

A Proof of the Yau Geometric Conjecture in 6
Dimensions

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Abstract

The subject of counting positive lattice points in n -dimensional simplexes has interested mathematicians for decades due to its applications in singularity theory and number theory. Enumerating the lattice points in a right-angled simplex is equivalent to determining the geometric genus of a singularity of a weighted homogeneous complex polynomial. It is also a method to shed insight into large gaps in the sequence of prime numbers. Seeking to contribute to these applications, this research project proves the Yau Geometric Conjecture in six dimensions, a sharp upper bound for the number of positive lattice points in a six-dimensional tetrahedron. The main method of proof is summing existing sharp upper bounds for the number of points in 5-dimensional simplexes over the cross sections of the six-dimensional simplex. This research project paves the way for the proof of a fully general sharp upper bound for the number of lattice points in a simplex. It also moves the mathematical community one step closer towards proving the Yau Geometric and Yau Number-Theoretic Conjectures in full generality.

1 Introduction

Let Δ_n be an n -dimensional real right-angled simplex defined by the inequality

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leq 1,$$

where $x_1, \dots, x_n \geq 0$ and $a_1 \geq a_2 \geq \dots \geq a_n \geq 1$. Define P_n to be the number of positive integral points in Δ_n , or

$$P_n = \# \left\{ (x_1, x_2, \dots, x_n) \in \mathbb{Z}_+^n \mid \frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leq 1 \right\}.$$

Similarly, we define Q_n to be the number of non-negative integral points in Δ_n , or

$$Q_n = \# \left\{ (x_1, x_2, \dots, x_n) \in (\mathbb{Z}_+ \cup \{0\})^n \mid \frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leq 1 \right\}.$$

According to Granville [3], the numbers P_n and Q_n are intimately related to a number theoretic function known as the *Dickman–de Bruijn function*.

Definition 1.1. *The Dickman–de Bruijn function $\psi(x, y)$ is defined as the number of positive integers n such that $n \leq x$, and all of the prime factors of n are at most y , where x and y are positive integers.*

The connection, described by Luo, et. al. [9], is most readily observed by noting that, if $p_1 < p_2 < \cdots < p_k$ are the primes less than or equal to y , then,

$$p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k} \leq x,$$

is equivalent to,

$$e_1 \log p_1 + e_2 \log p_2 + \cdots + e_k \log p_k \leq \log x,$$

which can be rewritten as,

$$\frac{e_1}{\frac{\log x}{\log p_1}} + \frac{e_2}{\frac{\log x}{\log p_2}} + \cdots + \frac{e_k}{\frac{\log x}{\log p_k}} \leq 1,$$

an expression in the format of the condition in the definition of Q_n . Hence, enumerating the Dickman–de Bruijn function is equivalent to calculating Q_n .

Granville [3] also describes connections between P_n and Q_n and other areas of number theory, including primality testing, determining large gaps in the sequence of the primes, and discovering new algorithms for prime factorization. Furthermore, Lin, et. al. [6] describes how determining the values of P_n and Q_n leads to insights in singularity theory.

Chen [1] tells us that the numbers P_n and Q_n are intimately linked through the equation

$$P_n(a_1, a_2, \dots, a_n) = Q_n(a_1(1-a), a_2(1-a), \dots, a_n(1-a)),$$

where a is defined as $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$. Hence, we can essentially treat the tasks of finding P_n and Q_n to be equivalent in general.

The quest to find and estimate P_n and Q_n dates back to 1899, when Pick [11] discovered the famous *Pick's theorem*, or a formula for Q_2 .

$$Q_2 = \text{area}(\Delta) + \frac{|\partial\Delta \cap \mathbb{Z}^2|}{2} - 1,$$

where Δ is a 2-dimensional tetrahedron, or a triangle, $\partial\Delta$ represents the boundary of the triangle, and $|\partial\Delta \cap \mathbb{Z}^2|$ represents the number of integral points on the boundary. Mordell [10] continued by discovering a formula for Q_3 using Dedekind sums. Erhart [2] followed with the discovery of *Ehrhart polynomials*, which facilitate the calculation of Q_n . However, these polynomials are only useful if every coefficient is known, a condition that is extremely difficult to meet in the general case.

The difficulty of this problem eventually led mathematicians to start trying to bound P_n and Q_n instead of finding precise formulas. Lehmer [5] found that if $a = a_1 = a_2 = \cdots = a_n$, then

$$Q_n = \binom{\lfloor a \rfloor + n}{n}.$$

This formula naturally yields a nice definition of sharpness of an estimate R_n of Q_n . We consider the estimate sharp if and only if

$$R_n|_{a_1=a_2=\cdots=a_n=a \in \mathbb{Z}} = \binom{a+n}{n}.$$

In other words, any upper or lower bound is sharp if and only if its estimate is exact when $a_1 = a_2 = \cdots = a_n \in \mathbb{Z}$.

Another important estimate is the two-part *GLY Conjecture*, an upper bound for P_n formulated by Lin, et. al. [8]. However, to state the GLY Conjecture, we need to first introduce the *signed Stirling numbers of the first kind* and a notation A_k^n for elementary symmetric polynomials.

Definition 1.2. *The (signed) Stirling numbers of the first kind $s(n, k)$ are defined by the following property:*

$$\prod_{i=0}^{n-1} (x - i) = \sum_{k=0}^n s(n, k) x^k.$$

Define $s(0, 0) = 1$ and $s(n, 0) = s(0, n) = 0$.

Definition 1.3. *Let a_1, a_2, \dots, a_n be positive real numbers. We denote*

$$A_{n-k}^n = \left(\prod_{i=1}^n a_i \right) \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \frac{1}{a_{i_1} a_{i_2} \cdots a_{i_k}}.$$

Thus, A_{n-k}^n is the elementary symmetric polynomial of a_1, a_2, \dots, a_n with degree $n - k$.

Conjecture 1.4 (GLY Conjecture). Let $P_n = \# \left\{ (x_1, x_2, \dots, x_n) \in \mathbb{Z}_+^n : \frac{x_1}{a_1} + \frac{x_2}{a_2} + \dots + \frac{x_n}{a_n} \leq 1 \right\}$, where $a_1 \geq a_2 \geq \dots \geq a_n \geq 1$ are real numbers. If $n \geq 3$, then:

1. *Rough (general) upper estimate: For all $a_n > 1$,*

$$n! P_n < q_n := \prod_{i=1}^n (a_i - 1).$$

2. *Sharp upper estimate: For $n \geq 3$, if $a_1 \geq a_2 \geq \dots \geq a_n \geq n - 1$, then*

$$n! P_n \leq A_0^n + \frac{s(n, n-1)}{n} A_1^n + \sum_{l=1}^{n-2} \frac{s(n, n-1-l)}{\binom{n-1}{l}} A_l^{n-1},$$

and equality holds if and only if $a_1 = a_2 = \dots = a_n \in \mathbb{Z}_+$.

The sharp GLY Conjecture has been proven to be true for $3 \leq n \leq 6$ [15, 4, 14, 13, 7]. The rough GLY upper estimate for all n was proven by Yau and Zhang [17]. In this paper, we will use the following theorem (the sharp GLY conjecture for $n = 5$):

Theorem 1.5 (GLY Conjecture for $n = 5$). Let $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq 4$ be real numbers and P_5 be the number of positive integral solutions of

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_5}{a_5} \leq 1.$$

Then,

$$\begin{aligned} 120P_5 \leq & a_1 a_2 a_3 a_4 a_5 - 2(a_1 a_2 a_3 a_4 + a_1 a_2 a_3 a_5 + a_1 a_2 a_4 a_5 + a_1 a_3 a_4 a_5 + a_2 a_3 a_4 a_5) \\ & + \frac{35}{4}(a_1 a_2 a_3 + a_1 a_2 a_4 + a_1 a_3 a_4 + a_2 a_3 a_4) \\ & - \frac{50}{6}(a_1 a_2 + a_1 a_3 + a_1 a_4 + a_2 a_3 + a_2 a_4 + a_3 a_4) + 6(a_1 + a_2 + a_3 + a_4). \end{aligned}$$

The quest to determine the general validity of the sharp GLY Conjecture has led to the formulation another conjecture, namely the *Yau Number-Theoretic Conjecture* (Conjecture 1.6).

Conjecture 1.6 (Yau Number-Theoretic Conjecture). *Let $n \geq 3$ be a positive integer, and let $a_1 \geq a_2 \dots \geq a_n > 1$ be real numbers. If $P_n > 0$, then*

$$n! P_n \leq (a_1 - 1) \dots (a_n - 1) - (a_n - 1)^n + a_n(a_n - 1) \dots (a_n - (n - 1)),$$

and equality holds if and only if $a_1 = \dots = a_n \in \mathbb{Z}$.

In this paper, we will use the $n = 5$ case of Conjecture 1.6, proven by Chen, et. al. [1], extensively. We reproduce it as a theorem below for easy reference.

Theorem 1.7 (Yau Number-Theoretic Conjecture for $n = 5$). *Let $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 > 1$ be real numbers. If $P_5 > 0$, then*

$$120P_n \leq \mu - (a_5 - 1)^5 + a_5(a_5 - 1)(a_5 - 2)(a_5 - 3)(a_5 - 4),$$

where $\mu = (a_1 - 1)(a_2 - 1)(a_3 - 1)(a_4 - 1)(a_5 - 1)$. Equality holds if and only if $a_1 = a_2 = a_3 = a_4 = a_5 \in \mathbb{Z}$.

Similar to the Yau Number-Theoretic Conjecture is Conjecture 1.9, or the Yau Geometric Conjecture. In order to state the Yau Geometric Conjecture, we must first define a *weighted homogeneous polynomial*:

Definition 1.8. *A polynomial $f(x_1, x_2, \dots, x_n)$ is a weighted homogeneous polynomial if it is a sum of monomials $x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$ such that, for some fixed positive rational numbers w_1, w_2, \dots, w_n ,*

$$\frac{i_1}{w_1} + \frac{i_2}{w_2} + \dots + \frac{i_n}{w_n} = 1,$$

for every monomial of f . The numbers w_1, w_2, \dots, w_n are known as the weights of the polynomial.

Conjecture 1.9 (Yau Geometric Conjecture). *Let $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be a weighted homogeneous polynomial with an isolated singularity at the origin. Let μ , p_g , and v be the Milnor number, geometric genus, and multiplicity of the singularity $V = \{z : f(z) = 0\}$. Then,*

$$\mu - p(v) \geq n! p_g,$$

where $p(v) = (v-1)^n - v(v-1) \dots (v-n+1)$. Equality holds if and only if f is a homogeneous polynomial.

Note that p_g counts the number of positive lattice points in the simplex

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \dots + \frac{x_n}{a_n} \leq 1,$$

where the a_i are the weights of the *weighted* homogeneous polynomial f and $a_1 \geq a_2 \geq \dots \geq a_n > 1$. Thus, the equality case of Conjecture 1.9 is $a_1 = a_2 = \dots = a_n \in \mathbb{Z}$. Furthermore, Saeki [12] tells us that $v = \lceil a_n \rceil$ and $\mu = (a_1 - 1)(a_2 - 1) \dots (a_n - 1)$. Chen, et. al [1] also proved that the fractional part of a_n has to be one of $\frac{a_n}{a_1}, \frac{a_n}{a_2}, \dots$ or $\frac{a_n}{a_{n-1}}$. Finally, we can also define the polynomial $p_n(v) = (v-1)^n - v(v-1) \dots (v-n+1)$. Thus, we have

$$p_5(v) = 5v^4 - 25v^3 + 40v^2 - 19v - 1 \tag{1}$$

$$p_6(v) = 1 + 114v - 259v^2 + 205v^3 - 70v^4 + 9v^5. \tag{2}$$

Conjecture 1.9 has been proven for $3 \leq n \leq 5$ [1, 15, 8]. In this paper, we prove Conjecture 1.9 for $n = 6$. Hence, our main theorem is:

Theorem 1.10 (Main Theorem). *Let $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 > 1$ be real numbers and let P_6 be the number of positive integral solutions of $\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_5}{a_5} + \frac{x_6}{a_6} \leq 1$.*

Define $\mu = (a_1 - 1)(a_2 - 1)(a_3 - 1)(a_4 - 1)(a_5 - 1)(a_6 - 1)$. If $P_6 > 0$, then

$$6! P_6 \leq \mu - (1 + 114v - 259v^2 + 205v^3 - 70v^4 + 9v^5),$$

where v is calculated as $v = \lceil a_6 \rceil$. Note that the fractional part β of a_6 is one of $\frac{a_6}{a_1}, \frac{a_6}{a_2}, \frac{a_6}{a_3}, \frac{a_6}{a_4}$, or $\frac{a_6}{a_5}$. Equality holds if and only if $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 \in \mathbb{Z}$.

2 Proof of Main Theorem (1.10)

Note that all computations in this paper were done using Mathematica 10.1 and Maple 2015.

We prove Theorem 1.10 by estimating P_5 on hyperplanes parallel to the $x_1x_2x_3x_4x_5$ -plane. We then sum up these estimates to get an upper bound for $6! P_6$. We must then only show that this upper bound for $6! P_6$ is less than or equal to the RHS of Theorem 1.10.

In the rest of this paper, we shall refer to the intersection of the simplex in Theorem 1.10 with the hyperplane $x_6 = k$ as the *level* $x_6 = k$. Hence, in our simplex, $x_6 = k$ points are in the 5-dimensional tetrahedron defined by

$$\frac{x_1}{a_1 \left(1 - \frac{k}{a_6}\right)} + \frac{x_2}{a_2 \left(1 - \frac{k}{a_6}\right)} + \frac{x_3}{a_3 \left(1 - \frac{k}{a_6}\right)} + \frac{x_4}{a_4 \left(1 - \frac{k}{a_6}\right)} + \frac{x_5}{a_5 \left(1 - \frac{k}{a_6}\right)} \leq 1.$$

We shall break our proof up into cases based on the ceiling of a_6 :

Case I $1 < a_6 \leq 2$. Thus, $\lceil a_6 \rceil = 2$.

Case II $2 < a_6 \leq 3$. Thus, $\lceil a_6 \rceil = 3$.

Case III $3 < a_6 \leq 4$. Thus, $\lceil a_6 \rceil = 4$.

Case IV $4 < a_6 \leq 5$. Thus, $\lceil a_6 \rceil = 5$.

Case V $5 < a_6 \leq 6$. Thus, $\lceil a_6 \rceil = 6$.

Case VI $6 < a_6$.

All of our cases will eventually reduce to proving some multivariate functions are positive over some domains. We will show a function is positive using a partial differentiation test, which involves calculating the partial derivative with respect to all the variables. We show this partial derivative is positive and then continue to partially differentiate with respect to one less variable for each consecutive step until only first-order partials remain. If we show that these are all positive through the domain of the function, and that the function is positive at the minimum, then we know that the function is positive throughout the domain.

2.1 Case I

In this case, $\lceil a_6 \rceil = 2$, so we can plug $v = 2$ into the statement of Theorem 1.10 to get the following theorem, which we prove in this case:

Theorem 2.1. *Let $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 > 1$ be real numbers and let P_6 be the number of positive integral solutions of $\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_5}{a_5} + \frac{x_6}{a_6} \leq 1$. If $P_6 > 0$ and $1 < a_6 \leq 2$, then*

$$6! P_6 \leq (a_1 - 1)(a_2 - 1)(a_3 - 1)(a_4 - 1)(a_5 - 1)(a_6 - 1) - 1.$$

Proof. In this case, $a_6 \in (1, 2]$, so the only level we have to consider is $x_6 = 1$. When $x_6 = 1$, $P_6 > 0$ implies that $(x_1, x_2, x_3, x_4, x_5, x_6) = (1, 1, 1, 1, 1, 1)$ is a solution to the inequality in Theorem 2.1. If

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} \leq 1 - \frac{1}{a_6} := \alpha,$$

then $\alpha \in (0, \frac{1}{2}]$ because $a_6 \in (1, 2]$. For simplicity, let $A_i = a_i \cdot \alpha$ for $i = 1, 2, 3, 4$, and 5 .

This yields the new inequality

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} \leq 1.$$

Thus, by Theorem 1.7, we have

$$6! P_6 = 6! P_5(x_6 = 1) \leq 6[(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1) - (A_5 - 1)^5 + A_5(A_5 - 1)(A_5 - 2)(A_5 - 3)(A_5 - 4)].$$

If we let Δ_1 be the difference obtained by subtracting the RHS of the above inequality from the RHS of Theorem 2.1, substituting in $a_i = \frac{A_i}{\alpha}$, then we merely have to apply the partial differentiation test for the expression

$$\begin{aligned} \Delta_1 &= \frac{1}{(1-\alpha)\alpha^4} (6\alpha^5 A_1 A_2 A_3 A_4 A_5 - 6\alpha^5 A_1 A_2 A_3 A_4 - 6\alpha^5 A_1 A_2 A_3 A_5 - 6\alpha^5 A_1 A_2 A_4 A_5 - \\ &6\alpha^5 A_1 A_3 A_4 A_5 - 6\alpha^5 A_2 A_3 A_4 A_5 - 30\alpha^5 A_5^4 - 6\alpha^4 A_1 A_2 A_3 A_4 A_5 + 6\alpha^5 A_1 A_2 A_3 + 6\alpha^5 A_1 A_2 A_4 \\ &+ 6\alpha^5 A_1 A_2 A_5 + 6\alpha^5 A_1 A_3 A_4 + 6\alpha^5 A_1 A_3 A_5 + 6\alpha^5 A_1 A_4 A_5 + 6\alpha^5 A_2 A_3 A_4 + 6\alpha^5 A_2 A_3 A_5 + \\ &6\alpha^5 A_2 A_4 A_5 + 6\alpha^5 A_3 A_4 A_5 + 150\alpha^5 A_5^3 + 6\alpha^4 A_1 A_2 A_3 A_4 + 6\alpha^4 A_1 A_2 A_3 A_5 + 6\alpha^4 A_1 A_2 A_4 A_5 \\ &+ 6\alpha^4 A_1 A_3 A_4 A_5 + 6\alpha^4 A_2 A_3 A_4 A_5 + 30\alpha^4 A_5^4 - 6\alpha^5 A_1 A_2 - 6\alpha^5 A_1 A_3 - 6\alpha^5 A_1 A_4 - 6\alpha^5 A_1 A_5 \\ &- 6\alpha^5 A_2 A_3 - 6\alpha^5 A_2 A_4 - 6\alpha^5 A_2 A_5 - 6\alpha^5 A_3 A_4 - 6\alpha^5 A_3 A_5 - 6\alpha^5 A_4 A_5 - 240\alpha^5 A_5^2 - \\ &6\alpha^4 A_1 A_2 A_3 - 6\alpha^4 A_1 A_2 A_4 - 6\alpha^4 A_1 A_2 A_5 - 6\alpha^4 A_1 A_3 A_4 - 6\alpha^4 A_1 A_3 A_5 - 6\alpha^4 A_1 A_4 A_5 - \\ &6\alpha^4 A_2 A_3 A_4 - 6\alpha^4 A_2 A_3 A_5 - 6\alpha^4 A_2 A_4 A_5 - 6\alpha^4 A_3 A_4 A_5 - 150\alpha^4 A_5^3 + 6\alpha^5 A_1 + 6\alpha^5 A_2 + 6\alpha^5 A_3 \\ &+ 6\alpha^5 A_4 + 120\alpha^5 A_5 + 6\alpha^4 A_1 A_2 + 6\alpha^4 A_1 A_3 + 6\alpha^4 A_1 A_4 + 6\alpha^4 A_1 A_5 + 6\alpha^4 A_2 A_3 + 6\alpha^4 A_2 A_4 \\ &+ 6\alpha^4 A_2 A_5 + 6\alpha^4 A_3 A_4 + 6\alpha^4 A_3 A_5 + 6\alpha^4 A_4 A_5 + 240\alpha^4 A_5^2 - 5\alpha^4 A_1 - 5\alpha^4 A_2 - 5\alpha^4 A_3 - \\ &5\alpha^4 A_4 - 119\alpha^4 A_5 - \alpha^3 A_1 A_2 - \alpha^3 A_1 A_3 - \alpha^3 A_1 A_4 - \alpha^3 A_1 A_5 - \alpha^3 A_2 A_3 - \alpha^3 A_2 A_4 - \alpha^3 A_2 A_5 \\ &- \alpha^3 A_3 A_4 - \alpha^3 A_3 A_5 - \alpha^3 A_4 A_5 + \alpha^2 A_1 A_2 A_3 + \alpha^2 A_1 A_2 A_4 + \alpha^2 A_1 A_2 A_5 + \alpha^2 A_1 A_3 A_4 + \\ &\alpha^2 A_1 A_3 A_5 + \alpha^2 A_1 A_4 A_5 + \alpha^2 A_2 A_3 A_4 + \alpha^2 A_2 A_3 A_5 + \alpha^2 A_2 A_4 A_5 + \alpha^2 A_3 A_4 A_5 - \alpha A_1 A_2 A_3 A_4 \\ &- \alpha A_1 A_2 A_3 A_5 - \alpha A_1 A_2 A_4 A_5 - \alpha A_1 A_3 A_4 A_5 - \alpha A_2 A_3 A_4 A_5 + A_1 A_2 A_3 A_4 A_5 - \alpha^4) \\ &= \frac{1}{(1-\alpha)\alpha^4} \Delta_2. \end{aligned}$$

To apply the partial differentiation test, we must determine the domain of Δ_2 . We note that $\frac{1}{A_5} < 1$ and $\frac{2}{A_4} \leq \frac{1}{A_4} + \frac{1}{A_5} \leq 1$. Similarly, $\frac{3}{A_3} \leq \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} \leq 1$, and a similar

statement is true involving A_2 and A_1 . Hence, we have

$$A_1 \geq 5, A_2 \geq 4, A_3 \geq 3, A_4 \geq 2, \text{ and } A_5 > 1.$$

Now that we have a domain established, we can begin applying the partial differentiation test to demonstrate that Δ_2 is positive.

We see that

$$\frac{\partial^5 \Delta_2}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5} = 6\alpha^5 - 6\alpha^4 + 1 > 0$$

for all $\alpha \in (0, 1)$. Thus the partial derivative of Δ_2 with respect to A_1, A_2, A_3, A_4 , and A_5 is positive and minimized at $A_5 = 1$.

$$\left. \frac{\partial^4 \Delta_2}{\partial A_1 \partial A_2 \partial A_3 \partial A_4} \right|_{A_5=1} = 1 - \alpha > 0.$$

It follows that the partial of Δ_2 with respect to A_1, A_2, A_3 , and A_5 is positive for all $A_4 \geq 1, \alpha \in (0, 1)$ because $\frac{\partial^3 \Delta_2}{\partial A_1 \partial A_2 \partial A_3}$ is symmetric in A_4 and A_5 . Hence, $\frac{\partial^3 \Delta_2}{\partial A_1 \partial A_2 \partial A_3}$ is an increasing function of A_4, A_5 with a minimum at $A_4 = A_5 = 1$.

$$\left. \frac{\partial^3 \Delta_2}{\partial A_1 \partial A_2 \partial A_3} \right|_{A_4=A_5=1} = (1 - \alpha)^2 > 0.$$

This is positive, so we know that the partial with respect to A_1, A_2 , and A_3 is positive. Since the partial with respect to A_1 and A_2 is symmetric with respect to A_3, A_4 , and A_5 , we know that $\frac{\partial^3 \Delta_2}{\partial A_1 \partial A_2 \partial A_4}$ and $\frac{\partial^3 \Delta_2}{\partial A_1 \partial A_2 \partial A_5}$ are positive over the given domain. Hence, $\frac{\partial^2 \Delta_2}{\partial A_1 \partial A_2}$ is an increasing function of A_3, A_4 , and A_5 for all $A_3, A_4, A_5 \geq 1$ and $\alpha \in (0, 1)$. The minimum is at $A_3 = A_4 = A_5 = 1$.

$$\left. \frac{\partial^2 \Delta_2}{\partial A_1 \partial A_2} \right|_{A_3=A_4=A_5=1} = (1 - \alpha)^3 > 0.$$

Because this is symmetric with respect to A_2, A_3, A_4 , and A_5 , we see that $\frac{\partial^2 \Delta_2}{\partial A_1 \partial A_3}, \frac{\partial^2 \Delta_2}{\partial A_1 \partial A_4}$, and $\frac{\partial^2 \Delta_2}{\partial A_1 \partial A_5}$ are positive over the given domain. Hence, $\frac{\partial \Delta_2}{\partial A_1}$ is an increasing function of A_2, A_3, A_4 , and A_5 that is minimized at $A_2 = A_3 = A_4 = A_5 = 1$.

$$\left. \frac{\partial \Delta_2}{\partial A_1} \right|_{A_2=A_3=A_4=A_5=1} = (1 - \alpha)^4 > 0.$$

Thus $\frac{\partial \Delta_2}{\partial A_1}$ is positive over the given domain. By the symmetry of Δ_2 in A_1, A_2, A_3 , and A_4 , we know that $\frac{\partial \Delta_2}{\partial A_2}, \frac{\partial \Delta_2}{\partial A_3}$, and $\frac{\partial \Delta_2}{\partial A_4}$ are positive over the given domain. Therefore, Δ_2 is an increasing function of A_1, A_2, A_3 , and A_4 . We can hence plug in the minimum values of A_1, A_2, A_3 , and A_4 to determine a new polynomial in A_5 and α that we want to show is positive.

We define

$$\begin{aligned} \Delta_3 &= \Delta_2|_{A_1=5, A_2=4, A_3=3, A_4=2} \\ &= -30\alpha^5 A_5^4 + 150\alpha^5 A_5^3 + 30\alpha^4 A_5^4 - 240\alpha^5 A_5^2 - 150\alpha^4 A_5^3 + 258\alpha^5 A_5 + 240\alpha^4 A_5^2 - 138\alpha^5 \\ &\quad - 257\alpha^4 A_5 + 151\alpha^4 - 14\alpha^3 A_5 - 71\alpha^3 + 71\alpha^2 A_5 + 154\alpha^2 - 154\alpha A_5 - 120\alpha + 120A_5. \end{aligned}$$

We must show that Δ_3 is positive over the interval $\alpha \in (0, \frac{1}{2}]$ and $A_5 \geq 1$. We split this into two subcases:

Subcase I (a) $A_5 \geq 1.73$.

Subcase I (b) $A_5 < 1.73$.

We calculated the number 1.73 numerically, by noting that the partial differentiation test works normally for $A_5 \geq 1.73$ because all the partial derivatives remain positive. When $A_5 < 1.73$ some partial derivatives become negative.

2.1.1 Subcase I (a)

In this subcase, we can apply the partial differentiation test normally. We begin by noting that

$$\frac{\partial^4 \Delta_3}{\partial A_5^4} = 720\alpha^4(1 - \alpha) > 0.$$

We then consider

$$\left. \frac{\partial^3 \Delta_3}{\partial A_5^3} \right|_{A_5=1.73} = 345.6\alpha^4(1 - \alpha) > 0.$$

Similarly,

$$\left. \frac{\partial^2 \Delta_3}{\partial A_5^2} \right|_{A_5=1.73} = 0.444\alpha^4(1 - \alpha) > 0.$$

We continue by considering

$$\left. \frac{\partial \Delta_3}{\partial A_5} \right|_{A_5=1.73} = 153.07896\alpha^5 - 152.07896\alpha^4 - 14\alpha^3 + 71\alpha^2 - 154\alpha + 120 > 0$$

for $\alpha \in (0, 1)$. This can also be proven by Lemma 1 in [16]. Finally, we evaluate Δ_3 at its minimum:

$$\Delta_3|_{A_5=1.73} = 97.9780377\alpha^5 - 83.2480377\alpha^4 - 95.22\alpha^3 + 276.83\alpha^2 - 386.42\alpha + 207.60 > 0.$$

This completes this subcase.

