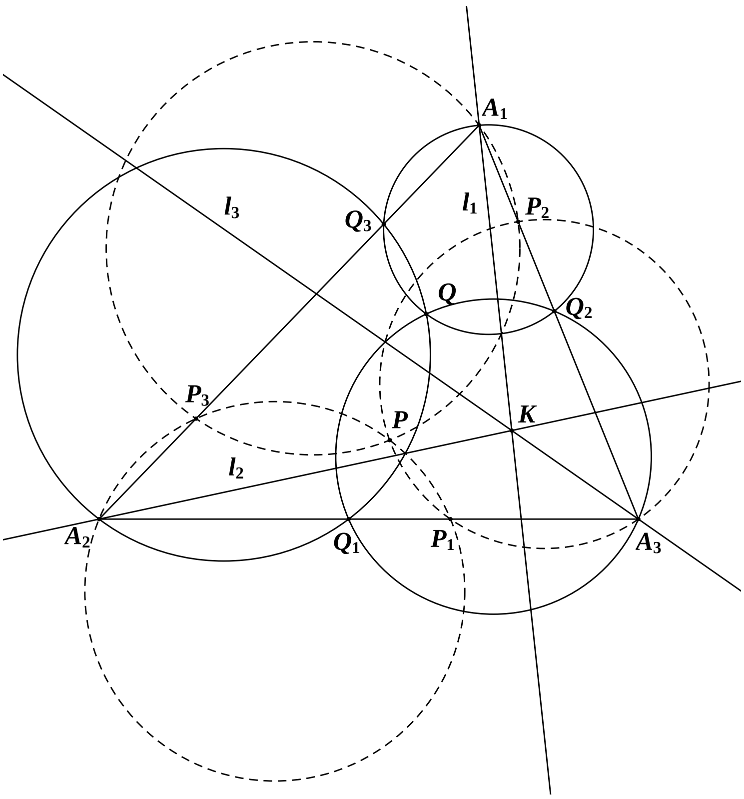

Figure 9 Two sets of Miquel Points and Miquel circles

We used complex number method to verify its validity for the very first.

Proof 1 See **Figure 10**. Here we consider i as the imaginary unit.
According to **Lemma 4**, we make a Miquel Point P of traid $P_1P_2P_3$ with respect to $\Delta A_1A_2A_3$ and another Miquel Point Q of traid $Q_1Q_2Q_3$ with respect to $\Delta A_1A_2A_3$.
Suppose the circumcenter of $\Delta A_kP_{k+1}P_{k+2}$ is S_k and the circumcenter of $\Delta A_kQ_{k+1}Q_{k+2}$ is T_k for $k\in\{1,2,3\}$. Because A_1,P_2,P_3,P are concyclic, we can obtain that $\angle PP_3A_1=\angle PP_1A_2=\angle PP_2A_3$, therefore we can get $\angle PS_1A_1=\angle PS_2A_2=\angle PS_3A_3$.

Using the same method, we can know that $\angle QT_1A_1 = \angle QT_2A_2 = \angle QT_3A_3$.

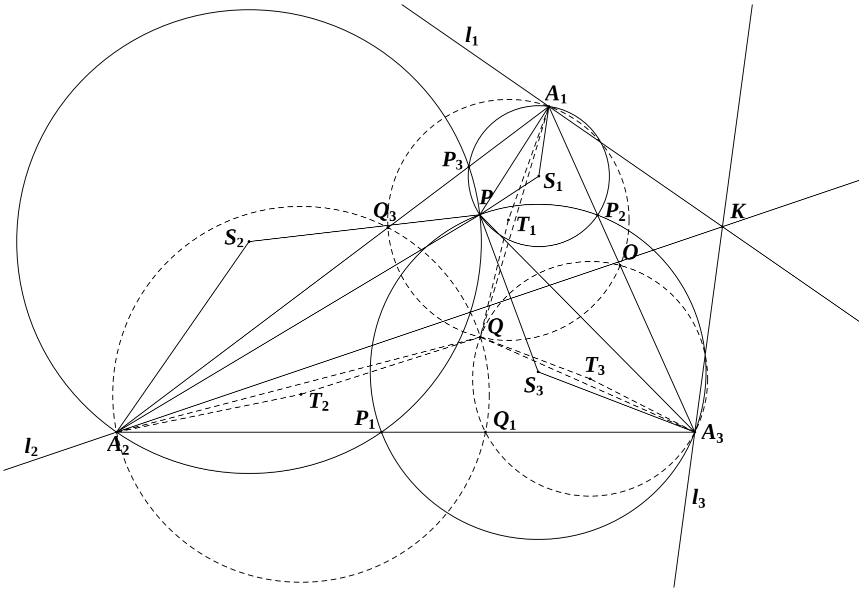

Figure 10 Proof of complex number method

Use one certain point as origin, a certain unit length and a certain direction of axes to set up the complex plane. We use the letters of different points to express the corresponding complex numbers.

According to **Property 2**, we know that $l_k \perp S_kT_k$ for $k \in \{1,2,3\}$.

In order to prove that l_1, l_2, l_3 are concurrent, we apply **Lemma 3** here. Owing to the perpendicularity, it is sufficient to prove

$$\cos\arg\frac{S_1-T_1}{A_1-A_2}\cdot\cos\arg\frac{S_2-T_2}{A_2-A_3}\cdot\cos\arg\frac{S_3-T_3}{A_3-A_1}$$

$$=-\cos\arg\frac{S_1-T_1}{A_1-A_3}\cdot\cos\arg\frac{S_2-T_2}{A_2-A_1}\cdot\cos\arg\frac{S_3-T_3}{A_3-A_2}$$

$$\Leftrightarrow \prod_{k=1}^{3}\frac{\mathrm{Re}\left\{(S_k-T_k)/(A_k-A_{k+1})\right\}}{|S_k-T_k|/|A_k-A_{k+1}|} = -\prod_{k=1}^{3}\frac{\mathrm{Re}\left\{(S_k-T_k)/(A_k-A_{k+2})\right\}}{|S_k-T_k|/|A_k-A_{k+2}|}$$

$$\Leftrightarrow \prod_{k=1}^{3}\mathrm{Re}\left\{\frac{S_k-T_k}{A_k-A_{k+1}}\right\} = -\prod_{k=1}^{3}\mathrm{Re}\left\{\frac{S_k-T_k}{A_k-A_{k+2}}\right\}$$

$$\Leftrightarrow \prod_{k=1}^{3}\mathrm{Re}\left\{(S_k-T_k)\left(\overline{A_k-A_{k+1}}\right)\right\} = -\prod_{k=1}^{3}\mathrm{Re}\left\{(S_k-T_k)\left(\overline{A_k-A_{k+2}}\right)\right\}.$$

On complex plane, we assume that

$$\frac{\angle PS_1A_1}{2}=\theta,\quad e^{i\theta}=\zeta \quad (\theta\in[0,2\pi)\text{ is a directed angle}),$$

$$\frac{\angle QT_1A_1}{2}=\varphi,\quad e^{i\varphi}=\xi \quad (\varphi\in[0,2\pi)\text{ is a directed angle}),$$

$$A_k = x_k + iy_k \quad (x_k, y_k\in\mathbb{R},\ k\in\{1,2,3\}),$$

$$P = x_P + iy_P,\ Q = x_Q + iy_Q \quad (x_P, y_P, x_Q, y_Q\in\mathbb{R}).$$

According to definition of ζ, and note that $\angle PS_1A_1 = \angle PS_2A_2 = \angle PS_3A_3$, we can obtain that for $k \in \{1,2,3\}$,

$$(A_k - S_k)\zeta = (P - S_k)\zeta^{-1}.$$

Then, we have for $k \in \{1,2,3\}$,

$$S_k = \frac{A_k\zeta - P\zeta^{-1}}{\zeta - \zeta^{-1}},$$

from which we can derive that for $k \in \{1,2,3\}$,

$$2S_k = \left((x_k + x_P) + (y_k - y_P)\cot\theta\right) + \left((x_P + x_k)\cot\theta + (y_k + y_P)\right)i.$$

