A study on the probability issue of "a frog jumping one meter a time"

Tianbo Qi, Yintai Zhang

The High School Attached to Tsinghua University

Supervisors: Wenbin Xu, Jingsong Li

Abstract: The problem we study in this paper originated from a simple high school math competition (AMC 12) question on a geometric probability model: "A frog makes 3 jumps, each exactly 1 meter long. The directions of the jumps are chosen independently at random. What is the probability that the frog's final position is no more than 1 meter from its starting position?"[1] From here, we changed the number of jumps the frog made, the length of each jump and the dimensions of the space. Using recursion, we obtained the recursive formula of the probability of a frog landing x meters within its original spot after jumping m 1-meter jumps in an N-dimension space by integrating the probability distribution of the former step. We also suggested a concept "the intensity of the probability field" to describe the relative probability of each spot and gave an expression of the intensity of each spot of a 2-dimensional space when the frog makes m jumps.

Keywords: probability, geometric probability model, function, N-dimensional space

1. Introduction

The problem we study in this paper originated from a simple high school math competition (AMC 12) question on a geometric probability model: "A frog makes 3 jumps, each exactly 1 meter long. The directions of the jumps are chosen independently at random. What is the probability that the frog's final position is no more than 1 meter from its starting position?" From here, we changed the number of jumps the frog made, the length of each jump and the dimensions it was in of the space. We would like to answer the question: what is the probability of a frog landing x meters within its original spot after jumping m 1-meter jumps in an N-dimension space?

2. Definitions

We denote the probability of a frog jumping m 1-meter jumps in an N-dimensional space and landing within x meters from where it started as $P_{NDm}(x)$. (Here, $x \in [0, m]$.) And we call it the probability function of m jumps in N-dimensional space.

Since the probability of a frog landing at any particular point is 0, it's rather difficult for us to compare whether the frog is more likely to land at this point or another. But regarding $P_{NDm}(x)$ as a function, we may calculate the probability of the frog landing on each circle, that is $dP_{NDm}(x) = P'_{NDm}(x) \cdot dx$. Thus, we can truthfully reflect the probability of the frog landing on each particular point using $P'_{NDm}(x)$. So we call it the intensity of the probability field of the frog jumping m 1-meter jumps in an n-dimensional space of the points precisely **x** meters from the starting point. And we write it as

$$Q_{NDm}(x) = \lim_{\Delta x \to 0} \frac{P_{NDm}(x + \Delta x) - P_{NDm}(x)}{\Delta x} = P_{NDm}'(x), \quad x \in [0, m].$$

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3. The mathematic calculation of two jumps in a 2-dimensional space

The first spot the frog landed must be on a circle with the starting spot as the center and a radius of 1 meter.
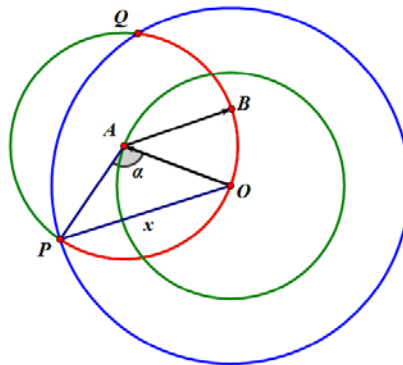

Figure 1

As shown in figure 1, we set the starting point as O and the first landing spot A. Making a circle with A as its center and 1 meter as its radius, set its intersecting points with the circle with O as its center and a radius of x (the target circle) as P and Q. The frog would be within the target circle if, and only if $\angle PAO = \alpha$ $\in \left[0, 2\arcsin\dfrac{x}{2} \right]$. So the probability of the frog landing within the target circle is $\dfrac{2\arcsin\dfrac{x}{2}}{\pi}$. Thus, $P_{2D2}(x) = \dfrac{2\arcsin\dfrac{x}{2}}{\pi}$ ($x \in [0,2]$). The plot of this function is as follows:
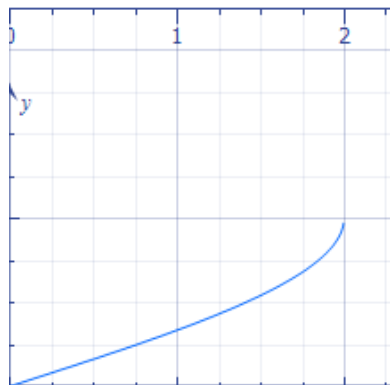

Figure 2

4. The computer simulation of two jumps in a 2-dimensional space

The greatest advantage of computer simulation is that we can make simulations of large number. However, the precision of the statistics are not exactly high, however, the general tendency can be shown.

