Three Generalizations of the Secretary Problem

Author: Zhe Kong, Yilin Ye

School: Shenzhen Middle School

Instructor: Wentao Zhang

Abstract

In this paper, we consider three generalizations of the classical secretary problem and get the corresponding optimal strategies by using the backward induction method:

1) Suppose that the interviewer meets the applicants in groups sequentially, which we call panel interview case, to find the k-th best of n applicants, what's the highest probability of success? For this problem we give a general optimal strategy.
2) Suppose that the interviewer meets the applicants in groups sequentially to find the best w out of n applicants, what's the highest probability of success? For this problem we give some optimal strategies with some special restrictions on the groups.
3) Suppose that we follow the setting of the standard secretary problem but allow the applicants to have a fixed probability $1-p$ ($0<p<1$) to reject an offer, we give an algorithm to compute the maximal success probability to find the t-th ($2 \le t < n$) best of n applicants, and also give a short remark on the last problem in panel interview case.

Keywords: secretary problem; panel interview; backward induction; optimal strategy; Hamilton-Jacobi-Bellman equation.

Notation:

n : Total number of the applicants

m : Total number of the groups

k : The specific rank of the desired applicant ($0< k \leq n$) **(Section 2)**

g_j : The j-th group ($0< j \leq m$)

α_j : Total number of the applicants in the j-th group

l_r : Total number of applicants in the first r group

w : The specific rank of the desired applicant (Function as k; $2 \leq w < n$) **(Section 3)**

t : The specific rank of the desired applicant (Function as k; $2 \leq t < n$) **(Section 4)**

p : The probability that an applicant will receive an offer ($0< p <1$)

g_r : First r groups that have been interviewed ($0< r <m$)

u_r : The probability of an applicant who is currently the second best continues to be the second best in the end **(Section 2)**

v_r : The probability of an applicant who is currently the best becomes the second best in the end **(Section 2)**

A_r : The probability of using the optimal stopping rule to find the exactly second best applicant with the assumption that the first r groups have been rejected (**Section 2)**

g_{r_0}: The group that contains the $[\frac{n+1}{2}]$-th interviewed candidates

$v_{i,r}$: The probability of the currently i-th best out of l_r candidates ends up to be the k-th best **(Section 2)**

j_i : The thresholds of groups ($1\leq i \leq w$; $1 \leq j_1 < j_2 < \cdots < j_w \leq m$)

z_r: The Probability of an applicant who is currently the second best continues to be the second best in the end **(Section 4)**

y_r: The probability of an applicant who is currently the best becomes the second best in the end in **(Section 4)**

B_r : The probability of using the optimal stopping rule to find the exactly second best applicant with the assumption that the first *r* groups have been rejected **(Section 4)**

$f_{i,r}$: The probability of the currently *i*-th best out of l_r candidates ends up to be the *k*-th best **(Section 4)**

1. Introduction

As a classical problem in optimal selection, the secretary problem is also known as the marriage problem, the googol game, the sultan's dowry problem, and the best choice problem.

Although the origin of the secretary problem is unclear, the consensus is that the standard secretary problem is first introduced by American mathematician Merrill M. Flood in 1949. Then, the problem was popularized by Martin Gardner[5]in February 1960, when he published the first research about the secretary problem in Scientific American. Shortly after the first publication, the secretary problem was taken up and developed by groups of illustrious mathematicians, among them are Lindley[11](1961), Dynkin[3](1963), Chow, Moriguti, Robins and Samuels[2](1964), and Gilbert and Mosteller[7](1966). Gradually, the problem appealed to all. In 1975, M. H. Smith[12]gave the way to find the best applicant with some kind of uncertainty, i.e., each applicant has a fixed probability 1-p ($0<p<1$) to reject an offer. It was not until 1984 that F. Thomas Bruss[1] came up with the first solution of the standard secretary problem: the 1/e-law of best choice, and the maximal win probability under the solution: 1/e as n goes to infinity.

F. Thomas Bruss's 1/e-law of best choice is an elegant solution: to reject the first [n/e] applicants, and then select the first candidate that is better than all the applicants have been interviewed so far, otherwise pick the last one. (e is the base of the natural logarithm, and $[\square]$ is the Gauss floor function, i.e., $[x]$ is the largest integer not greater than x.)

From then on, enormous amounts of variants of the secretary problem have been formulated and solved. In 1995, Robert J. Vanderbei[13] explored the problem of picking exactly the second best out of n applicants, and conjectured a general algorithm to find the k-th ($0< k\leq n$) best applicant. In addition, Robert J. Vanderbei[14] showed that for n is even, and $k = n/2$, the probability of using optimal stopping rule to find the k-th best student equals $1/(\frac{n}{2}+1)$. Later, in 2009, Shoou-Ren Hsiau and Jiing-Ru Yang[9] further extended the problem by studying how to find the best applicant out of n applicants with the assumption that the applicants are interviewed sequentially in groups.

In our paper, we slightly change the descriptions of the standard secretary problem, and place the problem in the situation that a university is enrolling students. This paper's assumptions of the secretary problem can be described as follows. Assume n applicants are applying, which can be ordered from the best to the worst with no ties. All n applicants can be randomly divided into m group $\{g_j\}_{j=1}^{m}$. In group g_j there are α_j students, $\sum_{j=1}^{m}\alpha_j = n$. Each time the interviewer meets one group of applicants. The interviewer cannot observe the absolute ranks. Also, immediately after the interview, the interviewer has to make the decision: to select some student or reject all the applicants in the group and start to interview the next group. Once an applicant is rejected, he or she cannot be recalled. The goal for the interviewer is to choose the k-th best applicant, i.e. the candidate that ranks k-th among total n applicants. The interview will be considered as a failure if the interviewer rejects all applicants, or picks up someone that turns out not to be the k-th best of all n candidates. In this paper, we also denote $l_r = \sum_{j=1}^{r}\alpha_j$,and then $l_m = n$.

Based on the previous works, we mainly study three generalizations of the secretary problem in the panel interview cases to find optimal solution, and the maximal win probability under that solution for each generalizations.

(1) Find the k-th best of n

The administrator finds that the best student can always receive admissions from other better universities and then drops the opportunity of enrolling into this university. Therefore, the interviewer prefers to select the second best student. What is the optimal strategy to maximize the probability of selecting the second best student in panel interview case? More generally, what is the optimal strategy to maximize the probability of selecting the k-th ($2 < k < n$) best student in panel interview case? We study this problem in this paper.

(2) Find the best w out of n

The interviewer would like to select the best two students out of n applicants. What is the optimal strategy to maximize the probability of success in panel interview case? More generally, what is the optimal strategy to maximize the probability of selecting the best w ($2 \le w < n$) students out of n in panel interview case? We study this problem in this paper.

(3) Find the t-th best of n with uncertainty

All the assumptions are the same as those of the standard secretary problem, besides that each of the applicants has a fixed probability $1-p$ ($0 < p < 1$) to refuse an offer. What is the optimal strategy to maximize the probability of finding the second best applicant in the classical case with uncertain employment? More generally, what is the optimal strategy to maximize the probability of selecting the t -th ($2 \le t < n$) best applicant out of n in such case? We study this problem in this paper.

Here the best strategy refers to the optimal stopping rule and the maximal probability to select the desired applicant or applicants.

2. Find the k-th best of n

In this section, we focus on the problem: what is the optimal strategy to find the k -th ($k > 2$) best student in panel interview case?

2.1 Panel interview case: $k = 2$

We follow R. J. Vanderbei's[13] algorithm. To find exactly the second best candidate in panel interview case, first assume the interviewer picks up student θ among the r interviewed groups $g_1, \cdots, g_r$, and finishes to interview all the m groups to see whether the choice is right.

Lemma 1.

(i) *If student θ is currently the second best, the probability u_r of student θ being the second best in the end is*

$$u_r = \frac{l_r(l_r - 1)}{n(n-1)}, \qquad 1 \le r \le m.$$

(ii) *If student θ is currently the best, the probability v_r of student θ being the second best in the end is*

$$v_r = \frac{l_r(n - l_r)}{n(n-1)}, \qquad 1 \le r \le m.$$

Theorem 1.

Let A_r be the probability of using the optimal stopping rule to find the exactly second best applicant with the assumption that the first r groups have been rejected.