Similarly, with symmetry, we can write that for $k \in \{1,2,3\}$,

$$2T_k = \left((x_k + x_Q) + (y_k - y_Q)\cot\varphi\right) + \left((x_Q + x_k)\cot\varphi + (y_k + y_Q)\right)i.$$

Thus, for $k \in \{1,2,3\}$,

$$\begin{aligned}&\operatorname{Re}\left\{(2S_k - 2T_k)\left(\overline{A_k - A_{k+1}}\right)\right\}\\ &= (x_k - x_{k+1})(x_P - x_Q) + (y_k - y_{k+1})(y_P - y_Q)\\ &+\cot\theta\left(y_P(x_{k+1} - x_k) + x_P(y_k - y_{k+1}) + x_k y_{k+1} - y_k x_{k+1}\right)\\ &-\cot\varphi\left(y_Q(x_{k+1} - x_k) + x_Q(y_k - y_{k+1}) + x_k y_{k+1} - y_k x_{k+1}\right).\end{aligned}$$

Similarly, we get that for $k \in \{1,2,3\}$,

$$\begin{aligned}&\operatorname{Re}\left\{(2S_{k+1} - 2T_{k+1})\left(\overline{A_{k+1} - A_k}\right)\right\}\\ &= (x_{k+1} - x_k)(x_P - x_Q) + (y_{k+1} - y_k)(y_P - y_Q)\\ &+\cot\theta\left(y_P(x_k - x_{k+1}) + x_P(y_{k+1} - y_k) + x_{k+1} y_k - y_{k+1} x_k\right)\\ &-\cot\varphi\left(y_Q(x_k - x_{k+1}) + x_Q(y_{k+1} - y_k) + x_{k+1} y_k - y_{k+1} x_k\right).\end{aligned}$$

Therefore, we can get that for $k \in \{1,2,3\}$,

$$\operatorname{Re}\left\{(2S_k - 2T_k)\left(\overline{A_k - A_{k+1}}\right)\right\} = -\operatorname{Re}\left\{(2S_{k+1} - 2T_{k+1})\left(\overline{A_{k+1} - A_k}\right)\right\},$$

that is for $k \in \{1,2,3\}$

$$\operatorname{Re}\left\{(S_k - T_k)\left(\overline{A_k - A_{k+1}}\right)\right\} = -\operatorname{Re}\left\{(S_{k+1} - T_{k+1})\left(\overline{A_{k+1} - A_k}\right)\right\}.$$

Thus, considering the symmetry, we obtain

$$\begin{aligned}\prod_{k=1}^{3}\operatorname{Re}\left\{(S_k - T_k)\left(\overline{A_k - A_{k+1}}\right)\right\} &= -\prod_{k=1}^{3}\operatorname{Re}\left\{(S_{k+1} - T_{k+1})\left(\overline{A_{k+1} - A_k}\right)\right\}\\ &= -\prod_{k=1}^{3}\operatorname{Re}\left\{(S_k - T_k)\left(\overline{A_k - A_{k+2}}\right)\right\}.\end{aligned}$$

According to equivalency, we get that l_1, l_2, l_3 are concurrent (or parallel). □

We are also enlightened by the question in introduction, which resulted in our further exploration over this figure. Finally, we find an easier pure geometric method to deal with this problem.

Proof 2 We only provide the proof when l_1, l_2, l_3 intersect inside the triangle. When

the three lines intersect outside the triangle, the proof is basically the same.
See **Figure 11**.

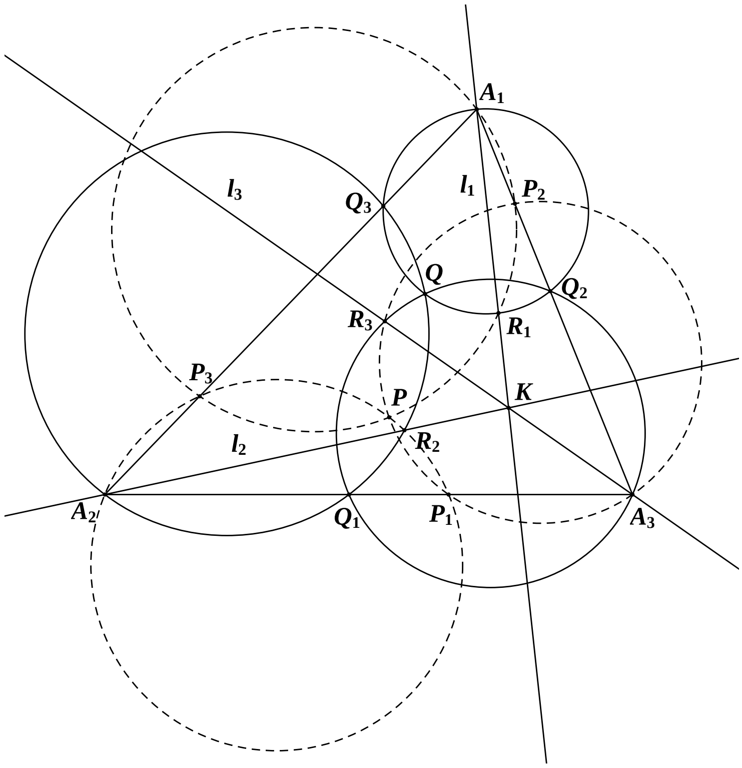

Figure 11 Proof of pure geometry method

According to **Lemma 4**, we assume the circumcircle of $\Delta A_1P_2P_3$ intersect the circumcircle of $\Delta A_1Q_2Q_3$ at point R_1, distinct to P.
Then we have

$$\Delta R_1Q_2P_2 \sim \Delta R_1Q_3P_3 .$$

So according to **Lemma 1**, we know that

$$\frac{\sin\angle Q_3A_1R_1}{\sin\angle Q_2A_1R_1} = -\frac{\left|\overline{R_1Q_3}\right|}{\left|\overline{R_1Q_2}\right|} = -\frac{\left|\overline{P_3Q_3}\right|}{\left|\overline{P_2Q_2}\right|} .$$

Similarly, we make an intersection R_2 of the circumcircles of $\Delta A_2P_3P_1$ and $\Delta A_2Q_3Q_1$, which is different from P, and then make another intersection R_3 of the circumcircles of $\Delta A_3P_1P_2$ and $\Delta A_3Q_1Q_2$, which is different from P.
Similarly, we can obtain two other identities, that is

$$\frac{\sin\angle Q_1A_2R_2}{\sin\angle Q_3A_2R_2} = -\frac{\left|\overline{R_2Q_1}\right|}{\left|\overline{R_2Q_3}\right|} = -\frac{\left|\overline{P_1Q_1}\right|}{\left|\overline{P_3Q_3}\right|} ,$$

and

$$\frac{\sin\angle Q_2A_3R_3}{\sin\angle Q_1A_3R_3} = -\frac{\left|\overline{R_3Q_2}\right|}{\left|\overline{R_3Q_1}\right|} = -\frac{\left|\overline{P_2Q_2}\right|}{\left|\overline{P_1Q_1}\right|} .$$

As a result,

$$\frac{\sin\angle Q_1A_2R_2}{\sin\angle Q_3A_2R_2}\cdot\frac{\sin\angle Q_3A_1R_1}{\sin\angle Q_2A_1R_1}\cdot\frac{\sin\angle Q_2A_3R_3}{\sin\angle Q_1A_3R_3}=\left(-\frac{\left|\overline{P_1Q_1}\right|}{\left|\overline{P_3Q_3}\right|}\right)\cdot\left(-\frac{\left|\overline{P_3Q_3}\right|}{\left|\overline{P_2Q_2}\right|}\right)\cdot\left(-\frac{\left|\overline{P_2Q_2}\right|}{\left|\overline{P_1Q_1}\right|}\right)=-1.$$

According to **Lemma 3**, l_1, l_2, l_3 are concurrent (or parallel). □

Based on the deduction above, we found that no matter whether we use complex number method, or use similar triangles, the ratio of sines or sides both have a mysterious symmetry. Therefore, we doubt whether these propositions are correct or not when we generalize the theorem to polygons. We make some further investigation, realizing that the theorem can be generalized actually. Eventually, we are able to prove the generalized theorem.