Here, we used C++ and Excel to simulate. (The program is attached in the

appendix. Due to the capacity of the computer, this program is accurate to 0.001 and generates 10000 data each time). The plot is as follows:
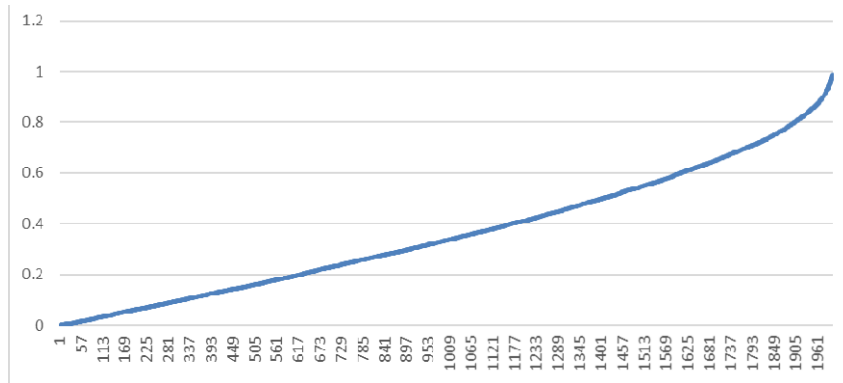

Figure 3

We may see that figure 3 fits well with figure 2.

5. The mathematic calculation of *m* jumps in a 2-dimensional space

We start the discussion from three jumps.

(i) When $0 \le r_3 < 1$, as in figure 4, if the frog started at O and landed at A of c_2 after the first two jumps, set $OA = r_2$, here we have $1 - r_3 \le r_2 \le r_3 + 1$.
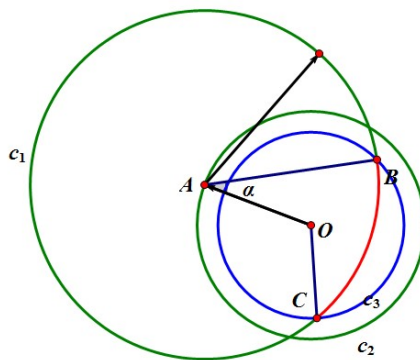
Figure 4

Here we see the probability of the frog landing at A is $Q_{2D2}(r_2) \cdot dr_2$. So using the cosine law, we have:

$$1 + r_2^2 - 2r_2 \cos \alpha = r_3^2 \Rightarrow \alpha = \arccos \frac{1 + r_2^2 - r_3^2}{2r_2}$$

Thus, we get the probability of landing with in c_2: $Q_{2D2}(r_2) \cdot dr_2 \cdot \dfrac{\arccos \dfrac{1 + r_2^2 - r_3^2}{2r_2}}{\pi}$,

and so we have:

$$P_{2D3}(r_3) = \int_{1-r_3}^{r_3+1} Q_{2D2}(r_2) \frac{\arccos \dfrac{1 + r_2^2 - r_3^2}{2r_2}}{\pi} dr_2, \ (0 \le r_3 < 1)$$

(ii) When $1 \le r_3 \le 3$,

(1) if $r_3 - 1 \le r_2 \le 2$, as shown in figure 5. Using similar methods, we may get

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the probability of the frog landing within c_3:

$$P_{2D2}(r_3) = \int_{r_3-1}^{2} Q_{2D2}(r_2)\frac{\arccos\dfrac{1+r_2^2-r_3^2}{2r_2}}{\pi}\,dr_2,\ (1 \le r_3 \le 3)$$

Figure 5

Using similar methods, we may get the probability of the frog landing within c_3:

$$P_{2D2}(r_3) = \int_{r_3-1}^{2} Q_{2D2}(r_2)\frac{\arccos\dfrac{1+r_2^2-r_3^2}{2r_2}}{\pi}\,dr_2,\ (1 \le r_3 \le 3)$$

(2) if $0 \le r_2 \le r_3 - 1$, as shown in figure 6, no matter with direction the third jump took, the frog must land within c_3:

Figure 6

Thus, the probability of the frog landing within c_3 equals the probability of the frog jumping to c_2:

$$P_{2D3}(r_3) = \int_{r_3-1}^{2} Q_{2D2}(r_2)\frac{\arccos\dfrac{1+r_2^2-r_3^2}{2r_2}}{\pi}\,dr_2 + P_{2D2}(r_3-1),(1 \le r_3 \le 3)$$
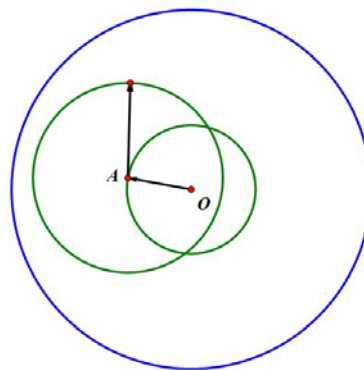
In summary, the probability of landing within r_3 from where the frog started after three jumps is:

$$P_{2D3}(r_3) = \begin{cases} \displaystyle\int_{1-r_3}^{r_3+1} Q_{2D2}(r_2)\dfrac{arccos\dfrac{1+r_2^2-r_3^2}{2r_2}}{\pi}\,dr_2, & (0 \le r_3 < 1) \\[4mm] \displaystyle\int_{r_3-1}^{2} Q_{2D2}(r_2)\dfrac{arccos\dfrac{1+r_2^2-r_3^2}{2r_2}}{\pi}\,dr_2 + P_{2D2}(r_3-1), & (1 \le r_3 \le 3) \end{cases}$$