2.1.2 Subcase I (b)

For Δ_3 , the partial derivative test does not work normally for A_5 — some of the derivatives end up becoming negative. However, we can plot the polynomial Δ_3 over the interval $A_5 \in [1, 1.73)$ and $\alpha \in (0, 1)$ using Mathematica and Maple and verify that it is non-negative in the region we consider. Its minimum value is 120, which occurs at $(A_5, \alpha) = (1, 0)$.

Hence, Subcase (b), and consequently Case I, is complete. \square

2.2 Case II

In this case, $\lceil a_6 \rceil = 3$, so we can plug $v = 3$ into the statement of Theorem 1.10 to get the following theorem, which we prove in this case:

Theorem 2.2. *Let $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 > 1$ be real numbers and let P_6 be the number of positive integral solutions of $\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_5}{a_5} + \frac{x_6}{a_6} \leq 1$. If $P_6 > 0$ and $2 < a_6 \leq 3$, then*

$$6! P_6 \leq (a_1 - 1)(a_2 - 1)(a_3 - 1)(a_4 - 1)(a_5 - 1)(a_6 - 1) - 64.$$

Proof. In this case, $a_6 \in (2, 3]$, so we have to consider two levels — $x_6 = 1$ and $x_6 = 2$. Since $P_6 > 0$, there must be solutions at the $x_6 = 1$ level, so our two subcases are:

Subcase II (a) $P_5(x_6 = 2) = 0$.

Subcase II (b) $P_5(x_6 = 2) > 0$.

2.2.1 Subcase II (a)

We are guaranteed that $(x_1, x_2, x_3, x_4, x_5, x_6) = (1, 1, 1, 1, 1, 1)$ is a solution to the inequality in Theorem 2.2. Thus, if

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} \leq 1 - \frac{1}{a_6} := \alpha,$$

then $\alpha \in (\frac{1}{2}, \frac{2}{3}]$ because $a_6 \in (2, 3]$. For simplicity, let $A_i = a_i \cdot \alpha$ for $i = 1, 2, 3, 4$, and 5.

This yields the new inequality

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} \leq 1.$$

Thus, by Theorem 1.7, we have

$$6! P_5(x_6 = 2) \leq 6 [(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1) - (A_5 - 1)^5 + A_5(A_5 - 1)(A_5 - 2)(A_5 - 3)(A_5 - 4)].$$

As before, we take the difference obtained by subtracting the RHS of the above inequality from the RHS of Theorem 2.2, substituting in $a_i = \frac{A_i}{\alpha}$. We observe that this difference is equal to $\Delta_1 - 63$, where Δ_1 is from Case I above. Since we applied the partial differentiation test on $\Delta_2 = \Delta_1 \cdot \alpha^4(1 - \alpha)$ in Case I, here we need to show that

$$\Delta_4 := \Delta_2 - 63\alpha^4(1 - \alpha)$$

is positive. Since $\Delta_2 - \Delta_4$ is a function in α only, we must check that the value of Δ_4 at its minimum is positive (because all of the partial derivatives in the test are the same for Δ_2 and Δ_4). As in Case I, we have

$$A_1 \geq 5, A_2 \geq 4, A_3 \geq 3, A_4 \geq 2, \text{ and } A_5 > 1,$$

so we must only check that

$$\Delta_4|_{A_1=5, A_2=4, A_3=3, A_4=2, A_5=1} = 63\alpha^5 - 49\alpha^4 - 85\alpha^3 + 225\alpha^2 - 274\alpha + 120 > 0.$$

Since this is true, Δ_4 is always positive, and this subcase is complete.

2.2.2 Subcase II (b)

In this subcase, we know that $P_5(x_6 = 2) > 0$, implying that $(1, 1, 1, 1, 1, 2)$ is a positive integral solution to the inequality in Theorem 2.2. Thus, if

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} \leq 1 - \frac{2}{a_6} := \alpha_1,$$

then $\alpha_1 \in (0, \frac{1}{3}]$ because $a_6 \in (2, 3]$. For simplicity, let $A_i = a_i \cdot \alpha_1$ for $i = 1, 2, 3, 4$, and 5 .

This yields the new inequality

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} \leq 1.$$

Thus, by Theorem 1.7, we have

$$6! P_5(x_6 = 2) \leq 6 [(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1) - (A_5 - 1)^5 + A_5(A_5 - 1)(A_5 - 2)(A_5 - 3)(A_5 - 4)],$$

and,

$$6! P_5(x_6 = 1) \leq 6 \left[\left(A_1 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right) \left(A_2 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right) \left(A_3 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right) \left(A_4 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right) \left(A_5 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right) - \left(A_5 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right)^5 + \left(A_5 \cdot \frac{1 + \alpha_1}{2\alpha_1} \right) \left(A_5 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right) \left(A_5 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 2 \right) \left(A_5 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 3 \right) \left(A_5 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 4 \right) \right]$$

Because $6! P_6 = 6! (P_5(x_6 = 1) + P_5(x_6 = 2))$, if we let Δ_5 be the difference obtained by subtracting the sum of the right hand sides of the above inequalities from the RHS of Theorem 2.2, substituting in $a_i = \frac{A_i}{\alpha_1}$, then we merely have to apply the partial differentiation test for the expression

$$\begin{aligned} \Delta_5 = & \frac{1}{(1 - \alpha_1)\alpha_1^5} (99A_1A_2A_3A_4A_5\alpha_1^6 - 84A_1A_2A_3A_4A_5\alpha_1^5 - 102A_1A_2A_3A_4\alpha_1^6 - 102A_1A_2A_3A_5\alpha_1^6 \\ & - 102A_1A_2A_4A_5\alpha_1^6 - 102A_1A_3A_4A_5\alpha_1^6 - 102A_2A_3A_4A_5\alpha_1^6 - 510A_5^4\alpha_1^6 + 15A_1A_2A_3A_4A_5\alpha_1^4 \\ & + 78A_1A_2A_3A_4\alpha_1^5 + 78A_1A_2A_3A_5\alpha_1^5 + 108A_1A_2A_3\alpha_1^6 + 78A_1A_2A_4A_5\alpha_1^5 + 108A_1A_2A_4\alpha_1^6 \\ & + 108A_1A_2A_5\alpha_1^6 + 78A_1A_3A_4A_5\alpha_1^5 + 108A_1A_3A_4\alpha_1^6 + 108A_1A_3A_5\alpha_1^6 + 108A_1A_4A_5\alpha_1^6 + 78A_2A_3A_4A_5\alpha_1^5 \\ & + 108A_2A_3A_4\alpha_1^6 + 108A_2A_3A_5\alpha_1^6 + 108A_2A_4A_5\alpha_1^6 + 108A_3A_4A_5\alpha_1^6 + 390A_5^4\alpha_1^5 + 2700A_5^3\alpha_1^6 \\ & - 12A_1A_2A_3A_4\alpha_1^4 - 12A_1A_2A_3A_5\alpha_1^4 - 72A_1A_2A_3\alpha_1^5 - 12A_1A_2A_4A_5\alpha_1^4 - 72A_1A_2A_4\alpha_1^5 \\ & - 72A_1A_2A_5\alpha_1^5 - 120A_1A_2\alpha_1^6 - 12A_1A_3A_4A_5\alpha_1^4 - 72A_1A_3A_4\alpha_1^5 - 72A_1A_3A_5\alpha_1^5 - 120A_1A_3\alpha_1^6 \\ & - 72A_1A_4A_5\alpha_1^5 - 120A_1A_4\alpha_1^6 - 120A_1A_5\alpha_1^6 - 12A_2A_3A_4A_5\alpha_1^4 - 72A_2A_3A_4\alpha_1^5 - 72A_2A_3A_5\alpha_1^5 \\ & - 120A_2A_3\alpha_1^6 - 72A_2A_4A_5\alpha_1^5 - 120A_2A_4\alpha_1^6 - 120A_2A_5\alpha_1^6 - 72A_3A_4A_5\alpha_1^5 - 120A_3A_4\alpha_1^6 - 120A_3A_5\alpha_1^6 \end{aligned}$$

$$\begin{aligned}
& -120A_4A_5\alpha_1^6 - 60A_5^4\alpha_1^4 - 1800A_5^3\alpha_1^5 - 4800A_5^2\alpha_1^6 - 15A_1A_2A_3A_4A_5\alpha_1^2 + 12A_1A_2A_3A_4\alpha_1^3 \\
& + 12A_1A_2A_3A_5\alpha_1^3 + 12A_1A_2A_4A_5\alpha_1^3 + 72A_1A_2\alpha_1^5 + 12A_1A_3A_4A_5\alpha_1^3 + 72A_1A_3\alpha_1^5 + 72A_1A_4\alpha_1^5 \\
& + 72A_1A_5\alpha_1^5 + 144A_1\alpha_1^6 + 12A_2A_3A_4A_5\alpha_1^3 + 72A_2A_3\alpha_1^5 + 72A_2A_4\alpha_1^5 + 72A_2A_5\alpha_1^5 + 144A_2\alpha_1^6 + 72A_3A_4\alpha_1^5 \\
& + 72A_3A_5\alpha_1^5 + 144A_3\alpha_1^6 + 72A_4A_5\alpha_1^5 + 144A_4\alpha_1^6 + 60A_5^4\alpha_1^3 + 2880A_5^2\alpha_1^5 + 2880A_5\alpha_1^6 + 4A_1A_2A_3A_4A_5\alpha_1 \\
& + 2A_1A_2A_3A_4\alpha_1^2 + 2A_1A_2A_3A_5\alpha_1^2 - 8A_1A_2A_3\alpha_1^3 + 2A_1A_2A_4A_5\alpha_1^2 - 8A_1A_2A_4\alpha_1^3 - 8A_1A_2A_5\alpha_1^3 \\
& + 8A_1A_2\alpha_1^4 + 2A_1A_3A_4A_5\alpha_1^2 - 8A_1A_3A_4\alpha_1^3 - 8A_1A_3A_5\alpha_1^3 + 8A_1A_3\alpha_1^4 - 8A_1A_4A_5\alpha_1^3 + 8A_1A_4\alpha_1^4 \\
& + 8A_1A_5\alpha_1^4 - 80A_1\alpha_1^5 + 2A_2A_3A_4A_5\alpha_1^2 - 8A_2A_3A_4\alpha_1^3 - 8A_2A_3A_5\alpha_1^3 + 8A_2A_3\alpha_1^4 - 8A_2A_4A_5\alpha_1^3 + 8A_2A_4\alpha_1^4 \\
& + 8A_2A_5\alpha_1^4 - 80A_2\alpha_1^5 - 8A_3A_4A_5\alpha_1^3 + 8A_3A_4\alpha_1^4 + 8A_3A_5\alpha_1^4 - 80A_3\alpha_1^5 + 8A_4A_5\alpha_1^4 - 80A_4\alpha_1^5 + 90A_5^4\alpha_1^2 \\
& - 600A_5^3\alpha_1^3 + 960A_5^2\alpha_1^4 - 1904A_5\alpha_1^5 + 1008\alpha_1^6 + 13A_1A_2A_3A_4A_5 - 10A_1A_2A_3A_4\alpha_1 - 10A_1A_2A_3A_5\alpha_1 \\
& + 4A_1A_2A_3\alpha_1^2 - 10A_1A_2A_4A_5\alpha_1 + 4A_1A_2A_4\alpha_1^2 + 4A_1A_2A_5\alpha_1^2 + 8A_1A_2\alpha_1^3 - 10A_1A_3A_4A_5\alpha_1 \\
& + 4A_1A_3A_4\alpha_1^2 + 4A_1A_3A_5\alpha_1^2 + 8A_1A_3\alpha_1^3 + 4A_1A_4A_5\alpha_1^2 + 8A_1A_4\alpha_1^3 + 8A_1A_5\alpha_1^3 - 32A_1\alpha_1^4 - 10A_2A_3A_4A_5\alpha_1 \\
& + 4A_2A_3A_4\alpha_1^2 + 4A_2A_3A_5\alpha_1^2 + 8A_2A_3\alpha_1^3 + 4A_2A_4A_5\alpha_1^2 + 8A_2A_4\alpha_1^3 + 8A_2A_5\alpha_1^3 - 32A_2\alpha_1^4 + 4A_3A_4A_5\alpha_1^2 \\
& + 8A_3A_4\alpha_1^3 + 8A_3A_5\alpha_1^3 - 32A_3\alpha_1^4 + 8A_4A_5\alpha_1^3 - 32A_4\alpha_1^4 + 30A_5^4\alpha_1 - 300A_5^3\alpha_1^2 + 960A_5^2\alpha_1^3 - 944A_5\alpha_1^4 - 1040\alpha_1^5) \\
& = \frac{1}{(1-\alpha_1)\alpha_1^5}\Delta_6.
\end{aligned}$$

Just like in Case I, we are trying to show that Δ_6 is positive for

$$A_1 \geq 5, A_2 \geq 4, A_3 \geq 3, A_4 \geq 2, \text{ and } A_5 > 1.$$

Although we only need to show this is true for $\alpha_1 \in (0, 1/3]$, we will demonstrate it true for the interval $\alpha_1 \in (0, 1/2]$ because it will aid us in a later case.

Thus, the first step in the partial differentiation test is determining that

$$\frac{\partial^5 \Delta_6}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5} = 99\alpha_1^6 - 84\alpha_1^5 + 15\alpha_1^4 - 15\alpha_1^2 + 4\alpha_1 + 13 > 0$$

for all $\alpha_1 \in (0, \frac{1}{2}]$. Thus the partial derivative of Δ_6 with respect to A_1, A_2, A_3, A_4 , and A_5 is positive and minimized at $A_5 = 1$.

$$\left. \frac{\partial^4 \Delta_6}{\partial A_1 \partial A_2 \partial A_3 \partial A_4} \right|_{A_5=1} = -3\alpha_1^6 - 6\alpha_1^5 + 3\alpha_1^4 + 12\alpha_1^3 - 13\alpha_1^2 - 6\alpha_1 + 13 > 0.$$

It follows that the partial of Δ_6 with respect to A_1 , A_2 , A_3 , and A_5 is positive for all $A_4 \geq 1, \alpha_1 \in (0, \frac{1}{2}]$ because $\frac{\partial^3 \Delta_6}{\partial A_1 \partial A_2 \partial A_3}$ is symmetric in A_4 and A_5 . Hence, $\frac{\partial^3 \Delta_6}{\partial A_1 \partial A_2 \partial A_3}$ is an increasing function of A_4 and A_5 with a minimum at $A_4 = A_5 = 1$.

$$\left. \frac{\partial^3 \Delta_6}{\partial A_1 \partial A_2 \partial A_3} \right|_{A_4=A_5=1} = 3\alpha_1^6 - 9\alpha_1^4 + 16\alpha_1^3 - 7\alpha_1^2 - 16\alpha_1 + 13 > 0.$$

This is positive over our domain, so we know that the partial with respect to A_1 , A_2 , and A_3 is positive. Since the partial with respect to A_1 and A_2 is symmetric with respect to A_3, A_4 , and A_5 , we know that $\frac{\partial^3 \Delta_6}{\partial A_1 \partial A_2 \partial A_4}$ and $\frac{\partial^3 \Delta_6}{\partial A_1 \partial A_2 \partial A_5}$ are also positive over the given domain. Hence, $\frac{\partial^2 \Delta_2}{\partial A_1 \partial A_2}$ is an increasing function of A_3, A_4 , and A_5 for all $A_3, A_4, A_5 \geq 1$ and $\alpha_1 \in (0, \frac{1}{2}]$. The minimum is at $A_3 = A_4 = A_5 = 1$.

$$\left. \frac{\partial^2 \Delta_6}{\partial A_1 \partial A_2} \right|_{A_3=A_4=A_5=1} = -3\alpha_1^6 + 6\alpha_1^5 - 13\alpha_1^4 + 20\alpha_1^3 + 3\alpha_1^2 - 26\alpha_1 + 13 > 0.$$

Because this is symmetric with respect to A_2, A_3, A_4 , and A_5 , we see that $\frac{\partial^2 \Delta_6}{\partial A_1 \partial A_3}, \frac{\partial^2 \Delta_6}{\partial A_1 \partial A_4}$, and $\frac{\partial^2 \Delta_6}{\partial A_1 \partial A_5}$ are positive over the given domain. Hence, $\frac{\partial \Delta_6}{\partial A_1}$ is an increasing function of A_2, A_3, A_4 and A_5 minimized at $A_2 = A_3 = A_4 = A_5 = 1$.

$$\left. \frac{\partial \Delta_6}{\partial A_1} \right|_{A_2=A_3=A_4=A_5=1} = 3\alpha_1^6 + 4\alpha_1^5 - 33\alpha_1^4 + 32\alpha_1^3 + 17\alpha_1^2 - 36\alpha_1 + 13 > 0.$$

Thus $\frac{\partial \Delta_6}{\partial A_1}$ is positive over the given domain. By the symmetry of Δ_6 in A_1, A_2, A_3 , and A_4 , we know that $\frac{\partial \Delta_6}{\partial A_2}, \frac{\partial \Delta_6}{\partial A_3}$, and $\frac{\partial \Delta_6}{\partial A_4}$ are positive over the given domain. Therefore, Δ_6 is an increasing function of A_1, A_2, A_3 , and A_4 . We can hence plug in the minimum values of A_1, A_2, A_3 , and A_4 to determine a new polynomial in A_5 and α_1 that we want to show is

positive. We define

$$\begin{aligned}\Delta_7 &= \Delta_6|_{A_1=5, A_2=4, A_3=3, A_4=2} \\ &= -510A_5^4\alpha_1^6 + 390A_5^4\alpha_1^5 + 2700A_5^3\alpha_1^6 - 60A_5^4\alpha_1^4 - 1800A_5^3\alpha_1^5 - 4800A_5^2\alpha_1^6 + 60A_5^4\alpha_1^3 \\ &\quad + 2880A_5^2\alpha_1^5 + 5040A_5\alpha_1^6 + 90A_5^4\alpha_1^2 - 600A_5^3\alpha_1^3 + 960A_5^2\alpha_1^4 - 4076A_5\alpha_1^5 - 1104\alpha_1^6 \\ &\quad + 30A_5^4\alpha_1 - 300A_5^3\alpha_1^2 + 960A_5^2\alpha_1^3 - 880A_5\alpha_1^4 + 1224\alpha_1^5 + 1392A_5\alpha_1^3 - 1320\alpha_1^4 \\ &\quad - 1208A_5\alpha_1^2 + 776\alpha_1^3 - 1060A_5\alpha_1 + 856\alpha_1^2 + 1560A_5 - 1200\alpha_1.\end{aligned}$$

We must show that Δ_7 is positive over the interval $\alpha_1 \in (0, \frac{1}{2}]$ and $A_5 \geq 1$.

We apply the partial differentiation test normally, beginning by noting that

$$\frac{\partial^4 \Delta_7}{\partial A_5^4} = -12240\alpha_1^6 + 9360\alpha_1^5 - 1440\alpha_1^4 + 1440\alpha_1^3 + 2160\alpha_1^2 + 720\alpha_1 > 0.$$

We then consider

$$\left. \frac{\partial^3 \Delta_7}{\partial A_5^3} \right|_{A_5=1} = 3960\alpha_1^6 - 1440\alpha_1^5 - 1440\alpha_1^4 - 2160\alpha_1^3 + 360\alpha_1^2 + 720\alpha_1 > 0.$$

Similarly,

$$\left. \frac{\partial^2 \Delta_7}{\partial A_5^2} \right|_{A_5=1} = 480\alpha_1^6 - 360\alpha_1^5 + 1200\alpha_1^4 - 960\alpha_1^3 - 720\alpha_1^2 + 360\alpha_1,$$

which is positive for $\alpha_1 \in (0, \frac{1}{3}]$. Although $\left. \frac{\partial^2 \Delta_7}{\partial A_5^2} \right|_{A_5=1}$ ends up becoming negative in the interval $\alpha_1 \in (\frac{1}{3}, \frac{1}{2}]$, this ends up being irrelevant because we can verify numerically that $\left. \frac{\partial \Delta_7}{\partial A_5} \right|_{A_5=1}$ is still positive. We do this by plotting $\left. \frac{\partial \Delta_7}{\partial A_5} \right|_{A_5=1}$ over that interval using Mathematica or Maple. Its minimum value is $\frac{16777}{32} = 524.281$, attained when $\alpha_1 = \frac{1}{2}$.

We continue by considering

$$\left. \frac{\partial \Delta_7}{\partial A_5} \right|_{A_5=1} = 1500\alpha_1^6 - 2156\alpha_1^5 + 800\alpha_1^4 + 1752\alpha_1^3 - 1748\alpha_1^2 - 940\alpha_1 + 1560 > 0$$

for $\alpha_1 \in \left(0, \frac{1}{2}\right)$. Finally, we evaluate Δ_7 at its minimum:

$$\Delta_7|_{A_5=1} = 1326\alpha_1^6 - 1382\alpha_1^5 - 1300\alpha_1^4 + 2588\alpha_1^3 - 562\alpha_1^2 - 2230\alpha_1 + 1560 > 0.$$

This completes this subcase, and hence Case II is complete. \square

2.3 Case III

In this case, $[a_6] = 4$, so we can plug $v = 4$ into the statement of Theorem 1.10 to get the following theorem, which we prove in this case:

Theorem 2.3. *Let $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 > 1$ be real numbers and let P_6 be the number of positive integral solutions of $\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_5}{a_5} + \frac{x_6}{a_6} \leq 1$. If $P_6 > 0$ and $3 < a_6 \leq 4$, then*

$$6! P_6 \leq (a_1 - 1)(a_2 - 1)(a_3 - 1)(a_4 - 1)(a_5 - 1)(a_6 - 1) - 729.$$

Proof. In this case, $a_6 \in (3, 4]$, so we have to consider three levels — $x_6 = 1$, $x_6 = 2$, and $x_6 = 3$. Since $P_6 > 0$, there must be solutions at the $x_6 = 1$ level, so our three subcases are:

Subcase III (a) $P_5(x_6 = 3) = P_5(x_6 = 2) = 0$.

Subcase III (b) $P_5(x_6 = 3) = 0, P_5(x_6 = 2) > 0$.

Subcase III (c) $P_5(x_6 = 3) > 0, P_5(x_6 = 2) > 0$.

2.3.1 Subcase III (a)

We are guaranteed that $(x_1, x_2, x_3, x_4, x_5, x_6) = (1, 1, 1, 1, 1, 1)$ is a solution to the inequality in Theorem 2.3. Thus, if

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} \leq 1 - \frac{1}{a_6} := \alpha,$$

then $\alpha \in (\frac{2}{3}, \frac{3}{4}]$ because $a_6 \in (3, 4]$. For simplicity, let $A_i = a_i \cdot \alpha$ for $i = 1, 2, 3, 4,$ and 5 .

This yields the new inequality

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} \leq 1.$$

Thus, by Theorem 1.7, we have

$$6! P_5(x_6 = 1) \leq 6 [(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1) - (A_5 - 1)^5 + A_5(A_5 - 1)(A_5 - 2)(A_5 - 3)(A_5 - 4)].$$

As before, we take the difference obtained by subtracting the RHS of the above inequality from the RHS of Theorem 2.3, substituting in $a_i = \frac{A_i}{\alpha}$. We observe that this difference is equal to $\Delta_1 - 728$, where Δ_1 is from Case I above. Since we applied the partial differentiation test on $\Delta_2 = \Delta_1 \cdot \alpha^4(1 - \alpha)$ in Case I, here we need to show that

$$\Delta_8 := \Delta_2 - 728\alpha^4(1 - \alpha)$$

is positive. To apply the partial differentiation test, we must determine the domain of Δ_8 .

By the same logic as in Cases I and II, we have

$$A_1 \geq 5, A_2 \geq 4, A_3 \geq 3, \text{ and } A_4 \geq A_5 \geq \frac{\alpha}{1 - \alpha} > 2,$$

because $A_5 = a_5 \cdot \alpha \geq a_6 \cdot \alpha = \frac{\alpha}{1-\alpha}$ and $\alpha \in \left(\frac{2}{3}, \frac{3}{4}\right]$. Now that we have a domain established, we can begin applying the partial differentiation test to demonstrate that Δ_8 is positive.

We see that

$$\frac{\partial^5 \Delta_8}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5} = 6\alpha^5 - 6\alpha^4 + 1 > 0$$

for all $\alpha \in (0, 1)$. Thus the partial derivative of Δ_8 with respect to A_1, A_2, A_3, A_4 , and A_5 is positive and minimized at $A_5 = \frac{\alpha}{1-\alpha}$.

$$\left. \frac{\partial^4 \Delta_8}{\partial A_1 \partial A_2 \partial A_3 \partial A_4} \right|_{A_5 = \frac{\alpha}{1-\alpha}} = \frac{\alpha^2(12\alpha^4 - 18\alpha^3 + 6\alpha^2 + 1)}{(\alpha - 1)} > 0.$$

It follows that the partial of Δ_8 with respect to A_1, A_2, A_3 , and A_5 is positive for all $A_4 \geq \frac{\alpha}{1-\alpha}, \alpha \in (0, 1)$ because $\frac{\partial^3 \Delta_8}{\partial A_1 \partial A_2 \partial A_3}$ is symmetric in A_4 and A_5 . Hence, $\frac{\partial^3 \Delta_8}{\partial A_1 \partial A_2 \partial A_3}$ is an increasing function of A_4, A_5 with a minimum at $A_4 = A_5 = \frac{\alpha}{1-\alpha}$.

$$\left. \frac{\partial^3 \Delta_8}{\partial A_1 \partial A_2 \partial A_3} \right|_{A_4 = A_5 = \frac{\alpha}{1-\alpha}} = \frac{\alpha^4(24\alpha^3 - 48\alpha^2 + 30\alpha - 5)}{(1-\alpha)^2} > 0.$$

This is positive, so we know that the partial with respect to A_1, A_2 , and A_3 is positive. Since the partial with respect to A_1 and A_2 is symmetric with respect to A_3, A_4 , and A_5 , we know that $\frac{\partial^3 \Delta_8}{\partial A_1 \partial A_2 \partial A_4}$ and $\frac{\partial^3 \Delta_8}{\partial A_1 \partial A_2 \partial A_5}$ are positive over the given domain. Hence, $\frac{\partial^2 \Delta_8}{\partial A_1 \partial A_2}$ is an increasing function of A_3, A_4 , and A_5 for all $A_3, A_4, A_5 \geq \frac{\alpha}{1-\alpha}$ and $\alpha \in \left(\frac{2}{3}, \frac{3}{4}\right]$. The minimum is at $A_3 = A_4 = A_5 = \frac{\alpha}{1-\alpha}$.

$$\left. \frac{\partial^2 \Delta_8}{\partial A_1 \partial A_2} \right|_{A_3 = A_4 = A_5 = \frac{\alpha}{1-\alpha}} = \frac{\alpha^4(48\alpha^4 - 120\alpha^3 + 109\alpha^2 - 42\alpha + 6)}{(1-\alpha)^3} > 0.$$

Because this is symmetric with respect to A_2, A_3, A_4 , and A_5 , we see that $\frac{\partial^2 \Delta_8}{\partial A_1 \partial A_3}$, $\frac{\partial^2 \Delta_8}{\partial A_1 \partial A_4}$, and $\frac{\partial^2 \Delta_8}{\partial A_1 \partial A_5}$ are positive over the given domain. Hence, $\frac{\partial \Delta_8}{\partial A_1}$ is an increasing function of A_2, A_3, A_4 and A_5 minimized at $A_2 = A_3 = A_4 = A_5 = \frac{\alpha}{1-\alpha}$.

$$\left. \frac{\partial \Delta_8}{\partial A_1} \right|_{A_2=A_3=A_4=A_5=\frac{\alpha}{1-\alpha}} = \frac{\alpha^4(96\alpha^5 - 287\alpha^4 + 336\alpha^3 - 192\alpha^2 + 54\alpha - 6)}{(1-\alpha)^4} > 0.$$

Thus $\frac{\partial \Delta_8}{\partial A_1}$ is positive over the given domain. By the symmetry of Δ_8 in A_1, A_2, A_3 , and A_4 , we know that $\frac{\partial \Delta_8}{\partial A_2}$, $\frac{\partial \Delta_8}{\partial A_3}$, and $\frac{\partial \Delta_8}{\partial A_4}$ are positive over the given domain. Therefore, Δ_8 is an increasing function of A_1, A_2, A_3 , and A_4 . We can hence plug in the minimum values of A_1, A_2, A_3 , and A_4 to determine a new polynomial in A_5 and α that we want to show is positive.