Theorem 3 As is shown in **Figure 12**, Given an arbitrary quadrilateral $A_1A_2A_3A_4$ (for notation convenience, we assume $A_{i+4}=A_i$, $i\in\mathbb{Z}$), and P_k, Q_k (for notation convenience, we assume $P_{i+4}=P_i$, $Q_{i+4}=Q_i$, $i\in\mathbb{Z}$) are points on A_kA_{k+1} respectively for $k\in\{1,2,3,4\}$ so that the circumcircles of $\Delta A_kP_kP_{k+3}$ are all concurrent at P and the circumcircles of $\Delta A_kQ_kQ_{k+3}$ are concurrent at Q, for $k\in\{1,2,3,4\}$. Let l_k be the radical axis of the circumcircles of $\Delta A_kP_kP_{k+3}$ and $\Delta A_kQ_kQ_{k+3}$ for $k\in\{1,2,3,4\}$, and then l_k are concurrent (or parallel) for $k\in\{1,2,3,4\}$.

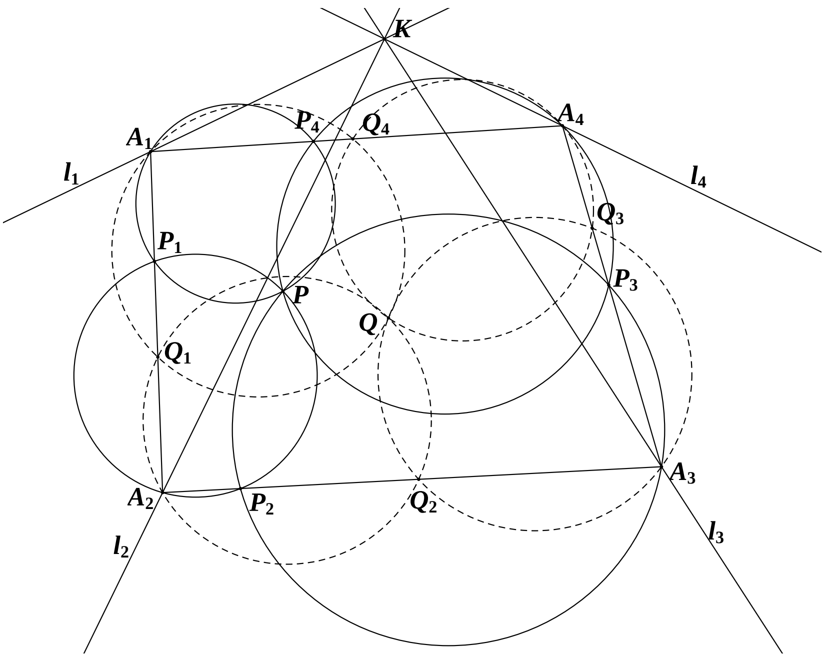

Figure 12 The figure of **Theorem 3**

Proof Make an intersection P' of the circumcircles $\Delta A_4P_4P_3$ and $\Delta A_2P_2P_1$, which is different from P.

According to **Lemma 4**, P is the Miquel Point of triad P_4P_1P' with respect to $\Delta A_4A_1A_2$.

Similarly, make an intersection Q' of circumcircle of $\Delta A_4Q_4Q_3$ and $\Delta A_2Q_2Q_1$, which is different from Q, then we obtain that Q is the Miquel Point of triad Q_4Q_1Q' with respect to $\Delta A_4A_1A_2$.

According to **Theorem 2**, with regard to $\Delta A_4A_1A_2$, l_4,l_1,l_2 are concurrent (or parallel).

In a similar way, we can obtain that, with respect to $\Delta A_1A_2A_3$, l_1,l_2,l_3 are concurrent (or parallel).

Therefore, we succeed in proving that l_1,l_2,l_3,l_4 are concurrent (or parallel). □

Theorem 4 Given an arbitrary n-polygon $A_1A_2\cdots A_n$ (for notation convenience, we assume $A_{n+i}=A_i$, $i\in\mathbb{Z}$), and P_k,Q_k (for notation convenience, we assume $P_{n+i}=P_i$, $Q_{n+i}=Q_i$, $i\in\mathbb{Z}$) are points on A_kA_{k+1} respectively for $k\in\{1,2,\cdots,n\}$ so that the circumcircles of $\Delta A_kP_kP_{k+n-1}$ are concurrent at P and the circumcircles of $\Delta A_kQ_kQ_{k+n-1}$ are concurrent at Q, for all $k\in\{1,2,\cdots,n\}$. Let l_k is the radical axis of circumcircles of $\Delta A_kP_kP_{k+n-1}$ and $\Delta A_kQ_kQ_{k+n-1}$ for $k\in\{1,2,\cdots,n\}$, and then l_k are all concurrent (or parallel) for $k\in\{1,2,\cdots,n\}$.

Proof With the exactly same method of proof to **Theorem 3**, we know P,Q are both Miquel Points of $\Delta A_kA_{k+1}A_{k+2}$ for $k\in\{1,2,\cdots,n\}$. According to **Theorem 2**, for any $\Delta A_kA_{k+1}A_{k+2}$, for $k\in\{1,2,\cdots,n\}$, and therefore for all $k\in\{1,2,\cdots,n\}$ we obtain l_k,l_{k+1},l_{k+2} are concurrent (or parallel).

Therefore, l_k are concurrent (or parallel) for all $k\in\{1,2,\cdots,n\}$. □

6 On More Sets of Miquel Points

According to the proof above, two sets of Miquel Points and Miquel circles have some special propoties. But how about more sets of Miquel Points and Miquel circles? We go further and discover a theorem as beautiful as **Theorem 2**.

Theorem 5 As is shown in **Figure 13**, given an arbitrary ΔABC, P,Q,R, are different Miquel Points for triads $P_1P_2P_3, Q_1Q_2Q_3, R_1R_2R_3$ with respect to $\Delta A_1A_2A_3$. According to **Theorem 2**, we assume that radical axes produced by Miquel circles of P and Q, Q and R, R and P meet at K_1, K_2, K_3 respectively, then K_1, K_2, K_3 are collinear.

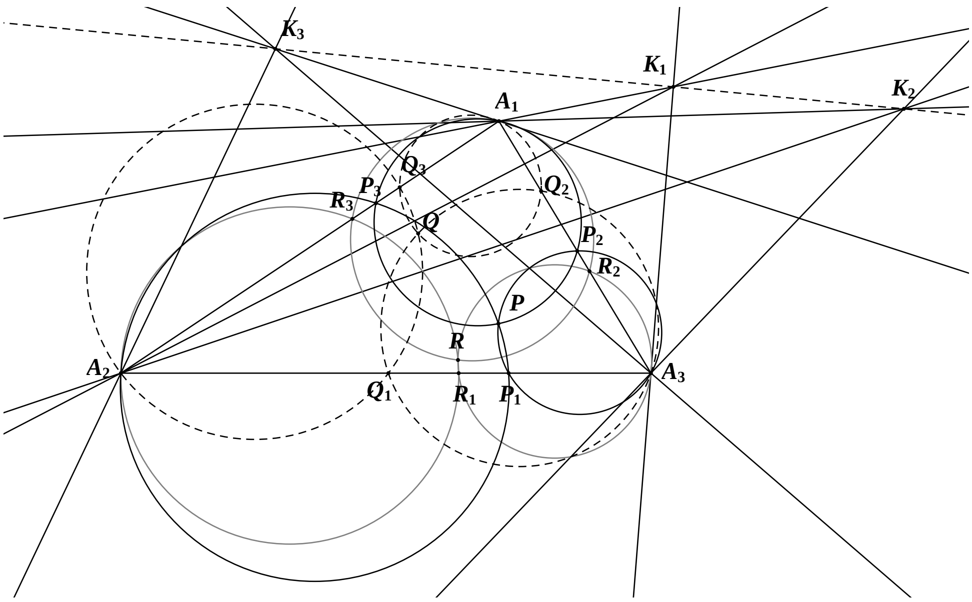

Figure 13 3 sets of Miquel Points and Miquel circles (triangle)

In **Figure 14**, we can feel that K_1, K_2, K_3 seem to be collinear.