In addition, the intensity of the probability field is:

$$Q_{2D3}(r_3) = \frac{d}{dr_3}P_{2D3}(r_3)$$

Similarly, we may get the probability of landing within r_4 from where the frog started after 4 jumps:

$$P_{2D4}(r_4) = \begin{cases} \displaystyle\int_{1-r_4}^{r_4+1} Q_{2D3}(r_3)\dfrac{arccos\dfrac{1+r_3^2-r_4^2}{2r_3}}{\pi}\,dr_3, & (0 \le r_4 < 1) \\[4mm] \displaystyle\int_{r_4-1}^{3} Q_{2D3}(r_3)\dfrac{arccos\dfrac{1+r_3^2-r_4^2}{2r_3}}{\pi}\,dr_3 + P_{2D3}(r_4-1), & (1 \le r_4 \le 4) \end{cases}$$

and the probability of landing within r_m from where the frog started after m jumps:

$$P_{2Dm}(r_m) = \begin{cases} \displaystyle\int_{1-r_m}^{r_m+1} Q_{2Dm-1}(r_{m-1})\dfrac{arccos\dfrac{1+r_{m-1}^2-r_m^2}{2r_{m-1}}}{\pi}\,dr_{m-1}, & (0 \le r_m < 1) \\[4mm] \displaystyle\int_{r_m-1}^{m-1} Q_{2Dm-1}(r_{m-1})\dfrac{arccos\dfrac{1+r_{m-1}^2-r_m^2}{2r_{m-1}}}{\pi}\,dr_{m-1} + P_{2Dm-1}(r_m-1), & (1 \le r_m \le m) \end{cases}$$

In addition, the intensity of probability field when making m jumps is:

$$Q_{2Dm}(r_m) = \frac{d}{dr_m}P_{2Dm}(r_m)$$

$$= \begin{cases} \dfrac{d}{dr_m}\displaystyle\int_{1-r_m}^{r_m+1} Q_{2Dm-1}(r_{m-1})\dfrac{arccos\dfrac{1+r_{m-1}^2-r_m^2}{2r_{m-1}}}{\pi}\,dr_{m-1}, & (0 \le r_m < 1) \\[4mm] \dfrac{d}{dr_m}\left[\displaystyle\int_{r_m-1}^{m-1} Q_{2Dm-1}(r_{m-1})\dfrac{arccos\dfrac{1+r_{m-1}^2-r_m^2}{2r_{m-1}}}{\pi}\,dr_{m-1} + P_{2Dm-1}(r_m-1)\right], & (1 \le r_m \le m) \end{cases}$$

6. Higher dimensions

We start the discussion here with 2 jumps in 3-dimensional space. To start with, it doesn't matter where the frog landed the first jump so we presume it landed at A_1, so the frog must have landed on the sphere A_1 as shown in figure 7:
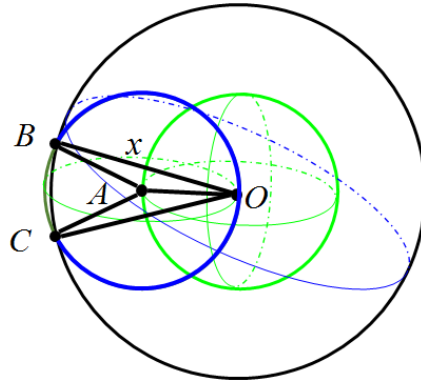
Figure 7

As where on the sphere the frog may land is random,

$$P_{3D2} = \frac{S_{\text{blue spherical cap}\,A-BOC}}{S_{\text{sphere}\,A}}.$$

If the frog landed x meters from O, we may have:

$$S_{\text{blue spherical cap}\,A-BOC} = 2\pi Rh = 2\pi\left(1+\cos\left(2\arccos\frac{x}{2}\right)\right).$$

Thus, the probability function here is:

$$P_{3D2}(x) = \frac{2\pi\left(1+\cos\left(2\arccos\frac{x}{2}\right)\right)}{4\pi} = \frac{1+\cos\left(2\arccos\frac{x}{2}\right)}{2}$$

$$= \frac{2\cos^2\left(\arccos\frac{x}{2}\right)}{2} = \frac{x^2}{4}, (0 \le x \le 2)$$

Next, we'll discuss the conditions when dimension is bigger than 3.

