The geometrical analysis of the path of free kick

Runda Ji, Hongyuan Fu, Zuoru Yang
Beijing Zhong Guancun Senior High School
Beijing, China
maxfu4869@gmail.com

Supervisor: Min Wei
Zhong Guancun Senior High School
Beijing, China

***Abstract*—Free kicks play an important role in a modern soccer match. A skillful soccer player usually uses the opportunity of a free kick to score a goal. There are mainly two ways to give a free kick, falling ball and banana ball. The last thesis analyzes the path of falling soccer in two-dimensional plane. This thesis mainly focuses on using geometrical ways to generate a graph in three-dimensional coordinate system to analyze the path of banana ball and find out the most suitable distance to give a free kick.**

***Index Terms*—soccer, free kick, path, geometry, calculus**

I. Nomenclature

Parameters

T	Temperature during the match
V_{wind}	Velocity of wind during the match
v_∞	Velocity of the soccer (ground as reference frame)
ω	Rotation speed of the soccer
P_{up}	Pressure on the upper surface of the soccer
P_{down}	Pressure on the bottom surface of the soccer
t	Time of the soccer contact with foot
θ_1	Vertical angle of emergence
θ_2	Horizontal angle of emergence
F	Force on the soccer
c	The circumference of the soccer
H	Displacement of the soccer

All of these parameters are in international unit

II. Introduction

A. Research Background

Free kicks play an important role in a modern soccer matches. A skillful soccer player usually uses the opportunity of a free kick to score a goal. There are mainly two ways to give a free kick, falling soccer and banana soccer. Banana soccer is a kind of free kick whose path is a curve. In a soccer match, player usually applies banana soccer to bypass the wall of players to score a goal.

B. Basic Assumptions

The soccer is a perfect sphere, which the pressure inside is 60.8—11.5kp (as shown in Fig.1). Ignore both the loss and the addition of the weight of the soccer during the match.

Fig.1. 2012 UEFA European Championship Official Soccer.

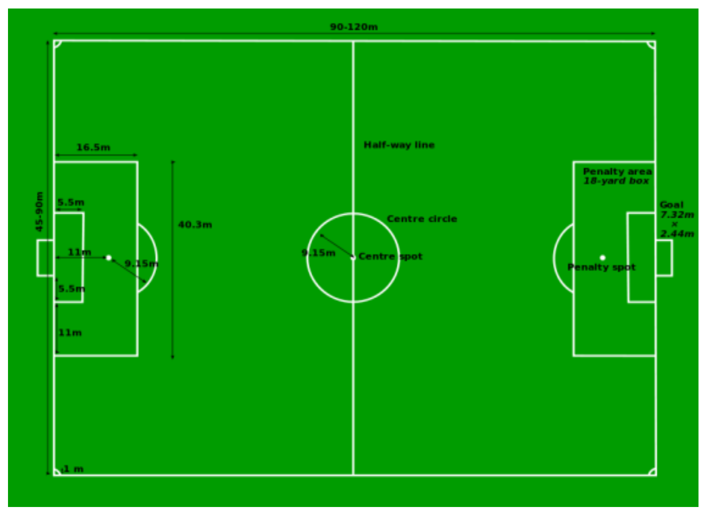

Fig.2. Soccer Pitch.

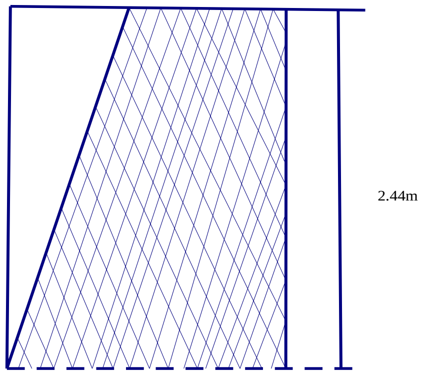

Fig.3. Goal.