Proof We designed a Maple program to verify this theorem, using coordinate method.

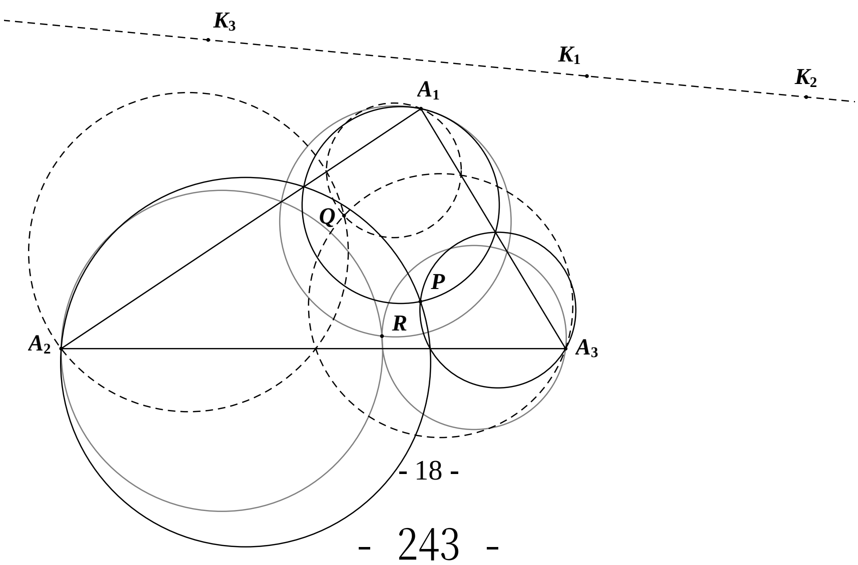

Figure 14 3 sets of Miquel Points (triangle, simplified)

```
(*Here list some procedure used later*)
Reduce3 := proc(P) (*Reduce the polynomial of a point or a line*)
   local g := gcd(gcd(P[1], P[2]), P[3]);
   return [simplify(P[1]/g), simplify(P[2]/g), simplify(P[3]/g)];
end proc:
Reduce4 := proc(P) (*Reduce the polynomial of a circle equation*)
   local g := gcd(gcd(gcd(P[1], P[2]), P[3]), P[4]);
   return [simplify(P[1]/g), simplify(P[2]/g),
      simplify(P[3]/g), simplify(P[4]/g)];
end proc:
Scale := proc(P, Q, t); (*Get the coordinate of a point on a segment *)
   return Reduce3([t*P[1]+(1-t)*Q[1], t*P[2]+(1-t)*Q[2],
      t*P[3]+(1-t)*Q[3]]);
end proc:
GetCircle := proc(P, Q, R); (*Get the equation of the circumcircle *)
   return Reduce4([
      -LinearAlgebra[Determinant](Matrix(
      [[P[1]*P[3], P[2]*P[3], P[3]^2],
      [Q[1]*Q[3], Q[2]*Q[3], Q[3]^2],
      [R[1]*R[3], R[2]*R[3], R[3]^2]])),
      +LinearAlgebra[Determinant](Matrix(
      [[P[1]^2+P[2]^2, P[2]*P[3], P[3]^2],
      [Q[1]^2+Q[2]^2, Q[2]*Q[3], Q[3]^2],
      [R[1]^2+R[2]^2, R[2]*R[3], R[3]^2]])),
      -LinearAlgebra[Determinant](Matrix(
      [[P[1]^2+P[2]^2, P[1]*P[3], P[3]^2],
      [Q[1]^2+Q[2]^2, Q[1]*Q[3], Q[3]^2],
      [R[1]^2+R[2]^2, R[1]*R[3], R[3]^2]])),
      +LinearAlgebra[Determinant](Matrix(
      [[P[1]^2+P[2]^2, P[1]*P[3], P[2]*P[3]],
      [Q[1]^2+Q[2]^2, Q[1]*Q[3], Q[2]*Q[3]],
      [R[1]^2+R[2]^2, R[1]*R[3], R[2]*R[3]]]))]);
end proc:
GetRadicalAxis := proc(E, F); (*Get the equation of radical axis*)
   return Reduce3([E[2]*F[1]-E[1]*F[2],
      E[3]*F[1]-E[1]*F[3], E[4]*F[1]-E[1]*F[4]]);
end proc:
GetIntersection := proc(L, M); (*Get the coordinate of intersection*)
   return Reduce3([L[2]*M[3]-L[3]*M[2],
      L[3]*M[1]-L[1]*M[3], L[1]*M[2]-L[2]*M[1]]);
end proc:
IsCoLine := proc(P, Q, R); (*Judge whether three point is collinear*)
   return P[1]*Q[2]*R[3]+P[2]*Q[3]*R[1]+P[3]*Q[1]*R[2]
      -P[3]*Q[2]*R[1]-P[1]*Q[3]*R[2]-P[2]*Q[1]*R[3];
end proc:
(*Assume A1, A2, A3*)
A1 := [a1x, a1y, 1]: A2 := [a2x, a2y, 1]: A3 := [a3x, a3y, 1]:
(*Assume and get the coordinates of P1, P2, P3, Q1, Q2, Q3, R1, R2, R3*)
P1 := Scale(A2, A3, t1): P2 := Scale(A3, A1, t2): P3 := Scale(A1, A2, t3):
Q1 := Scale(A2, A3, s1): Q2 := Scale(A3, A1, s2): Q3 := Scale(A1, A2, s3):
R1 := Scale(A2, A3, r1): R2 := Scale(A3, A1, r2): R3 := Scale(A1, A2, r3):
(*Get the equation of the circumcircles and radical axes*)
u1 := GetCircle(A1, P2, P3): u2 := GetCircle(A2, P3, P1):
u3 := GetCircle(A3, P1, P2): v1 := GetCircle(A1, Q2, Q3):
v2 := GetCircle(A2, Q3, Q1): v3 := GetCircle(A3, Q1, Q2):
w1 := GetCircle(A1, R2, R3): w2 := GetCircle(A2, R3, R1):
w3 := GetCircle(A3, R1, R2): L1 := GetRadicalAxis(u1, v1):
L2 := GetRadicalAxis(u2, v2): L3 := GetRadicalAxis(u3, v3):
M1 := GetRadicalAxis(v1, w1): M2 := GetRadicalAxis(v2, w2):
M3 := GetRadicalAxis(v3, w3): N1 := GetRadicalAxis(w1, u1):
N2 := GetRadicalAxis(w2, u2): N2 := GetRadicalAxis(w3, u3):
(*Get the coordinate of K1, K2, K3*)
K1 := GetIntersection(L1, L2):
K2 := GetIntersection(M1, M2):
K3 := GetIntersection(N1, N2):
(*Judge whether K1, K2, K3 is collinear*)
e := IsCoLine(K1, K2, K3):
(*Simplify the criterion and print the result*)
```

```
69  simplify(e);
```

After executing the program, line 69 give out the only result: 0, which stands for K_1, K_2, K_3 are collinear no matter where $P_1P_2P_3, Q_1Q_2Q_3, R_1R_2R_3$ lie and what shape $\Delta A_1A_2A_3$ is. □

We can also generalize **Theorem 5** to n-polygon.

Theorem 6 Based on **Theorem 4**, give an arbitrary n-polygon $A_1A_2\cdots A_n$ and n-point sets $P_1P_2\cdots P_n, Q_1Q_2\cdots Q_n, R_1R_2\cdots R_n$ to make three sets of n radical axes, which respectively meet at K_1, K_2, K_3, and then K_1, K_2, K_3 are collinear.