Lemma 1:

$$\int_0^\theta \sin^n x\, dx = \begin{cases} \dfrac{\int_0^\theta \sin^{n-2} x\, dx}{\int_0^{\frac{\pi}{2}} \sin^{n-2} x\, dx}\left(\pi \displaystyle\prod_{i=1}^{\frac{n}{2}} \frac{2i-1}{2i}\right) - \dfrac{\cos\theta \cdot \sin^{n-1}\theta}{n} & 2\mid n \\[4ex] \dfrac{\int_0^\theta \sin^{n-2} x\, dx}{\int_0^{\frac{\pi}{2}} \sin^{n-2} x\, dx}\left(2\displaystyle\prod_{i=1}^{\frac{n-1}{2}} \frac{2i}{2i+1}\right) - \dfrac{\cos\theta \cdot \sin^{n-1}\theta}{n} & 2\nmid n \end{cases}$$

Proof:

$$\int \sin^n x\, dx = -\int \sin^{n-1} x\, d\cos x = -\cos x \cdot \sin^{n-1} x + \int \cos x\, d\sin^{n-1} x$$

$$\int_0^\theta \sin^n x\, dx = -\int_0^\theta \sin^{n-1} x\, d\cos x = -\cos x \cdot \sin^{n-1} x \,\big|_0^\theta + \int_0^\theta \cos x\, d\sin^{n-1} x$$

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$$\int_0^\theta \sin^n x\, dx + \cos x \cdot \sin^{n-1} x \,\big|_0^\theta = \int_0^\theta \cos x \cdot d\sin^{n-1} x = \int_0^\theta (n-1)\cos^2 x \cdot \sin^{n-2} x \cdot dx$$

$$= \int_0^\theta (n-1)(1-\sin^2 x) \cdot \sin^{n-2} x \cdot dx$$

$$= (n-1)\int_0^\theta \sin^{n-2} x\, dx - (n-1)\int_0^\theta \sin^n x\, dx$$

Thus, $\dfrac{\displaystyle\int_0^\theta \sin^n x\, dx + \dfrac{\cos x \cdot \sin^{n-1} x\,\big|_0^\theta}{n}}{\displaystyle\int_0^\theta \sin^{n-2} x\, dx} = \dfrac{n-1}{n}$.

(i) When n is even,

$$\int_0^\theta \sin^n x\, dx + \frac{\cos x \cdot \sin^{n-1} x\,\big|_0^\theta}{n}$$

$$= \frac{\displaystyle\int_0^\theta \sin^n x\, dx + \frac{\cos x\, \sin^{n-1} x\,\big|_0^\theta}{n-1}}{\displaystyle\int_0^\theta \sin^{n-2} x\, dx} \cdot \frac{\displaystyle\int_0^\theta \sin^{n-2} x\, dx}{\displaystyle\int_0^\pi \sin^{n-2} x\, dx} \cdot \frac{\displaystyle\int_0^\pi \sin^{n-2} x\, dx}{\displaystyle\int_0^\pi \sin^{n-4} x\, dx} \cdots$$

$$\cdots \frac{\displaystyle\int_0^\pi \sin^4 x\, dx}{\displaystyle\int_0^\pi \sin^2 x\, dx} \cdot \frac{\displaystyle\int_0^\pi \sin^2 x\, dx}{\displaystyle\int_0^\pi \sin^0 x\, dx} \cdot \int_0^\pi \sin^0 x\, dx$$

$$= \frac{\displaystyle\int_0^\theta \sin^{n-2} x\, dx}{\displaystyle\int_0^\pi \sin^{n-2} x\, dx} \left(\frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \int_0^\pi \sin^0 x\, dx\right)$$

$$= \frac{\displaystyle\int_0^\theta \sin^{n-2} x\, dx}{\displaystyle\int_0^\pi \sin^{n-2} x\, dx} \left(\pi \prod_{i=1}^{\frac{n}{2}} \frac{2i-1}{2i}\right)$$

$$\int_0^\theta \sin^n x\, dx = \frac{\displaystyle\int_0^\theta \sin^{n-2} x\, dx}{\displaystyle\int_0^\pi \sin^{n-2} x\, dx}\left(\pi \prod_{i=1}^{\frac{n}{2}} \frac{2i-1}{2i}\right) - \frac{\cos x \cdot \sin^{n-1} x\,\big|_0^\theta}{n}$$

$$= \frac{\displaystyle\int_0^\theta \sin^{n-2} x\, dx}{\displaystyle\int_0^\pi \sin^{n-2} x\, dx}\left(\pi \prod_{i=1}^{\frac{n}{2}} \frac{2i-1}{2i}\right) - \frac{\cos x \cdot \sin^{n-1} x\,\big|_0^\theta}{n}$$

$$- \frac{\displaystyle\int_0^\theta \sin^{n-2} x\, dx}{\displaystyle\int_0^\pi \sin^{n-2} x\, dx}\left(\pi \prod_{i=1}^{\frac{n}{2}} \frac{2i-1}{2i}\right) - \frac{\cos\theta \cdot \sin^{n-1}\theta}{n}$$