We define

$$\begin{aligned} \Delta_9 &= \Delta_8|_{A_1=5, A_2=4, A_3=3, A_4=2} \\ &= -30\alpha^5 A_5^4 + 150\alpha^5 A_5^3 + 30\alpha^4 A_5^4 - 240\alpha^5 A_5^2 - 150\alpha^4 A_5^3 + 258\alpha^5 A_5 + 240\alpha^4 A_5^2 + 590\alpha^5 \\ &\quad - 257\alpha^4 A_5 - 577\alpha^4 - 14\alpha^3 A_5 - 71\alpha^3 + 71\alpha^2 A_5 + 154\alpha^2 - 154\alpha A_5 - 120\alpha + 120A_5. \end{aligned}$$

We must show that Δ_9 is positive over the interval $\alpha \in (\frac{2}{3}, \frac{3}{4}]$ and $A_5 \geq \frac{\alpha}{1-\alpha}$. We can apply the partial differentiation test normally. We begin by noting that

$$\frac{\partial^4 \Delta_9}{\partial A_5^4} = 720\alpha^4(1-\alpha) > 0.$$

We then consider

$$\left. \frac{\partial^3 \Delta_9}{\partial A_5^3} \right|_{A_5=\frac{\alpha}{1-\alpha}} = 180\alpha^4(9\alpha - 5) > 0.$$

Similarly,

$$\left. \frac{\partial^2 \Delta_9}{\partial A_5^2} \right|_{A_5=\frac{\alpha}{1-\alpha}} = \frac{60\alpha^4(29\alpha^2 - 31\alpha + 8)}{1-\alpha} > 0.$$

We continue by considering

$$\left. \frac{\partial \Delta_9}{\partial A_5} \right|_{A_5 = \frac{\alpha}{1-\alpha}} = \frac{1308\alpha^7 - 2183\alpha^6 + 1238\alpha^5 - 158\alpha^4 - 310\alpha^3 + 499\alpha^2 - 394\alpha + 120}{(1-\alpha)^2} > 0,$$

for $\alpha \in (\frac{2}{3}, \frac{3}{4}]$. Finally, we evaluate Δ_9 at its minimum:

$$\Delta_9|_{A_5 = \frac{\alpha}{1-\alpha}} = \frac{2\alpha^2(44\alpha^6 + 472\alpha^5 - 1216\alpha^4 + 898\alpha^3 - 46\alpha^2 - 197\alpha + 60)}{(1-\alpha)^3} > 0,$$

completing this subcase.

2.3.2 Subcase III (b)

In this subcase, we know that $P_5(x_6 = 2) > 0$, implying that $(1, 1, 1, 1, 1, 2)$ is a positive integral solution to the inequality in Theorem 2.3. Thus, if

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} \leq 1 - \frac{2}{a_6} := \alpha_1,$$

then $\alpha_1 \in (\frac{1}{3}, \frac{1}{2}]$ because $a_6 \in (3, 4]$. For simplicity, let $A_i = a_i \cdot \alpha_1$ for $i = 1, 2, 3, 4$, and 5.

This yields the new inequality

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} \leq 1.$$

Thus, by Theorem 1.7, we have

$$6! P_5(x_6 = 2) \leq 6 \left[(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1) - (A_5 - 1)^5 + A_5(A_5 - 1)(A_5 - 2)(A_5 - 3)(A_5 - 4) \right],$$

and,

$$6! P_5(x_6 = 1) \leq 6 \left[\left(A_1 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right) \left(A_2 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right) \left(A_3 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right) \left(A_4 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right) \left(A_5 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right) - \left(A_5 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right)^5 + \left(A_5 \cdot \frac{1 + \alpha_1}{2\alpha_1} \right) \left(A_5 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right) \left(A_5 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 2 \right) \left(A_5 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 3 \right) \left(A_5 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 4 \right) \right].$$

Because $6! P_6 = 6! (P_5(x_6 = 1) + P_5(x_6 = 2))$, as before, we take the difference obtained by subtracting the sums of the right hand sides of the above inequalities from the RHS of Theorem 2.3, substituting in $a_i = \frac{A_i}{\alpha_1}$. We observe that this difference is equal to $\Delta_5 - 665$, where Δ_5 is from Case II above. Since we applied the partial differentiation test on $\Delta_6 = \Delta_5 \cdot 16\alpha^5(1 - \alpha)$ in Case I, here we need to show that

$$\Delta_{10} := \Delta_6 - 665 \cdot 16\alpha_1^5(1 - \alpha_1)$$

is positive. Since $\Delta_{10} - \Delta_6$ is a function of α_1 only, we only need to check that the value of Δ_{10} at its minimum is positive (because all of the partial derivatives in the test are the same for Δ_6 and Δ_{10}). Recall that we have

$$A_1 \geq 5, A_2 \geq 4, A_3 \geq 3, A_4 \geq 2, \text{ and } A_5 > \frac{2\alpha_1}{1 - \alpha_1},$$

so we must only check that

$$\Delta_{10} \Big|_{A_1=5, A_2=4, A_3=3, A_4=2, A_5=\frac{2\alpha_1}{1-\alpha_1}} = \frac{1}{(1 - \alpha_1)^3} (16\alpha_1(3094\alpha_1^8 - 2093\alpha_1^7 - 1272\alpha_1^6 + 1770\alpha_1^5 - 738\alpha_1^4 + 351\alpha_1^3 - 28\alpha_1^2 - 244\alpha_1 + 120)) > 0.$$

Since this is true, Δ_{10} is always positive, and this subcase is complete.

2.3.3 Subcase III (c)

In this subcase, we know that $P_5(x_6 = 3) > 0$, implying that $(1, 1, 1, 1, 1, 3)$ is a positive integral solution to the inequality in Theorem 2.3. Thus, if

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} \leq 1 - \frac{3}{a_6} := \alpha_2,$$

then $\alpha_2 \in (0, \frac{1}{4}]$ because $a_6 \in (3, 4]$. For simplicity, let $A_i = a_i \cdot \alpha_2$ for $i = 1, 2, 3, 4$, and 5.

This yields the new inequality

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} \leq 1.$$

Thus, by Theorem 1.7, we have

$$6! P_5(x_6 = 3) \leq 6 \left[(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1) - (A_5 - 1)^5 + A_5(A_5 - 1)(A_5 - 2)(A_5 - 3)(A_5 - 4) \right],$$

as well as

$$6! P_5(x_6 = 2) \leq 6 \left[\left(A_1 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1 \right) \left(A_2 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1 \right) \left(A_3 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1 \right) \left(A_4 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1 \right) \left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1 \right) - \left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1 \right)^5 + \left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2} \right) \left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1 \right) \left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 2 \right) \left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 3 \right) \left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 4 \right) \right],$$

and

$$6! P_5(x_6 = 1) \leq 6 \left[\left(A_1 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1 \right) \left(A_2 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1 \right) \left(A_3 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1 \right) \left(A_4 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1 \right) \left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1 \right) - \left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1 \right)^5 + \left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2} \right) \left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1 \right) \left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2} - 2 \right) \left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2} - 3 \right) \left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2} - 4 \right) \right].$$

Because $6!P_6 = 6!(P_5(x_6 = 1) + P_5(x_6 = 2) + P_5(x_6 = 3))$, if we let Δ_{11} be the difference obtained by subtracting the sum of the right hand sides of the above inequalities from the RHS of Theorem 2.4 and substituting in $a_i = \frac{A_i}{\alpha_2}$, then we merely have to apply the partial differentiation test for the expression

$$\begin{aligned} \Delta_{11} &= \frac{1}{27\alpha_2^5(1-\alpha_2)} \left(-980A_5^4\alpha_2^6 + 5400A_3^3\alpha_2^6 - 10080A_5^2\alpha_2^6 + 324A_1\alpha_2^6 - 252A_1A_2\alpha_2^6 + 324A_2\alpha_2^6 \right. \\ &\quad - 252A_1A_3\alpha_2^6 + 216A_1A_2A_3\alpha_2^6 - 252A_2A_3\alpha_2^6 + 324A_3\alpha_2^6 - 252A_1A_4\alpha_2^6 + 216A_1A_2A_4\alpha_2^6 \\ &\quad - 252A_2A_4\alpha_2^6 + 216A_1A_3A_4\alpha_2^6 - 196A_1A_2A_3A_4\alpha_2^6 + 216A_2A_3A_4\alpha_2^6 - 252A_3A_4\alpha_2^6 + \\ &\quad 324A_4\alpha_2^6 - 252A_1A_5\alpha_2^6 + 216A_1A_2A_5\alpha_2^6 - 252A_2A_5\alpha_2^6 + 216A_1A_3A_5\alpha_2^6 - 196A_1A_2A_3A_5\alpha_2^6 \\ &\quad + 216A_2A_3A_5\alpha_2^6 - 252A_3A_5\alpha_2^6 + 216A_1A_4A_5\alpha_2^6 - 196A_1A_2A_4A_5\alpha_2^6 + 216A_2A_4A_5\alpha_2^6 - \\ &\quad 196A_1A_3A_4A_5\alpha_2^6 + 184A_1A_2A_3A_4A_5\alpha_2^6 - 196A_2A_3A_4A_5\alpha_2^6 + 216A_3A_4A_5\alpha_2^6 - 252A_4A_5\alpha_2^6 \\ &\quad + 6480A_5\alpha_2^6 + 19656\alpha_2^6 + 580A_5^4\alpha_2^5 - 2700A_3^3\alpha_2^5 + 4320A_5^2\alpha_2^5 - 135A_1\alpha_2^5 + 108A_1A_2\alpha_2^5 - \\ &\quad 135A_2\alpha_2^5 + 108A_1A_3\alpha_2^5 - 108A_1A_2A_3\alpha_2^5 + 108A_2A_3\alpha_2^5 - 135A_3\alpha_2^5 + 108A_1A_4\alpha_2^5 - \\ &\quad 108A_1A_2A_4\alpha_2^5 + 108A_2A_4\alpha_2^5 - 108A_1A_3A_4\alpha_2^5 + 116A_1A_2A_3A_4\alpha_2^5 - 108A_2A_3A_4\alpha_2^5 + \\ &\quad 108A_3A_4\alpha_2^5 - 135A_4\alpha_2^5 + 108A_1A_5\alpha_2^5 - 108A_1A_2A_5\alpha_2^5 + 108A_2A_5\alpha_2^5 - 108A_1A_3A_5\alpha_2^5 + \\ &\quad 116A_1A_2A_3A_5\alpha_2^5 - 108A_2A_3A_5\alpha_2^5 + 108A_3A_5\alpha_2^5 - 108A_1A_4A_5\alpha_2^5 + 116A_1A_2A_4A_5\alpha_2^5 - \\ &\quad 108A_2A_4A_5\alpha_2^5 + 116A_1A_3A_4A_5\alpha_2^5 - 124A_1A_2A_3A_4A_5\alpha_2^5 + 116A_2A_3A_4A_5\alpha_2^5 - \\ &\quad 108A_3A_4A_5\alpha_2^5 + 108A_4A_5\alpha_2^5 - 3213A_5\alpha_2^5 - 19737\alpha_2^5 - 80A_5^4\alpha_2^4 + 2160A_5^2\alpha_2^4 - 108A_1\alpha_2^4 + \\ &\quad 27A_1A_2\alpha_2^4 - 108A_2\alpha_2^4 + 27A_1A_3\alpha_2^4 + 27A_2A_3\alpha_2^4 - 108A_3\alpha_2^4 + 27A_1A_4\alpha_2^4 + 27A_2A_4\alpha_2^4 - \\ &\quad 16A_1A_2A_3A_4\alpha_2^4 + 27A_3A_4\alpha_2^4 - 108A_4\alpha_2^4 + 27A_1A_5\alpha_2^4 + 27A_2A_5\alpha_2^4 - 16A_1A_2A_3A_5\alpha_2^4 + \\ &\quad 27A_3A_5\alpha_2^4 - 16A_1A_2A_4A_5\alpha_2^4 - 16A_1A_3A_4A_5\alpha_2^4 + 20A_1A_2A_3A_4A_5\alpha_2^4 - 16A_2A_3A_4A_5\alpha_2^4 + \\ &\quad 27A_4A_5\alpha_2^4 - 3186A_5\alpha_2^4 + 80A_5^4\alpha_2^3 - 1350A_5^3\alpha_2^3 + 3600A_5^2\alpha_2^3 + 36A_1A_2\alpha_2^3 + 36A_1A_3\alpha_2^3 - \\ &\quad 27A_1A_2A_3\alpha_2^3 + 36A_2A_3\alpha_2^3 + 36A_1A_4\alpha_2^3 - 27A_1A_2A_4\alpha_2^3 + 36A_2A_4\alpha_2^3 - 27A_1A_3A_4\alpha_2^3 + \\ &\quad 16A_1A_2A_3A_4\alpha_2^3 - 27A_2A_3A_4\alpha_2^3 + 36A_3A_4\alpha_2^3 + 36A_1A_5\alpha_2^3 - 27A_1A_2A_5\alpha_2^3 + 36A_2A_5\alpha_2^3 - \\ &\quad 27A_1A_3A_5\alpha_2^3 + 16A_1A_2A_3A_5\alpha_2^3 - 27A_2A_3A_5\alpha_2^3 + 36A_3A_5\alpha_2^3 - 27A_1A_4A_5\alpha_2^3 + \\ &\quad 16A_1A_2A_4A_5\alpha_2^3 - 27A_2A_4A_5\alpha_2^3 + 16A_1A_3A_4A_5\alpha_2^3 + 16A_2A_3A_4A_5\alpha_2^3 - 27A_3A_4A_5\alpha_2^3 + \\ &\quad 36A_4A_5\alpha_2^3 + 230A_5^4\alpha_2^2 - 1350A_5^3\alpha_2^2 + 19A_1A_2A_3A_4\alpha_2^2 + 19A_1A_2A_3A_5\alpha_2^2 + 19A_1A_2A_4A_5\alpha_2^2 \\ &\quad + 19A_1A_3A_4A_5\alpha_2^2 - 20A_1A_2A_3A_4A_5\alpha_2^2 + 19A_2A_3A_4A_5\alpha_2^2 + 170A_5^4\alpha_2 - 20A_1A_2A_3A_4\alpha_2 \\ &\quad - 20A_1A_2A_3A_5\alpha_2 - 20A_1A_2A_4A_5\alpha_2 \\ &\quad - 20A_1A_3A_4A_5\alpha_2 - 11A_1A_2A_3A_4A_5\alpha_2 - 20A_2A_3A_4A_5\alpha_2 + 32A_1A_2A_3A_4A_5) \\ &= \frac{1}{27\alpha_2^5(1-\alpha_2)} \Delta_{12}. \end{aligned}$$

Our domain is

$$A_1 \geq 5, A_2 \geq 4, A_3 \geq 3, A_4 \geq 2, \text{ and } A_5 > 1.$$

Note also that we have $\alpha_2 \in \left(0, \frac{1}{4}\right]$.

To start the partial differentiation test, we see that

$$\frac{\partial^5 \Delta_{12}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5} = 184\alpha_2^6 - 124\alpha_2^5 + 20\alpha_2^4 - 20\alpha_2^2 - 11\alpha_2 + 32 > 0$$

for all $\alpha_2 \in (0, \frac{1}{4}]$. Thus the partial derivative of Δ_{12} with respect to A_1, A_2, A_3, A_4 , and A_5 is positive and minimized at $A_5 = 1$.

$$\left. \frac{\partial^4 \Delta_{12}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4} \right|_{A_5=1} = -12\alpha_2^6 - 8\alpha_2^5 + 4\alpha_2^4 + 16\alpha_2^3 - \alpha_2^2 - 31\alpha_2 + 32 > 0.$$

It follows that the partial of Δ_{12} with respect to A_1, A_2, A_3 , and A_5 is positive for all $A_4 \geq 1, \alpha_2 \in (0, \frac{1}{4}]$ because $\frac{\partial^3 \Delta_{12}}{\partial A_1 \partial A_2 \partial A_3}$ is symmetric in A_4 and A_5 . Hence, $\frac{\partial^3 \Delta_{12}}{\partial A_1 \partial A_2 \partial A_3}$ is an increasing function of A_4 and A_5 with a minimum at $A_4 = A_5 = 1$.

$$\left. \frac{\partial^3 \Delta_{12}}{\partial A_1 \partial A_2 \partial A_3} \right|_{A_4=A_5=1} = 8\alpha_2^6 - 12\alpha_2^4 + 5\alpha_2^3 + 18\alpha_2^2 - 51\alpha_2 + 32 > 0.$$

This is positive over our domain, so we know that the partial with respect to A_1, A_2 , and A_3 is positive. Since the partial with respect to A_1 and A_2 is symmetric with respect to A_3, A_4 , and A_5 , we know that $\frac{\partial^3 \Delta_{12}}{\partial A_1 \partial A_2 \partial A_4}$ and $\frac{\partial^3 \Delta_{12}}{\partial A_1 \partial A_2 \partial A_5}$ are also positive over the given domain. Hence, $\frac{\partial^2 \Delta_{12}}{\partial A_1 \partial A_2}$ is an increasing function of A_3, A_4 , and A_5 for all $A_3, A_4, A_5 \geq 1$ and $\alpha_2 \in (0, \frac{1}{4}]$. The minimum is at $A_3 = A_4 = A_5 = 1$.

$$\left. \frac{\partial^2 \Delta_{12}}{\partial A_1 \partial A_2} \right|_{A_3=A_4=A_5=1} = -8\alpha_2^6 + 8\alpha_2^5 - \alpha_2^4 + 3\alpha_2^3 + 37\alpha_2^2 - 71\alpha_2 + 32 > 0.$$

Because this is symmetric with respect to A_2, A_3, A_4 , and A_5 , we see that $\frac{\partial^2 \Delta_{12}}{\partial A_1 \partial A_3}, \frac{\partial^2 \Delta_{12}}{\partial A_1 \partial A_4}$, and $\frac{\partial^2 \Delta_{12}}{\partial A_1 \partial A_5}$ are positive over the given domain. Hence, $\frac{\partial \Delta_{12}}{\partial A_1}$ is an increasing function of

A_2, A_3, A_4 and A_5 minimized at $A_2 = A_3 = A_4 = A_5 = 1$.

$$\left. \frac{\partial \Delta_{12}}{\partial A_1} \right|_{A_2=A_3=A_4=A_5=1} = 12\alpha_2^6 - 11\alpha_2^5 - 44\alpha_2^4 + 46\alpha_2^3 + 56\alpha_2^2 - 91\alpha_2 + 32 > 0.$$

Thus $\frac{\partial \Delta_{12}}{\partial A_1}$ is positive over the given domain. By the symmetry of Δ_{12} in A_1, A_2, A_3 , and A_4 , we know that $\frac{\partial \Delta_{12}}{\partial A_2}$, $\frac{\partial \Delta_{12}}{\partial A_3}$, and $\frac{\partial \Delta_{12}}{\partial A_4}$ are positive over the given domain. Therefore, Δ_{12} is an increasing function of A_1, A_2, A_3 , and A_4 . We can hence plug in the minimum values of A_1, A_2, A_3 , and A_4 to determine a new polynomial in A_5 and α_2 that we want to show is positive. We define

$$\begin{aligned} \Delta_{13} &= \Delta_{12}|_{A_1=5, A_2=4, A_3=3, A_4=2} \\ &= -980A_5^4\alpha_2^6 + 580A_5^4\alpha_2^5 + 5400A_5^3\alpha_2^6 - 80A_5^4\alpha_2^4 - 2700A_5^3\alpha_2^5 - 10080A_5^2\alpha_2^6 + 80A_5^4\alpha_2^3 \\ &\quad + 4320A_5^2\alpha_2^5 + 10184A_5\alpha_2^6 + 230A_5^4\alpha_2^2 - 1350A_5^3\alpha_2^3 + 2160A_5^2\alpha_2^4 - 6385A_5\alpha_2^5 + 16044\alpha_2^6 \\ &\quad + 170A_5^4\alpha_2 - 1350A_5^3\alpha_2^2 + 3600A_5^2\alpha_2^3 - 2872A_5\alpha_2^4 - 16671\alpha_2^5 + 1051A_5\alpha_2^3 - 1515\alpha_2^4 \\ &\quad + 526A_5\alpha_2^2 + 318\alpha_2^3 - 4400A_5\alpha_2 + 2280\alpha_2^2 + 3840A_5 - 2400\alpha_2. \end{aligned}$$

We must show that Δ_{13} is positive over the interval $\alpha_2 \in (0, \frac{1}{4}]$ and $A_5 \geq 1$.

We apply the partial differentiation test normally, beginning by noting that

$$\frac{\partial^4 \Delta_{13}}{\partial A_5^4} = -23520\alpha_2^6 + 13920\alpha_2^5 - 1920\alpha_2^4 + 1920\alpha_2^3 + 5520\alpha_2^2 + 4080\alpha_2 > 0.$$

We then consider

$$\left. \frac{\partial^3 \Delta_{13}}{\partial A_5^3} \right|_{A_5=1} = 8880\alpha_2^6 - 2280\alpha_2^5 - 1920\alpha_2^4 - 6180\alpha_2^3 - 2580\alpha_2^2 + 4080\alpha_2 > 0.$$

Similarly,

$$\left. \frac{\partial^2 \Delta_{13}}{\partial A_5^2} \right|_{A_5=1} = 480\alpha_2^6 - 600\alpha_2^5 + 3360\alpha_2^4 + 60\alpha_2^3 - 5340\alpha_2^2 + 2040\alpha_2,$$

which is positive for $\alpha_2 \in (0, \frac{1}{4}]$. We continue by considering

$$\left. \frac{\partial \Delta_{13}}{\partial A_5} \right|_{A_5=1} = 2304\alpha_2^6 - 3525\alpha_2^5 + 1128\alpha_2^4 + 4521\alpha_2^3 - 2604\alpha_2^2 - 3720\alpha_2 + 3840 > 0.$$

Finally, we evaluate Δ_{13} at its minimum:

$$\Delta_{13}|_{A_5=1} = 20568\alpha_2^6 - 20856\alpha_2^5 - 2307\alpha_2^4 + 3699\alpha_2^3 + 1686\alpha_2^2 - 6630\alpha_2 + 3840 > 0.$$

This completes this subcase, and hence Case III is complete. \square

2.4 Case IV

In this case, $\lceil a_6 \rceil = 5$, so we can plug $v = 5$ into the statement of Theorem 1.10 to get the following theorem, which we prove in this case:

Theorem 2.4. *Let $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 > 1$ be real numbers and let P_6 be the number of positive integral solutions of $\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_5}{a_5} + \frac{x_6}{a_6} \leq 1$. If $P_6 > 0$ and $4 < a_6 \leq 5$, then*

$$6! P_6 \leq (a_1 - 1)(a_2 - 1)(a_3 - 1)(a_4 - 1)(a_5 - 1)(a_6 - 1) - 4096.$$

Proof. In this case, $a_6 \in (4, 5]$, so we have to consider four levels — $x_6 = 1$, $x_6 = 2$, $x_6 = 3$ and $x_6 = 4$. Since $P_6 > 0$, there must be solutions at the $x_6 = 1$ level, so our four subcases are:

Subcase IV (a) $P_5(x_6 = 4) = P_5(x_6 = 3) = P_5(x_6 = 2) = 0$.

Subcase IV (b) $P_5(x_6 = 4) = P_5(x_6 = 3) = 0, P_5(x_6 = 2) > 0$.

Subcase IV (c) $P_5(x_6 = 4) = 0, P_5(x_6 = 3) > 0, P_5(x_6 = 2) > 0$.

Subcase IV (d) $P_5(x_6 = 4) > 0, P_5(x_6 = 3) > 0, P_5(x_6 = 2) > 0$.

2.4.1 Subcase IV (a)

We are guaranteed that $(x_1, x_2, x_3, x_4, x_5, x_6) = (1, 1, 1, 1, 1, 1)$ is a solution to the inequality in Theorem 2.4. Thus, if

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} \leq 1 - \frac{1}{a_6} := \alpha,$$

then $\alpha \in (\frac{3}{4}, \frac{4}{5}]$ because $a_6 \in (4, 5]$. For simplicity, let $A_i = a_i \cdot \alpha$ for $i = 1, 2, 3, 4$, and 5.