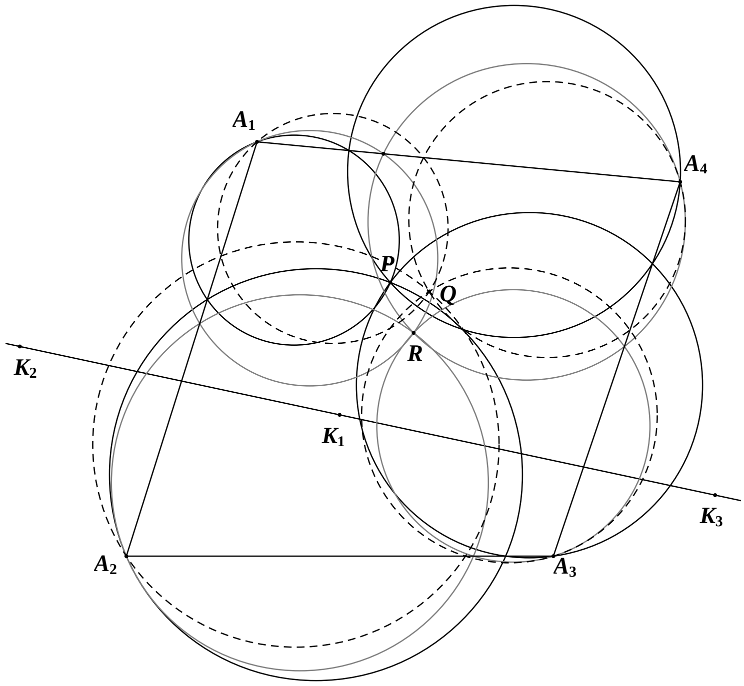

Figure 15 3 sets of Miquel Points and Miquel circles (quadrilateral as an example)

Proof Applying the same method in the proof of **Theorem 3**, we know that P, Q, R are also the Miquel Point with respect to $\Delta A_1A_2A_3$. Hence, based on the conclusion of **Theorem 5**, K_1, K_2, K_3 are collinear. □

After working on three sets of Miquel Points and Miquel circles, we continue delving on more sets of Miquel Points and Miquel circles.
If we have four sets of Miquel Points and Miquel circles, we found the shape like **Figure 16**. It's a complete quadrilateral.
As for five sets of Miquel Points and Miquel circles, it will be like **Figure 17**. It's the same shape as Desargues' theorem.

If we have n sets of Miquel Points and Miquel circles, there will be $\binom{n}{2}$ intersections of radical axes, and $\binom{n}{3}$ lines pass through these points, which satisfies that there are and only are 3 points on each line. It is hard to draw, but we believe it might be more amazing in figures of more Miquel Points and Miquel circles, indicating combinatorics properties between points and lines relating to Miquel Points.

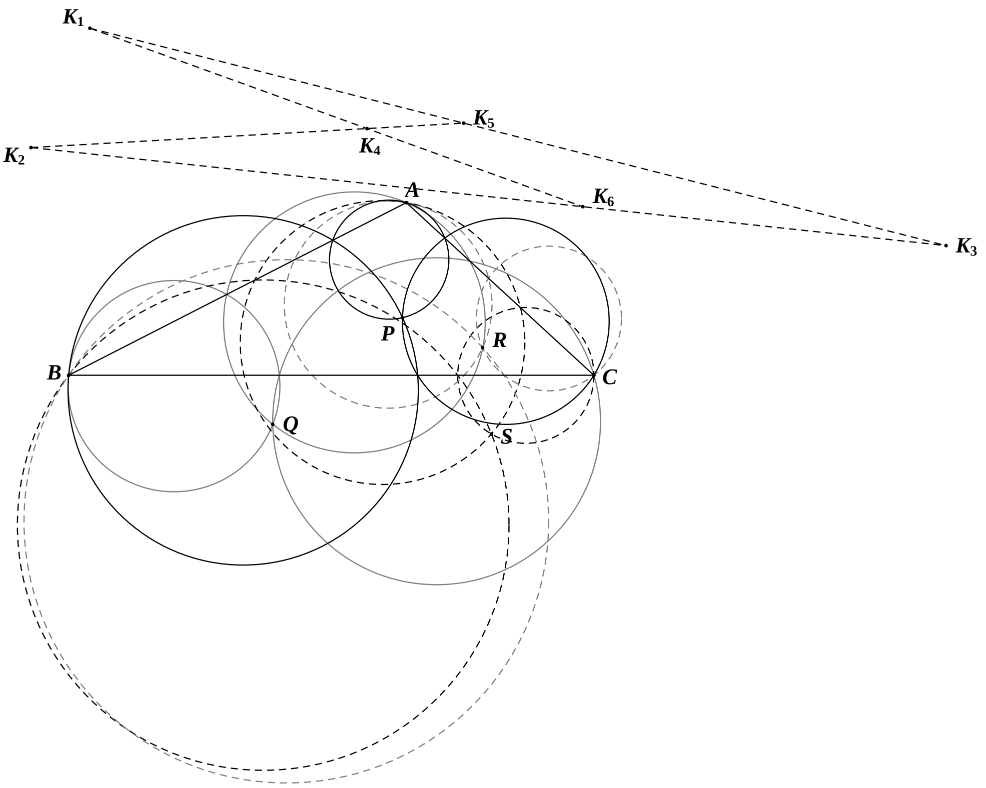

Figure 16 4 sets of Miquel Points and Miquel circles

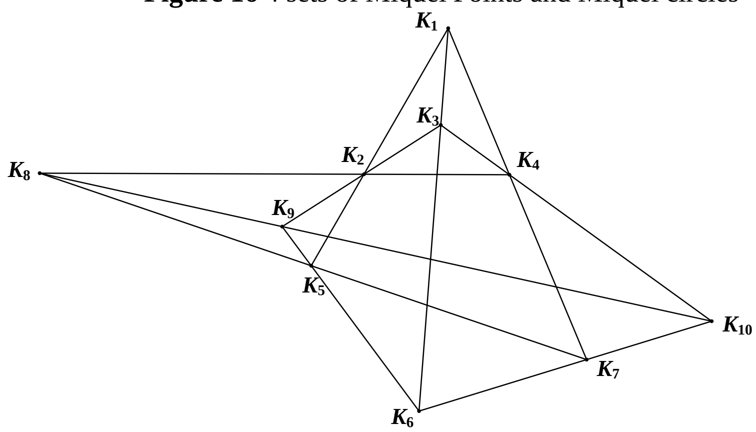

Figure 17 5 sets of Miquel Points and Miquel circles
(Note: As the figure is so complicated that we simplify it.)

7 On Higher Dimensions

It is well known that some theorems in plane geometry still work in solid geometry, indicating a magical connection from 2-dimension to 3-dimension. [7] In addition, some geometric elements in different dimensions have some relationship with each other, as the table describes below.

Dimension	$n=2$	$n=3$	$n\geq 4$
Object	Plane	Space	Hyperspace
	Line	Plane	Hyperplane
	Triangle	Tetrahedron	n-Simplex
	Circle	Sphere	$(n-1)$-Sphere
	Radical Axis	Radical Plane	Radical Hyperplane

Thus, we make some research on the Miquel Points on higher dimensions, wondering whether the Miquel's Theorem holds. Eventually, we find the Robert's Theorem, the corresponding theorem of Miquel's Theorem in 3-dimesional space.