(ii) When n is odd,

$$\int_0^\theta \sin^n x\,dx + \frac{\cos x \cdot \sin^{n-1} x\,\big|_0^\theta}{n}$$

$$= \frac{\int_0^\theta \sin^n x\,dx + \frac{\cos x \sin^{n-1} x\,\big|_0^\theta}{n-1}}{\int_0^\theta \sin^{n-2} x\,dx} \cdot \frac{\int_0^\theta \sin^{n-2} x\,dx}{\int_0^\pi \sin^{n-2} x\,dx} \cdot \frac{\int_0^\pi \sin^{n-2} x\,dx}{\int_0^\pi \sin^{n-4} x\,dx} \cdots$$

$$\cdot \frac{\int_0^\pi \sin^5 x\,dx}{\int_0^\pi \sin^5 x\,dx} \cdot \frac{\int_0^\pi \sin^5 x\,dx}{\int_0^\pi \sin^4 x\,dx} \cdot \int_0^\pi \sin^1 x\,dx$$

$$= \frac{\int_0^\theta \sin^{n-2} x\,dx}{\int_0^\pi \sin^{n-2} x\,dx}\left(\frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot \int_0^\pi \sin^1 x\,dx\right)$$

$$= \frac{\int_0^\theta \sin^{n-2} x\,dx}{\int_0^\pi \sin^{n-2} x\,dx}\left(2\prod_{i=1}^{\frac{n-1}{2}} \frac{2i}{2i+1}\right)$$

$$\int_0^\theta \sin^n x\,dx = \frac{\int_0^\theta \sin^{n-2} x\,dx}{\int_0^\pi \sin^{n-2} x\,dx}\left(2\prod_{i=1}^{\frac{n-1}{2}} \frac{2i}{2i+1}\right) - \frac{\cos x \cdot \sin^{n-1} x\,\big|_0^\theta}{n}$$

$$= \frac{\int_0^\theta \sin^{n-2} x\,dx}{\int_0^\pi \sin^{n-2} x\,dx}\left(2\prod_{i=1}^{\frac{n-1}{2}} \frac{2i}{2i+1}\right) - \frac{\cos x \cdot \sin^{n-1} x\,\big|_0^\theta}{n}$$

$$= \frac{\int_0^\theta \sin^{n-2} x\,dx}{\int_0^\pi \sin^{n-2} x\,dx}\left(2\prod_{i=1}^{\frac{n-1}{2}} \frac{2i}{2i+1}\right) - \frac{\cos\theta \cdot \sin^{n-1}\theta}{n}$$

Proof completed!

Lemma 2:

$$\int_0^\pi \sin^n x\,dx \cdot \int_0^\pi \sin^{n+1} y\,dy = \frac{2\pi}{n+1}, n \in N$$

Proof:

In lemma 1, put $\theta = \pi$, we have,

$$\int_0^\pi \sin^n x\,dx = \begin{cases} 2\prod\limits_{i=1}^{\frac{n-1}{2}} \dfrac{2i}{2i+1} & 2 \nmid n \\[4mm] \pi\prod\limits_{i=1}^{\frac{n}{2}} \dfrac{2i-1}{2i} & 2 \mid n \end{cases}$$

Thus, if n is even,

$$\int_0^\pi \sin^n x \cdot dx \cdot \int_0^\pi \sin^{n+1} y \cdot dy = \pi\prod_{i=1}^{\frac{n}{2}} \frac{2i-1}{2i} \cdot 2\prod_{i=1}^{\frac{n}{2}} \frac{2i}{2i+1}$$

$$= \frac{2\pi}{n+1} \prod_{i=1}^{\frac{n-1}{2}} \frac{2i-1}{2i} \cdot \prod_{i=1}^{\frac{n-1}{2}} \frac{2i}{2i-1} = \frac{2\pi}{n+1};$$

if *n* is odd,

$$\int_0^\pi sin^n x \cdot dx \cdot \int_0^\pi sin^{n+1} y \cdot dy = 2\prod_{i=1}^{\frac{n-1}{2}} \frac{2i}{2i+1} \cdot \pi \prod_{i=1}^{\frac{n+1}{2}} \frac{2i-1}{2i}$$

$$= \frac{2\pi}{n+1} \prod_{i=1}^{\frac{n-1}{2}} \frac{2i}{2i+1} \cdot \prod_{i=1}^{\frac{n-1}{2}} \frac{2i+1}{2i} = \frac{2\pi}{n+1}.$$

Proof completed!