This yields the new inequality

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} \leq 1.$$

Whereas we previously bounded the number of positive integral solutions to this inequality using the Yau Number Theoretic Conjecture for $n = 5$, we will now use the Yau Geometric Conjecture for $n = 5$, proven in [1]. This gives us the bound

$$6! P_5(x_6 = 1) \leq 6 [(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1) - p_5(\lceil A_5 \rceil)], \quad (3)$$

where p_5 is the function defined in (1). Note that since $p_5(v)$ is increasing for $v \geq 4$. Also, since $A_5 = a_5 \cdot \alpha > \frac{3}{4} \cdot 4 = 3$, we note that $\lceil A_5 \rceil \geq 4$. Thus, we maximize the RHS of (3) by substituting $p_5(4) = 243$ in for $p_5(\lceil A_5 \rceil)$. Hence, we have

$$6! P_5(x_6 = 1) \leq 6 [(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1) - 243]. \quad (4)$$

As before, we take the difference obtained by subtracting the RHS of (4) from the RHS of Theorem 2.4, substituting in $a_i = \frac{A_i}{\alpha}$ and $a_6 = \frac{1}{1 - \alpha}$, yielding

$$\begin{aligned}
\Delta_{14} &= \frac{1}{2\alpha^4(1-\alpha)} (5262\alpha^5 - 5264\alpha^4 + 12\alpha^5 A_1 - 10\alpha^4 A_1 - 12\alpha^5 A_1 A_2 + 12\alpha^4 A_1 A_2 - \\
& 2\alpha^3 A_1 A_2 + 12\alpha^5 A_1 A_2 A_3 - 12\alpha^4 A_1 A_2 A_3 + 2\alpha^2 A_1 A_2 A_3 - 12\alpha^5 A_1 A_2 A_3 A_4 + \\
& 12\alpha^4 A_1 A_2 A_3 A_4 - 2\alpha A_1 A_2 A_3 A_4 + 12\alpha^5 A_1 A_2 A_3 A_4 A_5 - 12\alpha^4 A_1 A_2 A_3 A_4 A_5 + \\
& 2A_1 A_2 A_3 A_4 A_5 - 12\alpha^5 A_1 A_2 A_3 A_5 + 12\alpha^4 A_1 A_2 A_3 A_5 - 2\alpha A_1 A_2 A_3 A_5 + 12\alpha^5 A_1 A_2 A_4 - \\
& 12\alpha^4 A_1 A_2 A_4 + 2\alpha^2 A_1 A_2 A_4 - 12\alpha^5 A_1 A_2 A_4 A_5 + 12\alpha^4 A_1 A_2 A_4 A_5 - 2\alpha A_1 A_2 A_4 A_5 + \\
& 12\alpha^5 A_1 A_2 A_5 - 12\alpha^4 A_1 A_2 A_5 + 2\alpha^2 A_1 A_2 A_5 - 12\alpha^5 A_1 A_3 + 12\alpha^4 A_1 A_3 - 2\alpha^3 A_1 A_3 + \\
& 12\alpha^5 A_1 A_3 A_4 - 12\alpha^4 A_1 A_3 A_4 + 2\alpha^2 A_1 A_3 A_4 - 12\alpha^5 A_1 A_3 A_4 A_5 + 12\alpha^4 A_1 A_3 A_4 A_5 - \\
& 2\alpha A_1 A_3 A_4 A_5 + 12\alpha^5 A_1 A_3 A_5 - 12\alpha^4 A_1 A_3 A_5 + 2\alpha^2 A_1 A_3 A_5 - 12\alpha^5 A_1 A_4 + 12\alpha^4 A_1 A_4 - \\
& 2\alpha^3 A_1 A_4 + 12\alpha^5 A_1 A_4 A_5 - 12\alpha^4 A_1 A_4 A_5 + 2\alpha^2 A_1 A_4 A_5 - 12\alpha^5 A_1 A_5 + 12\alpha^4 A_1 A_5 - \\
& 2\alpha^3 A_1 A_5 + 12\alpha^5 A_2 - 10\alpha^4 A_2 - 12\alpha^5 A_2 A_3 + 12\alpha^4 A_2 A_3 - 2\alpha^3 A_2 A_3 + 12\alpha^5 A_2 A_3 A_4 - \\
& 12\alpha^4 A_2 A_3 A_4 + 2\alpha^2 A_2 A_3 A_4 - 12\alpha^5 A_2 A_3 A_4 A_5 + 12\alpha^4 A_2 A_3 A_4 A_5 - 2\alpha A_2 A_3 A_4 A_5 + \\
& 12\alpha^5 A_2 A_3 A_5 - 12\alpha^4 A_2 A_3 A_5 + 2\alpha^2 A_2 A_3 A_5 - 12\alpha^5 A_2 A_4 + 12\alpha^4 A_2 A_4 - 2\alpha^3 A_2 A_4 + \\
& 12\alpha^5 A_2 A_4 A_5 - 12\alpha^4 A_2 A_4 A_5 + 2\alpha^2 A_2 A_4 A_5 - 12\alpha^5 A_2 A_5 + 12\alpha^4 A_2 A_5 - 2\alpha^3 A_2 A_5 + \\
& 12\alpha^5 A_3 - 10\alpha^4 A_3 - 12\alpha^5 A_3 A_4 + 12\alpha^4 A_3 A_4 - 2\alpha^3 A_3 A_4 + 12\alpha^5 A_3 A_4 A_5 - 12\alpha^4 A_3 A_4 A_5 \\
& + 2\alpha^2 A_3 A_4 A_5 - 12\alpha^5 A_3 A_5 + 12\alpha^4 A_3 A_5 - 2\alpha^3 A_3 A_5 + 12\alpha^5 A_4 - 10\alpha^4 A_4 - 12\alpha^5 A_4 A_5 \\
& + 12\alpha^4 A_4 A_5 - 2\alpha^3 A_4 A_5 + 12\alpha^5 A_5 - 10\alpha^4 A_5) \\
&= \frac{1}{2\alpha^4(1-\alpha)} \Delta_{15}.
\end{aligned}$$

We now proceed with the partial derivative test on Δ_{15} with $\alpha \in \left(\frac{3}{4}, \frac{4}{5}\right]$ and

$$A_1 \geq 5, A_2 \geq 4, A_3 \geq 3, A_4 \geq 2, \text{ and } A_5 > \frac{\alpha}{1-\alpha},$$

like in previous cases.

$$\frac{\partial^5 \Delta_{15}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5} = 12\alpha^5 - 12\alpha^4 + 2 > 0, \quad \alpha \in \left(\frac{3}{4}, \frac{4}{5}\right].$$

Thus the partial derivative of Δ_{15} with respect to A_1, A_2, A_3, A_4 , and A_5 is positive and

minimized at $A_5 = \frac{\alpha}{1-\alpha}$.

$$\left. \frac{\partial^4 \Delta_{15}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4} \right|_{A_5 = \frac{\alpha}{1-\alpha}} = \frac{2(12\alpha^6 - 18\alpha^5 + 6\alpha^4 + \alpha^2)}{1-\alpha} > 0.$$

We continue with,

$$\frac{\partial^3 \Delta_{15}}{\partial A_1 \partial A_2 \partial A_3} \Big|_{A_4=A_5=\frac{\alpha}{1-\alpha}} = \frac{2\alpha^4 (24\alpha^3 - 48\alpha^2 + 30\alpha - 5)}{(1-\alpha)^2} > 0.$$

This is positive over our domain, so we know that the partial with respect to A_1 , A_2 , and A_3 is positive. Furthermore,

$$\frac{\partial^2 \Delta_{15}}{\partial A_1 \partial A_2} \Big|_{A_3=A_4=A_5=\frac{\alpha}{1-\alpha}} = \frac{2\alpha^4 (48\alpha^4 - 120\alpha^3 + 109\alpha^2 - 42\alpha + 6)}{(1-\alpha)^3} > 0, \text{ and}$$

$$\frac{\partial \Delta_{15}}{\partial A_1} \Big|_{A_2=A_3=A_4=A_5=\frac{\alpha}{1-\alpha}} = \frac{2\alpha^4 (96\alpha^5 - 287\alpha^4 + 336\alpha^3 - 192\alpha^2 + 54\alpha - 6)}{(1-\alpha)^4} > 0.$$

We also observe that $A_5 = a_5\alpha \geq 5 \cdot \frac{3}{4} = \frac{15}{4}$, and since $A_1 \geq A_2 \geq A_3 \geq A_4 \geq A_5$, our minimum for Δ_{15} is

$$\Delta_{15} \Big|_{A_1=5, A_2=4, A_3=A_4=A_5=\frac{15}{4}} = \frac{264600\alpha^5 - 263368\alpha^4 - 10460\alpha^3 + 42075\alpha^2 - 84375\alpha + 67500}{32},$$

which is positive over our desired interval, completing this subcase.

2.4.2 Subcase IV (b)

In this subcase, we know that $P_5(x_6 = 2) > 0$, implying that $(1, 1, 1, 1, 1, 2)$ is a positive integral solution to the inequality in Theorem 2.4. Thus, if

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} \leq 1 - \frac{2}{a_6} := \alpha_1,$$

then $\alpha_1 \in (\frac{1}{2}, \frac{3}{4}]$ because $a_6 \in (4, 5]$. For simplicity, let $A_i = a_i \cdot \alpha_1$ for $i = 1, 2, 3, 4$, and 5.

This yields the new inequality

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} \leq 1.$$

Thus, by Theorem 1.7, we have

$$6! P_5(x_6 = 2) \leq 6 [(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1) - (A_5 - 1)^5 + A_5(A_5 - 1)(A_5 - 2)(A_5 - 3)(A_5 - 4)],$$

and,

$$6! P_5(x_6 = 1) \leq 6 \left[\left(A_1 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right) \left(A_2 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right) \left(A_3 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right) \left(A_4 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right) \left(A_5 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right) - \left(A_5 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right)^5 + \left(A_5 \cdot \frac{1 + \alpha_1}{2\alpha_1} \right) \left(A_5 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right) \left(A_5 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 2 \right) \left(A_5 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 3 \right) \left(A_5 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 4 \right) \right].$$

Because $6! P_6 = 6! (P_5(x_6 = 1) + P_5(x_6 = 2))$, as before, we take the difference obtained by subtracting the sums of the right hand sides of the above inequalities from the RHS of Theorem 2.4, substituting in $a_i = \frac{A_i}{\alpha_1}$, yielding

$$\begin{aligned} \Delta_{16} = & \frac{1}{16\alpha_1^5(1 - \alpha_1)} (-510A_5^4\alpha_1^6 + 2700A_5^3\alpha_1^6 - 4800A_5^2\alpha_1^6 + 144A_1\alpha_1^6 - 120A_1A_2\alpha_1^6 + 144A_2\alpha_1^6 \\ & - 120A_1A_3\alpha_1^6 + 108A_1A_2A_3\alpha_1^6 - 120A_2A_3\alpha_1^6 + 144A_3\alpha_1^6 - 120A_1A_4\alpha_1^6 + 108A_1A_2A_4\alpha_1^6 - 120A_2A_4\alpha_1^6 \\ & + 108A_1A_3A_4\alpha_1^6 - 102A_1A_2A_3A_4\alpha_1^6 + 108A_2A_3A_4\alpha_1^6 - 120A_3A_4\alpha_1^6 + 144A_4\alpha_1^6 - 120A_1A_5\alpha_1^6 \\ & + 108A_1A_2A_5\alpha_1^6 - 120A_2A_5\alpha_1^6 + 108A_1A_3A_5\alpha_1^6 - 102A_1A_2A_3A_5\alpha_1^6 + 108A_2A_3A_5\alpha_1^6 - 120A_3A_5\alpha_1^6 \\ & + 108A_1A_4A_5\alpha_1^6 - 102A_1A_2A_4A_5\alpha_1^6 + 108A_2A_4A_5\alpha_1^6 - 102A_1A_3A_4A_5\alpha_1^6 + 99A_1A_2A_3A_4A_5\alpha_1^6 \\ & - 102A_2A_3A_4A_5\alpha_1^6 + 108A_3A_4A_5\alpha_1^6 - 120A_4A_5\alpha_1^6 + 2880A_5\alpha_1^6 + 65520\alpha_1^6 + 390A_5^4\alpha_1^5 - 1800A_5^3\alpha_1^5 \\ & + 2880A_5^2\alpha_1^5 - 80A_1\alpha_1^5 + 72A_1A_2\alpha_1^5 - 80A_2\alpha_1^5 + 72A_1A_3\alpha_1^5 - 72A_1A_2A_3\alpha_1^5 + 72A_2A_3\alpha_1^5 - 80A_3\alpha_1^5 \\ & + 72A_1A_4\alpha_1^5 - 72A_1A_2A_4\alpha_1^5 + 72A_2A_4\alpha_1^5 - 72A_1A_3A_4\alpha_1^5 + 78A_1A_2A_3A_4\alpha_1^5 - 72A_2A_3A_4\alpha_1^5 + 72A_3A_4\alpha_1^5 \\ & - 80A_4\alpha_1^5 + 72A_1A_5\alpha_1^5 - 72A_1A_2A_5\alpha_1^5 + 72A_2A_5\alpha_1^5 - 72A_1A_3A_5\alpha_1^5 + 78A_1A_2A_3A_5\alpha_1^5 - 72A_2A_3A_5\alpha_1^5 \\ & + 72A_3A_5\alpha_1^5 - 72A_1A_4A_5\alpha_1^5 + 78A_1A_2A_4A_5\alpha_1^5 - 72A_2A_4A_5\alpha_1^5 + 78A_1A_3A_4A_5\alpha_1^5 - 84A_1A_2A_3A_4A_5\alpha_1^5 \end{aligned}$$

$$\begin{aligned}
& +78A_2A_3A_4A_5\alpha_1^5 - 72A_3A_4A_5\alpha_1^5 + 72A_4A_5\alpha_1^5 - 1904A_5\alpha_1^5 - 65552\alpha_1^5 - 60A_5^4\alpha_1^4 + 960A_5^2\alpha_1^4 - 32A_1\alpha_1^4 \\
& + 8A_1A_2\alpha_1^4 - 32A_2\alpha_1^4 + 8A_1A_3\alpha_1^4 + 8A_2A_3\alpha_1^4 - 32A_3\alpha_1^4 + 8A_1A_4\alpha_1^4 + 8A_2A_4\alpha_1^4 - 12A_1A_2A_3A_4\alpha_1^4 \\
& + 8A_3A_4\alpha_1^4 - 32A_4\alpha_1^4 + 8A_1A_5\alpha_1^4 + 8A_2A_5\alpha_1^4 - 12A_1A_2A_3A_5\alpha_1^4 + 8A_3A_5\alpha_1^4 - 12A_1A_2A_4A_5\alpha_1^4 \\
& - 12A_1A_3A_4A_5\alpha_1^4 + 15A_1A_2A_3A_4A_5\alpha_1^4 - 12A_2A_3A_4A_5\alpha_1^4 + 8A_4A_5\alpha_1^4 - 944A_5\alpha_1^4 + 60A_5^4\alpha_1^3 - 600A_5^3\alpha_1^3 \\
& + 960A_5^2\alpha_1^3 + 8A_1A_2\alpha_1^3 + 8A_1A_3\alpha_1^3 - 8A_1A_2A_3\alpha_1^3 + 8A_2A_3\alpha_1^3 + 8A_1A_4\alpha_1^3 - 8A_1A_2A_4\alpha_1^3 + 8A_2A_4\alpha_1^3 \\
& - 8A_1A_3A_4\alpha_1^3 + 12A_1A_2A_3A_4\alpha_1^3 - 8A_2A_3A_4\alpha_1^3 + 8A_3A_4\alpha_1^3 + 8A_1A_5\alpha_1^3 - 8A_1A_2A_5\alpha_1^3 + 8A_2A_5\alpha_1^3 \\
& - 8A_1A_3A_5\alpha_1^3 + 12A_1A_2A_3A_5\alpha_1^3 - 8A_2A_3A_5\alpha_1^3 + 8A_3A_5\alpha_1^3 - 8A_1A_4A_5\alpha_1^3 + 12A_1A_2A_4A_5\alpha_1^3 - 8A_2A_4A_5\alpha_1^3 \\
& + 12A_1A_3A_4A_5\alpha_1^3 + 12A_2A_3A_4A_5\alpha_1^3 - 8A_3A_4A_5\alpha_1^3 + 8A_4A_5\alpha_1^3 + 90A_5^4\alpha_1^2 - 300A_5^3\alpha_1^2 + 4A_1A_2A_3\alpha_1^2 \\
& + 4A_1A_2A_4\alpha_1^2 + 4A_1A_3A_4\alpha_1^2 + 2A_1A_2A_3A_4\alpha_1^2 + 4A_2A_3A_4\alpha_1^2 + 4A_1A_2A_5\alpha_1^2 + 4A_1A_3A_5\alpha_1^2 + 2A_1A_2A_3A_5\alpha_1^2 \\
& + 4A_2A_3A_5\alpha_1^2 + 4A_1A_4A_5\alpha_1^2 + 2A_1A_2A_4A_5\alpha_1^2 + 4A_2A_4A_5\alpha_1^2 + 2A_1A_3A_4A_5\alpha_1^2 - 15A_1A_2A_3A_4A_5\alpha_1^2 \\
& + 2A_2A_3A_4A_5\alpha_1^2 + 4A_3A_4A_5\alpha_1^2 + 30A_5^4\alpha_1 - 10A_1A_2A_3A_4\alpha_1 - 10A_1A_2A_3A_5\alpha_1 - 10A_1A_2A_4A_5\alpha_1 \\
& - 10A_1A_3A_4A_5\alpha_1 + 4A_1A_2A_3A_4A_5\alpha_1 - 10A_2A_3A_4A_5\alpha_1 + 13A_1A_2A_3A_4A_5) = \frac{1}{16\alpha_1^5(1-\alpha_1)}\Delta_{17}.
\end{aligned}$$

We now proceed with the partial derivative test on Δ_{17} with $\alpha_1 \in \left(\frac{1}{2}, \frac{3}{5}\right]$ and

$$A_1 \geq 5, A_2 \geq 4, A_3 \geq 3, A_4 \geq 2, \text{ and } A_5 > \frac{2\alpha_1}{1-\alpha_1},$$

like in previous cases.

$$\frac{\partial^5 \Delta_{17}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5} = 99\alpha_1^6 - 84\alpha_1^5 + 15\alpha_1^4 - 15\alpha_1^2 + 4\alpha_1 + 13 > 0, \quad \alpha_1 \in \left(\frac{1}{2}, \frac{3}{5}\right].$$

Thus the partial derivative of Δ_{17} with respect to $A_1, A_2, A_3, A_4,$ and A_5 is positive and minimized at $A_5 = \frac{2\alpha_1}{1-\alpha_1}$.

$$\left. \frac{\partial^4 \Delta_{17}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4} \right|_{A_5 = \frac{2\alpha_1}{1-\alpha_1}} = \frac{4\alpha_1(75\alpha_1^6 - 87\alpha_1^5 + 30\alpha_1^4 - 6\alpha_1^3 - 5\alpha_1^2 + 5\alpha_1 + 4)}{1-\alpha_1} > 0.$$

We continue with,

$$\left. \frac{\partial^3 \Delta_{17}}{\partial A_1 \partial A_2 \partial A_3} \right|_{A_4 = A_5 = \frac{2\alpha_1}{1-\alpha_1}} = \frac{16\alpha_1^2(57\alpha_1^6 - 84\alpha_1^5 + 42\alpha_1^4 - 11\alpha_1^3 + 3\alpha_1 + 1)}{(1-\alpha_1)^2} > 0.$$

This is positive over our domain, so we know that the partial with respect to A_1 , A_2 , and A_3 is positive. Furthermore,

$$\frac{\partial^2 \Delta_{17}}{\partial A_1 \partial A_2} \Big|_{A_3=A_4=A_5=\frac{2\alpha_1}{1-\alpha_1}} = \frac{16\alpha_1^3 (174\alpha_1^6 - 312\alpha_1^5 + 205\alpha_1^4 - 68\alpha_1^3 + 12\alpha_1^2 + 4\alpha_1 + 1)}{(1-\alpha_1)^3} > 0, \text{ and}$$

$$\frac{\partial \Delta_{17}}{\partial A_1} \Big|_{A_2=A_3=A_4=A_5=\frac{2\alpha_1}{1-\alpha_1}} = \frac{16\alpha_1^4 (534\alpha_1^6 - 1133\alpha_1^5 + 929\alpha_1^4 - 386\alpha_1^3 + 88\alpha_1^2 - \alpha_1 + 1)}{(1-\alpha_1)^4} > 0.$$

Finally, our minimum for Δ_{17} is

$$\begin{aligned} \Delta_{17} \Big|_{A_1=5, A_2=4, A_3=A_4=A_5=\frac{2\alpha_1}{1-\alpha_1}} &= \frac{16\alpha_1^2}{(1-\alpha_1)^3} (-2019\alpha_1^7 + 16182\alpha_1^6 - \\ &26485\alpha_1^5 + 17652\alpha_1^4 - 4241\alpha_1^3 - 262\alpha_1^2 + 73\alpha_1 + 60), \end{aligned}$$

which is positive over our desired interval, completing this subcase.

2.4.3 Subcase IV (c)

In this subcase, we know that $P_5(x_6 = 3) > 0$, implying that $(1, 1, 1, 1, 1, 3)$ is a positive integral solution to the inequality in Theorem 2.4. Thus, if

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} \leq 1 - \frac{3}{a_6} := \alpha_2,$$

then $\alpha_2 \in (\frac{1}{4}, \frac{2}{5}]$ because $a_6 \in (4, 5]$. For simplicity, let $A_i = a_i \cdot \alpha_2$ for $i = 1, 2, 3, 4$, and 5 .

This yields the new inequality

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} \leq 1.$$

Thus, by Theorem 1.7, we have

$$\begin{aligned} 6! P_5(x_6 = 3) &\leq 6 [(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1) \\ &\quad - (A_5 - 1)^5 + A_5(A_5 - 1)(A_5 - 2)(A_5 - 3)(A_5 - 4)], \end{aligned}$$

as well as

$$6! P_5(x_6 = 2) \leq 6 \left[\left(A_1 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1 \right) \left(A_2 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1 \right) \left(A_3 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1 \right) \left(A_4 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1 \right) \left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1 \right) - \left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1 \right)^5 + \left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2} \right) \left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1 \right) \left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 2 \right) \left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 3 \right) \left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 4 \right) \right],$$

and

$$6! P_5(x_6 = 1) \leq 6 \left[\left(A_1 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1 \right) \left(A_2 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1 \right) \left(A_3 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1 \right) \left(A_4 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1 \right) \left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1 \right) - \left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1 \right)^5 + \left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2} \right) \left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1 \right) \left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2} - 2 \right) \left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2} - 3 \right) \left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2} - 4 \right) \right].$$

Because $6! P_6 = 6! (P_5(x_6 = 1) + P_5(x_6 = 2) + P_5(x_6 = 3))$, if we let Δ_{11} be the difference obtained by subtracting the sum of the right hand sides of the above inequalities from the RHS of Theorem 2.4 and substituting in $a_i = \frac{A_i}{\alpha_2}$, then we merely have to apply the partial differentiation test for the expression

$$\begin{aligned} \Delta_{18} = & \frac{1}{27\alpha_2^5(1-\alpha_2)} (-980A_5^4\alpha_2^6 + 5400A_5^3\alpha_2^6 - 10080A_5^2\alpha_2^6 + 324A_1\alpha_2^6 - 252A_1A_2\alpha_2^6 + 324A_2\alpha_2^6 \\ & - 252A_1A_3\alpha_2^6 + 216A_1A_2A_3\alpha_2^6 - 252A_2A_3\alpha_2^6 + 324A_3\alpha_2^6 - 252A_1A_4\alpha_2^6 + 216A_1A_2A_4\alpha_2^6 - 252A_2A_4\alpha_2^6 \\ & + 216A_1A_3A_4\alpha_2^6 - 196A_1A_2A_3A_4\alpha_2^6 + 216A_2A_3A_4\alpha_2^6 - 252A_3A_4\alpha_2^6 + 324A_4\alpha_2^6 - 252A_1A_5\alpha_2^6 \\ & + 216A_1A_2A_5\alpha_2^6 - 252A_2A_5\alpha_2^6 + 216A_1A_3A_5\alpha_2^6 - 196A_1A_2A_3A_5\alpha_2^6 + 216A_2A_3A_5\alpha_2^6 - 252A_3A_5\alpha_2^6 \\ & + 216A_1A_4A_5\alpha_2^6 - 196A_1A_2A_4A_5\alpha_2^6 + 216A_2A_4A_5\alpha_2^6 - 196A_1A_3A_4A_5\alpha_2^6 + 184A_1A_2A_3A_4A_5\alpha_2^6 \\ & - 196A_2A_3A_4A_5\alpha_2^6 + 216A_3A_4A_5\alpha_2^6 - 252A_4A_5\alpha_2^6 + 6480A_5\alpha_2^6 + 110565\alpha_2^6 + 580A_5^4\alpha_2^5 - 2700A_5^3\alpha_2^5 \\ & + 4320A_5^2\alpha_2^5 - 135A_1\alpha_2^5 + 108A_1A_2\alpha_2^5 - 135A_2\alpha_2^5 + 108A_1A_3\alpha_2^5 - 108A_1A_2A_3\alpha_2^5 + 108A_2A_3\alpha_2^5 - 135A_3\alpha_2^5 \\ & + 108A_1A_4\alpha_2^5 - 108A_1A_2A_4\alpha_2^5 + 108A_2A_4\alpha_2^5 - 108A_1A_3A_4\alpha_2^5 + 116A_1A_2A_3A_4\alpha_2^5 - 108A_2A_3A_4\alpha_2^5 \\ & + 108A_3A_4\alpha_2^5 - 135A_4\alpha_2^5 + 108A_1A_5\alpha_2^5 - 108A_1A_2A_5\alpha_2^5 + 108A_2A_5\alpha_2^5 - 108A_1A_3A_5\alpha_2^5 + 116A_1A_2A_3A_5\alpha_2^5 \\ & - 108A_2A_3A_5\alpha_2^5 + 108A_3A_5\alpha_2^5 - 108A_1A_4A_5\alpha_2^5 + 116A_1A_2A_4A_5\alpha_2^5 - 108A_2A_4A_5\alpha_2^5 + 116A_1A_3A_4A_5\alpha_2^5 \\ & - 124A_1A_2A_3A_4A_5\alpha_2^5 + 116A_2A_3A_4A_5\alpha_2^5 - 108A_3A_4A_5\alpha_2^5 + 108A_4A_5\alpha_2^5 - 3213A_5\alpha_2^5 - 110646\alpha_2^5 \\ & - 80A_5^4\alpha_2^4 + 2160A_5^3\alpha_2^4 - 108A_1\alpha_2^4 + 27A_1A_2\alpha_2^4 - 108A_2\alpha_2^4 + 27A_1A_3\alpha_2^4 + 27A_2A_3\alpha_2^4 - 108A_3\alpha_2^4 \end{aligned}$$

$$\begin{aligned}
& + 27A_1A_4\alpha_2^4 + 27A_2A_4\alpha_2^4 - 16A_1A_2A_3A_4\alpha_2^4 + 27A_3A_4\alpha_2^4 - 108A_4\alpha_2^4 + 27A_1A_5\alpha_2^4 + 27A_2A_5\alpha_2^4 \\
& - 16A_1A_2A_3A_5\alpha_2^4 + 27A_3A_5\alpha_2^4 - 16A_1A_2A_4A_5\alpha_2^4 - 16A_1A_3A_4A_5\alpha_2^4 + 20A_1A_2A_3A_4A_5\alpha_2^4 \\
& - 16A_2A_3A_4A_5\alpha_2^4 + 27A_4A_5\alpha_2^4 - 3186A_5\alpha_2^4 + 80A_5^4\alpha_2^3 - 1350A_5^3\alpha_2^3 + 3600A_5^2\alpha_2^3 + 36A_1A_2\alpha_2^3 + 36A_1A_3\alpha_2^3 \\
& - 27A_1A_2A_3\alpha_2^3 + 36A_2A_3\alpha_2^3 + 36A_1A_4\alpha_2^3 - 27A_1A_2A_4\alpha_2^3 + 36A_2A_4\alpha_2^3 - 27A_1A_3A_4\alpha_2^3 + 16A_1A_2A_3A_4\alpha_2^3 \\
& - 27A_2A_3A_4\alpha_2^3 + 36A_3A_4\alpha_2^3 + 36A_1A_5\alpha_2^3 - 27A_1A_2A_5\alpha_2^3 + 36A_2A_5\alpha_2^3 - 27A_1A_3A_5\alpha_2^3 + 16A_1A_2A_3A_5\alpha_2^3 \\
& - 27A_2A_3A_5\alpha_2^3 + 36A_3A_5\alpha_2^3 - 27A_1A_4A_5\alpha_2^3 + 16A_1A_2A_4A_5\alpha_2^3 - 27A_2A_4A_5\alpha_2^3 + 16A_1A_3A_4A_5\alpha_2^3 \\
& + 16A_2A_3A_4A_5\alpha_2^3 - 27A_3A_4A_5\alpha_2^3 + 36A_4A_5\alpha_2^3 + 230A_5^4\alpha_2^2 - 1350A_5^3\alpha_2^2 + 19A_1A_2A_3A_4\alpha_2^2 \\
& + 19A_1A_2A_3A_5\alpha_2^2 + 19A_1A_2A_4A_5\alpha_2^2 + 19A_1A_3A_4A_5\alpha_2^2 - 20A_1A_2A_3A_4A_5\alpha_2^2 + 19A_2A_3A_4A_5\alpha_2^2 + 170A_5^4\alpha_2 \\
& - 20A_1A_2A_3A_4\alpha_2 - 20A_1A_2A_3A_5\alpha_2 - 20A_1A_2A_4A_5\alpha_2 - 20A_1A_3A_4A_5\alpha_2 - 11A_1A_2A_3A_4A_5\alpha_2 \\
& - 20A_2A_3A_4A_5\alpha_2 + 32A_1A_2A_3A_4A_5) = \frac{1}{27\alpha_2^5(1-\alpha_2)}\Delta_{19}.
\end{aligned}$$

Our domain is

$$A_1 \geq 5, A_2 \geq 4, A_3 \geq 3, A_4 \geq 2, \text{ and } A_5 > 1.$$

Note also that we have $\alpha_2 \in \left(\frac{1}{4}, \frac{2}{5}\right]$. We begin with

$$\frac{\partial^5 \Delta_{19}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5} = 184\alpha_2^6 - 124\alpha_2^5 + 20\alpha_2^4 - 20\alpha_2^2 - 11\alpha_2 + 32 > 0, \quad \alpha_2 \in \left(\frac{1}{4}, \frac{2}{5}\right].$$

Thus the partial derivative of Δ_{19} with respect to A_1, A_2, A_3, A_4 , and A_5 is positive and minimized at $A_5 = 1$.

$$\left. \frac{\partial^4 \Delta_{19}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4} \right|_{A_5=1} = -12\alpha_2^6 - 8\alpha_2^5 + 4\alpha_2^4 + 16\alpha_2^3 - \alpha_2^2 - 31\alpha_2 + 32 > 0.$$

We continue with,

$$\left. \frac{\partial^3 \Delta_{19}}{\partial A_1 \partial A_2 \partial A_3} \right|_{A_4=2, A_5=1} = -4\alpha_2^6 - 8\alpha_2^5 - 8\alpha_2^4 + 21\alpha_2^3 + 17\alpha_2^2 - 82\alpha_2 + 64 > 0.$$

This is positive over our domain, so we know that the partial with respect to A_1 , A_2 , and A_3 is positive. Furthermore,

$$\left. \frac{\partial^2 \Delta_{19}}{\partial A_1 \partial A_2} \right|_{A_3=3, A_4=2, A_5=1} = -8\alpha_2^6 - 8\alpha_2^5 - 29\alpha_2^4 + 50\alpha_2^3 + 89\alpha_2^2 - 286\alpha_2 + 192 > 0, \text{ and}$$

$$\left. \frac{\partial \Delta_{19}}{\partial A_1} \right|_{A_2=4, A_3=3, A_4=2, A_5=1} = -20\alpha_2^6 - 11\alpha_2^5 - 158\alpha_2^4 + 215\alpha_2^3 + 470\alpha_2^2 - 1264\alpha_2 + 768 > 0.$$

Finally, our minimum for Δ_{19} is

$$\begin{aligned} \Delta_{19}|_{A_1=5, A_2=4, A_3=3, A_4=2, A_5=1} &= 3(37159\alpha_2^6 - 37255\alpha_2^5 - \\ &769\alpha_2^4 + 1233\alpha_2^3 + 562\alpha_2^2 - 2210\alpha_2 + 1280), \end{aligned}$$

which is positive over our desired interval, completing this subcase.