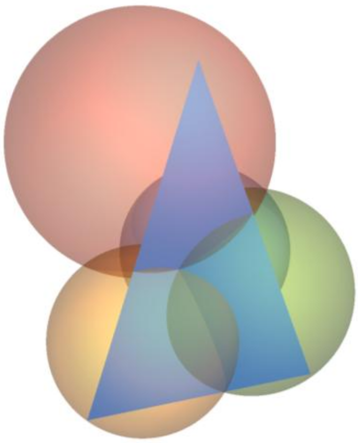
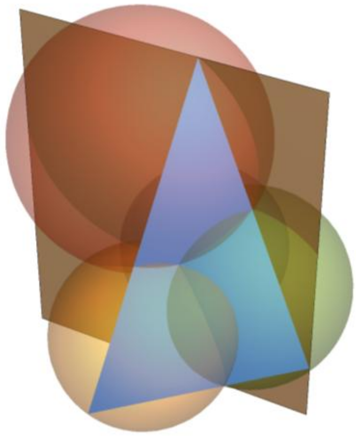

Figure 18(Left hand side) Miquel Point in 3-demension
Figure 19(Right hand side) Add a plane through Miquel Point

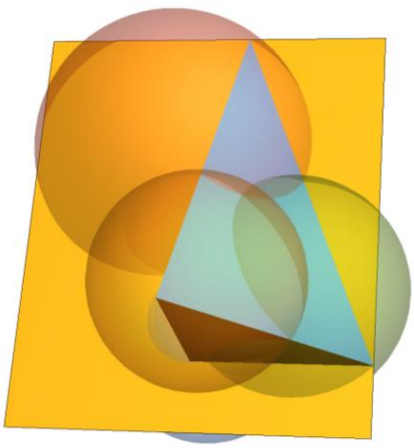
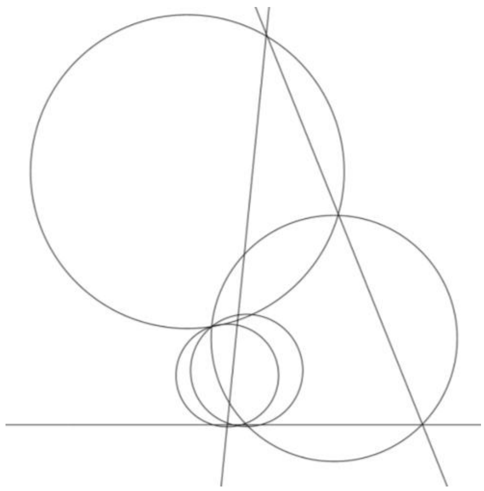

Figure 20(Left hand side) Another viewpoint to see the plane through Miquel Point
Figure 21(Right hand side) Sectional view of 3-dimension Miquel's Theorem

Robert's Theorem[8] Given a general tetrahedron, choose any point (but not vertex) on each edge and draw through each vertex a sphere passing through the three points on the edges which is adjacent to that vertex. Then these four spheres always have a point in common, and this point is denoted as Miquel Point or Robert Point.

See **Figure 18**, **Figure 19**, **Figure 20** and **Figure 21** above, we can clearly see that four spheres indeed meet at a point in **Figure 21**.

Furthermore, there is even a generalized Miquel's Theorem for high dimensions too, based on simplices in high-dimensional cases.

Miquel's Theorem for high dimensions[9] Let $d \in \mathbb{N}^*$ stands for the number of dimensions. For arbitrary linearly independent vectors $\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_d \in \mathbb{R}^d$ and real numbers λ_{ij}, $i, j \in \{0,1,\cdots,d\}$, satisfying $\lambda_{ij} + \lambda_{ji} = 1$ and $\lambda_{ij} \notin \{0,1\}$ for $i, j \in \{0,1,\cdots,d\}$, we have a d-simplex

$$S(\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_d) = \left\{ \mathbf{x}_0 + \sum_{i=1}^{d} \theta_i (\mathbf{x}_i - \mathbf{x}_0) : \sum_{i=1}^{d} \theta_i \le 1, \theta_i \ge 0, i = 1, \cdots, d \right\},$$

with positive volume and for $i \in \{0,1,\cdots,d\}$ a sphere S_i is drawn through each vertex $\mathbf{x}_i$ and the points $\lambda_{ij}\mathbf{x}_i + (1-\lambda_{ij})\mathbf{x}_j$, where $j \in \{0,1,\cdots,d\} \setminus \{i\}$. In this case, there exists a unique point $\mathbf{x}^*$, which is of the intersection $S_0 \cap S_1 \cap \cdots \cap S_d$, also denoted as Miquel Point M with respect to the d-simplex $S(\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_d)$.

With Miquel's Theorem being true in high-dimensional cases, it inspired us to consider whether our **Theorem 2** also holds for higher dimensions beyond 2. After a long period of tough calculation, we eventually found a generalized theorem of **Theorem 2**, which shows that in some cases, radical hyperplanes of two sets of spheres in Miquel's Theorem are still concurrent (or prarllel), and prove with a method a bit similar to that is used in [9].

Theorem 7 Let $d \in \mathbb{N}^*$, $2 | d$ stands for the number of dimensions. For arbitrary linearly independent vectors $\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_d \in \mathbb{R}^d$ and real numbers λ_{ij}, μ_{ij}, $i, j \in \{0,1,\cdots,d\}$, satisfying $\lambda_{ij} + \lambda_{ji} = 1$, $\mu_{ij} + \mu_{ji} = 1$ and $\lambda_{ij} \notin \{0,1\}$, $\mu_{ij} \notin \{0,1\}$ for $i, j \in \{0,1,\cdots,d\}$, we have a d-simplex

$$S(\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_d) = \left\{ \mathbf{x}_0 + \sum_{i=1}^{d} \theta_i (\mathbf{x}_i - \mathbf{x}_0) : \sum_{i=1}^{d} \theta_i \le 1, \theta_i \ge 0, i = 1, \cdots, d \right\},$$

with positive volume and for $i \in \{0,1,\cdots,d\}$, a sphere S_i is drawn through each vertex $\mathbf{x}_i$ passing the points $\lambda_{ij}\mathbf{x}_i + (1-\lambda_{ij})\mathbf{x}_j$, $j \in \{0,1,\cdots,d\} \setminus \{i\}$, and for $i \in \{0,1,\cdots,d\}$, a sphere T_i is drawn through each vertex $\mathbf{x}_i$ passing the points $\mu_{ij}\mathbf{x}_i + (1-\mu_{ij})\mathbf{x}_j$, $j \in \{0,1,\cdots,d\} \setminus \{i\}$ similarly. Then for $i \in \{0,1,\cdots,d\}$ let u_i be the radical (hyper)plane of S_i and T_i. In this case, for $i \in \{0,1,\cdots,d\}$ all the u_i are concurrent, that is to say they meet at the same point, (or parallel with each other).

Proof We denote the matrix

$$\mathbf{A}=\begin{pmatrix} x_{01} & \cdots & x_{0d} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ x_{(d-1)1} & \cdots & x_{(d-1)d} & 1 \\ x_{d1} & \cdots & x_{dd} & 1 \end{pmatrix}.$$

Because $\mathbf{x}_0,\cdots,\mathbf{x}_d$ are linearly independent and d-simplex $S(\mathbf{x}_0,\mathbf{x}_1,\cdots,\mathbf{x}_d)$ has a positive volume, we can obtain $\det\mathbf{A}\neq 0$. Denote A_{ij} as the cofactor of the element of the i-th row and j-th column in $\mathbf{A}$ for $i,j\in\{1,\cdots,d+1\}$. Therefore

$$\mathbf{A}^*=\begin{pmatrix} A_{11} & \cdots & A_{1d} & A_{1(d+1)} \\ \vdots & \ddots & \vdots & \vdots \\ A_{d1} & \cdots & A_{dd} & A_{d(d+1)} \\ A_{(d+1)1} & \cdots & A_{(d+1)d} & A_{(d+1)(d+1)} \end{pmatrix}.$$

We denote $\mathbf{y}_{ij}=\lambda_{ij}\mathbf{x}_i+(1-\lambda_{ij})\mathbf{x}_j$, $\mathbf{z}_{ij}=\mu_{ij}\mathbf{x}_i+(1-\mu_{ij})\mathbf{x}_j$, which represent the points on the $d(d+1)/2$ edges of $S(\mathbf{x}_0,\mathbf{x}_1,\cdots,\mathbf{x}_d)$ which S_i and T_i passes. Because we have $\lambda_{ij}+\lambda_{ji}=1$, $\mu_{ij}+\mu_{ji}=1$, we know that $\mathbf{y}_{ij}$ and $\mathbf{y}_{ji}$ represent the same point, and so do $\mathbf{z}_{ij}$ and $\mathbf{z}_{ji}$, for $i,j\in\{0,1,\cdots,d\}$.