Lemma 3:

The surface area of an *N*-dimensional sphere ($N \geq 2$) is

$$S_N(r) = \begin{cases} (2\pi r^2)^{\left(\frac{N-2}{2}\right)} \cdot \left(\prod_{i=1}^{\frac{N-2}{2}} \frac{1}{2i} \right) \cdot 2\pi r & 2 \mid n \\ \\ (2\pi r^2)^{\left(\frac{N-3}{2}\right)} \cdot \left(\prod_{i=1}^{\frac{N-3}{2}} \frac{1}{2i+1} \right) \cdot 4\pi r^2 & 2 \nmid n \end{cases}$$

Proof:

When *N*=2, $S_2(r) = 2\pi r$, lemma 3 holds true;

when *N*=3, $S_3(r) = 4\pi r^2$, lemma 3 holds true as well;

when *N*>3, as shown is figure 8:

Figure 8

Cutting the sphere with an (*N*-1)-dimensional space α, we may get an (*N*-1)-dimensional sphere (the part between *AB* in figure 8). And setting the

corresponding angle of that sphere as x, we may get the radius of that sphere: $r \cdot \sin x$.

And so the surface area of the little part (AC the figure 8) is $S_{N-1}(r \cdot \sin x) \cdot r dx$.

Thus, integrating these little parts, we get the surface area of a spherical cap:

$$Z_N(r,\theta) = \int_0^\theta S_{N-1}(r \cdot \sin x) \cdot r \cdot dx .$$

Putting $\theta = \pi$, we have the surface area of an N-dimensional sphere:

$$S_N(r) = \int_0^\pi S_{N-1}(r \cdot \sin x) \cdot r \cdot dx .$$

Presuming $S_{N-2}(r) = a_{N-2}\pi^{b_{N-2}}r^{N-2}$,

$$S_N(r) = \int_0^\pi S_{N-1}(r \cdot \sin x)r \cdot dx = \int_0^\pi \left(\int_0^\pi S_{N-2}(r \cdot \sin x \cdot \sin y)r \cdot \sin x \cdot dy \right)r \cdot dx$$

$$= r^2 \int_0^\pi \sin x \cdot \int_0^\pi S_{N-2}(r \cdot \sin y)dx \cdot dy$$

$$= r^2 \int_0^\pi \sin x \cdot \int_0^\pi a_{N-2}\pi^{b_{N-2}}(r \cdot \sin y)^{N-2}dx \cdot dy$$

$$= r^{N-1} \cdot a_{N-2}\pi^{b_{N-2}} \int_0^\pi \sin^{N-2} x \left(\int_0^\pi \sin^{N-2} y \cdot dy \right)dx$$

$$= r^{N-1} a_{N-2}\pi^{b_{N-2}} \left(\int_0^\pi \sin^{N-2} x \cdot dx \right) \cdot \left(\int_0^\pi \sin^{N-2} y \cdot dy \right)$$

Using lemma 2, we have,

$$\int_0^\pi \sin^{N-2} x \cdot dx \cdot \int_0^\pi \sin^{N-2} y \cdot dy = \frac{2\pi}{N-2}$$

$$\therefore S_N(r) = a_{N-2}\pi^{b_{N-2}}r^{N-2} \cdot \frac{2\pi}{N-2} = \frac{2\pi r^2}{N-2}S_{N-2}(r) .$$

And so when N is even:

$$S_N(r) = a_{N-2}\pi^{b_{N-2}}r^{N-2} \cdot \frac{2\pi}{N-2} = \frac{2\pi r^2}{N-2}S_{N-2}(r) = \frac{2\pi r^2}{N-2} \cdot \frac{2\pi r^2}{N-4} \cdot S_{N-4}(r) = \cdots \cdots$$

$$= (2\pi r^2)^{\left(\frac{N-2}{2}\right)} \cdot \left(\prod_{i=1}^{\frac{N-2}{2}} \frac{1}{2i} \right) \cdot S_2(r) = (2\pi r^2)^{\left(\frac{N-2}{2}\right)} \cdot \left(\prod_{i=1}^{\frac{N-2}{2}} \frac{1}{2i} \right) \cdot 2\pi r$$

;

when N is odd:

$$S_N(r) = a_{N-2}\pi^{b_{N-1}}r^{N-1} \cdot \frac{2\pi}{N-2} = \frac{2\pi r^2}{N-2}S_{N-2}(r) = \frac{2\pi r^2}{N-2} \cdot \frac{2\pi r^2}{N-4} \cdot S_{N-4}(r) = \cdots\cdots$$

$$= (2\pi r^2)^{\left(\frac{N-2}{2}\right)} \cdot \left(\prod_{i=1}^{\frac{N-2}{2}}\frac{1}{2i+1}\right) \cdot S_2(r) = (2\pi r^2)^{\left(\frac{N-2}{2}\right)} \cdot \left(\prod_{i=1}^{\frac{N-2}{2}}\frac{1}{2i+1}\right) \cdot 4\pi r^2$$

Proof completed!