Because of the influence of the weather, we make these assumptions:

1. T=20℃
2. Velocity of the wind is constantly 0 m/s
3. Moisture of the air does not influence the weight of the soccer
4. Characteristics of the soccer are constant
5. The soccer does not touch the ground before entering the goal
6. Shape of the soccer does not change
7. Diameter of the soccer is d=0.30m
8. Weight of the soccer is M=0.45kg

III. Model and Analysis

A. Basic Calculation

Assume the player kick the soccer with a force F=450N; analyze this force in a three-dimensional coordinate system as shown in Fig.4.

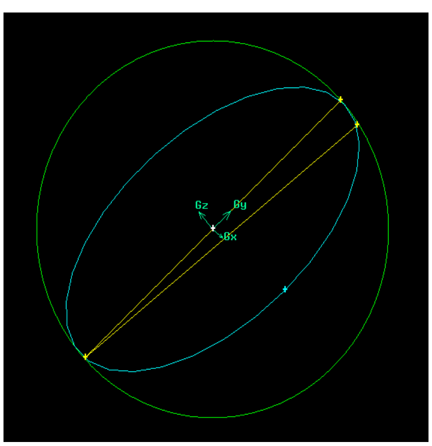

Fig.4. Three-dimensional coordinate system.

In this three-dimensional coordinate system, F is in the XOY coordinate plane, the angle θ between F and y-axis is 2°, the angle between XOY coordinate plane and ground is 30°.

$$|\vec{F_y}| = |\vec{F}|\cos\theta \approx 450N, \quad (1)$$
$$|\vec{F_x}| = |\vec{F}|\sin\theta \approx 16N, \quad (2)$$
$$|\overrightarrow{F_{net}}| = m\frac{|\overrightarrow{\Delta v}|}{\Delta t}, \quad (3)$$
$$|\overrightarrow{v_\infty}| = \frac{|\vec{F}|t}{m}, \quad (4)$$
$$|\vec{G}| = \sum mr^2\omega, \quad (5)$$
$$|\vec{G}| = \int_0^{\pi} M_{soccer}\,(R\sin\theta)^2\omega\, d\theta, \quad (6)$$
$$|\vec{F}|t = \frac{\pi}{2}M_{soccer}\,\omega R^2 . \quad (7)$$

Because the net force applied to an object is equal to the rate of change of momentum (3), the formula (3) can be written as (4). So the magnitude of v_∞ is approximately 20. This is the radial velocity of the soccer. Then according to the definition (5) of angular momentum, the angular momentum of the soccer during its movement can be presented by (6), which can be written as (7). So the magnitude of the angular velocity of the soccer during its movement is approximately 20.

B. Air Resistance

$$Re = \frac{\rho V d}{v}, \quad (8)$$
$$|\overrightarrow{F_1}| = \frac{1}{2}\rho|\overrightarrow{V_\infty}|^2 AC_d . \quad (9)$$

When analyzing the path of banana ball, air resistance is an important factor to consider. Air is one kind of fluids. In fluid dynamics, there are mainly two kinds of air resistances, viscous drag and pressure drag. The number Re is a dimensionless parameter that determines the behavior and characteristics of viscous flow patterns. It is defined by (8), where ρ is fluid density, V is stream velocity, d is a characteristic length scale (in this case is the diameter of the soccer), and v is fluid viscosity. For values of Re less than 2000, the dye streak remains smooth and undisturbed-that is, laminar flow resulted. The air resistance is mainly viscous drag. For Re greater than 3000, the dye streak seems to burst into turbulence and fill the whole tube with color.

Temperature (℃)	**v*10^5 (m^2/s)**
0	13.20
20	**15.00**
40	16.90
60	18.80
80	20.90
100	23.00
120	25.20
140	27.40

Fig.5. The relationship between temperature and the viscosity of air.

From Fig.5 we note that when the temperature is 20℃ the viscosity of air is 15.00*10^{-5}. V equals to v_∞. So in this case, the number Re is approximately 3*10^5.