2.4.4 Subcase IV (d)

In this subcase, we know that $P_5(x_6 = 4) > 0$, implying that $(1, 1, 1, 1, 1, 4)$ is a positive integral solution to the inequality in Theorem 2.4. Thus, if

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} \leq 1 - \frac{4}{a_6} := \alpha_3,$$

then $\alpha_3 \in (0, \frac{1}{5}]$ because $a_6 \in (4, 5]$. For simplicity, let $A_i = a_i \cdot \alpha_3$ for $i = 1, 2, 3, 4$, and 5.

This yields the new inequality

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} \leq 1.$$

Thus, by Theorem 1.7, we have

$$\begin{aligned} 6! P_5(x_6 = 4) &\leq 6 [(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1) \\ &\quad - (A_5 - 1)^5 + A_5(A_5 - 1)(A_5 - 2)(A_5 - 3)(A_5 - 4)], \end{aligned}$$

as well as

$$6! P_5(x_6 = 3) \leq 6 \left[\left(A_1 \cdot \frac{1+3\alpha_3}{4\alpha_3} - 1 \right) \left(A_2 \cdot \frac{1+3\alpha_3}{4\alpha_3} - 1 \right) \left(A_3 \cdot \frac{1+3\alpha_3}{4\alpha_3} - 1 \right) \left(A_4 \cdot \frac{1+3\alpha_3}{4\alpha_3} - 1 \right) \left(A_5 \cdot \frac{1+3\alpha_3}{4\alpha_3} - 1 \right) - \left(A_5 \cdot \frac{1+3\alpha_3}{4\alpha_3} - 1 \right)^5 + \left(A_5 \cdot \frac{1+3\alpha_3}{4\alpha_3} \right) \left(A_5 \cdot \frac{1+3\alpha_3}{4\alpha_3} - 1 \right) \left(A_5 \cdot \frac{1+3\alpha_3}{4\alpha_3} - 2 \right) \left(A_5 \cdot \frac{1+3\alpha_3}{4\alpha_3} - 3 \right) \left(A_5 \cdot \frac{1+3\alpha_3}{4\alpha_3} - 4 \right) \right],$$

and

$$6! P_5(x_6 = 2) \leq 6 \left[\left(A_1 \cdot \frac{2+2\alpha_3}{4\alpha_3} - 1 \right) \left(A_2 \cdot \frac{2+2\alpha_3}{4\alpha_3} - 1 \right) \left(A_3 \cdot \frac{2+2\alpha_3}{4\alpha_3} - 1 \right) \left(A_4 \cdot \frac{2+2\alpha_3}{4\alpha_3} - 1 \right) \left(A_5 \cdot \frac{2+2\alpha_3}{4\alpha_3} - 1 \right) - \left(A_5 \cdot \frac{2+2\alpha_3}{4\alpha_3} - 1 \right)^5 + \left(A_5 \cdot \frac{2+2\alpha_3}{4\alpha_3} \right) \left(A_5 \cdot \frac{2+2\alpha_3}{4\alpha_3} - 1 \right) \left(A_5 \cdot \frac{2+2\alpha_3}{4\alpha_3} - 2 \right) \left(A_5 \cdot \frac{2+2\alpha_3}{4\alpha_3} - 3 \right) \left(A_5 \cdot \frac{2+2\alpha_3}{4\alpha_3} - 4 \right) \right],$$

and

$$6! P_5(x_6 = 1) \leq 6 \left[\left(A_1 \cdot \frac{3+\alpha_3}{4\alpha_3} - 1 \right) \left(A_2 \cdot \frac{3+\alpha_3}{4\alpha_3} - 1 \right) \left(A_3 \cdot \frac{3+\alpha_3}{4\alpha_3} - 1 \right) \left(A_4 \cdot \frac{3+\alpha_3}{4\alpha_3} - 1 \right) \left(A_5 \cdot \frac{3+\alpha_3}{4\alpha_3} - 1 \right) - \left(A_5 \cdot \frac{3+\alpha_3}{4\alpha_3} - 1 \right)^5 + \left(A_5 \cdot \frac{3+\alpha_3}{4\alpha_3} \right) \left(A_5 \cdot \frac{3+\alpha_3}{4\alpha_3} - 1 \right) \left(A_5 \cdot \frac{3+\alpha_3}{4\alpha_3} - 2 \right) \left(A_5 \cdot \frac{3+\alpha_3}{4\alpha_3} - 3 \right) \left(A_5 \cdot \frac{3+\alpha_3}{4\alpha_3} - 4 \right) \right].$$

Because $6! P_6 = 6! (P_5(x_6 = 1) + P_5(x_6 = 2) + P_5(x_6 = 3) + P_5(x_6 = 4))$, if we let Δ_{20} be the difference obtained by subtracting the sum of the right hand sides of the above inequalities from the RHS of Theorem 2.4 and substituting in $a_i = \frac{A_i}{\alpha_3}$, then we merely have to apply the partial differentiation test for the expression

$$\begin{aligned} \Delta_{20} = & \frac{1}{128\alpha_3^5(1-\alpha_3)} (-5310A_5^4\alpha_3^6 + 30000A_5^3\alpha_3^6 - 57600A_5^2\alpha_3^6 + 1920A_1\alpha_3^6 - 1440A_1A_2\alpha_3^6 \\ & + 1920A_2\alpha_3^6 - 1440A_1A_3\alpha_3^6 + 1200A_1A_2A_3\alpha_3^6 - 1440A_2A_3\alpha_3^6 + 1920A_3\alpha_3^6 - 1440A_1A_4\alpha_3^6 + 1200A_1A_2A_4\alpha_3^6 \\ & - 1440A_2A_4\alpha_3^6 + 1200A_1A_3A_4\alpha_3^6 - 1062A_1A_2A_3A_4\alpha_3^6 + 1200A_2A_3A_4\alpha_3^6 - 1440A_3A_4\alpha_3^6 + 1920A_4\alpha_3^6 \\ & - 1440A_1A_5\alpha_3^6 + 1200A_1A_2A_5\alpha_3^6 - 1440A_2A_5\alpha_3^6 + 1200A_1A_3A_5\alpha_3^6 - 1062A_1A_2A_3A_5\alpha_3^6 + 1200A_2A_3A_5\alpha_3^6 \end{aligned}$$

$$\begin{aligned}
& -1440A_3A_5\alpha_3^6 + 1200A_1A_4A_5\alpha_3^6 - 1062A_1A_2A_4A_5\alpha_3^6 + 1200A_2A_4A_5\alpha_3^6 - 1062A_1A_3A_4A_5\alpha_3^6 \\
& + 975A_1A_2A_3A_4A_5\alpha_3^6 - 1062A_2A_3A_4A_5\alpha_3^6 + 1200A_3A_4A_5\alpha_3^6 - 1440A_4A_5\alpha_3^6 + 38400A_5\alpha_3^6 + 524160\alpha_3^6 \\
& + 2550A_5^4\alpha_3^5 - 12000A_3^3\alpha_3^5 + 19200A_5^2\alpha_3^5 - 640A_1\alpha_3^5 + 480A_1A_2\alpha_3^5 - 640A_2\alpha_3^5 + 480A_1A_3\alpha_3^5 - 480A_1A_2A_3\alpha_3^5 \\
& + 480A_2A_3\alpha_3^5 - 640A_3\alpha_3^5 + 480A_1A_4\alpha_3^5 - 480A_1A_2A_4\alpha_3^5 + 480A_2A_4\alpha_3^5 - 480A_1A_3A_4\alpha_3^5 + 510A_1A_2A_3A_4\alpha_3^5 \\
& - 480A_2A_3A_4\alpha_3^5 + 480A_3A_4\alpha_3^5 - 640A_4\alpha_3^5 + 480A_1A_5\alpha_3^5 - 480A_1A_2A_5\alpha_3^5 + 480A_2A_5\alpha_3^5 - 480A_1A_3A_5\alpha_3^5 \\
& + 510A_1A_2A_3A_5\alpha_3^5 - 480A_2A_3A_5\alpha_3^5 + 480A_3A_5\alpha_3^5 - 480A_1A_4A_5\alpha_3^5 + 510A_1A_2A_4A_5\alpha_3^5 - 480A_2A_4A_5\alpha_3^5 \\
& + 510A_1A_3A_4A_5\alpha_3^5 - 540A_1A_2A_3A_4A_5\alpha_3^5 + 510A_2A_3A_4A_5\alpha_3^5 - 480A_3A_4A_5\alpha_3^5 + 480A_4A_5\alpha_3^5 - 15232A_5\alpha_3^5 \\
& - 524672\alpha_3^5 - 300A_5^4\alpha_3^4 + 11520A_5^2\alpha_3^4 - 768A_1\alpha_3^4 + 160A_1A_2\alpha_3^4 - 768A_2\alpha_3^4 + 160A_1A_3\alpha_3^4 + 160A_2A_3\alpha_3^4 \\
& - 768A_3\alpha_3^4 + 160A_1A_4\alpha_3^4 + 160A_2A_4\alpha_3^4 - 60A_1A_2A_3A_4\alpha_3^4 + 160A_3A_4\alpha_3^4 - 768A_4\alpha_3^4 + 160A_1A_5\alpha_3^4 \\
& + 160A_2A_5\alpha_3^4 - 60A_1A_2A_3A_5\alpha_3^4 + 160A_3A_5\alpha_3^4 - 60A_1A_2A_4A_5\alpha_3^4 - 60A_1A_3A_4A_5\alpha_3^4 + 75A_1A_2A_3A_4A_5\alpha_3^4 \\
& - 60A_2A_3A_4A_5\alpha_3^4 + 160A_4A_5\alpha_3^4 - 22656A_5\alpha_3^4 + 300A_5^4\alpha_3^3 - 7200A_5^3\alpha_3^3 + 26880A_5^2\alpha_3^3 + 288A_1A_2\alpha_3^3 \\
& + 288A_1A_3\alpha_3^3 - 160A_1A_2A_3\alpha_3^3 + 288A_2A_3\alpha_3^3 + 288A_1A_4\alpha_3^3 - 160A_1A_2A_4\alpha_3^3 + 288A_2A_4\alpha_3^3 - 160A_1A_3A_4\alpha_3^3 \\
& + 60A_1A_2A_3A_4\alpha_3^3 - 160A_2A_3A_4\alpha_3^3 + 288A_3A_4\alpha_3^3 + 288A_1A_5\alpha_3^3 - 160A_1A_2A_5\alpha_3^3 + 288A_2A_5\alpha_3^3 \\
& - 160A_1A_3A_5\alpha_3^3 + 60A_1A_2A_3A_5\alpha_3^3 - 160A_2A_3A_5\alpha_3^3 + 288A_3A_5\alpha_3^3 - 160A_1A_4A_5\alpha_3^3 + 60A_1A_2A_4A_5\alpha_3^3 \\
& - 160A_2A_4A_5\alpha_3^3 + 60A_1A_3A_4A_5\alpha_3^3 + 60A_2A_3A_4A_5\alpha_3^3 - 160A_3A_4A_5\alpha_3^3 + 288A_4A_5\alpha_3^3 + 1290A_5^4\alpha_3^2 \\
& - 10800A_5^3\alpha_3^2 - 48A_1A_2A_3\alpha_3^2 - 48A_1A_2A_4\alpha_3^2 - 48A_1A_3A_4\alpha_3^2 + 130A_1A_2A_3A_4\alpha_3^2 - 48A_2A_3A_4\alpha_3^2 \\
& - 48A_1A_2A_5\alpha_3^2 - 48A_1A_3A_5\alpha_3^2 + 130A_1A_2A_3A_5\alpha_3^2 - 48A_2A_3A_5\alpha_3^2 - 48A_1A_4A_5\alpha_3^2 + 130A_1A_2A_4A_5\alpha_3^2 \\
& - 48A_2A_4A_5\alpha_3^2 + 130A_1A_3A_4A_5\alpha_3^2 - 75A_1A_2A_3A_4A_5\alpha_3^2 + 130A_2A_3A_4A_5\alpha_3^2 - 48A_3A_4A_5\alpha_3^2 + 1470A_5^4\alpha_3 \\
& - 90A_1A_2A_3A_4\alpha_3 - 90A_1A_2A_3A_5\alpha_3 - 90A_1A_2A_4A_5\alpha_3 - 90A_1A_3A_4A_5\alpha_3 - 100A_1A_2A_3A_4A_5\alpha_3 \\
& - 90A_2A_3A_4A_5\alpha_3 + 177A_1A_2A_3A_4A_5) = \frac{1}{128\alpha_3^5(1-\alpha_3)}\Delta_{21}.
\end{aligned}$$

Once again, our domain is

$$A_1 \geq 5, A_2 \geq 4, A_3 \geq 3, A_4 \geq 2, \text{ and } A_5 > 1.$$

Note also that we have $\alpha_3 \in \left(0, \frac{1}{5}\right]$. We begin with

$$\frac{\partial^5 \Delta_{21}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5} = 975\alpha_3^6 - 540\alpha_3^5 + 75\alpha_3^4 - 75\alpha_3^2 - 100\alpha_3 + 177 > 0, \quad \alpha_3 \in \left(0, \frac{1}{5}\right].$$

Thus the partial derivative of Δ_{21} with respect to A_1, A_2, A_3, A_4 , and A_5 is positive and minimized at $A_5 = 1$.

$$\left. \frac{\partial^4 \Delta_{21}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4} \right|_{A_5=1} = -87\alpha_3^6 - 30\alpha_3^5 + 15\alpha_3^4 + 60\alpha_3^3 + 55\alpha_3^2 - 190\alpha_3 + 177 > 0.$$

We continue with,

$$\left. \frac{\partial^3 \Delta_{21}}{\partial A_1 \partial A_2 \partial A_3} \right|_{A_4=2, A_5=1} = -36\alpha_3^6 - 30\alpha_3^5 - 30\alpha_3^4 + 20\alpha_3^3 + 192\alpha_3^2 - 470\alpha_3 + 354 > 0.$$

This is positive over our domain, so we know that the partial with respect to A_1, A_2 , and A_3 is positive. Furthermore,

$$\left. \frac{\partial^2 \Delta_{21}}{\partial A_1 \partial A_2} \right|_{A_3=3, A_4=2, A_5=1} = -2(36\alpha_3^6 + 15\alpha_3^5 + 25\alpha_3^4 + 6\alpha_3^3 - 346\alpha_3^2 + 795\alpha_3 - 531) > 0, \text{ and}$$

$$\left. \frac{\partial \Delta_{21}}{\partial A_1} \right|_{A_2=4, A_3=3, A_4=2, A_5=1} = -4(45\alpha_3^6 + 25\alpha_3^5 + 92\alpha_3^4 - 70\alpha_3^3 - 755\alpha_3^2 + 1725\alpha_3 - 1062) > 0.$$

Finally, our minimum for Δ_{21} is

$$\begin{aligned} \Delta_{21}|_{A_1=5, A_2=4, A_3=3, A_4=2, A_5=1} &= 530142\alpha_3^6 - 531374\alpha_3^5 \\ &\quad - 16028\alpha_3^4 + 24900\alpha_3^3 + 6310\alpha_3^2 - 35190\alpha_3 + 21240, \end{aligned}$$

which is positive over our desired interval, completing this subcase, and thus completing Case IV. \square

2.5 Case V

In this case, $[a_6] = 6$, so we can plug $v = 6$ into the statement of Theorem 1.10 to get the following theorem, which we prove in this case:

Theorem 2.5. *Let $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 > 1$ be real numbers and let P_6 be the number of positive integral solutions of $\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_5}{a_5} + \frac{x_6}{a_6} \leq 1$. If $P_6 > 0$ and*

$5 < a_6 \leq 6$, then

$$6! P_6 \leq (a_1 - 1)(a_2 - 1)(a_3 - 1)(a_4 - 1)(a_5 - 1)(a_6 - 1) - 14905.$$

Proof. In this case, $a_6 \in (5, 6]$, so we have to consider five levels — $x_6 = 1$, $x_6 = 2$, $x_6 = 3$, $x_6 = 4$, and $x_6 = 5$. Since $P_6 > 0$, there must be solutions at the $x_6 = 1$ level, so our five subcases are:

Subcase V (a) $P_5(x_6 = 5) = P_5(x_6 = 4) = P_5(x_6 = 3) = P_5(x_6 = 2) = 0.$

Subcase V (b) $P_5(x_6 = 5) = P_5(x_6 = 4) = P_5(x_6 = 3) = 0, P_5(x_6 = 2) > 0.$

Subcase V (c) $P_5(x_6 = 5) = P_5(x_6 = 4) = 0, P_5(x_6 = 3) > 0, P_5(x_6 = 2) > 0.$

Subcase V (d) $P_5(x_6 = 5) = 0, P_5(x_6 = 4) > 0, P_5(x_6 = 3) > 0, P_5(x_6 = 2) > 0.$

Subcase V (e) $P_5(x_6 = 5) > 0, P_5(x_6 = 4) > 0, P_5(x_6 = 3) > 0, P_5(x_6 = 2) > 0.$

2.5.1 Subcase V (a)

We are guaranteed that $(x_1, x_2, x_3, x_4, x_5, x_6) = (1, 1, 1, 1, 1, 1)$ is a solution to the inequality in Theorem 2.5. Thus, if

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} \leq 1 - \frac{1}{a_6} := \alpha,$$

then $\alpha \in (\frac{4}{5}, \frac{5}{6}]$ because $a_6 \in (5, 6]$. For simplicity, let $A_i = a_i \cdot \alpha$ for $i = 1, 2, 3, 4$, and 5.

This yields the new inequality

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} \leq 1.$$

Just like in Subcase IV (a), we will now use the Yau Geometric Conjecture for $n = 5$,

proven in [1]. This gives us the bound

$$6! P_5(x_6 = 1) \leq 6 [(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1) - p_5(\lceil A_5 \rceil)], \quad (5)$$

where p_5 is the function defined in (1). Note that since $p_5(v)$ is increasing for $v \geq 4$. Also, since $A_5 = a_5 \cdot \alpha > \frac{4}{5} \cdot 5 = 4$, we note that $\lceil A_5 \rceil \geq 5$. Thus, we maximize the RHS of (5) by substituting $p_5(5) = 904$ in for $p_5(\lceil A_5 \rceil)$. Hence, we have

$$6! P_5(x_6 = 1) \leq 6 [(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1) - 904]. \quad (6)$$

As before, we take the difference obtained by subtracting the RHS of (6) from the RHS of Theorem 2.4, substituting in $a_i = \frac{A_i}{\alpha}$ and $a_6 = \frac{1}{1 - \alpha}$, yielding

$$\begin{aligned} \Delta_{22} &= \frac{1}{2\alpha^4(1 - \alpha)} (18948\alpha^5 - 18950\alpha^4 + 12\alpha^5 A_1 - 10\alpha^4 A_1 - 12\alpha^5 A_1 A_2 + 12\alpha^4 A_1 A_2 - \\ &\quad 2\alpha^3 A_1 A_2 + 12\alpha^5 A_1 A_2 A_3 - 12\alpha^4 A_1 A_2 A_3 + 2\alpha^2 A_1 A_2 A_3 - 12\alpha^5 A_1 A_2 A_3 A_4 + \\ &\quad 12\alpha^4 A_1 A_2 A_3 A_4 - 2\alpha A_1 A_2 A_3 A_4 + 12\alpha^5 A_1 A_2 A_3 A_4 A_5 - 12\alpha^4 A_1 A_2 A_3 A_4 A_5 + \\ &\quad 2A_1 A_2 A_3 A_4 A_5 - 12\alpha^5 A_1 A_2 A_3 A_5 + 12\alpha^4 A_1 A_2 A_3 A_5 - 2\alpha A_1 A_2 A_3 A_5 + 12\alpha^5 A_1 A_2 A_4 - \\ &\quad 12\alpha^4 A_1 A_2 A_4 + 2\alpha^2 A_1 A_2 A_4 - 12\alpha^5 A_1 A_2 A_4 A_5 + 12\alpha^4 A_1 A_2 A_4 A_5 - 2\alpha A_1 A_2 A_4 A_5 + \\ &\quad 12\alpha^5 A_1 A_2 A_5 - 12\alpha^4 A_1 A_2 A_5 + 2\alpha^2 A_1 A_2 A_5 - 12\alpha^5 A_1 A_3 + 12\alpha^4 A_1 A_3 - 2\alpha^3 A_1 A_3 + \\ &\quad 12\alpha^5 A_1 A_3 A_4 - 12\alpha^4 A_1 A_3 A_4 + 2\alpha^2 A_1 A_3 A_4 - 12\alpha^5 A_1 A_3 A_4 A_5 + 12\alpha^4 A_1 A_3 A_4 A_5 - \\ &\quad 2\alpha A_1 A_3 A_4 A_5 + 12\alpha^5 A_1 A_3 A_5 - 12\alpha^4 A_1 A_3 A_5 + 2\alpha^2 A_1 A_3 A_5 - 12\alpha^5 A_1 A_4 + 12\alpha^4 A_1 A_4 - \\ &\quad 2\alpha^3 A_1 A_4 + 12\alpha^5 A_1 A_4 A_5 - 12\alpha^4 A_1 A_4 A_5 + 2\alpha^2 A_1 A_4 A_5 - 12\alpha^5 A_1 A_5 + 12\alpha^4 A_1 A_5 - \\ &\quad 2\alpha^3 A_1 A_5 + 12\alpha^5 A_2 - 10\alpha^4 A_2 - 12\alpha^5 A_2 A_3 + 12\alpha^4 A_2 A_3 - 2\alpha^3 A_2 A_3 + 12\alpha^5 A_2 A_3 A_4 - \\ &\quad 12\alpha^4 A_2 A_3 A_4 + 2\alpha^2 A_2 A_3 A_4 - 12\alpha^5 A_2 A_3 A_4 A_5 + 12\alpha^4 A_2 A_3 A_4 A_5 - 2\alpha A_2 A_3 A_4 A_5 + \\ &\quad 12\alpha^5 A_2 A_3 A_5 - 12\alpha^4 A_2 A_3 A_5 + 2\alpha^2 A_2 A_3 A_5 - 12\alpha^5 A_2 A_4 + 12\alpha^4 A_2 A_4 - 2\alpha^3 A_2 A_4 + \\ &\quad 12\alpha^5 A_2 A_4 A_5 - 12\alpha^4 A_2 A_4 A_5 + 2\alpha^2 A_2 A_4 A_5 - 12\alpha^5 A_2 A_5 + 12\alpha^4 A_2 A_5 - 2\alpha^3 A_2 A_5 + \\ &\quad 12\alpha^5 A_3 - 10\alpha^4 A_3 - 12\alpha^5 A_3 A_4 + 12\alpha^4 A_3 A_4 - 2\alpha^3 A_3 A_4 + 12\alpha^5 A_3 A_4 A_5 - 12\alpha^4 A_3 A_4 A_5 \\ &\quad + 2\alpha^2 A_3 A_4 A_5 - 12\alpha^5 A_3 A_5 + 12\alpha^4 A_3 A_5 - 2\alpha^3 A_3 A_5 + 12\alpha^5 A_4 - 10\alpha^4 A_4 - 12\alpha^5 A_4 A_5 \\ &\quad + 12\alpha^4 A_4 A_5 - 2\alpha^3 A_4 A_5 + 12\alpha^5 A_5 - 10\alpha^4 A_5) \\ &= \frac{1}{2\alpha^4(1 - \alpha)} \Delta_{23}. \end{aligned}$$

We now proceed with the partial derivative test on Δ_{23} with $\alpha \in (\frac{4}{5}, \frac{5}{6}]$ and

$$A_1 \geq 5, A_2 \geq 4, A_3 \geq 3, A_4 \geq 2, \text{ and } A_5 > \frac{\alpha}{1 - \alpha},$$

like in previous cases.

$$\frac{\partial^5 \Delta_{23}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5} = 12\alpha^5 - 12\alpha^4 + 2 > 0, \quad \alpha \in (\frac{4}{5}, \frac{5}{6}].$$

Thus the partial derivative of Δ_{23} with respect to A_1, A_2, A_3, A_4 , and A_5 is positive and minimized at $A_5 = \frac{\alpha}{1-\alpha}$.

$$\frac{\partial^4 \Delta_{23}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4} \Big|_{A_5 = \frac{\alpha}{1-\alpha}} = \frac{2(12\alpha^6 - 18\alpha^5 + 6\alpha^4 + \alpha^2)}{1-\alpha} > 0.$$

We continue with,

$$\frac{\partial^3 \Delta_{23}}{\partial A_1 \partial A_2 \partial A_3} \Big|_{A_4 = A_5 = \frac{\alpha}{1-\alpha}} = \frac{2\alpha^4(24\alpha^3 - 48\alpha^2 + 30\alpha - 5)}{(1-\alpha)^2} > 0.$$

This is positive over our domain, so we know that the partial with respect to A_1, A_2 , and A_3 is positive. Furthermore,

$$\frac{\partial^2 \Delta_{23}}{\partial A_1 \partial A_2} \Big|_{A_3 = A_4 = A_5 = \frac{\alpha}{1-\alpha}} = \frac{2\alpha^4(48\alpha^4 - 120\alpha^3 + 109\alpha^2 - 42\alpha + 6)}{(1-\alpha)^3} > 0, \text{ and}$$

$$\frac{\partial \Delta_{23}}{\partial A_1} \Big|_{A_2 = A_3 = A_4 = A_5 = \frac{\alpha}{1-\alpha}} = \frac{2\alpha^4(96\alpha^5 - 287\alpha^4 + 336\alpha^3 - 192\alpha^2 + 54\alpha - 6)}{(1-\alpha)^4} > 0.$$

We must now only test Δ_{23} at its minimum. We observe that if $A_5 \geq 5$, then we have

$$\Delta_{23} \Big|_{A_1=5, A_2=A_3=A_4=A_5=\frac{\alpha}{1-\alpha}} = 2(5-6\alpha)^2(434\alpha^3 + 290\alpha^2 + 175\alpha + 125),$$

which is positive over our desired interval. We must now only consider the minimum of Δ_{23} when we have $4 \leq \frac{\alpha}{1-\alpha} \leq A_5 < 5$. We observe that since the partial differentiation test told us that all the partial derivatives of Δ_{23} are positive, indicating that to minimize Δ_{23} , we set $A_2 = A_3 = A_4 = A_5 = x$ for some $x \in (4, 5]$. Thus, we have $\frac{1}{A_1} \leq 1 - \frac{4}{x}$, or that

$$A_1 \geq \frac{x}{4-x}.$$

Hence, we consider

$$\begin{aligned}\Delta_{24} &= \Delta_{23}|_{A_1=\frac{x}{4-x}, A_2=A_3=A_4=A_5=x} \\ &= \frac{1}{4-x} \left(-75792\alpha^5 + 75800\alpha^4 - 2\alpha x^5 + 2x^5 + 48\alpha^5 x^4 - 48\alpha^4 x^4 + 8\alpha^2 x^4 - 192\alpha^5 x^3 + \right. \\ &\quad \left. 192\alpha^4 x^3 - 12\alpha^3 x^3 - 20\alpha^2 x^3 + 288\alpha^5 x^2 - 280\alpha^4 x^2 + 40\alpha^3 x^2 + 18768\alpha^5 x - 18800\alpha^4 x \right).\end{aligned}$$

It can easily be numerically verified that $\Delta_{24} \geq 0$ over the interval $x \in (4, 5)$, $\alpha \in \left(\frac{4}{5}, \frac{5}{6}\right)$, completing this subcase.