Noting that $\mathbf{y}_{ii}=\lambda_{ii}\mathbf{x}_i+(1-\lambda_{ii})\mathbf{x}_i=\mathbf{x}_i$, then from **Lemma 6** we can easily obtain that for $i\in\{0,1,\cdots,d\}$, the sphere S_i satisfies the equation

$$\Delta_i(\mathbf{x})=\begin{vmatrix} \|\mathbf{y}_{i0}\|^2 & \mathbf{y}'_{i0} & 1 \\ \vdots & \vdots & \vdots \\ \|\mathbf{y}_{id}\|^2 & \mathbf{y}'_{id} & 1 \\ \|\mathbf{x}\|^2 & \mathbf{x}' & 1 \end{vmatrix}=0.$$

where the determinant is for a $(d+2)\times(d+2)$ one, the $(i+1)$-th row of which is $\begin{pmatrix}\|\mathbf{y}_{ii}\|^2 & \mathbf{y}'_{ii} & 1\end{pmatrix}=\begin{pmatrix}\|\mathbf{x}_i\|^2 & \mathbf{x}'_i & 1\end{pmatrix}$, representing a matrix of $1\times(d+2)$.

Noting that $\|\mathbf{y}_{ij}\|^2=\lambda_{ij}^2\|\mathbf{x}_i\|^2+2\lambda_{ij}(1-\lambda_{ij})\langle\mathbf{x}_i,\mathbf{x}_j\rangle+(1-\lambda_{ij})^2\|\mathbf{x}_j\|^2$, then by applying the well-known transformation rules for determinants (subtracting λ_{ij} times the $(i+1)$-th row from the $(j+1)$-th row, and then divide the $(j+1)$-th row by $(1-\lambda_{ij})$, for $j\in\{0,1,\cdots,d\}\setminus\{i\}$), we derive that $\Delta_i(\mathbf{x})=\Delta'_i(\mathbf{x})\prod_{k=0}^{d}(1-\lambda_{ik})$, where

$$\Delta'_i(\mathbf{x})=\begin{vmatrix} -\lambda_{i0}\|\mathbf{x}_i\|^2+2\lambda_{i0}\langle\mathbf{x}_i,\mathbf{x}_0\rangle+(1-\lambda_{i0})\|\mathbf{x}_0\|^2 & \mathbf{x}_0' & 1 \\ \vdots & \vdots & \vdots \\ -\lambda_{id}\|\mathbf{x}_i\|^2+2\lambda_{id}\langle\mathbf{x}_i,\mathbf{x}_d\rangle+(1-\lambda_{id})\|\mathbf{x}_d\|^2 & \mathbf{x}_d' & 1 \\ \|\mathbf{x}\|^2 & \mathbf{x}' & 1 \end{vmatrix}.$$

Because $\lambda_{ij} \notin \{0,1\}$ for $i,j \in \{0,1,\cdots,d\}$, $\Delta_i(\mathbf{x})=0$ iff $\Delta_i'(\mathbf{x})=0$. Thus, the sphere S_i satisfies the equation

$$\begin{vmatrix} -\lambda_{i0}\|\mathbf{x}_i\|^2+2\lambda_{i0}\langle \mathbf{x}_i,\mathbf{x}_0\rangle+(1-\lambda_{i0})\|\mathbf{x}_0\|^2 & \mathbf{x}_0{}' & 1 \\ \vdots & \vdots & \vdots \\ -\lambda_{id}\|\mathbf{x}_i\|^2+2\lambda_{id}\langle \mathbf{x}_i,\mathbf{x}_d\rangle+(1-\lambda_{id})\|\mathbf{x}_d\|^2 & \mathbf{x}_d{}' & 1 \\ \|\mathbf{x}\|^2 & \mathbf{x}' & 1 \end{vmatrix}=0,$$

in which the coefficient of $\|\mathbf{x}\|^2$ happen to be $\det\mathbf{A}\neq 0$, $i\in\{0,1,\cdots,d\}$.

By substituting λ_{ij} with μ_{ij} in the equation of S_i, we can get the equation of T_i, in which the coefficient of $\|\mathbf{x}\|^2$ is still $\det\mathbf{A}$, $i\in\{0,1,\cdots,d\}$.

Thus, the equation of u_i, the radical (hyper)plane of S_i and T_i can be represented as

$$\begin{vmatrix} (\mu_{i0}-\lambda_{i0})\|\mathbf{x}_i-\mathbf{x}_0\|^2 & \mathbf{x}_0{}' & 1 \\ \vdots & \vdots & \vdots \\ (\mu_{id}-\lambda_{id})\|\mathbf{x}_i-x_d\|^2 & \mathbf{x}_d{}' & 1 \\ 0 & \mathbf{x}' & 1 \end{vmatrix}=0,$$

simply subtracting the equation of S_i from that of T_i, and applying the identity $\|\mathbf{x}_i\|^2-2\langle \mathbf{x}_i,\mathbf{x}_0\rangle+\|\mathbf{x}_0\|^2=\|\mathbf{x}_i-\mathbf{x}_0\|^2$, $i\in\{0,1,\cdots,d\}$.

Denote $(\mu_{ij}-\lambda_{ij})\|\mathbf{x}_i-\mathbf{x}_j\|^2=Q_{ij}$ for $i,j\in\{0,1,\cdots,d\}$, and after expanding the determinant the equation of u_i can be represented as

$$\begin{pmatrix} Q_{i0} & \cdots & Q_{i(d-1)} & Q_{id} \end{pmatrix}\begin{pmatrix} A_{11} & \cdots & A_{1d} & A_{1(d+1)} \\ \vdots & \ddots & \vdots & \vdots \\ A_{d1} & \cdots & A_{dd} & A_{d(d+1)} \\ A_{(d+1)1} & \cdots & A_{(d+1)d} & A_{(d+1)(d+1)} \end{pmatrix}\mathbf{x}=\mathbf{0},$$

which is equivalent to $\begin{pmatrix} Q_{i0} & \cdots & Q_{i(d-1)} & Q_{id} \end{pmatrix}\mathbf{A}^*\mathbf{x}=\mathbf{0}$, for $i\in\{0,1,\cdots,d\}$.

According to **Lemma 9**, $u_0,\cdots,u_d$ are concurrent (or parallel) iff

$$\det\begin{pmatrix} \begin{pmatrix} Q_{00} & \cdots & Q_{0(d-1)} & Q_{0d} \end{pmatrix}\mathbf{A}^* \\ \vdots \\ \begin{pmatrix} Q_{d0} & \cdots & Q_{d(d-1)} & Q_{0d} \end{pmatrix}\mathbf{A}^* \end{pmatrix}=\det\left(\begin{pmatrix} Q_{00} & \cdots & Q_{0d} \\ \vdots & \ddots & \vdots \\ Q_{d0} & \cdots & Q_{dd} \end{pmatrix}\mathbf{A}^*\right)$$

$$=\det\begin{pmatrix} Q_{00} & \cdots & Q_{0d} \\ \vdots & \ddots & \vdots \\ Q_{d0} & \cdots & Q_{dd} \end{pmatrix}\det\mathbf{A}^*=0.$$

Noting that $\det\mathbf{A}\neq 0 \Rightarrow \det\mathbf{A}^*\neq 0$, thus $u_0,\cdots,u_d$ are concurrent (or parallel) iff $\det\mathbf{Q}=0$, where

$$\mathbf{Q}=\begin{pmatrix} Q_{00} & \cdots & Q_{0d} \\ \vdots & \ddots & \vdots \\ Q_{d0} & \cdots & Q_{dd} \end{pmatrix}=\begin{pmatrix} 0 & Q_{01} & \cdots & Q_{0(d-1)} & Q_{0d} \\ -Q_{01} & 0 & \cdots & Q_{1(d-1)} & Q_{1d} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -Q_{0(d-1)} & -Q_{1(d-1)} & \cdots & 0 & Q_{(d-1)d} \\ -Q_{0d} & -Q_{1d} & \cdots & -Q_{(d-1)d} & 0 \end{pmatrix}.$$

is a skey-symmetric matrix, because $Q_{ij}=\left(\mu_{ij}-\lambda_{ij}\right)\left\|\mathbf{x}_i-\mathbf{x}_j\right\|^2=-Q_{ij}$, $Q_{ii}=0$ for all $i,j\in\{0,1,\cdots,d\}$.