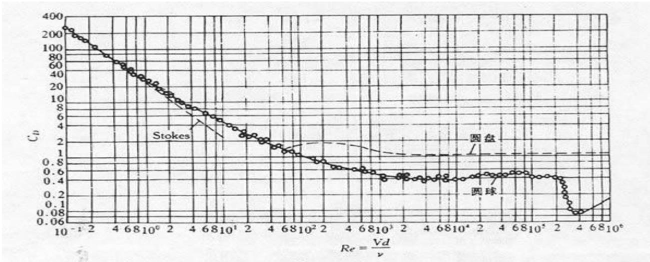

Fig.6. The curve of relationship between resistance constant (C_d) of soccer and disk and Reynolds number.

From Fig.6 we note that when the number Re is about $3*10^5$, the resistance constant (C_d) is nearly 0.2. A is the front face area of the soccer. So the pressure drag can be expressed as $F_1 \approx 9.118 \times 10^{-3} V_\infty^2$, the relationship between the magnitudes of F_1 and V_∞^2 is direct proportion. So the magnitude of acceleration a_1 is approximately $0.020 V_\infty^2$. Because the direction of acceleration a_1 is opposite to the direction of its velocity, the velocity of the soccer tends to decrease.

C. Vertical Force

The axis of the rotation of the soccer passes the center of the soccer and is perpendicular to the direction of the velocity of the soccer. When the soccer is rotating, due to the velocity of the surface of the soccer, the velocity of air around the soccer is different. According to the equation in Bernoulli's theorem, if velocity rises at a given elevation, pressure must fall. In this case, because the velocity of air is greater on the right side of the soccer, the pressure on the right side of the soccer is less than on the left side, so the path of the soccer tends to slightly move to right.

To analyze the vertical force acting on the soccer, we apply the idea of differentiation and divide the soccer into infinite small aspects as in Fig.7.

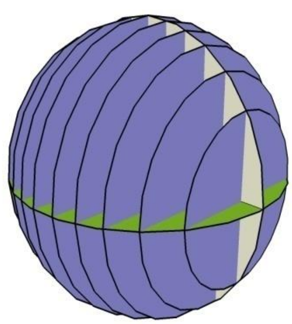

Fig.7. Diagrammatic Sketch of the division of the soccer.

Establish grid in gambit as Fig.8. Each cell in the grid is the unit of iteration. Use Fluent to iteration from left of the grid to the right.

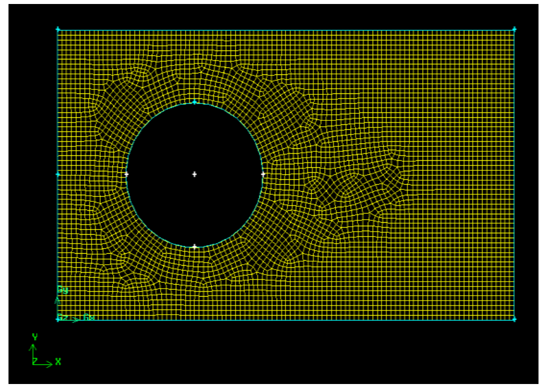

Fig.8. Gambit.

Define the boundary condition v=20m/s, and the outcome of iteration is as Fig.9 and Fig 10.

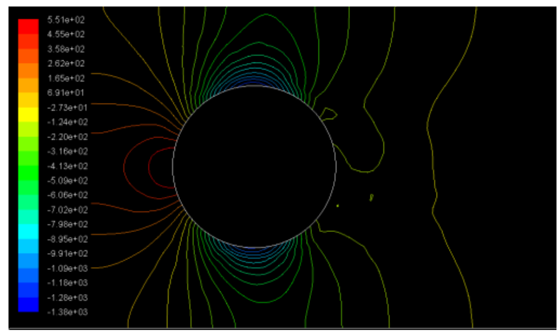

Fig.9. Distribution of static pressure around the soccer.

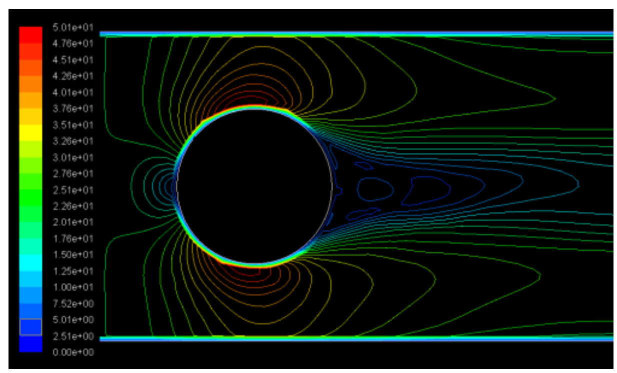

Fig.10. Distribution of velocity of the air around the soccer.