2.5.2 Subcase V (b)

In this subcase, we know that $P_5(x_6 = 2) > 0$, implying that $(1, 1, 1, 1, 1, 2)$ is a positive integral solution to the inequality in Theorem 2.5. Thus, if

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} \leq 1 - \frac{2}{a_6} := \alpha_1,$$

then $\alpha_1 \in \left(\frac{3}{5}, \frac{2}{3}\right]$ because $a_6 \in (5, 6]$. For simplicity, let $A_i = a_i \cdot \alpha_1$ for $i = 1, 2, 3, 4$, and 5.

This yields the new inequality

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} \leq 1.$$

Thus, by Theorem 1.7, we have

$$\begin{aligned}6! P_5(x_6 = 2) &\leq 6 [(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1) \\ &\quad - (A_5 - 1)^5 + A_5(A_5 - 1)(A_5 - 2)(A_5 - 3)(A_5 - 4)],\end{aligned}$$

and,

$$\begin{aligned}6! P_5(x_6 = 1) &\leq 6 \left[\left(A_1 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right) \left(A_2 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right) \left(A_3 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right) \left(A_4 \cdot \frac{1 + \alpha_1}{2\alpha_1} - \right. \right. \\ &\quad \left. \left. 1 \right) \left(A_5 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right) - \left(A_5 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right)^5 + \left(A_5 \cdot \frac{1 + \alpha_1}{2\alpha_1} \right) \left(A_5 \right. \right. \\ &\quad \left. \left. \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right) \left(A_5 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 2 \right) \left(A_5 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 3 \right) \left(A_5 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 4 \right) \right].\end{aligned}$$

Because $6!P_6 = 6!(P_5(x_6 = 1) + P_5(x_6 = 2))$, as before, we take the difference obtained by subtracting the sums of the right hand sides of the above inequalities from the RHS of Theorem 2.5, substituting in $a_i = \frac{A_i}{\alpha_1}$, yielding

$$\begin{aligned} \Delta_{25} &= \frac{1}{16\alpha_1^5(1 - \alpha_1)} \left(-510A_5^4a_1^6 + 2700A_5^3a_1^6 - 4800A_5^2a_1^6 + 144A_1a_1^6 - 120A_1A_2a_1^6 + 144A_2a_1^6 \right. \\ &\quad - 120A_1A_3a_1^6 + 108A_1A_2A_3a_1^6 - 120A_2A_3a_1^6 + 144A_3a_1^6 - 120A_1A_4a_1^6 + 108A_1A_2A_4a_1^6 - \\ &\quad 120A_2A_4a_1^6 + 108A_1A_3A_4a_1^6 - 102A_1A_2A_3A_4a_1^6 + 108A_2A_3A_4a_1^6 - 120A_3A_4a_1^6 + 144A_4a_1^6 \\ &\quad - 120A_1A_5a_1^6 + 108A_1A_2A_5a_1^6 - 120A_2A_5a_1^6 + 108A_1A_3A_5a_1^6 - 102A_1A_2A_3A_5a_1^6 + \\ &\quad 108A_2A_3A_5a_1^6 - 120A_3A_5a_1^6 + 108A_1A_4A_5a_1^6 - 102A_1A_2A_4A_5a_1^6 + 108A_2A_4A_5a_1^6 - \\ &\quad 102A_1A_3A_4A_5a_1^6 + 99A_1A_2A_3A_4A_5a_1^6 - 102A_2A_3A_4A_5a_1^6 + 108A_3A_4A_5a_1^6 - 120A_4A_5a_1^6 + \\ &\quad 2880A_5a_1^6 + 238464a_1^6 + 390A_5^4a_1^5 - 1800A_5^3a_1^5 + 2880A_5^2a_1^5 - 80A_1a_1^5 + 72A_1A_2a_1^5 - 80A_2a_1^5 \\ &\quad + 72A_1A_3a_1^5 - 72A_1A_2A_3a_1^5 + 72A_2A_3a_1^5 - 80A_3a_1^5 + 72A_1A_4a_1^5 - 72A_1A_2A_4a_1^5 + 72A_2A_4a_1^5 \\ &\quad - 72A_1A_3A_4a_1^5 + 78A_1A_2A_3A_4a_1^5 - 72A_2A_3A_4a_1^5 + 72A_3A_4a_1^5 - 80A_4a_1^5 + 72A_1A_5a_1^5 - \\ &\quad 72A_1A_2A_5a_1^5 + 72A_2A_5a_1^5 - 72A_1A_3A_5a_1^5 + 78A_1A_2A_3A_5a_1^5 - 72A_2A_3A_5a_1^5 + 72A_3A_5a_1^5 - \\ &\quad 72A_1A_4A_5a_1^5 + 78A_1A_2A_4A_5a_1^5 - 72A_2A_4A_5a_1^5 + 78A_1A_3A_4A_5a_1^5 - 84A_1A_2A_3A_4A_5a_1^5 + \\ &\quad 78A_2A_3A_4A_5a_1^5 - 72A_3A_4A_5a_1^5 + 72A_4A_5a_1^5 - 1904A_5a_1^5 - 238496a_1^5 - 60A_5^4a_1^4 + 960A_5^3a_1^4 \\ &\quad - 32A_1a_1^4 + 8A_1A_2a_1^4 - 32A_2a_1^4 + 8A_1A_3a_1^4 + 8A_2A_3a_1^4 - 32A_3a_1^4 + 8A_1A_4a_1^4 + 8A_2A_4a_1^4 - \\ &\quad 12A_1A_2A_3A_4a_1^4 + 8A_3A_4a_1^4 - 32A_4a_1^4 + 8A_1A_5a_1^4 + 8A_2A_5a_1^4 - 12A_1A_2A_3A_5a_1^4 + 8A_3A_5a_1^4 \\ &\quad - 12A_1A_2A_4A_5a_1^4 - 12A_1A_3A_4A_5a_1^4 + 15A_1A_2A_3A_4A_5a_1^4 - 12A_2A_3A_4A_5a_1^4 + 8A_4A_5a_1^4 - \\ &\quad 944A_5a_1^4 + 60A_5^4a_1^3 - 600A_5^3a_1^3 + 960A_5^2a_1^3 + 8A_1A_2a_1^3 + 8A_1A_3a_1^3 - 8A_1A_2A_3a_1^3 + 8A_2A_3a_1^3 + \\ &\quad 8A_1A_4a_1^3 - 8A_1A_2A_4a_1^3 + 8A_2A_4a_1^3 - 8A_1A_3A_4a_1^3 + 12A_1A_2A_3A_4a_1^3 - 8A_2A_3A_4a_1^3 + 8A_3A_4a_1^3 \\ &\quad + 8A_1A_5a_1^3 - 8A_1A_2A_5a_1^3 + 8A_2A_5a_1^3 - 8A_1A_3A_5a_1^3 + 12A_1A_2A_3A_5a_1^3 - 8A_2A_3A_5a_1^3 + \\ &\quad 8A_3A_5a_1^3 - 8A_1A_4A_5a_1^3 + 12A_1A_2A_4A_5a_1^3 - 8A_2A_4A_5a_1^3 + 12A_1A_3A_4A_5a_1^3 + 12A_2A_3A_4A_5a_1^3 \\ &\quad - 8A_3A_4A_5a_1^3 + 8A_4A_5a_1^3 + 90A_5^4a_1^2 - 300A_5^3a_1^2 + 4A_1A_2A_3a_1^2 + 4A_1A_2A_4a_1^2 + 4A_1A_3A_4a_1^2 \\ &\quad + 2A_1A_2A_3A_4a_1^2 + 4A_2A_3A_4a_1^2 + 4A_1A_2A_5a_1^2 + 4A_1A_3A_5a_1^2 + 2A_1A_2A_3A_5a_1^2 + 4A_2A_3A_5a_1^2 \\ &\quad + 4A_1A_4A_5a_1^2 + 2A_1A_2A_4A_5a_1^2 + 4A_2A_4A_5a_1^2 + 2A_1A_3A_4A_5a_1^2 - 15A_1A_2A_3A_4A_5a_1^2 + \\ &\quad 2A_2A_3A_4A_5a_1^2 + 4A_3A_4A_5a_1^2 + 30A_5^4a_1 - 10A_1A_2A_3A_4a_1 - 10A_1A_2A_3A_5a_1 - \\ &\quad 10A_1A_2A_4A_5a_1 - 10A_1A_3A_4A_5a_1 + 4A_1A_2A_3A_4A_5a_1 - 10A_2A_3A_4A_5a_1 + 13A_1A_2A_3A_4A_5) \\ &= \frac{1}{16\alpha_1^5(1 - \alpha_1)} \Delta_{26}. \end{aligned}$$

We now proceed with the partial derivative test on Δ_{26} with $\alpha_1 \in \left(\frac{3}{5}, \frac{2}{3}\right]$ and

$$A_1 \geq 5, A_2 \geq 4, A_3 \geq 3, A_4 \geq 2, \text{ and } A_5 > \frac{2\alpha_1}{1 - \alpha_1},$$

like in previous cases.

$$\frac{\partial^5 \Delta_{26}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5} = 99\alpha_1^6 - 84\alpha_1^5 + 15\alpha_1^4 - 15\alpha_1^2 + 4\alpha_1 + 13 > 0, \quad \alpha_1 \in \left(\frac{3}{5}, \frac{2}{3}\right].$$

Thus the partial derivative of Δ_{26} with respect to A_1, A_2, A_3, A_4 , and A_5 is positive and minimized at $A_5 = \frac{2\alpha_1}{1 - \alpha_1}$.

$$\frac{\partial^4 \Delta_{26}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4} \Big|_{A_5 = \frac{2\alpha_1}{1 - \alpha_1}} = \frac{4\alpha_1 (75\alpha_1^6 - 87\alpha_1^5 + 30\alpha_1^4 - 6\alpha_1^3 - 5\alpha_1^2 + 5\alpha_1 + 4)}{1 - \alpha_1} > 0.$$

We continue with,

$$\frac{\partial^3 \Delta_{26}}{\partial A_1 \partial A_2 \partial A_3} \Big|_{A_4 = A_5 = \frac{2\alpha_1}{1 - \alpha_1}} = \frac{16\alpha_1^2 (57\alpha_1^6 - 84\alpha_1^5 + 42\alpha_1^4 - 11\alpha_1^3 + 3\alpha_1 + 1)}{(1 - \alpha_1)^2} > 0.$$

This is positive over our domain, so we know that the partial with respect to A_1, A_2 , and A_3 is positive. Furthermore,

$$\frac{\partial^2 \Delta_{26}}{\partial A_1 \partial A_2} \Big|_{A_3 = A_4 = A_5 = \frac{2\alpha_1}{1 - \alpha_1}} = \frac{16\alpha_1^3 (174\alpha_1^6 - 312\alpha_1^5 + 205\alpha_1^4 - 68\alpha_1^3 + 12\alpha_1^2 + 4\alpha_1 + 1)}{(1 - \alpha_1)^3} > 0, \text{ and}$$

$$\frac{\partial \Delta_{26}}{\partial A_1} \Big|_{A_2 = A_3 = A_4 = A_5 = \frac{2\alpha_1}{1 - \alpha_1}} = \frac{16\alpha_1^4 (534\alpha_1^6 - 1133\alpha_1^5 + 929\alpha_1^4 - 386\alpha_1^3 + 88\alpha_1^2 - \alpha_1 + 1)}{(1 - \alpha_1)^4} > 0.$$

We must now only test Δ_{26} at its minimum. We observe that if $A_5 \geq 3.9$, then we have

$$\begin{aligned} \Delta_{26}|_{A_1=5, A_2=4, A_3=A_4=A_5=3.9} &= 246714a_1^6 - 245222a_1^5 - 1692.76a_1^4 \\ &+ 5984.03a_1^3 - 9061.53a_1^2 - 2778.87a_1 + 15422.9, \end{aligned}$$

which is positive over our desired interval. We must now only consider the minimum of Δ_{26}

when we have $3 \leq \frac{2\alpha_1}{1 - \alpha_1} \leq A_5 < 3.9$.

If $A_5 < 3.9$, we have

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} \leq \frac{29}{39},$$

which must have positive integral solutions. Thus, $A_1 \geq 4 \cdot \frac{39}{29}$ and $A_2 \geq 3 \cdot \frac{39}{29}$, with $A_3 \geq A_4 \geq A_5 \geq \frac{2\alpha_1}{1-\alpha_1} \geq 3$ still true from before, and $A_5 < 3.9$ by assumption.

With these new bounds on A_1, A_2, A_3, A_4 , and A_5 , we can use the Yau Geometric Conjecture for $n = 5$ to bound $P_6 = P_5(x_6 = 1) + P_5(x_6 = 2)$, like in Subcase V (a). Thus, we get

$$6! P_5(x_6 = 2) \leq 6 [(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1) - p_5(\lceil A_5 \rceil)]$$

and,

$$6! P_5(x_6 = 1) \leq 6 \left[\left(A_1 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right) \left(A_2 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right) \left(A_3 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right) \right. \\ \left. \left(A_4 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right) \left(A_5 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right) - p_5 \left(\left\lceil A_5 \cdot \frac{1 + \alpha_1}{2\alpha_1} \right\rceil \right) \right].$$

Note that $\lceil A_5 \rceil = \left\lceil \frac{2\alpha_1}{1 - \alpha_1} \right\rceil = 4$, so $p_5(\lceil A_5 \rceil) = p_5(4) = 243$. Similarly, $\left\lceil A_5 \cdot \frac{1 + \alpha_1}{2\alpha_1} \right\rceil = \left\lceil \frac{2\alpha_1}{1 - \alpha_1} \cdot \frac{1 + \alpha_1}{2\alpha_1} \right\rceil = \left\lceil \frac{1 + \alpha_1}{1 - \alpha_1} \right\rceil = 5$, so $p_5 \left(\left\lceil A_5 \cdot \frac{1 + \alpha_1}{2\alpha_1} \right\rceil \right) = p_5(5) = 904$. Hence, our bounds are actually

$$6! P_5(x_6 = 2) \leq 6 [(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1) - 243],$$

and,

$$6! P_5(x_6 = 1) \leq 6 \left[\left(A_1 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right) \left(A_2 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right) \left(A_3 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right) \right. \\ \left. \left(A_4 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right) \left(A_5 \cdot \frac{1 + \alpha_1}{2\alpha_1} - 1 \right) - 904 \right].$$

Because $6! P_6 = 6! (P_5(x_6 = 1) + P_5(x_6 = 2))$, as before, we take the difference obtained by subtracting the sums of the right hand sides of the above inequalities from the RHS of Theorem 2.5, substituting in $a_i = \frac{A_i}{\alpha_1}$, yielding

$$\begin{aligned}
\Delta_{27} &= \frac{1}{16\alpha_1^5(1-\alpha_1)} \left(-510A_5^4\alpha_1^6 + 2700A_5^3\alpha_1^6 - 4800A_5^2\alpha_1^6 + 144A_1\alpha_1^6 - 120A_1A_2\alpha_1^6 + 144A_2\alpha_1^6 \right. \\
&\quad - 120A_1A_3\alpha_1^6 + 108A_1A_2A_3\alpha_1^6 - 120A_2A_3\alpha_1^6 + 144A_3\alpha_1^6 - 120A_1A_4\alpha_1^6 + 108A_1A_2A_4\alpha_1^6 \\
&\quad - 120A_2A_4\alpha_1^6 + 108A_1A_3A_4\alpha_1^6 - 102A_1A_2A_3A_4\alpha_1^6 + 108A_2A_3A_4\alpha_1^6 - 120A_3A_4\alpha_1^6 + \\
&\quad 144A_4\alpha_1^6 - 120A_1A_5\alpha_1^6 + 108A_1A_2A_5\alpha_1^6 - 120A_2A_5\alpha_1^6 + 108A_1A_3A_5\alpha_1^6 - 102A_1A_2A_3A_5\alpha_1^6 \\
&\quad + 108A_2A_3A_5\alpha_1^6 - 120A_3A_5\alpha_1^6 + 108A_1A_4A_5\alpha_1^6 - 102A_1A_2A_4A_5\alpha_1^6 + 108A_2A_4A_5\alpha_1^6 - \\
&\quad 102A_1A_3A_4A_5\alpha_1^6 + 99A_1A_2A_3A_4A_5\alpha_1^6 - 102A_2A_3A_4A_5\alpha_1^6 + 108A_3A_4A_5\alpha_1^6 - 120A_4A_5\alpha_1^6 \\
&\quad + 2880A_5\alpha_1^6 + 238464\alpha_1^6 + 390A_5^4\alpha_1^5 - 1800A_5^3\alpha_1^5 + 2880A_5^2\alpha_1^5 - 80A_1\alpha_1^5 + 72A_1A_2\alpha_1^5 - \\
&\quad 80A_2\alpha_1^5 + 72A_1A_3\alpha_1^5 - 72A_1A_2A_3\alpha_1^5 + 72A_2A_3\alpha_1^5 - 80A_3\alpha_1^5 + 72A_1A_4\alpha_1^5 - 72A_1A_2A_4\alpha_1^5 \\
&\quad + 72A_2A_4\alpha_1^5 - 72A_1A_3A_4\alpha_1^5 + 78A_1A_2A_3A_4\alpha_1^5 - 72A_2A_3A_4\alpha_1^5 + 72A_3A_4\alpha_1^5 - 80A_4\alpha_1^5 + \\
&\quad 72A_1A_5\alpha_1^5 - 72A_1A_2A_5\alpha_1^5 + 72A_2A_5\alpha_1^5 - 72A_1A_3A_5\alpha_1^5 + 78A_1A_2A_3A_5\alpha_1^5 - 72A_2A_3A_5\alpha_1^5 \\
&\quad + 72A_3A_5\alpha_1^5 - 72A_1A_4A_5\alpha_1^5 + 78A_1A_2A_4A_5\alpha_1^5 - 72A_2A_4A_5\alpha_1^5 + 78A_1A_3A_4A_5\alpha_1^5 - \\
&\quad 84A_1A_2A_3A_4A_5\alpha_1^5 + 78A_2A_3A_4A_5\alpha_1^5 - 72A_3A_4A_5\alpha_1^5 + 72A_4A_5\alpha_1^5 - 1904A_5\alpha_1^5 - 238496\alpha_1^5 \\
&\quad - 60A_5^4\alpha_1^4 + 960A_5^2\alpha_1^4 - 32A_1\alpha_1^4 + 8A_1A_2\alpha_1^4 - 32A_2\alpha_1^4 + 8A_1A_3\alpha_1^4 + 8A_2A_3\alpha_1^4 - 32A_3\alpha_1^4 \\
&\quad + 8A_1A_4\alpha_1^4 + 8A_2A_4\alpha_1^4 - 12A_1A_2A_3A_4\alpha_1^4 + 8A_3A_4\alpha_1^4 - 32A_4\alpha_1^4 + 8A_1A_5\alpha_1^4 + 8A_2A_5\alpha_1^4 \\
&\quad - 12A_1A_2A_3A_5\alpha_1^4 + 8A_3A_5\alpha_1^4 - 12A_1A_2A_4A_5\alpha_1^4 - 12A_1A_3A_4A_5\alpha_1^4 + 15A_1A_2A_3A_4A_5\alpha_1^4 \\
&\quad - 12A_2A_3A_4A_5\alpha_1^4 + 8A_4A_5\alpha_1^4 - 944A_5\alpha_1^4 + 60A_5^4\alpha_1^3 - 600A_5^3\alpha_1^3 + 960A_5^2\alpha_1^3 + 8A_1A_2\alpha_1^3 + \\
&\quad 8A_1A_3\alpha_1^3 - 8A_1A_2A_3\alpha_1^3 + 8A_2A_3\alpha_1^3 + 8A_1A_4\alpha_1^3 - 8A_1A_2A_4\alpha_1^3 + 8A_2A_4\alpha_1^3 - 8A_1A_3A_4\alpha_1^3 \\
&\quad + 12A_1A_2A_3A_4\alpha_1^3 - 8A_2A_3A_4\alpha_1^3 + 8A_3A_4\alpha_1^3 + 8A_1A_5\alpha_1^3 - 8A_1A_2A_5\alpha_1^3 + 8A_2A_5\alpha_1^3 - \\
&\quad 8A_1A_3A_5\alpha_1^3 + 12A_1A_2A_3A_5\alpha_1^3 - 8A_2A_3A_5\alpha_1^3 + 8A_3A_5\alpha_1^3 - 8A_1A_4A_5\alpha_1^3 + 12A_1A_2A_4A_5\alpha_1^3 \\
&\quad - 8A_2A_4A_5\alpha_1^3 + 12A_1A_3A_4A_5\alpha_1^3 + 12A_2A_3A_4A_5\alpha_1^3 - 8A_3A_4A_5\alpha_1^3 + 8A_4A_5\alpha_1^3 + 90A_5^4\alpha_1^2 \\
&\quad - 300A_5^3\alpha_1^2 + 4A_1A_2A_3\alpha_1^2 + 4A_1A_2A_4\alpha_1^2 + 4A_1A_3A_4\alpha_1^2 + 2A_1A_2A_3A_4\alpha_1^2 + 4A_2A_3A_4\alpha_1^2 \\
&\quad + 4A_1A_2A_5\alpha_1^2 + 4A_1A_3A_5\alpha_1^2 + 2A_1A_2A_3A_5\alpha_1^2 + 4A_2A_3A_5\alpha_1^2 + 4A_1A_4A_5\alpha_1^2 + \\
&\quad 2A_1A_2A_4A_5\alpha_1^2 + 4A_2A_4A_5\alpha_1^2 + 2A_1A_3A_4A_5\alpha_1^2 - 15A_1A_2A_3A_4A_5\alpha_1^2 + 2A_2A_3A_4A_5\alpha_1^2 + \\
&\quad 4A_3A_4A_5\alpha_1^2 + 30A_5^4\alpha_1 - 10A_1A_2A_3A_4\alpha_1 - 10A_1A_2A_3A_5\alpha_1 - 10A_1A_2A_4A_5\alpha_1 - \\
&\quad 10A_1A_3A_4A_5\alpha_1 + 4A_1A_2A_3A_4A_5\alpha_1 - 10A_2A_3A_4A_5\alpha_1 + 13A_1A_2A_3A_4A_5) \\
&= \frac{1}{16\alpha_1^5(1-\alpha_1)} \Delta_{28}.
\end{aligned}$$

We note that Δ_{28} and Δ_{26} are defined similarly, with their only difference being an α_1 polynomial that is subtracted. Since $\Delta_{28} - \Delta_{26}$ is a function of α_1 only, all the partial derivatives have already been shown positive. Hence, we only need to deal with showing that the minimum value of Δ_{28} is non-negative. We note that

$$\begin{aligned}
\Delta_{28}|_{A_1=4, A_2=3, A_3=A_4=\frac{7}{2}, A_5=\frac{2\alpha_1}{1-\alpha_1}} &= \frac{2\alpha_1}{841(\alpha_1-1)} (43835559\alpha_1^6 - 95014566\alpha_1^5 + \\
&48838016\alpha_1^4 - 130694\alpha_1^3 + 2463073\alpha_1^2 - 437892\alpha_1 - 1788696),
\end{aligned}$$

which is non-negative for $\alpha_1 \in \left(\frac{3}{5}, \frac{2}{3}\right]$. Thus, we need only consider when $A_4 < \frac{7}{2}$. This leaves us with,

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} \leq 1 - \frac{10}{39} - \frac{2}{7} = \frac{125}{273},$$

which must have positive integral solutions, implying that $A_1 \geq 3 \cdot \frac{273}{125}$ and $A_2 \geq 2 \cdot \frac{273}{125}$.

Thus, Δ_{28} is minimized at

$$\begin{aligned} \Delta_{28} \Big|_{A_1=3 \cdot \frac{273}{125}, A_2=2 \cdot \frac{273}{125}, A_3=A_4=A_5=\frac{2\alpha_1}{1-\alpha_1}} &= \frac{16\alpha_1^3}{15625(\alpha_1-1)^3} (75990474\alpha_1^6 - 411281087\alpha_1^5 + \\ &690459330\alpha_1^4 - 479368418\alpha_1^3 + 121590162\alpha_1^2 - 1618071\alpha_1 - 447174), \end{aligned}$$

which is zero over our desired interval, completing this subcase.

2.5.3 Subcase V (c)

In this subcase, we know that $P_5(x_6 = 3) > 0$, implying that $(1, 1, 1, 1, 1, 3)$ is a positive integral solution to the inequality in Theorem 2.5. Thus, if

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} \leq 1 - \frac{3}{a_6} := \alpha_2,$$

then $\alpha_2 \in \left(\frac{2}{5}, \frac{1}{2}\right]$ because $a_6 \in (5, 6]$. For simplicity, let $A_i = a_i \cdot \alpha_2$ for $i = 1, 2, 3, 4$, and 5.