Choose suitable r_0 *such that the group* g_{r_0} *contains the* $[\frac{n+1}{2}]$*-th candidates, i.e.,* $r_0 = \min\{r \mid l_r \geq \frac{n+1}{2}\}$*, and set*

$$r_1 = \min\{p \mid \frac{1}{l_p - 1}\sum_{j=p+1}^{m}\frac{\alpha_j(l_j-1)}{l_{j-1}} \leq 1\},\ \textit{where}\ \ l_p = \sum_{j=1}^{p}\alpha_j. \qquad (1)$$

We have $r_1 \geq r_0$*. Furthermore,*

$$A_r = \begin{cases} \dfrac{l_{r_1-1}}{n(n-1)}\displaystyle\sum_{j=r_1}^{m}\frac{\alpha_j(l_j-1)}{l_{j-1}}, & 0 \leq r < r_1, \\ \dfrac{l_r}{n(n-1)}\displaystyle\sum_{j=r+1}^{m}\frac{\alpha_j(l_j-1)}{l_{j-1}}, & r_1 \leq r < m. \end{cases} \qquad (2)$$

The functions A_r*,* $0 \leq r \leq m$*, satisfy*

$$v_r \leq A_r \leq u_r\ \ , \textit{for}\ \ r_1 \leq r \leq m,$$

$$A_r > u_r,\ A_r > v_r,\ \textit{else.}$$

Proof for Lemma 1. For case (i), when the group g_{r+1} is coming, the probability that all the α_{r+1} applicants in $(r+1)$-th group have lower relative ranks than θ is $\frac{l_r(l_r-1)}{l_{r+1}(l_{r+1}-1)}$, and we have

$$u_r = \begin{cases} \dfrac{l_r(l_r-1)}{l_{r+1}(l_{r+1}-1)}u_{r+1}, & 1 \leq r < m, \\ 1, & r = m. \end{cases} \qquad (3)$$

Then

$$u_r = \frac{l_r(l_r-1)}{l_{r+1}(l_{r+1}-1)}\frac{l_{r+1}(l_{r+1}-1)}{l_{r+2}(l_{r+2}-1)}\cdots\frac{l_{m-1}(l_{m-1}-1)}{l_m(l_m-1)}u_m = \frac{l_r(l_r-1)}{n(n-1)}.$$

For case (ii), when the group g_{r+1} is coming, the probability that all the α_{r+1} applicants in $(r+1)$-th group have lower relative ranks than θ is $\frac{l_r}{l_{r+1}}$, and the probability that there is one and only one candidate in $(r+1)$-th group better than θ

is $\frac{\alpha_{r+1}l_r}{l_{r+1}(l_{r+1}-1)}$. Hence we have the recurrence formula

$$v_r = \begin{cases} \frac{\alpha_{r+1}l_r}{l_{r+1}(l_{r+1}-1)}u_{r+1} + \frac{l_r}{l_{r+1}}v_{r+1}, & 1 \le r < m, \\ 0, & r = m. \end{cases} \quad (4)$$

Plug (3) into (4), we have

$$\frac{v_r}{l_r} = \frac{\alpha_{r+1}}{n(n-1)} + \frac{v_{r+1}}{l_{r+1}}.$$

Denote $h_r = \frac{v_r}{l_r}$, then $h_r = \frac{\alpha_{r+1}}{n(n-1)} + h_{r+1}$. Since $h_m = 0$, by induction

$$h_r = \frac{\alpha_{r+1} + \alpha_{r+2} + \cdots + \alpha_m}{n(n-1)} = \frac{n - l_r}{n(n-1)},$$

that is, $v_r = \frac{l_r(n-l_r)}{n(n-1)}$.

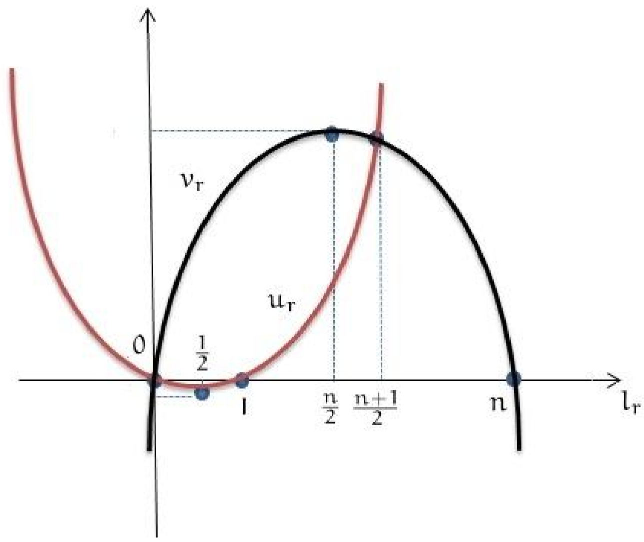

Figure 1: u_r and v_r, though we plot u_r and v_r as continuous functions, they are actually discrete.

Proof for Theorem 1. Use the principle of dynamic programming, it is sufficient to do the right thing at each stage. Suppose r groups have been interviewed and no one has been accepted so far. Now comes the $(r+1)$-th group g_{r+1}, the probability that all students in g_{r+1} are worse than the best and the second best in first r groups is

$\frac{l_r(l_r-1)}{l_{r+1}(l_{r+1}-1)}$, then no students will be matriculated in such group.

Moreover, suppose one applicant in group g_{r+1} to be the currently the best while currently the second best is in the first r groups, and the probability of this situation is $\frac{\alpha_{r+1}l_r}{l_{r+1}(l_{r+1}-1)}$. If the interviewer rejects him or her with the whole group g_{r+1} and then the probability of success is A_{r+1}, if the interviewer accepts him or her and the probability of success is v_{r+1}. We should pick the case with larger probability.

Furthermore, suppose one applicant in group g_{r+1} to be currently the second best while currently the best is in the first r groups, and the probability of this situation is $\frac{\alpha_{r+1}l_r}{l_{r+1}(l_{r+1}-1)}$. If the interviewer rejects him or her with the whole group g_{r+1}, then the probability of success is A_{r+1}. If the interviewer accepts him or her, then the probability of success is u_{r+1}. We should pick the case with larger probability.

Finally, suppose two applicants in group g_{r+1} respectively to be currently the best and currently the second best, and the probability of this situation is $\frac{\alpha_{r+1}(\alpha_{r+1}-1)}{l_{r+1}(l_{r+1}-1)}$. If the interviewer rejects the whole group g_{r+1}, then the probability of success is A_{r+1}. If the interviewer accepts currently the best, then the probability of success is v_{r+1}. If the interviewer accepts currently the second best, then the probability of success is u_{r+1}. We should pick the case with larger probability.

To summarize, we have

$$A_r=\begin{cases}\max(A_{r+1},v_{r+1}), & r=0,\\ \dfrac{l_r(l_r-1)}{l_{r+1}(l_{r+1}-1)}A_{r+1}+\dfrac{\alpha_{r+1}l_r}{l_{r+1}(l_{r+1}-1)}\left(\max(A_{r+1},v_{r+1})+\max(A_{r+1},u_{r+1})\right) & \\ +\dfrac{\alpha_{r+1}(\alpha_{r+1}-1)}{l_{r+1}(l_{r+1}-1)}\max(A_{r+1},u_{r+1},v_{r+1}), & 1\le r<m,\\ 0, & r=m.\end{cases} \quad (5)$$

Notice that if $l_r<\dfrac{n+1}{2}$, then $u_r<v_r$, and if $l_r\ge\dfrac{n+1}{2}$, then $u_r\ge v_r$.

To simplify the formula（5）,we first consider $l_r\ge\dfrac{n+1}{2}$, then $u_r\ge v_r$. Assume $v_r\le A_r\le u_r$, from the second sub-formula of （5）, we have

$$A_{r-1}=\frac{l_{r-1}}{l_r}A_r+\frac{\alpha_r(l_r-1)}{n(n-1)}.$$

Then

$$A_r=\frac{l_r}{n(n-1)}\sum_{j=r+1}^{m}\frac{\alpha_j(l_j-1)}{l_{j-1}}. \quad (6)$$

Furthermore, since

$$\frac{\alpha_j(l_j-1)}{l_{j-1}}=\frac{\alpha_j(l_{j-1}+\alpha_j-1)}{l_{j-1}}=\alpha_j+\frac{\alpha_j(\alpha_j-1)}{l_{j-1}}>\alpha_j,$$

Hence

$$\sum_{j=r+1}^{m}\frac{\alpha_j(l_j-1)}{l_{j-1}}>\sum_{j=r+1}^{m}\alpha_j=n-l_r,$$

we have $A_r>v_r$ for $r_0\le r<m$, and $A_r=v_r$ for $r=m$. It assures our assumption.