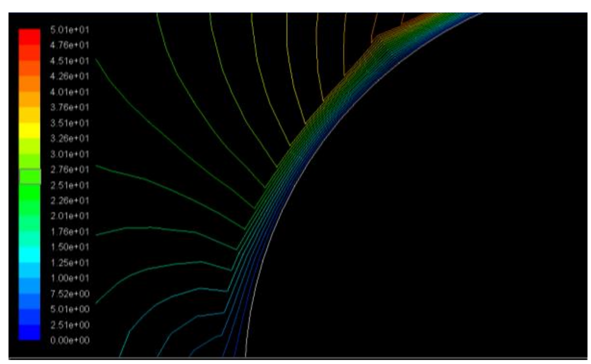

Fig.11. Magnified of distribution of the velocity of air around the soccer.

Find the relationship between V and θ. Connect the dot that has a distinct velocity and the center of the ball. The angle between that line and the radius of the ball is θ. It is shown in Fig.12.

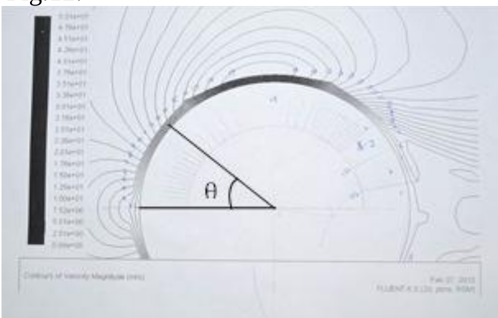

Fig.12. Diagrammatic sketch of angle θ.

When the velocity of the soccer is 20m/s, $\theta - v_\theta$ data is in Appendix [A]. Then define $C_p = \frac{P - P_\infty}{\frac{1}{2}\rho v_\infty^2} = 1 - (\frac{v_\theta}{v_\infty})^2$, use C++ to program and calculate. The code is in Appendix [B]. The outcome data is in Appendix [C]. Enter these data into Matlab; the graph generated is as Fig.13.

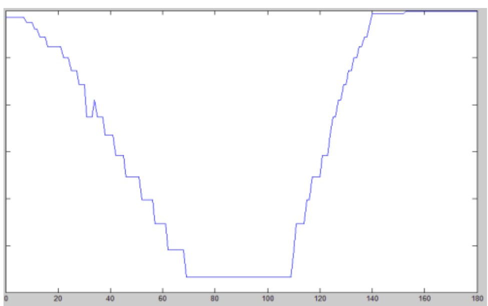

Fig.13. The graph of $\theta - C_{p\theta}$.

$$\overrightarrow{|v_\theta|} = \overrightarrow{|v_\infty|}\sqrt{1 - C_{p\theta}}\ , \tag{10}$$

$$P + \frac{1}{2}\rho\overrightarrow{|v|}^2 = constant\ , \tag{11}$$

$$P_{up\theta} = P_0 + \frac{1}{2}\rho(\overrightarrow{|v_\theta|} + \omega r)^2\ , \tag{12}$$

$$P_{down\theta} = P_0 + \frac{1}{2}\rho(\overrightarrow{|v_\theta|} - \omega r)^2\ , \tag{13}$$

$$\triangle P_\theta = \frac{1}{2}\rho[\left(\overrightarrow{|v_\theta|} + \omega r\right)^2 + (\overrightarrow{|v_\theta|} - \omega r)^2]\ , \tag{14}$$

$$\triangle P_{\theta x} = \frac{1}{2}\rho[\left(\overrightarrow{|v_\theta|} + \omega r\right)^2 + (\overrightarrow{|v_\theta|} - \omega r)^2]\sin\theta\ , \tag{15}$$

$$\overrightarrow{|F_2|} = 2S\iint_D 2\rho\overrightarrow{|v_\theta|}\omega r\sin\theta\, dr d\theta\ , \tag{16}$$

$$\overrightarrow{|F_2|} = 4\pi R^2\rho\omega\overrightarrow{|v_\infty|} \cdot \sum_{r=0}^{1500}\sum_{\theta=0}^{180}\sqrt{1 - C_{p\theta}}\frac{r}{10000}\sin\theta\,\Delta\frac{r}{10000}\Delta\theta\ . \tag{17}$$