This yields the new inequality

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} \leq 1.$$

Thus, by Theorem 1.7, we have

$$\begin{aligned} 6! P_5(x_6 = 3) &\leq 6 [(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1) \\ &\quad - (A_5 - 1)^5 + A_5(A_5 - 1)(A_5 - 2)(A_5 - 3)(A_5 - 4)], \end{aligned}$$

as well as

$$6! P_5(x_6 = 2) \leq 6 \left[\left(A_1 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1 \right) \left(A_2 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1 \right) \left(A_3 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1 \right) \left(A_4 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1 \right) \left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1 \right) - \left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1 \right)^5 + \left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2} \right) \left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 1 \right) \left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 2 \right) \left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 3 \right) \left(A_5 \cdot \frac{1+2\alpha_2}{3\alpha_2} - 4 \right) \right],$$

and

$$6! P_5(x_6 = 1) \leq 6 \left[\left(A_1 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1 \right) \left(A_2 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1 \right) \left(A_3 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1 \right) \left(A_4 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1 \right) \left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1 \right) - \left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1 \right)^5 + \left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2} \right) \left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2} - 1 \right) \left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2} - 2 \right) \left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2} - 3 \right) \left(A_5 \cdot \frac{2+\alpha_2}{3\alpha_2} - 4 \right) \right].$$

Because $6! P_6 = 6! (P_5(x_6 = 1) + P_5(x_6 = 2) + P_5(x_6 = 3))$, if we let Δ_{29} be the difference obtained by subtracting the sum of the right hand sides of the above inequalities from the RHS of Theorem 2.5 and substituting in $a_i = \frac{A_i}{\alpha_2}$, then we merely have to apply the partial differentiation test for the expression

$$\begin{aligned} \Delta_{29} = & \frac{1}{27\alpha_2^5(1-\alpha_2)} (-980A_5^4\alpha_2^6 + 5400A_5^3\alpha_2^6 - 10080A_5^2\alpha_2^6 + 324A_1\alpha_2^6 - 252A_1A_2\alpha_2^6 + 324A_2\alpha_2^6 \\ & - 252A_1A_3\alpha_2^6 + 216A_1A_2A_3\alpha_2^6 - 252A_2A_3\alpha_2^6 + 324A_3\alpha_2^6 - 252A_1A_4\alpha_2^6 + 216A_1A_2A_4\alpha_2^6 - 252A_2A_4\alpha_2^6 \\ & + 216A_1A_3A_4\alpha_2^6 - 196A_1A_2A_3A_4\alpha_2^6 + 216A_2A_3A_4\alpha_2^6 - 252A_3A_4\alpha_2^6 + 324A_4\alpha_2^6 - 252A_1A_5\alpha_2^6 \\ & + 216A_1A_2A_5\alpha_2^6 - 252A_2A_5\alpha_2^6 + 216A_1A_3A_5\alpha_2^6 - 196A_1A_2A_3A_5\alpha_2^6 + 216A_2A_3A_5\alpha_2^6 - 252A_3A_5\alpha_2^6 \\ & + 216A_1A_4A_5\alpha_2^6 - 196A_1A_2A_4A_5\alpha_2^6 + 216A_2A_4A_5\alpha_2^6 - 196A_1A_3A_4A_5\alpha_2^6 + 184A_1A_2A_3A_4A_5\alpha_2^6 \\ & - 196A_2A_3A_4A_5\alpha_2^6 + 216A_3A_4A_5\alpha_2^6 - 252A_4A_5\alpha_2^6 + 6480A_5\alpha_2^6 + 402408\alpha_2^6 + 580A_5^4\alpha_2^5 - 2700A_5^3\alpha_2^5 \\ & + 4320A_5^2\alpha_2^5 - 135A_1\alpha_2^5 + 108A_1A_2\alpha_2^5 - 135A_2\alpha_2^5 + 108A_1A_3\alpha_2^5 - 108A_1A_2A_3\alpha_2^5 + 108A_2A_3\alpha_2^5 - 135A_3\alpha_2^5 \\ & + 108A_1A_4\alpha_2^5 - 108A_1A_2A_4\alpha_2^5 + 108A_2A_4\alpha_2^5 - 108A_1A_3A_4\alpha_2^5 + 116A_1A_2A_3A_4\alpha_2^5 - 108A_2A_3A_4\alpha_2^5 \\ & + 108A_3A_4\alpha_2^5 - 135A_4\alpha_2^5 + 108A_1A_5\alpha_2^5 - 108A_1A_2A_5\alpha_2^5 + 108A_2A_5\alpha_2^5 - 108A_1A_3A_5\alpha_2^5 + 116A_1A_2A_3A_5\alpha_2^5 \\ & - 108A_2A_3A_5\alpha_2^5 + 108A_3A_5\alpha_2^5 - 108A_1A_4A_5\alpha_2^5 + 116A_1A_2A_4A_5\alpha_2^5 - 108A_2A_4A_5\alpha_2^5 + 116A_1A_3A_4A_5\alpha_2^5 \\ & - 124A_1A_2A_3A_4A_5\alpha_2^5 + 116A_2A_3A_4A_5\alpha_2^5 - 108A_3A_4A_5\alpha_2^5 + 108A_4A_5\alpha_2^5 - 3213A_5\alpha_2^5 - 402489\alpha_2^5 \\ & - 80A_5^4\alpha_2^4 + 2160A_5^3\alpha_2^4 - 108A_1\alpha_2^4 + 27A_1A_2\alpha_2^4 - 108A_2\alpha_2^4 + 27A_1A_3\alpha_2^4 + 27A_2A_3\alpha_2^4 - 108A_3\alpha_2^4 + 27A_1A_4\alpha_2^4 \end{aligned}$$

$$\begin{aligned}
& +27A_2A_4\alpha_2^4 - 16A_1A_2A_3A_4\alpha_2^4 + 27A_3A_4\alpha_2^4 - 108A_4\alpha_2^4 + 27A_1A_5\alpha_2^4 + 27A_2A_5\alpha_2^4 - 16A_1A_2A_3A_5\alpha_2^4 \\
& + 27A_3A_5\alpha_2^4 - 16A_1A_2A_4A_5\alpha_2^4 - 16A_1A_3A_4A_5\alpha_2^4 + 20A_1A_2A_3A_4A_5\alpha_2^4 - 16A_2A_3A_4A_5\alpha_2^4 + 27A_4A_5\alpha_2^4 \\
& - 3186A_5\alpha_2^4 + 80A_5^4\alpha_2^3 - 1350A_5^3\alpha_2^3 + 3600A_5^2\alpha_2^3 + 36A_1A_2\alpha_2^3 + 36A_1A_3\alpha_2^3 - 27A_1A_2A_3\alpha_2^3 + 36A_2A_3\alpha_2^3 \\
& + 36A_1A_4\alpha_2^3 - 27A_1A_2A_4\alpha_2^3 + 36A_2A_4\alpha_2^3 - 27A_1A_3A_4\alpha_2^3 + 16A_1A_2A_3A_4\alpha_2^3 - 27A_2A_3A_4\alpha_2^3 + 36A_3A_4\alpha_2^3 \\
& + 36A_1A_5\alpha_2^3 - 27A_1A_2A_5\alpha_2^3 + 36A_2A_5\alpha_2^3 - 27A_1A_3A_5\alpha_2^3 + 16A_1A_2A_3A_5\alpha_2^3 - 27A_2A_3A_5\alpha_2^3 + 36A_3A_5\alpha_2^3 \\
& - 27A_1A_4A_5\alpha_2^3 + 16A_1A_2A_4A_5\alpha_2^3 - 27A_2A_4A_5\alpha_2^3 + 16A_1A_3A_4A_5\alpha_2^3 + 16A_2A_3A_4A_5\alpha_2^3 - 27A_3A_4A_5\alpha_2^3 \\
& + 36A_4A_5\alpha_2^3 + 230A_5^4\alpha_2^2 - 1350A_5^3\alpha_2^2 + 19A_1A_2A_3A_4\alpha_2^2 + 19A_1A_2A_3A_5\alpha_2^2 + 19A_1A_2A_4A_5\alpha_2^2 \\
& + 19A_1A_3A_4A_5\alpha_2^2 - 20A_1A_2A_3A_4A_5\alpha_2^2 + 19A_2A_3A_4A_5\alpha_2^2 + 170A_5^4\alpha_2 - 20A_1A_2A_3A_4\alpha_2 - 20A_1A_2A_3A_5\alpha_2 \\
& - 20A_1A_2A_4A_5\alpha_2 - 20A_1A_3A_4A_5\alpha_2 - 11A_1A_2A_3A_4A_5\alpha_2 - 20A_2A_3A_4A_5\alpha_2 + 32A_1A_2A_3A_4A_5) \\
& = \frac{1}{27\alpha_2^5(1-\alpha_2)}\Delta_{30}.
\end{aligned}$$

Our domain is

$$A_1 \geq 5, A_2 \geq 4, A_3 \geq 3, A_4 \geq A_5 \geq \frac{3\alpha_2}{1-\alpha_2}.$$

Note also that we have $\alpha_2 \in \left(\frac{2}{5}, \frac{1}{2}\right]$. We begin with

$$\frac{\partial^5 \Delta_{30}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5} = 184\alpha_2^6 - 124\alpha_2^5 + 20\alpha_2^4 - 20\alpha_2^2 - 11\alpha_2 + 32 > 0, \quad \alpha_2 \in \left(\frac{2}{5}, \frac{1}{2}\right].$$

Thus the partial derivative of Δ_{30} with respect to A_1, A_2, A_3, A_4 , and A_5 is positive and minimized at $A_5 = 1$.

$$\left. \frac{\partial^4 \Delta_{30}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4} \right|_{A_5=1} = -12\alpha_2^6 - 8\alpha_2^5 + 4\alpha_2^4 + 16\alpha_2^3 - \alpha_2^2 - 31\alpha_2 + 32 > 0.$$

We continue with,

$$\left. \frac{\partial^3 \Delta_{30}}{\partial A_1 \partial A_2 \partial A_3} \right|_{A_4=2, A_5=1} = -4\alpha_2^6 - 8\alpha_2^5 - 8\alpha_2^4 + 21\alpha_2^3 + 17\alpha_2^2 - 82\alpha_2 + 64 > 0.$$

This is positive over our domain, so we know that the partial with respect to A_1 , A_2 , and A_3 is positive. Furthermore,

$$\frac{\partial^2 \Delta_{30}}{\partial A_1 \partial A_2} \Big|_{A_3=3, A_4=2, A_5=1} = -8\alpha_2^6 - 8\alpha_2^5 - 29\alpha_2^4 + 50\alpha_2^3 + 89\alpha_2^2 - 286\alpha_2 + 192 > 0, \text{ and}$$

$$\frac{\partial \Delta}{\partial A_1} \Big|_{A_2=4, A_3=3, A_4=2, A_5=1} = -20\alpha_2^6 - 11\alpha_2^5 - 158\alpha_2^4 + 215\alpha_2^3 + 470\alpha_2^2 - 1264\alpha_2 + 768 > 0.$$

Finally, our minimum for Δ_{30} is

$$\begin{aligned} \Delta_{30} \Big|_{A_1=5, A_2=4, A_3=3, A_4=A_5=\frac{3\alpha_2}{1-\alpha_2}} &= \frac{9\alpha_2^2}{(\alpha_2 - 1)^3} (15156\alpha_2^7 - 171949\alpha_2^6 + \\ &272493\alpha_2^5 - 182673\alpha_2^4 + 44099\alpha_2^3 + 972\alpha_2^2 + 1152\alpha_2 - 1120), \end{aligned}$$

which is positive over our desired interval, completing this subcase.

2.5.4 Subcase V (d)

In this subcase, we know that $P_5(x_6 = 4) > 0$, implying that $(1, 1, 1, 1, 1, 4)$ is a positive integral solution to the inequality in Theorem 2.5. Thus, if

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} \leq 1 - \frac{4}{a_6} := \alpha_3,$$

then $\alpha_3 \in (\frac{1}{5}, \frac{1}{3}]$ because $a_6 \in (5, 6]$. For simplicity, let $A_i = a_i \cdot \alpha_3$ for $i = 1, 2, 3, 4$, and 5.

This yields the new inequality

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} \leq 1.$$

Thus, by Theorem 1.7, we have

$$\begin{aligned} 6! P_5(x_6 = 4) &\leq 6 [(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1) \\ &\quad - (A_5 - 1)^5 + A_5(A_5 - 1)(A_5 - 2)(A_5 - 3)(A_5 - 4)], \end{aligned}$$

as well as

$$6! P_5(x_6 = 3) \leq 6 \left[\left(A_1 \cdot \frac{1+3\alpha_3}{4\alpha_3} - 1 \right) \left(A_2 \cdot \frac{1+3\alpha_3}{4\alpha_3} - 1 \right) \left(A_3 \cdot \frac{1+3\alpha_3}{4\alpha_3} - 1 \right) \left(A_4 \cdot \frac{1+3\alpha_3}{4\alpha_3} - 1 \right) \left(A_5 \cdot \frac{1+3\alpha_3}{4\alpha_3} - 1 \right) - \left(A_5 \cdot \frac{1+3\alpha_3}{4\alpha_3} - 1 \right)^5 + \left(A_5 \cdot \frac{1+3\alpha_3}{4\alpha_3} \right) \left(A_5 \cdot \frac{1+3\alpha_3}{4\alpha_3} - 1 \right) \left(A_5 \cdot \frac{1+3\alpha_3}{4\alpha_3} - 2 \right) \left(A_5 \cdot \frac{1+3\alpha_3}{4\alpha_3} - 3 \right) \left(A_5 \cdot \frac{1+3\alpha_3}{4\alpha_3} - 4 \right) \right],$$

and

$$6! P_5(x_6 = 2) \leq 6 \left[\left(A_1 \cdot \frac{2+2\alpha_3}{4\alpha_3} - 1 \right) \left(A_2 \cdot \frac{2+2\alpha_3}{4\alpha_3} - 1 \right) \left(A_3 \cdot \frac{2+2\alpha_3}{4\alpha_3} - 1 \right) \left(A_4 \cdot \frac{2+2\alpha_3}{4\alpha_3} - 1 \right) \left(A_5 \cdot \frac{2+2\alpha_3}{4\alpha_3} - 1 \right) - \left(A_5 \cdot \frac{2+2\alpha_3}{4\alpha_3} - 1 \right)^5 + \left(A_5 \cdot \frac{2+2\alpha_3}{4\alpha_3} \right) \left(A_5 \cdot \frac{2+2\alpha_3}{4\alpha_3} - 1 \right) \left(A_5 \cdot \frac{2+2\alpha_3}{4\alpha_3} - 2 \right) \left(A_5 \cdot \frac{2+2\alpha_3}{4\alpha_3} - 3 \right) \left(A_5 \cdot \frac{2+2\alpha_3}{4\alpha_3} - 4 \right) \right],$$

and

$$6! P_5(x_6 = 1) \leq 6 \left[\left(A_1 \cdot \frac{3+\alpha_3}{4\alpha_3} - 1 \right) \left(A_2 \cdot \frac{3+\alpha_3}{4\alpha_3} - 1 \right) \left(A_3 \cdot \frac{3+\alpha_3}{4\alpha_3} - 1 \right) \left(A_4 \cdot \frac{3+\alpha_3}{4\alpha_3} - 1 \right) \left(A_5 \cdot \frac{3+\alpha_3}{4\alpha_3} - 1 \right) - \left(A_5 \cdot \frac{3+\alpha_3}{4\alpha_3} - 1 \right)^5 + \left(A_5 \cdot \frac{3+\alpha_3}{4\alpha_3} \right) \left(A_5 \cdot \frac{3+\alpha_3}{4\alpha_3} - 1 \right) \left(A_5 \cdot \frac{3+\alpha_3}{4\alpha_3} - 2 \right) \left(A_5 \cdot \frac{3+\alpha_3}{4\alpha_3} - 3 \right) \left(A_5 \cdot \frac{3+\alpha_3}{4\alpha_3} - 4 \right) \right].$$

Because $6! P_6 = 6!(P_5(x_6 = 1) + P_5(x_6 = 2) + P_5(x_6 = 3) + P_5(x_6 = 4))$, if we let Δ_{20} be the difference obtained by subtracting the sum of the right hand sides of the above inequalities from the RHS of Theorem 2.4 and substituting in $a_i = \frac{A_i}{\alpha_3}$, then we merely have to apply the partial differentiation test for the expression

$$\begin{aligned}
\Delta_{31} &= \frac{1}{128\alpha_3^5(1-\alpha_3)} \left(-5310A_5^4\alpha_3^6 + 30000A_5^3\alpha_3^6 - 57600A_5^2\alpha_3^6 + 1920A_1\alpha_3^6 - 1440A_1A_2\alpha_3^6 + \right. \\
&1920A_2\alpha_3^6 - 1440A_1A_3\alpha_3^6 + 1200A_1A_2A_3\alpha_3^6 - 1440A_2A_3\alpha_3^6 + 1920A_3\alpha_3^6 - 1440A_1A_4\alpha_3^6 + \\
&1200A_1A_2A_4\alpha_3^6 - 1440A_2A_4\alpha_3^6 + 1200A_1A_3A_4\alpha_3^6 - 1062A_1A_2A_3A_4\alpha_3^6 + 1200A_2A_3A_4\alpha_3^6 \\
&- 1440A_3A_4\alpha_3^6 + 1920A_4\alpha_3^6 - 1440A_1A_5\alpha_3^6 + 1200A_1A_2A_5\alpha_3^6 - 1440A_2A_5\alpha_3^6 + \\
&1200A_1A_3A_5\alpha_3^6 - 1062A_1A_2A_3A_5\alpha_3^6 + 1200A_2A_3A_5\alpha_3^6 - 1440A_3A_5\alpha_3^6 + 1200A_1A_4A_5\alpha_3^6 \\
&- 1062A_1A_2A_4A_5\alpha_3^6 + 1200A_2A_4A_5\alpha_3^6 - 1062A_1A_3A_4A_5\alpha_3^6 + 975A_1A_2A_3A_4A_5\alpha_3^6 - \\
&1062A_2A_3A_4A_5\alpha_3^6 + 1200A_3A_4A_5\alpha_3^6 - 1440A_4A_5\alpha_3^6 + 38400A_5\alpha_3^6 + 1907712\alpha_3^6 + 2550A_5^4\alpha_3^5 \\
&- 12000A_5^3\alpha_3^5 + 19200A_5^2\alpha_3^5 - 640A_1\alpha_3^5 + 480A_1A_2\alpha_3^5 - 640A_2\alpha_3^5 + 480A_1A_3\alpha_3^5 - \\
&480A_1A_2A_3\alpha_3^5 + 480A_2A_3\alpha_3^5 - 640A_3\alpha_3^5 + 480A_1A_4\alpha_3^5 - 480A_1A_2A_4\alpha_3^5 + 480A_2A_4\alpha_3^5 - \\
&480A_1A_3A_4\alpha_3^5 + 510A_1A_2A_3A_4\alpha_3^5 - 480A_2A_3A_4\alpha_3^5 + 480A_3A_4\alpha_3^5 - 640A_4\alpha_3^5 + 480A_1A_5\alpha_3^5 \\
&- 480A_1A_2A_5\alpha_3^5 + 480A_2A_5\alpha_3^5 - 480A_1A_3A_5\alpha_3^5 + 510A_1A_2A_3A_5\alpha_3^5 - 480A_2A_3A_5\alpha_3^5 + \\
&480A_3A_5\alpha_3^5 - 480A_1A_4A_5\alpha_3^5 + 510A_1A_2A_4A_5\alpha_3^5 - 480A_2A_4A_5\alpha_3^5 + 510A_1A_3A_4A_5\alpha_3^5 - \\
&540A_1A_2A_3A_4A_5\alpha_3^5 + 510A_2A_3A_4A_5\alpha_3^5 - 480A_3A_4A_5\alpha_3^5 + 480A_4A_5\alpha_3^5 - 15232A_5\alpha_3^5 - \\
&1908224\alpha_3^5 - 300A_5^4\alpha_3^4 + 11520A_5^2\alpha_3^4 - 768A_1\alpha_3^4 + 160A_1A_2\alpha_3^4 - 768A_2\alpha_3^4 + 160A_1A_3\alpha_3^4 \\
&+ 160A_2A_3\alpha_3^4 - 768A_3\alpha_3^4 + 160A_1A_4\alpha_3^4 + 160A_2A_4\alpha_3^4 - 60A_1A_2A_3A_4\alpha_3^4 + 160A_3A_4\alpha_3^4 - \\
&768A_4\alpha_3^4 + 160A_1A_5\alpha_3^4 + 160A_2A_5\alpha_3^4 - 60A_1A_2A_3A_5\alpha_3^4 + 160A_3A_5\alpha_3^4 - 60A_1A_2A_4A_5\alpha_3^4 \\
&- 60A_1A_3A_4A_5\alpha_3^4 + 75A_1A_2A_3A_4A_5\alpha_3^4 - 60A_2A_3A_4A_5\alpha_3^4 + 160A_4A_5\alpha_3^4 - 22656A_5\alpha_3^4 + \\
&300A_5^4\alpha_3^3 - 7200A_5^3\alpha_3^3 + 26880A_5^2\alpha_3^3 + 288A_1A_2\alpha_3^3 + 288A_1A_3\alpha_3^3 - 160A_1A_2A_3\alpha_3^3 \\
&+ 288A_2A_3\alpha_3^3 + 288A_1A_4\alpha_3^3 - 160A_1A_2A_4\alpha_3^3 + 288A_2A_4\alpha_3^3 - 160A_1A_3A_4\alpha_3^3 + \\
&60A_1A_2A_3A_4\alpha_3^3 - 160A_2A_3A_4\alpha_3^3 + 288A_3A_4\alpha_3^3 + 288A_1A_5\alpha_3^3 - 160A_1A_2A_5\alpha_3^3 + \\
&288A_2A_5\alpha_3^3 - 160A_1A_3A_5\alpha_3^3 + 60A_1A_2A_3A_5\alpha_3^3 - 160A_2A_3A_5\alpha_3^3 + 288A_3A_5\alpha_3^3 - \\
&160A_1A_4A_5\alpha_3^3 + 60A_1A_2A_4A_5\alpha_3^3 - 160A_2A_4A_5\alpha_3^3 + 60A_1A_3A_4A_5\alpha_3^3 + 60A_2A_3A_4A_5\alpha_3^3 - \\
&160A_3A_4A_5\alpha_3^3 + 288A_4A_5\alpha_3^3 + 1290A_5^4\alpha_3^2 - 10800A_5^3\alpha_3^2 - 48A_1A_2A_3\alpha_3^2 - 48A_1A_2A_4\alpha_3^2 - \\
&48A_1A_3A_4\alpha_3^2 + 130A_1A_2A_3A_4\alpha_3^2 - 48A_2A_3A_4\alpha_3^2 - 48A_1A_2A_5\alpha_3^2 - 48A_1A_3A_5\alpha_3^2 + \\
&130A_1A_2A_3A_5\alpha_3^2 - 48A_2A_3A_5\alpha_3^2 - 48A_1A_4A_5\alpha_3^2 + 130A_1A_2A_4A_5\alpha_3^2 - 48A_2A_4A_5\alpha_3^2 + \\
&130A_1A_3A_4A_5\alpha_3^2 - 75A_1A_2A_3A_4A_5\alpha_3^2 + 130A_2A_3A_4A_5\alpha_3^2 - 48A_3A_4A_5\alpha_3^2 + 1470A_5^4\alpha_3 - \\
&90A_1A_2A_3A_4\alpha_3 - 90A_1A_2A_3A_5\alpha_3 - 90A_1A_2A_4A_5\alpha_3 \\
&- 90A_1A_3A_4A_5\alpha_3 - 100A_1A_2A_3A_4A_5\alpha_3 - 90A_2A_3A_4A_5\alpha_3 + 177A_1A_2A_3A_4A_5) \\
&= \frac{1}{128\alpha_3^5(1-\alpha_3)} \Delta_{32}.
\end{aligned}$$

Our domain is

$$A_1 \geq 5, A_2 \geq 4, A_3 \geq 3, A_4 \geq 2, \text{ and } A_5 > 1.$$

Note also that we have $\alpha_3 \in \left(\frac{1}{5}, \frac{1}{3}\right]$. We begin with

$$\frac{\partial^5 \Delta_{32}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5} = 975\alpha_3^6 - 540\alpha_3^5 + 75\alpha_3^4 - 75\alpha_3^2 - 100\alpha_3 + 177 > 0, \quad \alpha_3 \in \left(\frac{1}{5}, \frac{1}{3}\right].$$

Thus the partial derivative of Δ_{32} with respect to A_1, A_2, A_3, A_4 , and A_5 is positive and minimized at $A_5 = 1$.

$$\left. \frac{\partial^4 \Delta_{32}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4} \right|_{A_5=1} = -87\alpha_3^6 - 30\alpha_3^5 + 15\alpha_3^4 + 60\alpha_3^3 + 55\alpha_3^2 - 190\alpha_3 + 177 > 0.$$

We continue with,

$$\left. \frac{\partial^3 \Delta_{32}}{\partial A_1 \partial A_2 \partial A_3} \right|_{A_4=2, A_5=1} = -36\alpha_3^6 - 30\alpha_3^5 - 30\alpha_3^4 + 20\alpha_3^3 + 192\alpha_3^2 - 470\alpha_3 + 354 > 0.$$

This is positive over our domain, so we know that the partial with respect to A_1, A_2 , and A_3 is positive. Furthermore,

$$\left. \frac{\partial^2 \Delta_{32}}{\partial A_1 \partial A_2} \right|_{A_3=3, A_4=2, A_5=1} = -2(36\alpha_3^6 + 15\alpha_3^5 + 25\alpha_3^4 + 6\alpha_3^3 - 346\alpha_3^2 + 795\alpha_3 - 531) > 0, \text{ and}$$

$$\left. \frac{\partial \Delta_{32}}{\partial A_1} \right|_{A_2=4, A_3=3, A_4=2, A_5=1} = -4(45\alpha_3^6 + 25\alpha_3^5 + 92\alpha_3^4 - 70\alpha_3^3 - 755\alpha_3^2 + 1725\alpha_3 - 1062) > 0.$$

Finally, our minimum for Δ_{32} is

$$\Delta_{32}|_{A_1=5, A_2=4, A_3=3, A_4=2, A_5=1} = 2(956847\alpha_3^6 - 957463\alpha_3^5 - 8014\alpha_3^4 + 12450\alpha_3^3 + 3155\alpha_3^2 - 17595\alpha_3 + 10620),$$

which is positive over our desired interval, completing this subcase.

2.5.5 Subcase V (e)

In this subcase, we know that $P_5(x_6 = 5) > 0$, implying that $(1, 1, 1, 1, 1, 5)$ is a positive integral solution to the inequality in Theorem 2.5. Thus, if

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} \leq 1 - \frac{5}{a_6} := \alpha_4,$$

then $\alpha_4 \in (0, \frac{1}{6}]$ because $a_6 \in (5, 6]$. For simplicity, let $A_i = a_i \cdot \alpha_4$ for $i = 1, 2, 3, 4$, and 5.