Moreover, to satisfy $A_r\le u_r$, we need

$$\sum_{j=r+1}^{m}\frac{\alpha_j(l_j-1)}{l_{j-1}}\le l_r-1.$$

That is

$$n-l_r+\sum_{j=r+1}^{m}\frac{\alpha_j(\alpha_j-1)}{l_{j-1}}\le l_r-1. \quad (7)$$

We denote

$$r_1 = \min\{p \mid \frac{1}{l_p - 1}\sum_{j=p+1}^{m}\frac{\alpha_j(l_j-1)}{l_{j-1}} \le 1\}, \text{ where } l_p = \sum_{j=1}^{p}\alpha_j .$$

Because $\frac{n-l_r}{l_r-1} \ge 1$ for $r < r_0$,we have $r_1 \ge r_0$.

(i) In the case $r_1 > r_0$, we have $v_r < A_r \le u_r$ for $r_1 \le r < m$, and $A_r > u_r$ for $r_0 \le r < r_1$. We need to discuss the value of A_r for $r_0 \le r < r_1$. Since we have $A_{r_1-1} \ge u_{r_1-1}$, apply to the second sub-formula of（5), then $A_{r_1-2} = A_{r_1-1} \ge u_{r_1-1}$.Moreover, for $r_0 \le r < r_1$, $A_{r-1} = A_r$, we have

$$A_{r_0-1} = A_{r_0} = \cdots = A_{r_1-2} = A_{r_1-1} \ge u_{r_1-1}.$$

Now consider $l_r < \frac{n+1}{2}$, and then $u_r < v_r$. We need to compare the value of A_{r_0-1} and that of v_{r_0-1}. On one hand, if $l_{r_0} = \frac{n+1}{2}$, then $v_{r_0-1} \le v_{r_0}$, and $A_{r_1-1} > u_{r_1-1} \ge u_{r_0} \ge v_{r_0}$.Then we have $A_{r_0-1} > v_{r_0-1}$. On the other hand, if $l_{r_0} > \frac{n+1}{2}$, then $l_{r_0} \ge \frac{n+2}{2}$.Then

$$A_{r_1-1} \ge u_{r_0} \ge u_r\,|_{l_r=\frac{n+2}{2}} > v_r\,|_{l_r=\frac{n}{2}} \ge v_{r_0-1}.$$

That is, $A_{r_1-1} > v_{r_0-1}$. Here one can get $u_r\,|_{l_r=\frac{n+2}{2}} > v_r\,|_{l_r=\frac{n}{2}}$ from **Lemma 1** by direct computation, $u_r\,|_{l_r=\frac{n+2}{2}}$ means the value of u_r when $l_r = \frac{n+2}{2}$, and $v_r\,|_{l_r=\frac{n}{2}}$ means the value of v_r when $l_r = \frac{n}{2}$. Therefore we have $A_{r_0-1} > v_{r_0-1}$.

In sum we have $A_{r_0-1} > v_{r_0-1} > u_{r_0-1}$. From the second sub-formula of （5）, for $1 \le r < r_0$, $A_{r-1} = A_r$, we have

$$A_0 = A_1 = \cdots = A_{r_0-2} = A_{r_0-1}.$$

(ii) In the case $r_1 = r_0$, we have $v_r < A_r \le u_r$ for $r_1 \le r < m$.

If $l_{r_0} = \frac{n+1}{2}$, then $A_{r_0} > v_{r_0} = u_{r_0}$, that is $r_1 > r_0$, which is contradicted. It means that when $l_{r_0} = \frac{n+1}{2}$, there won't be $r_1 = r_0$.

If $l_{r_0} > \frac{n+1}{2}$, then $v_{r_0} < A_{r_0} < u_{r_0}$. Applying it to the second sub-formula of （5）, we have

$$A_{r_0} = \frac{l_{r_0-1}}{n(n-1)} \sum_{j=r_0}^{m} \frac{\alpha_j (l_j - 1)}{l_{j-1}}.$$

By direct computation we have $A_{r_0-1} > v_{r_0-1}$ (similar as the discussion of （6）). Applying it to the second sub-formula of （5）, we have $A_{r-1} = A_r$ for $1 \le r < r_0$, that is

$$A_0 = A_1 = \cdots = A_{r_0-2} = A_{r_0-1}.$$

Remark 1. The recursive formula (5) gives a partial differential equation that is called Hamilton–Jacobi–Bellman (HJB) equation, and A_r can be viewed as the "value function", which gives the minimum cost (or maximal revenue) for a given dynamical system with an associated cost function. The optimal strategy is determined by values of the three max-functions of last two terms in the second sub-formula.

2.2 Example

In this section we give a concrete example of the **Section 2.1**.

Theorem 2.

Assume n is even, and in each group there are two students， i.e., $n = 2m$, $\alpha_j = 2$, $1 \le j \le m$, set $r_0 = \min\{r \mid l_r \ge \frac{n+1}{2}\}$, then

$$u_r = \frac{2r(2r-1)}{n(n-1)},$$

$$v_r = \frac{2r(n-2r)}{n(n-1)}.$$

Moreover,

$$A_r = \begin{cases} \dfrac{2(r_0-1)}{n(n-1)} \displaystyle\sum_{j=r_0}^{m} \dfrac{2j-1}{j-1}, & 0 \le r < r_0, \\ \dfrac{2r}{n(n-1)} \displaystyle\sum_{j=r+1}^{m} \dfrac{2j-1}{j-1}, & r_0 \le r < m. \end{cases} \quad (8)$$

Proof for Theorem 2. We have

$$A_r = \begin{cases} \max(A_{r+1}, v_{r+1}), & r = 0, \\ \dfrac{r(2r-1)}{(r+1)(2r+1)} A_{r+1} + \dfrac{2r}{(r+1)(2r+1)} \left(\max(A_{r+1}, v_{r+1}) + \max(A_{r+1}, u_{r+1}) \right) & \\ + \dfrac{1}{(r+1)(2r+1)} \max(A_{r+1}, u_{r+1}, v_{r+1}), & 1 \le r < m, \\ 0, & r = m. \end{cases} \quad (9)$$

Since n is even, if $2r \le \frac{n}{2}$, then $u_r < v_r$, and if $2r > \frac{n}{2}$, then $u_r > v_r$.

To simplify the formula (9), first we consider $l_r > \frac{n+1}{2}$, then $u_r > v_r$. We have $2r \ge \frac{n}{2} + 1$, assume $v_r < A_r < u_r$. From the second sub-formula of (9), we have

$$A_{r-1} = \frac{r-1}{r} A_r + \frac{2r(2r-1)}{n(n-1)}. \quad (10)$$

Then

$$A_r = \frac{2r}{n(n-1)} \sum_{j=r+1}^{m} \frac{2j-1}{j-1}. \quad (11)$$

Since

$$\frac{2j-1}{j-1} = 2 + \frac{1}{j-1} > 2,$$

hence

$$\sum_{j=r+1}^{m} \frac{2j-1}{j-1} > n - 2r.$$

That is, $A_r > v_r$, which assures our assumption.

Moreover, to satisfy $A_r < u_r$, we need

$$\sum_{j=r+1}^{m} \frac{2j-1}{j-1} < 2r - 1.$$

Write it as

$$\sum_{j=r+1}^{m}\frac{1}{j-1}<4r-n-1.$$

Since $4r-n-1\geq 1$ is true for all $2r\geq\frac{n}{2}+1$, and $\sum_{j=r+1}^{m}\frac{1}{j-1}\leq 1$ is true for all $2r\geq\frac{n}{2}+1$. Hence $A_r<u_r$ is true for $r>r_0$.

Furthermore, if $2r<\frac{n}{2}+1$, then （11） tells $A_r>u_r$, that is at least we have $A_{r_0-1}>u_{r_0-1}$. Notice that even though $A_{r_0-1}>u_{r_0-1}$ does not satisfy our assumption $v_r<A_r<u_r$, we can induce the value of A_{r_0-1} from A_{r_0} by applying the second sub-formula of （9）. Moreover, $A_{r_0-1}>v_{r_0-1}$ (the proof is similar as the above discussion of formula (11)).

Now we consider $l_r<\frac{n+1}{2}$, that is $u_r<v_r$. First we have $A_{r_0-1}>v_{r_0-1}>u_{r_0-1}$, then we have $A_{r-1}=A_r$ for $0<r<r_0$ by applying the second sub-formula of （9）. That is

$$A_0=A_1=\cdots=A_{r_0-2}=A_{r_0-1}.$$

Remark 2. From the proof of **Theorem 2**, we have the optimal stopping rules:

Reject the first r_0-1 groups of applicants, then accept currently second best student that comes along. When n goes to infinite, the maximal win probability is nearly

$$\lim_{n\to\infty}\frac{2(r_0-1)}{n(n-1)}\sum_{j=r_0}^{m}\frac{2j-1}{j-1}=\frac{1}{4}.$$

2.3 Panel interview case: $k>2$

Assume the interviewer has met r groups of candidates, and selects the currently i-th best, who is actually the k-th best of all candidates. Let $v_{i,r}$ be the probability of the currently i-th best out of l_r candidates ends up to be the k-th best.