Because $C_{p\theta}$ and the radius of the soccer r are not correlated, v_∞ is constant, so v_θ can be expressed in terms of (10). Applying the Bernoulli's theorem, pressure on the top of the soccer can be expressed in terms of (12), and pressure on the bottom of the soccer can be expressed in terms of (13). Subtracting (13) from (12) gives the difference of pressure between two dots on the top and bottom surfaces of the soccer (14), so the difference of pressure on x-axis can be expressed in terms of (15). Apply the idea of integration to calculate the vertical force. In the domain $D: \theta \in (0, \pi), r \in (0, 0.15)$, integrate each difference of pressure between the dots on the top and bottom to get the whole difference of pressure between the top surface and bottom surface. So the vertical force can be expressed as (16), which can be written as (17). Because this equation is too complicated to calculate, we program by C++ to calculate

$$\sum_{r=0}^{1500}\sum_{\theta=0}^{180}\sqrt{1 - C_{p\theta}}\frac{r}{10000}\sin\theta\,\Delta\frac{r}{10000}\Delta\theta.$$

Assume that $\Delta\frac{r}{10000} = 0.0001$, $\Delta\theta = 1$, and

$$\int_a^b f(x) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})\ ,$$

the code of C++ is in Appendix [D]. After we get the approximate magnitude of it, we substitute it into (17), and then we obtain vertical force $\overrightarrow{|F_2|} \approx 0.284466438\overrightarrow{|v_\infty|}$, so $\overrightarrow{|a_2|} \approx 0.632\overrightarrow{|v_\infty|}$.

IV. TRAJECTORY OF THE SOCCER

A. *Function & Equations*

$$f(x, y, z, t) = 0\ , \tag{18}$$

$$\vec{v} = \left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}\right)\ , \tag{19}$$

$$\overrightarrow{a_1} = \frac{9.362 \times 10^{-3}\left(\frac{\partial^2 x}{\partial t^2} + \frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 z}{\partial t^2}\right)}{\sqrt{\frac{\partial^2 x}{\partial t^2} + \frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 z}{\partial t^2}}}\left(-\frac{\partial x}{\partial t}, -\frac{\partial y}{\partial t}, -\frac{\partial z}{\partial t}\right), \tag{20}$$

$$\overrightarrow{a_2} = \frac{0.632\sqrt{\frac{\partial^2 x}{\partial t^2} + \frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 z}{\partial t^2}}}{\sqrt{\frac{\partial^2 x}{\partial t^2} + \frac{\partial^2 y}{\partial t^2}}}\left(\frac{\partial y}{\partial t}, -\frac{\partial x}{\partial t}, 0\right)\ , \tag{21}$$

$$\vec{g} = (0, 0, -10)\ , \tag{22}$$

$$\vec{a} = \frac{\partial x}{\partial t^2} + \frac{\partial y}{\partial t^2} + \frac{\partial z}{\partial t^2} = \overrightarrow{a_1} + \overrightarrow{a_2} + \vec{g}\ , \tag{23}$$

$$\begin{cases} x = f(t) \\ y = g(t) \\ z = h(t) \end{cases}, \begin{cases} v_x = f'(t) \\ v_y = g'(t) \\ v_z = h'(t) \end{cases}, \begin{cases} a_x = f''(t) \\ a_y = g''(t) \\ a_z = h''(t) \end{cases} \tag{24}$$