Therefore, when conditioned $d\in\mathbb{N}^*$, $2\,|\,d$, we have

$$\det\mathbf{Q}=\det\mathbf{Q}'=\det\left(-\mathbf{Q}\right)=\left(-1\right)^{d+1}\det\mathbf{Q}=-\det\mathbf{Q}=0,$$

and thus we can conclude that $u_0,\cdots,u_d$ are concurrent (or parallel) no matter what λ_{ij}, μ_{ij}, where $i,j\in\{0,1,\cdots,d\}$, exactly are. So we have done. □

Theorem 8 Given the same condition in **Theorem 7** except $2\nmid d$, we have that $u_0,\cdots,u_d$ pass the same point (or parallel) iff

$$\sum_{\sigma\in S_{2n}}\operatorname{sgn}\left(\sigma\right)\prod_{i=1}^{n}Q_{\sigma(2i-1)\sigma(2i)}=0,$$

where $n=\frac{d+1}{2}$, $\left(\mu_{ij}-\lambda_{ij}\right)\left\|\mathbf{x}_i-\mathbf{x}_j\right\|^2=Q_{ij}$ for $i,j\in\{0,1,\cdots,d\}$, and S_{2n} is a set including all the permutations of $\left(1,2,\cdots,2n\right)$.

Proof Similarly, $u_0,\cdots,u_d$ pass the same point (or parallel) iff $\det\mathbf{Q}=0$.
According to **Lemma 11**, $\det\mathbf{Q}=\operatorname{pf}^2\mathbf{Q}=0$, thus we obtain

$$\operatorname{pf}\mathbf{Q}=\frac{1}{2^n n!}\sum_{\sigma\in S_{2n}}\operatorname{sgn}\left(\sigma\right)\prod_{i=1}^{n}Q_{\sigma(2i-1)\sigma(2i)}=0,$$

which is equivalent to what needs to be proved. □

Corollary 4 Given the same condition in **Theorem 8**, if $u_0,\cdots,u_d$ pass the same point, then there exists a line l such that $u_0,\cdots,u_d$ all pass l.

Proof Similarly, $u_0,\cdots,u_d$ pass the same point only if $\det\mathbf{Q}=0$.
According to **Lemma 12**, we obtain that $2\,|\,\operatorname{rank}\left(\mathbf{Q}\right)$. Because $\det\mathbf{Q}=0$, therefore $\operatorname{rank}\left(\mathbf{Q}\right)<d+1$. Adding that $2\,|\left(d+1\right)$, we can derive that $\operatorname{rank}\left(\mathbf{Q}\right)\le d-1$.
Noting that $\left|\mathbf{A}^*\right|\neq 0\Rightarrow\operatorname{rank}\left(\mathbf{QA}^*\right)=\operatorname{rank}\left(\mathbf{Q}\right)\le d+1$, and according to **Lemma 10** that there exists a line l such that $u_0,\cdots,u_d$ all pass l. □
The case that $d=3$ is quite interesting, for 3-dimesional world is what we can imagine and the case $d=3$ is rather astonishing for the four plane all pass the same line as long as the condition is satisfied to make them meet at a point.

Figure 22 below shows the spheres and the Tetrahedron (although mostly hidden by the spheres). and **Figure 23** below shows that the four planes pass the same line.

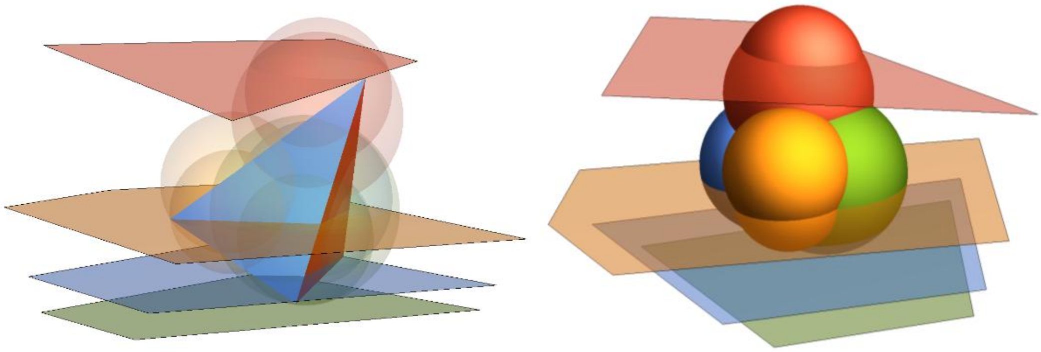

Figure 22 Tetrahedron and spheres along with their radical planes
(Left-hand side figure highlights the tetrahedron,
while right-hand side highlights the spheres)

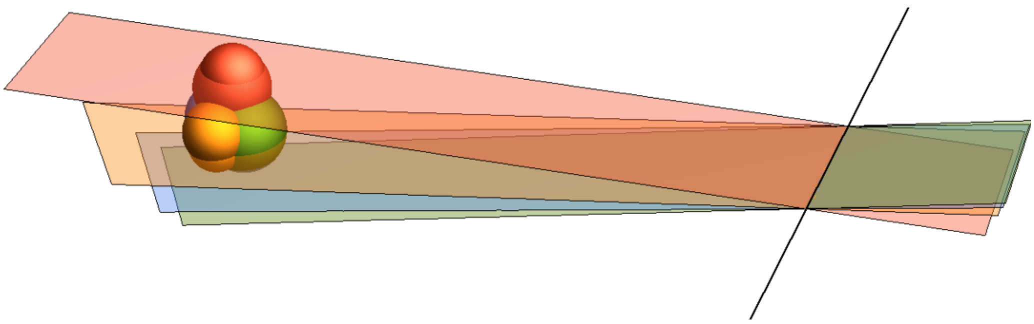

Figure 23 A far viewpoint to see that four radical planes are indeed meet at one line

In conclusion, we put forward the following corollary, directly derived by assuming $d=3$ in **Corollary 4**.

Corollary 5 Given a tetrahedron $A_1A_2A_3A_4$, and points D_{ij}, E_{ij} are on the edges A_iA_j, where $i,j\in\{1,2,3,4\}$, $i<j$. For each $i\in\{1,2,3,4\}$, a sphere S_i was drawn through A_i and D_{ij} (or D_{ji}), $j\in\{1,2,3,4\}\setminus\{i\}$. So are spheres T_i, $i\in\{1,2,3,4\}$ constructed, with respect to E_{ij}, $i,j\in\{1,2,3,4\}$, $i<j$. Let u_i be the radical plane of S_i and T_i, $i\in\{1,2,3,4\}$, and then u_1,u_2,u_3,u_4 meet at a same point iff

$$\overline{D_{12}E_{12}}\cdot\overline{A_1A_2}\cdot\overline{D_{34}E_{34}}\cdot\overline{A_3A_4}+\overline{D_{14}E_{14}}\cdot\overline{A_1A_4}\cdot\overline{D_{23}E_{23}}\cdot\overline{A_2A_3}$$
$$-\overline{D_{13}E_{13}}\cdot\overline{A_1A_3}\cdot\overline{D_{24}E_{24}}\cdot\overline{A_2A_4}=0,$$

and if this condition is satisfied, u_1,u_2,u_3,u_4 all pass the same line.

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In the whole process of writing this paper, we use some informational technology to help us. We apply Sketchpad 5.06 to draw figures of plane geometry, apply Maple 18 to handle with coordinates and equations and use Mathematica 10 to do some visualization work. In other words, all the 3-D figures are generated by Mathematica.

9 Postscript

Miquel Points of complete quadrilateral and triangle has been studied a lot. Few people, however, have studied two series of Miquel Points and Miquel circles even among triangles. So this paper is an attempt. Step by step, we study from triangles to polygons, from two sets of Miquel Points and Miquel circles to the more sets, from 2-dimensional cases to high-dimensional cases. We not only find a 'Fixed Point', which never changes (the lines are always concurrent at this point) when we put in more random points, but also find 'Fixed Lines' as well as more beautiful theorems and propoties, which is also the highlight of this paper. We will go on with our research and more elegant conclusions are waiting for us to exvacate.

10 Reference

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