This yields the new inequality

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} \leq 1.$$

Thus, by Theorem 1.7, we have

$$6! P_5(x_6 = 5) \leq 6 \left[(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1) - (A_5 - 1)^5 + A_5(A_5 - 1)(A_5 - 2)(A_5 - 3)(A_5 - 4) \right],$$

as well as

$$6! P_5(x_6 = 4) \leq 6 \left[\left(A_1 \cdot \frac{1+4\alpha_4}{5\alpha_4} - 1 \right) \left(A_2 \cdot \frac{1+4\alpha_4}{5\alpha_4} - 1 \right) \left(A_3 \cdot \frac{1+4\alpha_4}{5\alpha_4} - 1 \right) \left(A_4 \cdot \frac{1+4\alpha_4}{5\alpha_4} - 1 \right) \left(A_5 \cdot \frac{1+4\alpha_4}{5\alpha_4} - 1 \right) - \left(A_5 \cdot \frac{1+4\alpha_4}{5\alpha_4} - 1 \right)^5 + \left(A_5 \cdot \frac{1+4\alpha_4}{5\alpha_4} \right) \left(A_5 \cdot \frac{1+4\alpha_4}{5\alpha_4} - 1 \right) \left(A_5 \cdot \frac{1+4\alpha_4}{5\alpha_4} - 2 \right) \left(A_5 \cdot \frac{1+4\alpha_4}{5\alpha_4} - 3 \right) \left(A_5 \cdot \frac{1+4\alpha_4}{5\alpha_4} - 4 \right) \right],$$

and

$$6! P_5(x_6 = 3) \leq 6 \left[\left(A_1 \cdot \frac{2+3\alpha_4}{5\alpha_4} - 1 \right) \left(A_2 \cdot \frac{2+3\alpha_4}{5\alpha_4} - 1 \right) \left(A_3 \cdot \frac{2+3\alpha_4}{5\alpha_4} - 1 \right) \left(A_4 \cdot \frac{2+3\alpha_4}{5\alpha_4} - 1 \right) \left(A_5 \cdot \frac{2+3\alpha_4}{5\alpha_4} - 1 \right) - \left(A_5 \cdot \frac{2+3\alpha_4}{5\alpha_4} - 1 \right)^5 + \left(A_5 \cdot \frac{2+3\alpha_4}{5\alpha_4} \right) \left(A_5 \cdot \frac{2+3\alpha_4}{5\alpha_4} - 1 \right) \left(A_5 \cdot \frac{2+3\alpha_4}{5\alpha_4} - 2 \right) \left(A_5 \cdot \frac{2+3\alpha_4}{5\alpha_4} - 3 \right) \left(A_5 \cdot \frac{2+3\alpha_4}{5\alpha_4} - 4 \right) \right],$$

and

$$6! P_5(x_6 = 2) \leq 6 \left[\left(A_1 \cdot \frac{3+2\alpha_4}{5\alpha_4} - 1 \right) \left(A_2 \cdot \frac{3+2\alpha_4}{5\alpha_4} - 1 \right) \left(A_3 \cdot \frac{3+2\alpha_4}{5\alpha_4} - 1 \right) \left(A_4 \cdot \frac{3+2\alpha_4}{5\alpha_4} - 1 \right) \left(A_5 \cdot \frac{3+2\alpha_4}{5\alpha_4} - 1 \right) - \left(A_5 \cdot \frac{3+2\alpha_4}{5\alpha_4} - 1 \right)^5 + \left(A_5 \cdot \frac{3+2\alpha_4}{5\alpha_4} \right) \left(A_5 \cdot \frac{3+2\alpha_4}{5\alpha_4} - 1 \right) \left(A_5 \cdot \frac{3+2\alpha_4}{5\alpha_4} - 2 \right) \left(A_5 \cdot \frac{3+2\alpha_4}{5\alpha_4} - 3 \right) \left(A_5 \cdot \frac{3+2\alpha_4}{5\alpha_4} - 4 \right) \right],$$

and finally,

$$6! P_5(x_6 = 1) \leq 6 \left[\left(A_1 \cdot \frac{4+\alpha_4}{5\alpha_4} - 1 \right) \left(A_2 \cdot \frac{4+\alpha_4}{5\alpha_4} - 1 \right) \left(A_3 \cdot \frac{4+\alpha_4}{5\alpha_4} - 1 \right) \left(A_4 \cdot \frac{4+\alpha_4}{5\alpha_4} - 1 \right) \left(A_5 \cdot \frac{4+\alpha_4}{5\alpha_4} - 1 \right) - \left(A_5 \cdot \frac{4+\alpha_4}{5\alpha_4} - 1 \right)^5 + \left(A_5 \cdot \frac{4+\alpha_4}{5\alpha_4} \right) \left(A_5 \cdot \frac{4+\alpha_4}{5\alpha_4} - 1 \right) \left(A_5 \cdot \frac{4+\alpha_4}{5\alpha_4} - 2 \right) \left(A_5 \cdot \frac{4+\alpha_4}{5\alpha_4} - 3 \right) \left(A_5 \cdot \frac{4+\alpha_4}{5\alpha_4} - 4 \right) \right].$$

Because $6! P_6 = 6! (P_5(x_6 = 1) + P_5(x_6 = 2) + P_5(x_6 = 3) + P_5(x_6 = 4) + P_5(x_6 = 5))$,

if we let Δ_{33} be the difference obtained by subtracting the sum of the right hand sides of the above inequalities from the RHS of Theorem 2.5 and substituting in $a_i = \frac{A_i}{\alpha_4}$, then we merely have to apply the partial differentiation test for the expression

$$\begin{aligned} \Delta_{33} = & \frac{1}{3125\alpha_4^5(1-\alpha_4)} (-146850A_5^4\alpha_4^6 + 843750A_5^3\alpha_4^6 - 1650000A_5^2\alpha_4^6 + 56250A_1\alpha_4^6 - 41250A_1A_2\alpha_4^6 \\ & + 56250A_2\alpha_4^6 - 41250A_1A_3\alpha_4^6 + 33750A_1A_2A_3\alpha_4^6 - 41250A_2A_3\alpha_4^6 + 56250A_3\alpha_4^6 - 41250A_1A_4\alpha_4^6 \\ & + 33750A_1A_2A_4\alpha_4^6 - 41250A_2A_4\alpha_4^6 + 33750A_1A_3A_4\alpha_4^6 - 29370A_1A_2A_3A_4\alpha_4^6 + 33750A_2A_3A_4\alpha_4^6 \\ & - 41250A_3A_4\alpha_4^6 + 56250A_4\alpha_4^6 - 41250A_1A_5\alpha_4^6 + 33750A_1A_2A_5\alpha_4^6 - 41250A_2A_5\alpha_4^6 + 33750A_1A_3A_5\alpha_4^6 \\ & - 29370A_1A_2A_3A_5\alpha_4^6 + 33750A_2A_3A_5\alpha_4^6 - 41250A_3A_5\alpha_4^6 + 33750A_1A_4A_5\alpha_4^6 - 29370A_1A_2A_4A_5\alpha_4^6 \\ & + 33750A_2A_4A_5\alpha_4^6 - 29370A_1A_3A_4A_5\alpha_4^6 + 26550A_1A_2A_3A_4A_5\alpha_4^6 - 29370A_2A_3A_4A_5\alpha_4^6 + 33750A_3A_4A_5\alpha_4^6 \\ & - 41250A_4A_5\alpha_4^6 + 1125000A_5\alpha_4^6 + 46575000\alpha_4^6 + 59250A_5^4\alpha_4^5 - 281250A_5^3\alpha_4^5 + 450000A_5^2\alpha_4^5 - 15625A_1\alpha_4^5 \\ & + 11250A_1A_2\alpha_4^5 - 15625A_2\alpha_4^5 + 11250A_1A_3\alpha_4^5 - 11250A_1A_2A_3\alpha_4^5 + 11250A_2A_3\alpha_4^5 - 15625A_3\alpha_4^5 \\ & + 11250A_1A_4\alpha_4^5 - 11250A_1A_2A_4\alpha_4^5 + 11250A_2A_4\alpha_4^5 - 11250A_1A_3A_4\alpha_4^5 + 11850A_1A_2A_3A_4\alpha_4^5 \\ & - 11250A_2A_3A_4\alpha_4^5 + 11250A_3A_4\alpha_4^5 - 15625A_4\alpha_4^5 + 11250A_1A_5\alpha_4^5 - 11250A_1A_2A_5\alpha_4^5 + 11250A_2A_5\alpha_4^5 \\ & - 11250A_1A_3A_5\alpha_4^5 + 11850A_1A_2A_3A_5\alpha_4^5 - 11250A_2A_3A_5\alpha_4^5 + 11250A_3A_5\alpha_4^5 - 11250A_1A_4A_5\alpha_4^5 \\ & + 11850A_1A_2A_4A_5\alpha_4^5 - 11250A_2A_4A_5\alpha_4^5 + 11850A_1A_3A_4A_5\alpha_4^5 - 12450A_1A_2A_3A_4A_5\alpha_4^5 \end{aligned}$$

$$\begin{aligned}
& +11850A_2A_3A_4A_5\alpha_4^5 - 11250A_3A_4A_5\alpha_4^5 + 11250A_4A_5\alpha_4^5 - 371875A_5\alpha_4^5 - 46590625\alpha_4^5 - 6000A_5^4\alpha_4^4 \\
& + 300000A_5^2\alpha_4^4 - 25000A_1\alpha_4^4 + 4375A_1A_2\alpha_4^4 - 25000A_2\alpha_4^4 + 4375A_1A_3\alpha_4^4 + 4375A_2A_3\alpha_4^4 - 25000A_3\alpha_4^4 \\
& + 4375A_1A_4\alpha_4^4 + 4375A_2A_4\alpha_4^4 - 1200A_1A_2A_3A_4\alpha_4^4 + 4375A_3A_4\alpha_4^4 - 25000A_4\alpha_4^4 + 4375A_1A_5\alpha_4^4 \\
& + 4375A_2A_5\alpha_4^4 - 1200A_1A_2A_3A_5\alpha_4^4 + 4375A_3A_5\alpha_4^4 - 1200A_1A_2A_4A_5\alpha_4^4 - 1200A_1A_3A_4A_5\alpha_4^4 \\
& + 1500A_1A_2A_3A_4A_5\alpha_4^4 - 1200A_2A_3A_4A_5\alpha_4^4 + 4375A_4A_5\alpha_4^4 - 737500A_5\alpha_4^4 + 6000A_5^4\alpha_4^3 - 187500A_5^3\alpha_4^3 \\
& + 900000A_5^2\alpha_4^3 + 10000A_1A_2\alpha_4^3 + 10000A_1A_3\alpha_4^3 - 4375A_1A_2A_3\alpha_4^3 + 10000A_2A_3\alpha_4^3 + 10000A_1A_4\alpha_4^3 \\
& - 4375A_1A_2A_4\alpha_4^3 + 10000A_2A_4\alpha_4^3 - 4375A_1A_3A_4\alpha_4^3 + 1200A_1A_2A_3A_4\alpha_4^3 - 4375A_2A_3A_4\alpha_4^3 + 10000A_3A_4\alpha_4^3 \\
& + 10000A_1A_5\alpha_4^3 - 4375A_1A_2A_5\alpha_4^3 + 10000A_2A_5\alpha_4^3 - 4375A_1A_3A_5\alpha_4^3 + 1200A_1A_2A_3A_5\alpha_4^3 - 4375A_2A_3A_5\alpha_4^3 \\
& + 10000A_3A_5\alpha_4^3 - 4375A_1A_4A_5\alpha_4^3 + 1200A_1A_2A_4A_5\alpha_4^3 - 4375A_2A_4A_5\alpha_4^3 + 1200A_1A_3A_4A_5\alpha_4^3 \\
& + 1200A_2A_3A_4A_5\alpha_4^3 - 4375A_3A_4A_5\alpha_4^3 + 10000A_4A_5\alpha_4^3 + 34500A_5^4\alpha_4^2 - 375000A_5^3\alpha_4^2 - 2500A_1A_2A_3\alpha_4^2 \\
& - 2500A_1A_2A_4\alpha_4^2 - 2500A_1A_3A_4\alpha_4^2 + 3775A_1A_2A_3A_4\alpha_4^2 - 2500A_2A_3A_4\alpha_4^2 - 2500A_1A_2A_5\alpha_4^2 \\
& - 2500A_1A_3A_5\alpha_4^2 + 3775A_1A_2A_3A_5\alpha_4^2 - 2500A_2A_3A_5\alpha_4^2 - 2500A_1A_4A_5\alpha_4^2 + 3775A_1A_2A_4A_5\alpha_4^2 \\
& - 2500A_2A_4A_5\alpha_4^2 + 3775A_1A_3A_4A_5\alpha_4^2 - 1500A_1A_2A_3A_4A_5\alpha_4^2 + 3775A_2A_3A_4A_5\alpha_4^2 - 2500A_3A_4A_5\alpha_4^2 \\
& + 53100A_5^4\alpha_4 - 1880A_1A_2A_3A_4\alpha_4 - 1880A_1A_2A_3A_5\alpha_4 - 1880A_1A_2A_4A_5\alpha_4 - 1880A_1A_3A_4A_5\alpha_4 \\
& - 3175A_1A_2A_3A_4A_5\alpha_4 - 1880A_2A_3A_4A_5\alpha_4 + 4700A_1A_2A_3A_4A_5) = \frac{1}{3125\alpha_4^5(1-\alpha_4)}\Delta_{34}.
\end{aligned}$$

Our domain is

$$A_1 \geq 5, A_2 \geq 4, A_3 \geq 3, A_4 \geq 2, \text{ and } A_5 > 1.$$

Note also that we have $\alpha_4 \in \left(0, \frac{1}{6}\right]$. We begin with

$$\frac{\partial^5 \Delta_{34}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5} = 25(1062\alpha_4^6 - 498\alpha_4^5 + 60\alpha_4^4 - 60\alpha_4^2 - 127\alpha_4 + 188) > 0, \quad \alpha_4 \in \left(0, \frac{1}{6}\right].$$

Thus the partial derivative of Δ_{34} with respect to A_1, A_2, A_3, A_4 , and A_5 is positive and minimized at $A_5 = 1$.

$$\left. \frac{\partial^4 \Delta_{34}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4} \right|_{A_5=1} = -5(564\alpha_4^6 + 120\alpha_4^5 - 60\alpha_4^4 - 240\alpha_4^3 - 455\alpha_4^2 + 1011\alpha_4 - 940) > 0.$$

We continue with,

$$\left. \frac{\partial^3 \Delta_{34}}{\partial A_1 \partial A_2 \partial A_3} \right|_{A_4=2, A_5=1} = -5(252\alpha_4^6 + 120\alpha_4^5 + 120\alpha_4^4 + 155\alpha_4^3 - 1165\alpha_4^2 + 2398\alpha_4 - 1880) > 0.$$

This is positive over our domain, so we know that the partial with respect to A_1 , A_2 , and A_3 is positive. Furthermore,

$$\frac{\partial^2 \Delta_{34}}{\partial A_1 \partial A_2} \Big|_{A_3=3, A_4=2, A_5=1} = -5 (504\alpha_4^6 + 120\alpha_4^5 - 35\alpha_4^4 + 610\alpha_4^3 - 3505\alpha_4^2 + 7946\alpha_4 - 5640) > 0, \text{ and}$$

$$\frac{\partial \Delta_{34}}{\partial A_1} \Big|_{A_2=4, A_3=3, A_4=2, A_5=1} = -25 (252\alpha_4^6 + 127\alpha_4^5 + 210\alpha_4^4 - 275\alpha_4^3 - 2610\alpha_4^2 + 6808\alpha_4 - 4512) > 0.$$

Finally, our minimum for Δ_{34} is

$$\begin{aligned} \Delta_{34}|_{A_1=5, A_2=4, A_3=3, A_4=2, A_5=1} &= 5 (9352104\alpha_4^6 - 9355070\alpha_4^5 - \\ &114085\alpha_4^4 + 182585\alpha_4^3 - 9730\alpha_4^2 - 168604\alpha_4 + 112800), \end{aligned}$$

which is positive over our desired interval, completing this subcase, and concluding Case V. □

2.6 Case VI

In this case, we are trying to prove:

Theorem 2.6. *Let $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 > 6$ be real numbers and let P_6 be the number of positive integral solutions of $\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_5}{a_5} + \frac{x_6}{a_6} \leq 1$. Define $\mu = (a_1 - 1)(a_2 - 1)(a_3 - 1)(a_4 - 1)(a_5 - 1)(a_6 - 1)$. If $P_6 > 0$, then*

$$6! P_6 \leq \mu - (1 + 114v - 259v^2 + 205v^3 - 70v^4 + 9v^5)|_{v=a_6-\beta+1},$$

where $v = [a_6]$ and β is the fractional part of a_6 . Note that the fractional part β of a_6 is one of $\frac{a_6}{a_1}, \frac{a_6}{a_2}, \frac{a_6}{a_3}, \frac{a_6}{a_4}$, or $\frac{a_6}{a_5}$. Equality holds if and only if $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 \in \mathbb{Z}$.

Proof. Before we begin the proof, it will be beneficial to explicitly write out the sharp GLY Conjecture for $n = 6$, which was proven in [13].

Theorem 2.7 (Sharp GLY Conjecture for $n = 6$). *Let $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 \geq 5$ be real numbers. Then,*

$$\begin{aligned}
720P_6 \leq & a_1a_2a_3a_4a_5a_6 \\
& - \frac{5}{2}(a_1a_2a_3a_4a_5 + a_1a_2a_3a_4a_6 + a_1a_2a_3a_5a_6 + a_1a_2a_4a_5a_6 + a_1a_3a_4a_5a_6 + a_2a_3a_4a_5a_6) \\
& - 24(a_1 + a_2 + a_3 + a_4 + a_5) \\
& + \frac{137}{5}(a_1a_2 + a_1a_3 + a_1a_4 + a_1a_5 + a_2a_3 + a_2a_4 + a_2a_5 + a_3a_4 + a_3a_5 + a_4a_5) \\
& - \frac{45}{2}(a_1a_2a_3 + a_1a_2a_4 + a_1a_2a_5 + a_1a_3a_4 + a_1a_3a_5 + a_1a_4a_5 + a_2a_3a_4 + a_2a_3a_5 \\
& \quad + a_2a_4a_5 + a_3a_4a_5) + 17(a_1a_2a_3a_4 + a_1a_2a_3a_5 + a_1a_2a_4a_5 + a_1a_3a_4a_5 + a_2a_3a_4a_5).
\end{aligned}$$

Equality holds if and only if $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 \in \mathbb{Z}_+$.

To prove Theorem 2.6, we want to show that the RHS of its inequality is greater than the RHS of the inequality in Theorem 2.7. Taking their difference and substituting in $A_i = \frac{a_i}{a_6}$ for $i = 1, 2, 3, 4,$ and $5,$ we get

$$\begin{aligned}
\Delta_{35} = & \frac{1}{10} (15A_1A_2A_3A_4A_5a_6^5 + 15A_1A_2A_3A_4a_6^5 - 160A_1A_2A_3A_4a_6^4 + 15A_1A_2A_3A_5a_6^5 \\
& - 160A_1A_2A_3A_5a_6^4 + 10A_1A_2A_3a_6^4 + 215A_1A_2A_3a_6^3 + 15A_1A_2A_4A_5a_6^5 - 160A_1A_2A_4A_5a_6^4 \\
& + 10A_1A_2A_4a_6^4 + 215A_1A_2A_4a_6^3 + 10A_1A_2A_5a_6^4 + 215A_1A_2A_5a_6^3 - 10A_1A_2a_6^3 - 264A_1A_2a_6^2 \\
& + 15A_1A_3A_4A_5a_6^5 - 160A_1A_3A_4A_5a_6^4 + 10A_1A_3A_4a_6^4 + 215A_1A_3A_4a_6^3 + 10A_1A_3A_5a_6^4 + \\
& 215A_1A_3A_5a_6^3 - 10A_1A_3a_6^3 - 264A_1A_3a_6^2 + 10A_1A_4A_5a_6^4 + 215A_1A_4A_5a_6^3 - 10A_1A_4a_6^3 - \\
& 264A_1A_4a_6^2 - 10A_1A_5a_6^3 - 264A_1A_5a_6^2 + 10A_1a_6^2 + 230A_1a_6 + 15A_2A_3A_4A_5a_6^5 \\
& - 160A_2A_3A_4A_5a_6^4 + 10A_2A_3A_4a_6^4 + 215A_2A_3A_4a_6^3 + 10A_2A_3A_5a_6^4 + 215A_2A_3A_5a_6^3 \\
& - 10A_2A_3a_6^3 - 264A_2A_3a_6^2 + 10A_2A_4A_5a_6^4 + 215A_2A_4A_5a_6^3 - 10A_2A_4a_6^3 - 264A_2A_4a_6^2 - \\
& 10A_2A_5a_6^3 - 264A_2A_5a_6^2 + 10A_2a_6^2 + 230A_2a_6 + 10A_3A_4A_5a_6^4 + 215A_3A_4A_5a_6^3 - 10A_3A_4a_6^3 \\
& - 264A_3A_4a_6^2 - 10A_3A_5a_6^3 - 264A_3A_5a_6^2 + 10A_3a_6^2 + 230A_3a_6 - 10A_4A_5a_6^3 - 264A_4A_5a_6^2 \\
& + 10A_4a_6^2 + 230A_4a_6 + 10A_5a_6^2 + 230A_5a_6 - 90a_6^5 + 450a_6^4\beta + 250a_6^4 - 900a_6^3\beta^2 - 1000a_6^3\beta \\
& - 150a_6^3 + 900a_6^2\beta^3 + 1500a_6^2\beta^2 + 450a_6^2\beta - 260a_6^2 - 450a_6\beta^4 \\
& - 1000a_6\beta^3 - 450a_6\beta^2 + 520a_6\beta + 230a_6 + 90\beta^5 + 250\beta^4 + 150\beta^3 - 260\beta^2 - 240\beta + 10),
\end{aligned}$$

which is symmetric in $A_1, A_2, A_3, A_4,$ and $A_5.$ Thus, without loss of generality, let $\beta = \frac{a_6}{a_5}$ and substitute it into $\Delta_{35},$ yielding

$$\begin{aligned}
\Delta_{36} &= \frac{1}{10A_5^5} (15A_1A_2A_3a_6^5A_5^6 + 15A_1A_2A_4a_6^5A_5^6 + 15A_1A_3A_4a_6^5A_5^6 + 15A_1A_2A_3A_4a_6^5A_5^6 + \\
&15A_2A_3A_4a_6^5A_5^6 + 10A_1A_2a_6^4A_5^6 + 10A_1A_3a_6^4A_5^6 - 160A_1A_2A_3a_6^4A_5^6 + 10A_2A_3a_6^4A_5^6 + \\
&10A_1A_4a_6^4A_5^6 - 160A_1A_2A_4a_6^4A_5^6 + 10A_2A_4a_6^4A_5^6 - 160A_1A_3A_4a_6^4A_5^6 - 160A_2A_3A_4a_6^4A_5^6 + \\
&10A_3A_4a_6^4A_5^6 - 10A_1a_6^3A_5^6 + 215A_1A_2a_6^3A_5^6 - 10A_2a_6^3A_5^6 + 215A_1A_3a_6^3A_5^6 + 215A_2A_3a_6^3A_5^6 \\
&- 10A_3a_6^3A_5^6 + 215A_1A_4a_6^3A_5^6 + 215A_2A_4a_6^3A_5^6 + 215A_3A_4a_6^3A_5^6 - 10A_4a_6^3A_5^6 - 264A_1a_6^2A_5^6 \\
&- 264A_2a_6^2A_5^6 - 264A_3a_6^2A_5^6 - 264A_4a_6^2A_5^6 + 10a_6^2A_5^6 + 230a_6A_5^6 + 15A_1A_2A_3A_4a_6^5A_5^5 - \\
&90a_6^5A_5^5 + 10A_1A_2A_3a_6^4A_5^5 + 10A_1A_2A_4a_6^4A_5^5 + 10A_1A_3A_4a_6^4A_5^5 - 160A_1A_2A_3A_4a_6^4A_5^5 + \\
&10A_2A_3A_4a_6^4A_5^5 + 250a_6^4A_5^5 - 10A_1A_2a_6^3A_5^5 - 10A_1A_3a_6^3A_5^5 + 215A_1A_2A_3a_6^3A_5^5 - 10A_2A_3a_6^3A_5^5 \\
&- 10A_1A_4a_6^3A_5^5 + 215A_1A_2A_4a_6^3A_5^5 - 10A_2A_4a_6^3A_5^5 + 215A_1A_3A_4a_6^3A_5^5 + 215A_2A_3A_4a_6^3A_5^5 \\
&- 10A_3A_4a_6^3A_5^5 - 150a_6^3A_5^5 + 10A_1a_6^2A_5^5 - 264A_1A_2a_6^2A_5^5 + 10A_2a_6^2A_5^5 - 264A_1A_3a_6^2A_5^5 - \\
&264A_2A_3a_6^2A_5^5 + 10A_3a_6^2A_5^5 - 264A_1A_4a_6^2A_5^5 - 264A_2A_4a_6^2A_5^5 - 264A_3A_4a_6^2A_5^5 + 10A_4a_6^2A_5^5 \\
&- 260a_6^2A_5^5 + 230A_1a_6A_5^5 + 230A_2a_6A_5^5 + 230A_3a_6A_5^5 + 230A_4a_6A_5^5 + 230a_6A_5^5 + 10A_5^5 + \\
&450a_6^4A_5^4 - 1000a_6^3A_5^4 + 450a_6^2A_5^4 + 520a_6A_5^4 - 240A_5^4 - 900a_6^3A_5^3 + 1500a_6^2A_5^3 - \\
&450a_6A_5^3 - 260A_5^3 + 900a_6^2A_5^2 - 1000a_6A_5^2 + 150A_5^2 - 450a_6A_5 + 250A_5 + 90) \\
&= \frac{1}{10A_5^5} \Delta_{37}.
\end{aligned}$$

We apply the partial differentiation test to Δ_{37} over the interval $A_1 \geq A_2 \geq A_3 \geq A_4 \geq A_5 \geq 1$ and $a_6 > 6$. We begin with,

$$\frac{\partial^{10} \Delta_{37}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5^6} = 10800a_6^5 > 0.$$

We continue:

$$\left. \frac{\partial^9 \Delta_{37}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5^5} \right|_{A_5=1} = 600a_6^4(21a_6 - 32) > 0,$$

$$\left. \frac{\partial^8 \Delta_{37}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5^4} \right|_{A_5=1} = 2400a_6^4(3a_6 - 8) > 0,$$

$$\left. \frac{\partial^7 \Delta_{37}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5^3} \right|_{A_5=1} = 300a_6^4(9a_6 - 32) > 0,$$

$$\left. \frac{\partial^6 \Delta_{37}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5^2} \right|_{A_5=1} = 50a_6^4(15a_6 - 64) > 0,$$

$$\left. \frac{\partial^5 \Delta_{37}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5^1} \right|_{A_5=1} = 5a_6^4(33a_6 - 160) > 0,$$

$$\begin{aligned} \frac{\partial^4 \Delta_{37}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4} \Big|_{A_4=A_5=1} &= 10a_6^4(3a_6 - 16) > 0, \\ \frac{\partial^3 \Delta_{37}}{\partial A_1 \partial A_2 \partial A_3} \Big|_{A_3=A_4=A_5=1} &= 5a_6^3(9a_6^2 - 62a_6 + 43), \end{aligned}$$

which briefly dips under zero (for $a_6 < 6.15$), but this is a non-issue because:

$$\frac{\partial^2 \Delta_{37}}{\partial A_1 \partial A_2} \Big|_{A_3=A_4=A_5=1} = a_6^2(60a_6^3 - 450a_6^2 + 635a_6 - 264),$$

is always positive. We continue, noting that,

$$\frac{\partial \Delta_{37}}{\partial A_1} \Big|_{A_2=A_3=A_4=A_5=1} = a_6(75a_6^4 - 580a_6^3 + 1250a_6^2 - 1046a_6 + 230) > 0.$$

Thus, we must only test the minimum value of Δ_{37} , which is:

$$\Delta_{37}|_{A_1=A_2=A_3=A_4=A_5=1} = 0,$$

indicating that Δ_{37} is non-negative, and that this case is complete. \square

3 Conclusion

We have thus proven the six-dimensional case of the Yau Geometric Conjecture. The statement of the Conjecture contains a condition that $P_6 > 0$ for it to hold. However, Yau and Zuo [18] proved a similar statement in the case that $p_g = 0$ (the geometric genus of the singularity is zero). Thus, this paper completely solves the problem of finding a sharp upper bound in the six-dimensional case.

This research project raises a number of interesting questions regarding further study of P_n and Q_n . For instance, can the Yau Geometric Conjecture be proven in general using the methods outlined in this paper (splitting the simplex into smaller levels and summing upper

estimates)? Can the general case of the Yau Number-Theoretic Conjecture be approached similarly? Can these methods be extended to shed some insight into the validity of the Rough GLY Upper Estimate?

It is also important to consider lower bounds of P_n and Q_n . Currently, upper bounds have been extensively studied (as evidenced by the number of conjectures and theorems on the topic), but relatively little is known about lower bounds. A lower bound for P_n and Q_n is still interesting for the applications outlined in the Introduction of this paper.

The author of this project plans to continue pursuing research in the topics and areas mentioned above.

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