For simplicity, assume $\alpha_j = 2$, $1 \le j \le m$ and n is even, then $l_r = 2r$ and $n = 2m$.

Lemma 2. *For* $i < k$*,*

$$v_{i,r} = \frac{\binom{2r}{i}\binom{n-2r}{k-i}}{\binom{n}{k}}\frac{i}{k}. \qquad (12)$$

Proof for Lemma 2. We have

$$v_{i,r} = \begin{cases} \frac{(2r+2-i)(2r+1-i)}{(2r+2)(2r+1)}v_{i,r+1} + \frac{i(2r+1-i)}{(2r+2)(2r+1)}v_{i+1,r+1} \\ \frac{i(2r+2-(i+1))}{(2r+2)(2r+1)}v_{i+1,r+1} + \frac{i(i+1)}{(2r+2)(2r+1)}v_{i+2,r+1}, & i \le 2r < n, \\ 1, & i = k, r = m, \\ 0, & i \ne k, r = m. \end{cases} \qquad (13)$$

The first sub-formula of (13) can be written as

$$v_{i,r} = \frac{(2r+2-i)(2r+1-i)}{(2r+2)(2r+1)}v_{i,r+1} + \frac{2i(2r+1-i)}{(2r+2)(2r+1)}v_{i+1,r+1} + \frac{i(i+1)}{(2r+2)(2r+1)}v_{i+2,r+1}.$$

Backward induction on i and r proves the uniqueness of $v_{i,r}$.

Moreover,

$$\frac{(2r+2-i)(2r+1-i)}{(2r+2)(2r+1)}v_{i,r+1}+\frac{2i(2r+1-i)}{(2r+2)(2r+1)}v_{i+1,r+1}+\frac{i(i+1)}{(2r+2)(2r+1)}v_{i+2,r+1}$$

$$=\frac{(2r+2-i)(2r+1-i)}{(2r+2)(2r+1)}\frac{\binom{2r+2}{i}\binom{n-2r-2}{k-i}}{\binom{n}{k}}\frac{i}{k}$$

$$+\frac{2i(2r+1-i)}{(2r+2)(2r+1)}\frac{\binom{2r+2}{i+1}\binom{n-2r-2}{k-i-1}}{\binom{n}{k}}\frac{i+1}{k}$$

$$+\frac{i(i+1)}{(2r+2)(2r+1)}\frac{\binom{2r+2}{i+2}\binom{n-2r-2}{k-i-2}}{\binom{n}{k}}\frac{i+2}{k}$$

$$=\frac{\binom{2r}{i}\binom{n-2r-2}{k-i}}{\binom{n}{k}}\frac{i}{k}+2\frac{\binom{2r}{i}\binom{n-2r-2}{k-i-1}}{\binom{n}{k}}\frac{i}{k}+\frac{\binom{2r}{i}\binom{n-2r-2}{k-i-2}}{\binom{n}{k}}\frac{i}{k}$$

$$=\frac{\binom{2r}{i}\binom{n-2r}{k-i}}{\binom{n}{k}}\frac{i}{k}.$$

Here, notice that we used the combinatorial identity:

$$\binom{n-2r}{k-i}=\binom{n-2r-2}{k-i}+2\binom{n-2r-2}{k-i-1}+\binom{n-2r-2}{k-i-2}.$$

Theorem 3.

Let A_r denote the optimal probability of success given that r groups of applicants have been interviewed and rejected.

$$A_r=\frac{(2r+2-k)(2r+1-k)}{(2r+2)(2r+1)}A_{r+1}+\sum_{i=1}^{k}\left(\frac{1}{2r+1}+\frac{1}{2r+2}-\frac{2i-1}{(2r+2)(2r+1)}\right)\max(A_{r+1},v_{i,r+1}).$$

It is not possible to have explicit solutions except the four cases $k=1,2,n-1,n$. However, with the above equation, the result is easy to compute numerically.

Using computer program, we have the figures of cases when $k=3$ and $k=5$.

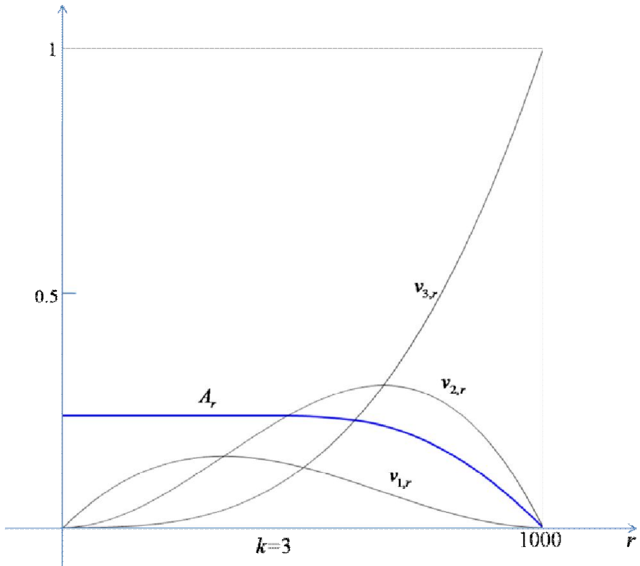

FIGURE 2. A plot of A_r, and $v_{i,r}$ $(1 \le r < m)$ for m=1000

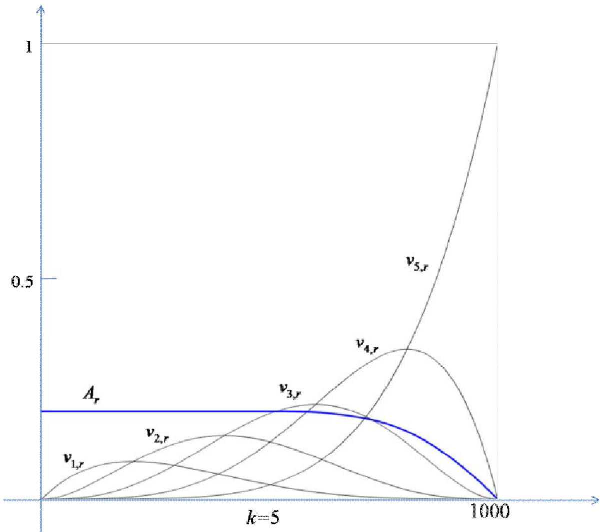

FIGURE 3. A plot of A_r, and $v_{i,r}$ $(1 \le r < m)$ for m=1000

From the figure above, we notice that A_r remains constant when r is small, and decrease when r approaches m. Notice $v_{k,m} = 1$. $v_{k,r}$ increases rapidly when r approaches m. For $v_{i,r}\,(1 \le i < k)$, they first increase, then decrease and finally reach 0.

3. Find the best w out of n

In this section, the interviewer randomly divides all the n candidates into m groups $\{g_j\}_{j=1}^{m}$. In group g_j there are α_j students, $\sum_{j=1}^{m} \alpha_j = n$. Each time the interviewer meets one group. Then the interviewer either selects some students or rejects all the candidates in the group. What is the optimal strategy to find the best w student?

We follow the algorithm of F. Thomas Bruss and Guy Louchard[1]. Similarly we denote w thresholds of groups: $1 \le j_1 < j_2 < \cdots < j_w \le m$, and define two sorts of candidates:

- ◇ **Compulsory candidates** i.e., candidates who should be accepted by the interviewer based on the current information. For example, the interviewer has accepted so far k candidates, and then he should accept the coming student who is better than some of the k candidates.
- ◇ **Marginal candidates** i.e., candidates who are relative better than or equal to the $(k+1)$-th best.

For all possible values of j_1 with $1 \le j_1 \le m$, first choose the relative best (if any) located at position $p_1 > j_1$, then for each possible position p_1 compute an optimal threshold $j_2(p_1) \ge p_1$. If there is a compulsory candidate at $p_2 < j_2$, the interview should accept and start the strategy at p_2 with a new threshold $j_3(p_2)$. If there is no compulsory candidates before j_2, then from j_2 on, choose a compulsory candidate or a marginal candidate (if any) at position p_2. And then start again the strategy at p_2 with a new threshold $j_3(p_2)$. And we continue such process to the threshold $j_i(p_{w-1})$. Here $p_1, \cdots, p_w$ are groups that contain selected people.