4

$$\begin{cases} f''(t) = \dfrac{0.632g'(t)}{\sqrt{(f'(t))^2 + (g'(t))^2}} - \dfrac{9.362 \times 10^{-3}(f'(t))^2}{\sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}} \\ g''(t) = -\dfrac{0.632f'(t)}{\sqrt{(f'(t))^2 + (g'(t))^2}} - \dfrac{9.362 \times 10^{-3}(g'(t))^2}{\sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}} \\ h''(t) = -\dfrac{9.362 \times 10^{-3}(h'(t))^2}{\sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}} - 10, \end{cases} \quad (25)$$

$$\begin{cases} \lim_{\delta t \to 0} f(t + \delta t) = f(x) + f'(t)\delta t \text{ , } f(0) = 0 \text{ ,} \\ \lim_{\delta t \to 0} g(t + \delta t) = g(x) + g'(t)\delta t \text{ , } g(0) = 0 \text{ ,} \\ \lim_{\delta t \to 0} h(t + \delta t) = h(x) + h'(t)\delta t \text{ , } h(0) = 0 \text{ ,} \end{cases} \quad (26)$$

$$\begin{cases} \lim_{\delta t \to 0} f'(t + \delta t) = f'(x) + f''(t)\delta t \text{ , } f'(0) = 0 \text{ ,} \\ \lim_{\delta t \to 0} g'(t + \delta t) = g'(x) + g''(t)\delta t \text{ , } g'(0) = 10\sqrt{3} \text{ ,} \\ \lim_{\delta t \to 0} h'(t + \delta t) = h'(x) + h''(t)\delta t \text{ , } h'(0) = 10. \end{cases} \quad (27)$$

The movement of the soccer is a movement with velocity and accelerator affect each other continuously, so we need the idea of differentiation. Divide the time of this movement into infinite small parts, so during each of these instants, the change of velocity and accelerator is approximately zero. Appling basic rules of projectile motion, we can work out the instantaneous acceleration and instantaneous velocity of the soccer in every instant. Thus, a graph can be generated from all the data we get.

Apply differential-algebraic equation to express the relationship (18) between the position of the soccer and time. Use x, y, and z to represent the location of the soccer in three-dimensional coordinate system, and t represents the time. Combination of (18)-(23) yields a system of partial differential equations. In order to reach the solution of this complicated system of equations, we transform this system of partial differential equations into parametric equations by substituting x, y, and z through (24). The equations can be written as (25)-(27). To solve the system of equations through C++ programming, we assume that the change of time $\delta t = 0.01s$, which means in $\delta t = 0.01s$, the soccer is in uniform rectilinear motion (C++ code is in Appendix [E]). Calculate by programming, there are many sets of solutions (solutions are in Appendix [F]), which represent the different locations the soccer is in during the movement. Enter these data into Matlab, and then a graph of the trajectory of the soccer is generated as Fig.14.

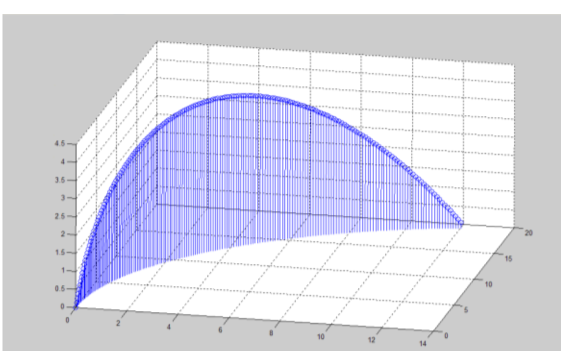

Fig.14. The trajectory of the soccer.

V. Conclusion

X	Y	Z
8.46431	17.4701	2.65175
8.55762	17.535	2.59916
8.65114	17.5991	2.54573
8.74487	17.6624	2.49145
8.83882	**17.7249**	**2.43633**
8.93296	17.7865	2.38037
9.02731	17.8474	2.32357
9.12185	17.9074	2.26593
9.21658	17.9666	2.20745

Fig.15. Part of data from Appendix [F].