Thus, if

$$S_N(r) = a_N \pi^{b_N}(r)^{N-1},$$

we have

$$Z_N(r,\vartheta) = \int_0^\vartheta a_{N-1}\pi^{b_{N-1}}(r \cdot \sin x)^{N-2}r \cdot dx = a_{N-1}\pi^{b_{N-1}}r^{n-1}\int_0^\vartheta \sin^{N-2}x\, dx,$$

$$S_N(r) = \int_0^\pi S_{N-1}(r \cdot \sin x)r \cdot dx = a_{N-1}\pi^{b_{N-1}}r^{n-1}\int_0^\pi \sin^{N-2}x\, dx.$$

And because the frog jumps one meter a time,

$$r = 1,$$

$$P_{N D 2}(x) = \frac{S_N(1) - Z_N(1,\vartheta)}{S_N(1)}$$

$$= \frac{a_{N-1}\pi^{b_{N-1}}r^{N-1}\int_0^\pi \sin^{N-2}x\, dx - a_{N-1}\pi^{b_{N-1}}r^{N-1}\int_0^\vartheta \sin^{N-2}x\, dx}{a_{N-1}\pi^{b_{N-1}}r^{N-1}\int_0^\pi \sin^{N-2}x\, dx}$$

$$= \frac{\int_0^\pi \sin^{N-2}x\, dx - \int_0^\vartheta \sin^{N-2}x\, dx}{\int_0^\pi \sin^{N-2}x\, dx} = 1 - \frac{\int_0^\vartheta \sin^{N-2}x\, dx}{\int_0^\pi \sin^{N-2}x\, dx}$$

Using lemmas 1 and 2,

(i) When N is even,

$$\int_0^\vartheta \sin^{N-2}x\, dx = \frac{\int_0^\vartheta \sin^{N-4}x\, dx}{\int_0^\pi \sin^{N-4}x\, dx}(\pi\prod_{i=1}^{\frac{N-2}{2}}\frac{2i-1}{2i}) - \frac{\cos\vartheta \cdot \sin^{N-2}\vartheta}{N-2}$$

$$\frac{\int_0^\vartheta \sin^{N-4}x\, dx}{\int_0^\pi \sin^{N-4}x\, dx} = P_{(N-2)D2}(x)$$

$$\int_0^\vartheta \sin^{N-2}x\, dx = P_{(N-2)D2}(x)(\pi\prod_{i=1}^{\frac{N-2}{2}}\frac{2i-1}{2i}) - \frac{\cos\vartheta \cdot \sin^{N-2}\vartheta}{N-2}$$

$$P_{ND2}(x) = 1 - \frac{P_{(N-2)D2}(x)\left(\pi\prod_{i=1}^{\frac{N-2}{2}}\frac{2i-1}{2i}\right) - \frac{\cos\theta\cdot\sin^{N-2}\theta}{N-2}}{\int_0^{\pi}\sin^{N-2}x\,dx}$$

$$= 1 - \frac{P_{(N-2)D2}(x)\left(\pi\prod_{i=1}^{\frac{N-2}{2}}\frac{2i-1}{2i}\right) - \frac{\cos\theta\cdot\sin^{N-2}\theta}{N-2}}{\pi\prod_{i=1}^{\frac{N}{2}}\frac{2i-1}{2i}}$$

(ii) When N is odd, similarly, we may have:

$$P_{ND2}(x) = 1 - \frac{P_{(N-2)D2}(x)\left(2\prod_{i=1}^{\frac{N-3}{2}}\frac{2i}{2i+1}\right) - \frac{\cos\theta\cdot\sin^{N-2}\theta}{N-2}}{2\prod_{i=1}^{\frac{N-1}{2}}\frac{2i}{2i+1}}$$

In summary, the probability function of 2 jumps in an N-dimensional space is

$$P_{ND2}(x) = \begin{cases} 1 - \dfrac{P_{(N-2)D2}(x)\left(\pi\prod_{i=1}^{\frac{N-2}{2}}\frac{2i-1}{2i}\right) - \frac{\cos\theta\sin^{N-2}\theta}{N-2}}{\pi\prod_{i=1}^{\frac{N}{2}}\frac{2i-1}{2i}}, N = 2k, (k\in N^+, k \geq 2) \\[4mm] 1 - \dfrac{P_{(N-2)D2}(x)\left(2\prod_{i=1}^{\frac{N-3}{2}}\frac{2i}{2i+1}\right) - \frac{\cos\theta\sin^{N-2}\theta}{N-2}}{2\prod_{i=1}^{\frac{N-1}{2}}\frac{2i}{2i+1}}, N = 2k+1 (k\in N^+, k \geq 2) \end{cases}$$

Here,

$$\theta = \arccos\frac{x}{2}, x\in[0,2].$$

Using similar methods as that in 2-dimensional space, we may deduce the probability function of m jumps in an N-dimensional space.