3.1 Panel interview case: $w = 2$

For the panel interview case with $w = 2$, define a (j_1, j_2)-strategy as following:

Reject the first j_1 groups of candidates, and then there are four possibilities:

- ◇ From j_1 on, the interviewer accepts the currently best two candidates up to a certain index j_2, and they are in the same group;

◇ From j_1 on, the interviewer accepts the currently best two candidates up to a certain index j_2, but they are not in the same group.

◇ From j_1 on, choose the currently best (if any) compared to all previous candidates. And if the interviewer accepts only one relative best candidate up to a certain index j_2, then selects a currently best or second best after j_2.

◇ From j_1 to j_2 , no one satisfies the condition of enrollment, and after j_2 the interviewer accepts the currently best two candidates, which are possibly in or not in the same group.

For simplicity, we consider the case n is even, and $\alpha_j = 2$ for all j and then $m = \frac{n}{2}$. With those assumptions, we have

Theorem 4. *The optimal thresholds* j_1^* *and* j_2^* *for the panel interview case with* $w = 2$ *satisfy the asymptotic relationship as m goes to infinity:*

$$\frac{j_1^*}{m} \to -e^{-\frac{1}{2}} W(-e^{-3+e^{\frac{1}{2}}}) = 0.2291147285\cdots,$$

$$\frac{j_2^*}{m} \to e^{-\frac{1}{2}} = 0.6065306596\cdots,$$

where $W(x)$ *is the Lambert's function. The asymptotic success probability of the* (j_1^*, j_2^*)*-strategy equals* $0.2254366561\cdots$.

Proof for Theorem 4. Given *p*, first we compute $j_2(p_1) \geq p_1$. We have three possibilities: $p_1 < j_2 \leq p_2$, $p_1 < p_2 < j_2$ and $p_1 = p_2 < j_2$.

The probability of success of the first case $p_1 < j_2 \leq p_2$ is given by

$$P_1 = \sum_{p_2=j_2}^{m} \prod_{k=p_1+1}^{j_2-1} \frac{2k-2}{2k-1}\frac{2k-1}{2k} \prod_{l=j_2}^{p_2-1} \frac{2l-3}{2l-1}\frac{2l-2}{2l} \cdot \frac{4(2p_2-2)}{2p_2(2p_2-1)} \cdot \prod_{s=p_2+1}^{m} \frac{2s-3}{2s-1}\frac{2s-2}{2s}$$

$$= \sum_{p_2=j_2}^{m} \prod_{k=p_1+1}^{j_2-1} \left(1-\frac{1}{k}\right) \prod_{l=j_2}^{p_2-1} \left(1-\frac{3}{l}+\frac{2}{2l-1}\right) \cdot \left(\frac{4}{p_2}-\frac{4}{2p_2-1}\right) \cdot \prod_{s=p_2+1}^{m} \left(1-\frac{3}{s}+\frac{2}{2s-1}\right)$$

Notice $\frac{1}{2k-1} > \frac{1}{2k}$, $\frac{1}{k+1} \le \frac{2}{k} - \frac{2}{2k-1}$ for $k \ge 2$, $-\frac{3}{k} + \frac{2}{2k-1} \le -\frac{2}{k+1}$ for $k \ge 1$.
Then we have

$$\begin{aligned} P_1 &> \sum_{p_2=j_2}^{m} \prod_{k=p_1+1}^{j_2-1} \left(1-\frac{1}{k}\right) \prod_{l=j_2}^{p_2-1} \left(1-\frac{2}{l}\right) \cdot \frac{2}{p_2+1} \cdot \prod_{s=p_2+1}^{m} \left(1-\frac{2}{s}\right) \\ &= \sum_{p_2=j_2}^{m} \frac{2p_1p_2(j_2-2)}{m(m-1)(p_2-2)(p_2+1)} \triangleq p_1S_1(p_1, j_2), \end{aligned}$$

and

$$\begin{aligned} P_1 &< \sum_{p_2=j_2}^{m} \prod_{k=p_1+1}^{j_2-1} \left(1-\frac{1}{k}\right) \prod_{l=j_2}^{p_2-1} \left(1-\frac{2}{l+1}\right) \cdot \frac{2}{p_2} \cdot \prod_{s=p_2+1}^{m} \left(1-\frac{2}{s+1}\right) \\ &= \sum_{p_2=j_2}^{m} \frac{2j_2p_1(p_2+1)}{m(m+1)(p_2-1)p_2} \triangleq p_1T_1(p_1, j_2). \end{aligned}$$

The probability of success of the second case $p_1 < p_2 < j_2$ is given by

$$\begin{aligned} P_2 &= \sum_{p_2=p_1+1}^{j_2-1} \prod_{k=p_1+1}^{p_2-1} \frac{2k-2}{2k-1}\frac{2k-1}{2k} \cdot \frac{2(2p_2-2)}{2p_2(2p_2-1)} \cdot \prod_{s=p_2+1}^{m} \frac{2s-3}{2s-1}\frac{2s-2}{2s} \\ &= \sum_{p_2=p_1+1}^{j_2-1} \prod_{k=p_1+1}^{p_2-1} \left(1-\frac{1}{k}\right) \cdot \left(\frac{2}{p_2} - \frac{2}{2p_2-1}\right) \cdot \prod_{s=p_2+1}^{m} \left(1-\frac{3}{s}+\frac{2}{2s-1}\right) \end{aligned}$$

Similarly,

$$\begin{aligned} P_2 &> \sum_{p_2=p_1+1}^{j_2-1} \prod_{k=p_1+1}^{p_2-1} \left(1-\frac{1}{k}\right) \cdot \frac{1}{p_2+1} \cdot \prod_{s=p_2+1}^{m} \left(1-\frac{2}{s}\right) \\ &= \sum_{p_2=p_1+1}^{j_2-1} \frac{p_1p_2}{m(m-1)(p_2+1)} \triangleq p_1S_2(p_1, j_2), \end{aligned}$$

P_2 is monotone in j_2, so does S_2 and T_2.

$$\begin{aligned} P_2 &< \sum_{p_2=p_1+1}^{j_2-1} \prod_{k=p_1+1}^{p_2-1} \left(1-\frac{1}{k}\right) \cdot \frac{1}{p_2} \cdot \prod_{s=p_2+1}^{m} \left(1-\frac{2}{s+1}\right) \\ &= \sum_{p_2=p_1+1}^{j_2-1} \frac{p_1(p_2+1)}{m(m+1)p_2} \triangleq p_1T_2(p_1, j_2). \end{aligned}$$

Notice that $p_1S_1(p_1, j_2) + p_1S_2(p_1, j_2)$ and $p_1T_1(p_1, j_2) + p_1T_2(p_1, j_2)$ are unimodal in j_2, i.e.,

$$\begin{aligned} & p_1S_1(p_1, j_2) + p_1S_2(p_1, j_2) - p_1S_1(p_1, j_2-1) - p_1S_2(p_1, j_2-1) \\ &= -\frac{p_1}{m(m-1)}\left(1-\frac{1}{j_2} - \sum_{p_2=j_2}^{m} \frac{2}{(p_2-2)(p_2+1)}\right) \end{aligned} \quad (14)$$

changes sign at most once.

The probability of success of the third case $p_1 = p_2 < j_2$ is given by

$$P_3 = \prod_{s=p_1+1}^{m} \frac{2s-3}{2s-1}\frac{2s-2}{2s} = \prod_{s=p_1+1}^{m}\left(1-\frac{3}{s}+\frac{2}{2s-1}\right).$$

Moreover,

$$\prod_{s=p_1+1}^{m}\left(1-\frac{2}{s}\right) < P_3 < \prod_{s=p_1+1}^{m}\left(1-\frac{2}{s+1}\right).$$

And we denote

$$\prod_{s=p_1+1}^{m}\left(1-\frac{2}{s}\right) = \frac{p_1(p_1-1)}{m(m-1)} \triangleq p_1 S_3(p_1, j_2),$$

$$\prod_{s=p_1+1}^{m}\left(1-\frac{2}{s+1}\right) = \frac{p_1(p_1+1)}{m(m+1)} \triangleq p_1 T_3(p_1, j_2).$$

Here S_3 and T_3 are affine linear in j_2 (actually do not involve with j_2), then we have

$$p_1 S_1(p_1, j_2) + p_1 S_2(p_1, j_2) + p_1 S_3(p_1, j_2)$$

and

$$p_1 T_1(p_1, j_2) + p_1 T_2(p_1, j_2) + p_1 T_3(p_1, j_2)$$

are unimodal in j_2.