The height of the goalpost is about 2.44 meters, from Fig.15 we find the value of x and y when the value of z is approximately 2.44. The displacement of the soccer $H = \sqrt{x^2 + y^2}$. So the displacement of the soccer is approximately 19.8m. However, because the strides of different players are different, so players need to adjust their steps before shooting. After doing a research about the strides of players, we assume that the step of the player is 1m. So the most suitable distance to shoot is about 19.8 ± 1m. Assuming the top left corner is the center of a circle, 18.8m and 20.8m are radius draw two circles. Then assume the top right corner is the center of a circle, 18.8m and 20.8 are radius draw two circles as in Fig.16.

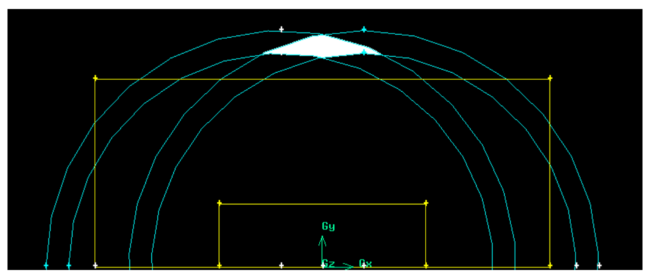

Fig.16. Diagrammatic sketch of the best area to shoot.

The white area in Fig.16 is the area where the arcs cross. In this area, with right amount of force, player can score a goal by shooting towards either the top left corner or the top right corner of the goalpost. It is much more difficult for the goal keeper to stop the soccer. So the white area is the most suitable area to shoot.

In this thesis, we combine the theory of fluid dynamics and two fundamental operations of calculus, differentiation and integration. We also apply methods of modeling, partial differential equation, and iterated function. We analyze the motion and trajectory of soccer. The analysis of soccer's path is almost the same with the trajectory analysis of bullet and other projectile motions, which requires various mathematical and physical knowledge. This knowledge and analysis can be applied to both military affairs and criminal investigation.

VI. Verification of Conclusion

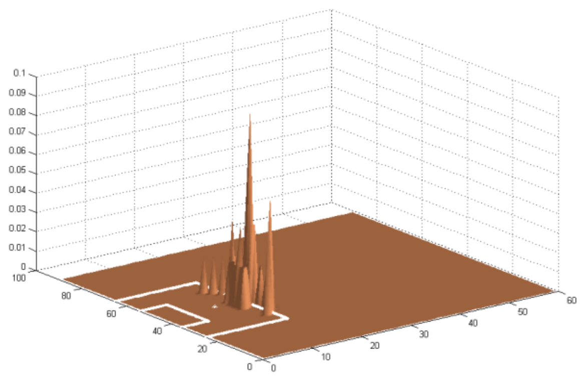

Fig. 17 Distribution of the ratio of goals.

To verify the result, we obtain data from the official website of UEFA. (Original data are in Appendix [G]) The data include all of the goals in the soccer matches in recent years and the distance of the goals. From the data we reach the rate of scoring goals in different distances from the goalpost. By using C++， we processed the data (C++ code is in Appendix [H]), and generate a graph by Matlab as Fig.17. From Fig. 17 we note that the area where the rate of scoring goals is the highest almost overlaps the white area in our conclusion, so our result is credible and applicable to real life.

VII. Future Research

In this thesis we assume that the force that a player kicks the soccer with is constant. However, in real life, the magnitude of that force should be a range. Thus the trajectory of the soccer should be a system of curves, and the displacement of the soccer should be an equation about the force. In the future research we will endeavor to solve this problem.

In fluid dynamics, a Kármán vortex street is a repeating pattern of swirling vortices caused by the unsteady separation of flow of a fluid around blunt bodies as shown in Fig.18.

Fig.18. Kármán vortex street.

A vortex street will only form at a certain range of flow velocities, specified by a range of Reynolds numbers. Vortex Street can be created by steady winds blowing past smokestacks, transmission line. The street gives rise to oscillating lateral forces on the shedding body. This will obviously influence the motion of the soccer. So in the future we will also do a research about it.

Acknowledgment

The authors gratefully acknowledge the contributions of Mrs. Min Wei and Mrs. Fengyi Pan, who help us in the basic preparation of the thesis and calculation.