If the frog is y meters from where it started after m-1 jumps, the probability of that frog jumping precisely onto that sphere is:

$$dP_{NDm-1}(y) - P'_{NDm-1}(y)\cdot dx$$

Thus, the probability of it landing within x meters from its starting point is:

$$p(y) = \frac{\int_0^{\arccos\frac{1+y^2-x^2}{2y}} S_{N-1}(\sin z)\cdot dz}{\int_0^{\pi}S_{N-1}(\sin z)\cdot dz}$$

Knowing that $\max(x-1,0) \leq y \leq \min(x+1, m-1)$, we have the probability function of m jumps in an N-dimensional space:

$$P_{NDm}(x) - \begin{cases} \int_{1-x}^{x+1} dP_{NDm-1}(y)\cdot p(y), (0 \leq x \leq 1) \\[3mm] \int_{x-1}^{m-1} dP_{NDm-1}(y)\cdot p(y) + P_{NDm-1}(x-1), (1 < x < m) \end{cases}$$

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that is,

$$P_{NDm}(x) = \begin{cases} \int_{1-x}^{x+1} \dfrac{\int_0^{\arccos\frac{1+y^2-x^2}{2y}} S_{N-1}(\sin z)\cdot dz}{\int_0^\pi S_{N-1}(\sin z)\cdot dz} \cdot dP_{NDm-1}(y), & (0 \le x \le 1) \\[2em] \int_{x-1}^{m-1} \dfrac{\int_0^{\arccos\frac{1+y^2-x^2}{2y}} S_{N-1}(\sin z)\cdot dz}{\int_0^\pi S_{N-1}(\sin z)\cdot dz} \cdot dP_{NDm-1}(y) + P_{NDm-1}(x-1), & (1 < x \le m) \end{cases} .$$

References

[1] 2010 AMC 12B Problems 18, http://www.artofproblemsolving.com/Wiki /index.php/2010_AMC_12B_Problems/Problem_18.

Appendix

Program1: Generating 10000 data at random

```
#include <iostream>
#include <math.h>
#include <time.h>
#include <stdlib.h>
#include <stdio.h>
const long int TIME=10000;
const int STEP=2;
using namespace std;
double dis[TIME];
int count1=0,count2=0;
Int main()
{
    freopen("data.txt","w",stdout);
    double angle,x,y;
    long int i,j;
    srand(time(0));
     for (i=0;i<TIME;i++)
    {
        x=0;y=0;
        for (j=0;j<STEP;j++)
        {
            angle=(rand()%6283)/pow(10,3);
            x+=cos(angle);
            y+=sin(angle);
        }
```

```
            dis[i]=sqrt(x*x+y*y);
            printf("%.3lf", dis[i]);
            cout<<endl;
        }
        fclose(stdout);
        return 0;
}
```

Program 2: on the probability of 2 jumps in 2-D spaces

```
#include <iostream>
#include <stdio.h>
#include <map>
using namespace std;
const long int TIME=10000;
map<float,int> a;
int main()
{
        freopen("data.txt","r",stdin);
        freopen("result.txt","w",stdout);
        int i,j;
        float temp;
        a.clear();
        for (i=0;i<TIME;i++)
        {
            cin>>temp;
            for (j=0;j<=int(temp*1000);j++)
                a[j]--;
        }
        for (i=0;i<2000;i++)
        {
            cout<<float(TIME+a[i])/float(TIME)<<endl;
        }
        fclose(stdin);
        fclose(stdout);
        return 0;
}
```

Program 3: on the probability of 2 jumps in 3-D spaces

```
#include <iostream>
```

```cpp
#include <math.h>
#include <time.h>
#include <stdlib.h>
#include <stdio.h>
const long int TIME=10000;
const int STEP=2;
using namespace std;
double dis[TIME];
int count1=0,count2=0;
int main()
{
    freopen("data.txt","w",stdout);
    double angle,x,y,z;
    long int i,j;
    srand(time(0));
    for (i=0;i<TIME;i++)
    {
        x=0;y=0;z=0;
        for (j=0;j<STEP;j++)
        {
            angle=(rand()%6283)/pow(10,3);
            //cout<<angle<<' ';
            double a;
            a=sin(angle);
            x+=a;
            angle=(rand()%6283)/pow(10,3);
            //cout<<angle<<' ';
            z+=sin(angle)*sqrt(1-a*a);
            y+=cos(angle)*sqrt(1-a*a);
        }
        dis[i]=sqrt(x*x+y*y+z*z);
        printf("%.3lf", dis[i]);
        cout<<endl;
    }
    fclose(stdout);
    return 0;
}
```