We want to maximize the value

$$p_1 S_1(p_1, j_2) + p_1 S_2(p_1, j_2) + p_1 S_3(p_1, j_2).$$

As $m \to \infty$, we would like to replace each term of sums by integrals, and use continuous variables by normalizing:

$$p_1' \triangleq \frac{p_1}{m},\ p_2' \triangleq \frac{p_2}{m},\ j_2' \triangleq \frac{j_2}{m} \tag{15}$$

And we continue to use p_1, p_2 and j_2 (a bit abuse of notations) as continuous variables. It is equivalent to maximize the asymptotic expression

$$V_1(p_1, j_2) \triangleq p_1\left(\int_{p_2=j_2}^{1} \frac{2j_2}{p_2} dp_2 + \int_{p_2=p_1}^{j_2} 1 dp_2\right) + p_1^2$$
$$= \left(-2j_2 \ln(j_2) + j_2\right) p_1,$$

here notice that $\frac{p_2-2}{m}, \frac{p_2+1}{m}$ and $\frac{p_2}{m}$ are asymptotic equivalent as $m \to \infty$.

Moreover, write $V_1(p_1, j_2)$ as a product $p_1\varphi(j_2)$. Since

$$\frac{d\varphi(j_2)}{dj_2} = -2\ln(j_2) - 1$$

is monotone in j_2, $\varphi(j_2)$ is unimodal.

Let $\frac{dV_1(p_1, j_2)}{dj_2} = 0$, then $\ln(j_2) = -\frac{1}{2}$, which gives

$$j_2^* \to e^{-\frac{1}{2}} = 0.6065306596\cdots.$$

Now we want to consider

$$p_1T_1(p_1, j_2) + p_1T_2(p_1, j_2) + p_1T_3(p_1, j_2).$$

Similar as the above discussion, we replace terms of the sum by integrals and obtain $V_1'(p_1, j_2)$, which is the same as $V_1(p_1, j_2)$. Hence for our original problem, to maximize $P_1 + P_2 + P_3$ as $m \to \infty$, we should take

$$j_2^* \to e^{-\frac{1}{2}} = 0.6065306596\cdots.$$

Now we compute j_1. First notice that the first successful pick may happen after j_2^* (here we abuse[1] the notation of j_2^*, and it actually means $me^{-\frac{1}{2}}$). For $\varphi(j_2)$ is unimodal, $j_2(p_1)$ is exactly given by p_1, i.e.,

$$j_2(p_1) = \begin{cases} j_2^*, & p_1 < j_2^*, \\ p_1, & p_1 \geq j_2^*. \end{cases} \qquad (16)$$

There are three cases. Denote Q_1, Q_2 and Q_3 to be the win probabilities of the cases $p_1 < j_2^* < p_2$, $j_2^* \leq p_1 < p_2$ and $j_2^* \leq p_1 = p_2$, respectively.

We need to compute the aggregation of success probabilities

[1] When we transform the sum to integral, all j_1, j_2 are normalized by $j_1' \square \frac{j_1}{m}$, $j_2' \square \frac{j_2}{m}$, but we abuse the notations and still write them as j_1, j_2. After we got the desired values j_1^* and j_2^* by taking derivatives, the order numbers of the thresholds should be mj_1^* and mj_2^*, while we abuse the notations again and still write them as j_1^* and j_2^*.

$$Q(j_1)=Q_1(j_1)+Q_2(j_1)+Q_3(j_1)\,. \tag{17}$$

The probability Q_1 of success of the case $p_1 < j_2^* < p_2$ is given by

$$\begin{aligned}
Q_1 &= \sum_{p_1=j_1+1}^{j_2^*-1} \prod_{i=j_1+1}^{p_1-1} \frac{2i-2}{2i-1}\frac{2i-1}{2i} \cdot \frac{2(2p_1-2)}{2p_1(2p_1-1)} \cdot \left(P_1+P_2+P_3\right)(p_1, j_2^*) \\
&= \sum_{p_1=j_1+1}^{j_2^*-1} \prod_{i=j_1+1}^{p_1-1} \left(1-\frac{1}{i}\right) \cdot \left(\frac{2}{p_1}-\frac{2}{2p_1-1}\right) \cdot \left(P_1+P_2+P_3\right)(p_1, j_2^*) \\
&\triangleq R_1 \cdot \left(P_1+P_2+P_3\right)(p_1, j_2^*).
\end{aligned}$$

Notice that if $p_1 = j_1+1$, we do not have the term

$$\prod_{i=j_1+1}^{p_1-1} \frac{2i-2}{2i-1}\frac{2i-1}{2i}\,,$$

and we have

$$\sum_{p_1=j_1+1}^{j_2^*-1} \prod_{i=j_1+1}^{p_1-1} \left(1-\frac{1}{i}\right) \cdot \frac{1}{p_1+1} < R_1 < \sum_{p_1=j_1+1}^{j_2^*-1} \prod_{i=j_1+1}^{p_1-1} \left(1-\frac{1}{i}\right) \cdot \frac{1}{p_1}\,.$$

That is

$$\sum_{p_1=j_1+1}^{j_2^*-1} \frac{j_1}{(p_1+1)(p_1-1)} < R_1 < \sum_{p_1=j_1+1}^{j_2^*-1} \frac{j_1}{p_1(p_1-1)}\,.$$

The probability Q_2 of success of the case $j_2^* \le p_1 < p_2$ is given by

$$\begin{aligned}
Q_2 &= \sum_{p_1=j_2^*}^{m-1} \sum_{p_2=p_1+1}^{m} \prod_{i=j_1+1}^{p_1-1} \frac{2i-2}{2i-1}\frac{2i-1}{2i} \cdot \frac{2(2p_1-2)}{2p_1(2p_1-1)} \\
&\cdot \prod_{l=j_1+1}^{p_2-1} \frac{2l-2}{2l-1}\frac{2l-1}{2l} \cdot \frac{4(2p_2-2)}{2p_2(2p_2-1)} \cdot \prod_{s=p_2+1}^{m} \frac{2s-3}{2s-1}\frac{2i-2}{2s} \\
&= \sum_{p_1=j_2^*}^{m-1} \sum_{p_2=p_1+1}^{m} \prod_{i=j_1+1}^{p_1-1} \left(1-\frac{1}{i}\right) \cdot \left(\frac{2}{p_1}-\frac{2}{2p_1-1}\right) \\
&\cdot \prod_{l=j_1+1}^{p_2-1} \left(1-\frac{3}{l}+\frac{2}{2l-1}\right) \cdot \left(\frac{4}{p_2}-\frac{4}{2p_2-1}\right) \cdot \prod_{s=p_2+1}^{m} \left(1-\frac{3}{s}+\frac{2}{2s-1}\right)
\end{aligned}$$

Q_2 is monotone in j_1, and we have

$$\begin{aligned}
Q_2 &> \sum_{p_1=j_2^*}^{m-1} \sum_{p_2=p_1+1}^{m} \prod_{i=j_1+1}^{p_1-1} \left(1-\frac{1}{i}\right) \cdot \frac{1}{p_1+1} \cdot \prod_{l=j_1+1}^{p_2-1} \left(1-\frac{2}{l}\right) \cdot \frac{2}{p_2+1} \cdot \prod_{s=p_2+1}^{m} \left(1-\frac{2}{s}\right) \\
&= \sum_{p_1=j_2^*}^{m-1} \sum_{p_2=p_1+1}^{m} \frac{2p_1p_2j_1}{m(m+1)(p_1+1)(p_2-2)(p_2+1)},
\end{aligned}$$

and

$$Q_2 < \sum_{p_1=j_2^*}^{m-1} \sum_{p_2=p_1+1}^{m} \prod_{i=j_1+1}^{p_1-1} \left(1-\frac{1}{i}\right) \cdot \frac{1}{p_1} \cdot \prod_{l=j_1+1}^{p_2-1} \left(1-\frac{2}{l+1}\right) \cdot \frac{2}{p_2} \cdot \prod_{s=p_2+1}^{m} \left(1-\frac{2}{s+1}\right)$$
$$= \sum_{p_1=j_2^*}^{m-1} \sum_{p_2=p_1+1}^{m} \frac{2j_1(p_1+1)(p_2+1)}{m(m+1)(p_1-1)p_2(p_2-1)}.$$

The probability Q_3 of success of the case $j_2^* \le p_1 = p_2$ is given by,

$$Q_3 = \sum_{p_2=p_1=j_2^*}^{m} \prod_{i=j_1+1}^{p_1-1} \frac{2i-2}{2i-1}\frac{2i-1}{2i} \cdot \frac{2}{2p_1(2p_1-1)} \cdot \prod_{s=p_1+1}^{m} \frac{2s-3}{2s-1}\frac{2i-2}{2s}$$
$$= \sum_{p_2=p_1=j_2^*}^{m} \prod_{i=j_1+1}^{p_1-1} \left(1-\frac{1}{i}\right) \cdot \frac{2}{2p_1(2p_1-1)} \cdot \prod_{s=p_1+1}^{m} \left(1-\frac{3}{s}+\frac{2}{2s-1}\right).$$

Q_3 is also monotone in j_1, and we have

$$\sum_{p_2=p_1=j_2^*}^{m} \frac{2j_1}{m(m-1)(2p_1-1)} < Q_3 < \sum_{p_2=p_1=j_2^*}^{m} \frac{2j_1(p_1+1)}{m(m+1)(p_1-1)(2p_1-1)}.$$

In sum, we can control the sum

$$Q(j_1) = Q_1(j_1) + Q_2(j_1) + Q_3(j_1)$$

by two sides and it is easy to check that two bounds of $Q(j_1)$ are unimodal in j_1 (similar as (14), one can show the difference changes the sign at most once). Similar as we compute the value $V_1(p_1, j_2)$, we normalize p_1, p_2 and j_1 as (15), and replace the terms of (17) by integrals, and then compute the asymptotic success probability of $Q(j_1)$ by two sides, which gives the same

$$V_2(j_1) \approx \int_{p_1=j_1}^{j_2^*} \frac{j_1}{p_1^2} V_1(p_1, j_2^*) dp_1 + \int_{p_1=j_2^*}^{1} \int_{p_2=p_1}^{1} \frac{2j_1}{p_2} dp_2 dp_1 + o(1) \int_{p_1=j_2^*}^{1} \frac{j_1}{p_1} dp_1$$
$$= j_1 \left[j_1 - 2\ln(j_1) e^{-\frac{1}{2}} - 5e^{-\frac{1}{2}} + 2 \right].$$

To maximize $V_2(j_1)$, let $\frac{dV_2(j_1)}{dj_1} = 0$, that is

$$2j_1 - 2\ln(j_1)e^{-\frac{1}{2}} - 7e^{-\frac{1}{2}} + 2 = 0,$$

$$-e^{-3+e^{\frac{1}{2}}} = -j_1 e^{\frac{1}{2}} e^{-j_1 e^{\frac{1}{2}}},$$

Then, we solve the equation, and gives

$$j_1^* = -e^{-\frac{1}{2}} W(-e^{-3+e^{\frac{1}{2}}}),$$

where $W(x)$ is the Lambert function, i.e., $x = ye^{y}$, then $y = W(x)$.

With Matlab, we have $j_1^* = 0.2291147285\cdots$

Furthermore, we have $V_2(j_1) = 0.2254366561\cdots$, which is the maximal probability of success of (j_1^*, j_2^*)-strategy. The unimodality of $V_2(j_1)$ implies the uniqueness of j_1^*. Transform this normalized j_1^* to the original threshold by multiplying m, still denote as j_1^* (again here we abuse the notation a bit), and we have finished the proof.

3.2 Panel interview case: $w = 3$

For simplicity, we only consider the case $n = 3m$ and $\alpha_j = 3$ for $1 \le j \le m$. For $w = 3$, define a (j_1, j_2, j_3)-strategy similar as the case $w = 2$, then we have

Theorem 4. *The optimal thresholds j_1^*, j_2^* and j_3^* for the panel interview case with $w = 3$ satisfy the asymptotic relationship as m goes to infinity:*

$$\frac{j_1^*}{m} \to 0.1666171752\cdots,$$

$$\frac{j_2^*}{m} \to -e^{-\frac{1}{3}} W(-e^{-\frac{5}{2}+e^{\frac{1}{3}}}) = 0.4369818602\cdots,$$

$$\frac{j_3^*}{m} \to e^{-\frac{1}{3}} = 0.7165313106\cdots.$$

$W(x)$ is the Lambert's function, and the asymptotic success probability of the (j_1^*, j_2^*, j_3^*)-strategy equals $0.1625200069\cdots$.

Proof for Theorem 4. The proof is quite similar as the proof of case $w = 2$ but a bit more complicated. However, one can finish it by using techniques of the proof of the case $w = 2$ and **section 3** of Bruss and Louchard[1].

Remark 3. One can generalize the computational technique to any $w > 3$ with the restriction $n = wm$ and $\alpha_j = w$ for $1 \le j \le m$. It is harder to discuss the general case with no restriction on the number of students in each group.

4. Find the t-th best of n with uncertainty

M. H. Smith[12] studied the problem of finding the best with the assumptions of standard secretary problem and an additional condition that each applicant has a fixed probability $1-p$ to refuse an offer. In the section, we extend his work from $t=1$ to $t \ge 2$, and also give a remark on the problem in panel interview case at the end.

4.1 Matriculate with uncertainty: $t=2$

To find exactly the second best candidate, we assume the interviewer picks up student θ among the r interviewed applicants, and finishes interviewing all the n applicants to see whether the choice is right.

Lemma 3.

(i) *If student θ is currently the second best, the probability z_r of the student θ being the second best at the end is*

$$z_r = \frac{pr(r-1)}{n(n-1)}, \qquad 1 \le r \le n. \qquad (18)$$

(ii) *If student θ is currently the best, the probability y_r of the student θ being the second best at the end is*

$$y_r = \frac{pr(n-r)}{n(n-1)}, \quad 1 \le r \le n. \qquad (19)$$

Proof for Lemma 3. For case (i), we have

$$z_r = \begin{cases} \dfrac{r-1}{r+1} z_{r+1}, & 1 \le r < n, \\ p, & r = n. \end{cases}$$

For case (ii), we have the recurrence formula

$$y_r = \begin{cases} \dfrac{r}{r+1} y_{r+1} + \dfrac{1}{r+1} y_{r+1}, & 1 \le r < n, \\ 0, & r = n. \end{cases}$$

Let B_r be the probability of success under the condition that the first r applicants have been rejected, and then we have the related HJB equation:

$$B_r = \begin{cases} \max(B_{r+1}, py_{r+1}), & r = 0, \\ \dfrac{r-1}{r+1} B_{r+1} + \dfrac{1}{r+1}\left(\max(B_{r+1}, py_{r+1}) + \max(B_{r+1}, pz_{r+1})\right) & 1 \le r < n \\ 0, & r = n. \end{cases} \quad (20)$$

Here, for the second sub-formula of (20), when the $(r+1)$-th applicant is coming, an candidate may have $\frac{r-1}{r+1}$ possibility worse than all previous r candidates, and the interviewer should reject that candidate. He or she has $\frac{1}{r+1}$ possibility to be currently the second best. If the interviewer rejects that candidate, the win probability will be B_{r+1}.If the interviewer accepts the applicant, he or she has p possibility to accept the offer, and the win probability is z_{r+1}. The $(r+1)$-th applicant has $\frac{1}{r+1}$ possibility to be currently the best. If the interviewer rejects him or her, the win probability will be B_{r+1}.If the interviewer accepts the applicant, he or she has p possibility to accept the offer, and the win probability is y_{r+1}.

Set $r_0 = [\frac{n}{2}]$, then $r_0 = \frac{n}{2}$ if n is even and $r_0 = \frac{n-1}{2}$ if n is odd. Then we have

Theorem 5. *Let* $r_0 = [\frac{n}{2}]$, *then*

$$B_r = \begin{cases} \dfrac{p^2 r_0(n-r_0)}{n(n-1)}, & 0 \le r < r_0, \\ \dfrac{p^2 r(n-r)}{n(n-1)}, & r_0 \le r \le n. \end{cases} \quad (21)$$

Proof for Theorem 5. It is clear that (21) is true for $r=n$. For $r>r_0$, we have $y_r<z_r$, and assume $py_r<B_r<pz_r$ for $r_0<r<n$, then from the second sub-formula of the above HJB equation (20), we have

$$B_r=\frac{p^2r(n-r)}{n(n-1)}=py_r.$$

That means (21) is true for $r_0<r<n$. From the second sub-formula of (20), (21) is also true for $r=r_0$.

Now we want to show (21) is correct for $0\le r<r_0$. No matter n is even or odd, we have $B_{r_0}=py_{r_0}\ge py_r\ge pz_r$ for $0<r\le r_0$.Applying the second sub-formula of the equation (20), we have $B_{r-1}=B_r$, that is for $0\le r<r_0$,
we have

$$B_0=B_1=\cdots=B_{r_0-2}=B_{r_0-1}=B_{r_0}.$$

Here we give one optimal strategy:

Reject the first r_0 applicants. After that, accept the first currently second best applicant that comes sequentially.

With this optimal strategy, the win probability of finding the second best student is nearly

$$\lim_{n\to\infty}B_{r_0}=\lim_{n\to\infty}\frac{p^2r_0(n-r_0)}{n(n-1)}=\frac{p^2}{4}.$$

Remark 4. The maximal win probability of finding the best student with uncertainty is $p^{\frac{1}{1-p}}$, as n goes to infinity (see Smith[12]), which is always larger than the maximal win probability of finding the second best student with uncertainty, i.e., $\frac{p^2}{4}$.

4.2 Matriculate with uncertainty: $t > 2$

Assume the interviewer has met r candidates, and selects the currently i-th best, who is actually the t-th best of all candidates at the end. Let $f_{i,r}$ be the probability of the currently i-th best ends up to be the t-th best.

Theorem 6. *For* $i < t$,

$$f_{i,r} = \frac{\binom{r}{i}\binom{n-r}{t-i}}{\binom{n}{t}} \frac{ip}{t}.$$

Proof for Theorem 6. We have the recurrence formula

$$f_{i,r} = \begin{cases} \frac{r-i+1}{r+1} f_{i,r+1} + \frac{i}{r+1} f_{i+1,r+1} & i \le r < n \\ p & i = t, r = n \\ 0 & i \ne t, r = n \end{cases}$$

Similar as the proof of **Lemma 2**, one can use backward induction to finish the proof.

Remark 5. Let B_r be the optimal probability of success given r applicants have been interviewed and rejected.
The Hamilton-Jacobi-Bellman (HJB) equation is

$$B_r = \frac{r-t+1}{r+1} B_{r+1} + \sum_{i=1}^{t} \frac{\max(B_{r+1}, pf_{i,r+1})}{r+1}.$$

Besides the cases $t = 1, 2, n-1, n$, there is no explicit solution for other cases. However, the HJB equation gives the algorithm and one can easily compute the results numerically.

Remark 6. We can consider the problem of finding the t-th best of n with uncertainty in panel interview case. All the techniques can be found in **Section 2** and **Section 4**, the results are similar. It is easy to check that for $t = 2$, the relative z_r, y_r and B_r are:

$$z_r = \frac{pl_r(l_r - 1)}{n(n-1)}, \qquad 1 \le r \le m,$$

$$y_r = \frac{pl_r(n-l_r)}{n(n-1)}, \qquad 1 \le r \le m,$$

$$B_r = \begin{cases} \dfrac{p^2 l_{r_1-1}}{n(n-1)} \displaystyle\sum_{j=r_1}^{m} \frac{\alpha_j(l_j-1)}{l_{j-1}}, & 0 \le r < r_1, \\ \dfrac{p^2 l_r}{n(n-1)} \displaystyle\sum_{j=r+1}^{m} \frac{\alpha_j(l_j-1)}{l_{j-1}}, & r_1 \le r < m. \end{cases}$$

Here, all the setting is the same as that in **Section 2** except each applicants has a fixed probability $1-p$ to refuse an offer. The definitions of z_r, y_r, B_r and r_1 are the same as those in **Lemma 1** and **Theorem 1**.

Similarly we can discuss for $t > 2$ in panel interview case with special restrictions on the groups that are same as those in **Theorem 1** of **Section 2**. And the relative $f_{i,r}$ should be the result in **Lemma 2** times p, i.e.,

$$f_{i,r} = \frac{\binom{2r}{i}\binom{n-2r}{t-i}}{\binom{n}{t}} \frac{ip}{t}, \qquad i < t.$$

Notice that k in (12) is replaced by t here.

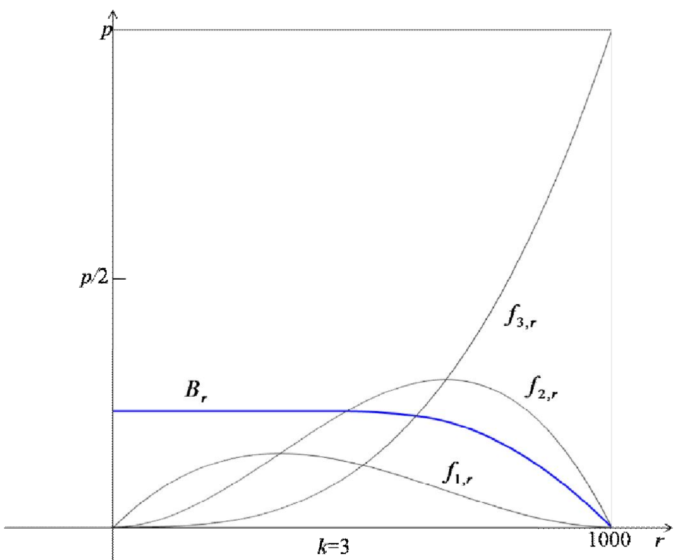

FIGURE 4. A plot of B_r, and $f_{i,r}(1 \le r < m)$ for m=1000

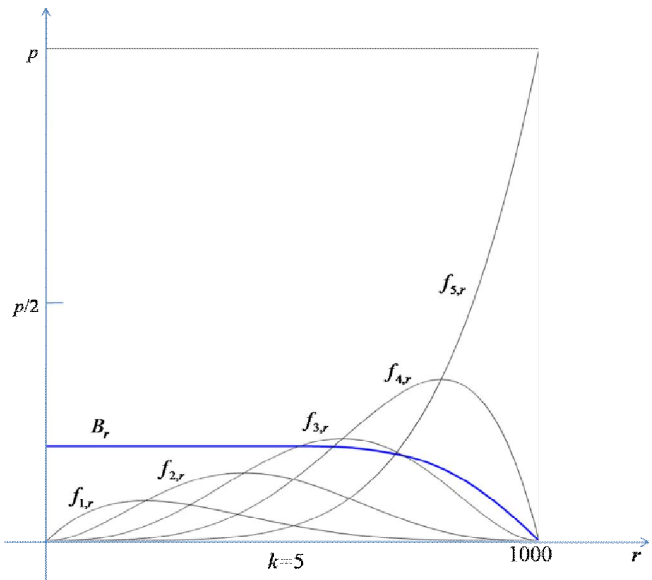

FIGURE 5. A plot of B_r, and $f_{i,r}$ $(1 \le r < m)$ for m=1000

From the figure above, we notice that B_r remains constant when r is small, and decrease when r approaches m. Notice $f_{k,m} = p \cdot f_{k,r}$ increases rapidly when r approaches m. For $f_{i,r}\left(1 \le i < k\right)$, they first increase, then decrease and finally reach 0.

5. Conclusions and Discussions

In the first two problems, we consider the panel interview case instead of one-by-one interview case, and the results of maximal probabilities of success of panel interview case are not worse than those of classical case. That is, using panel interview is more convenient and more realistic, which saves plenty of time of the university administrators. In addition, the techniques of proof in panel interview case can be used to generalize massive other previous works.

In the first problem of finding the second-best from a pool of n applicants by panel interview, we have given an explicit solution for general case. Specifically, the optimal strategy is to reject the groups that the sum of which nearly contains half of all applicants and then to accept the first currently-second-best applicant that arrives sequentially. In an special example with suitable restrictions, we give the probability of success using this strategy, which is about $\frac{1}{4}$. Apparently, it is harder to select the second-best from a pool of applicants than it is to select the best. Moreover, we give a general algorithm to compute the maximal win probability of selecting the k-th best students for $2 < k < n$.

In the second problem of finding best m out of n applicants by panel interview, we follow the algorithm of Bruss and Louchard[1] and discuss the problem in special setting of groups, in order to simplify the computation. And we prove that in the panel interview case, if each group contains the same number of people then the result of panel interview case is the same as that of the standard case that interviewer interviews applicants one by one for large enough n. Specially, the asymptotic win probabilities for $w=2$ and $w=3$ are $0.2254366561\cdots$ and $0.1625200069\cdots$, respectively.

In the last problem, we study the problem with all assumptions that are the same as those of the standard secretary problem besides each applicant has a fixed probability $1-p\,(0<p<1)$ to refuse an offer. The optimal strategy of finding the t-th ($2\le t<n$) is giving. Specifically, the optimal strategy is to reject (nearly) the first half applicants and then to accept the first second-best-so-far applicant that arrives sequentially. When n goes to infinity, the probability of success using this strategy is $\frac{p^2}{4}$, which is less than the win probability of finding the best applicant. We also give a short comment for the problem in panel interview case.

Of course, the results in our paper raise series of questions: what about the number of n is unknown in panel interview case or in matriculate with uncertainty case? What about the probability of rejection is not fixed but satisfies some kind of distribution in the matriculation with uncertainty case? What about in panel interview case the interviewer does not have full memory of the relative ranks of all interviewed students but only have partial memory of several latest interviewed students? Furthermore, there are various applications of secretary problem, for instance, applications to online auction by Robert Kleinberg[10](2005) and combinatorial auction. We leave these investigations to future work.

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