References

[1]Haoqiang Tan, "C++ Program Design," (2nd ed.) Beijing: Tsinghua University Press, pp. 60-180.

[2]Sybil P.Parker, "McGraw-Hill Encyclopedia of Physics," in *Hardcover*, (2nd ed.) New York: McGraw-Hill, January 1993, pp. 654-700.

[3]Shuangshuang Ting, "Teaching Material Analysis of University: Calculus," (1st ed.) Beijing: China Ocean University Press, August 2011, pp. 30-80.

[4]Yingchuan Fan, "The Teaching Materials of Higher Mathematics" (1st ed.) Beijing: People's Education Press, March 1958, pp. 34-85.

[5]Hongjun Jiang, "Engineering Fluid Mechanics," (1st ed.) Beijing: Electric Industry Press, July 1979, pp. 45-120.

[6]Tongsheng Hao, "Theoretical Mechanics," (3rd ed.) Beijing: People's Education Press, January 2003, pp. 37-82.

[7]Zurong Ting, "Fluid Dynamics," (1st ed.) Beijing: Higher Education Press, December 2003, pp. 58-134.

[8]Mathematic Teaching Group of Sichuan University of Mining and Technology, "Mathematics Handbook," (1st ed.) Sichuan: Science Press, January 1978, pp. 24-60.

[9]Youzhong Guo, Tsingxi Li, "Mathematical and Physical Methods," (1st ed.) Wuhan: Wuhan University Press, January 1997, pp. 78-102.

[10]Hao Zhan, Wanping Li, "The Emulation of Column Detour Flow at Different Reynolds Number by FLUENT," *Journal of Wuhan University of Technology*, vol.12, pp. 64-73, December 2008.

[11]Zuoru yang, Runda Ji, and Hongyuan Fu, "Geometrical Analysis of Path of Falling ball," unpublished, presented at 5th Shing-Tung Yau High School Mathematics Awards, Beijing, China, 2012.

APPENDIX

[A]

0	7.52194	31	30.0877	62	45.1316	93	47.6389	124	32.5951	156	2.50731
1	7.52194	32	30.0877	63	45.1316	94	47.6389	125	30.0877	157	2.50731
2	7.52194	33	30.0877	64	45.1316	95	47.6389	126	30.0877	158	2.50731
3	7.52194	34	27.5804	65	45.1316	96	47.6389	127	27.5804	159	2.50731
4	7.52194	35	30.0877	66	45.1316	97	47.6389	128	27.5804	160	2.50731
5	7.52194	36	30.0877	67	45.1316	98	47.6389	129	25.0731	161	2.50731
6	7.52194	37	30.0877	68	45.1316	99	47.6389	130	25.0731	162	2.50731
7	7.52194	38	32.5951	69	47.6389	100	47.6389	131	22.5658	163	2.50731
8	10.0292	39	32.5951	70	47.6389	101	47.6389	132	22.5658	164	2.50731
9	10.0292	40	32.5951	71	47.6389	102	47.6389	133	20.0585	165	2.50731
10	10.0292	41	32.5951	72	47.6389	103	47.6389	134	20.0585	166	2.50731
11	12.5366	42	35.1024	73	47.6389	104	47.6389	135	17.5512	167	2.50731
12	12.5366	43	35.1024	74	47.6389	105	47.6389	136	17.5512	168	2.50731
13	15.0439	44	35.1024	75	47.6389	106	47.6389	137	15.0439	169	2.50731
14	15.0439	45	35.1024	76	47.6389	107	47.6389	138	15.0439	170	2.50731
15	15.0439	46	37.6097	77	47.6389	108	47.6389	140	5.01462	171	2.50731
16	17.5512	47	37.6097	78	47.6389	109	47.6389	141	5.01462	172	2.50731
17	17.5512	48	37.6097	79	47.6389	110	45.1316	142	5.01462	173	2.50731
18	17.5512	49	37.6097	80	47.6389	111	42.6243	143	5.01462	174	2.50731
19	17.5512	50	37.6097	81	47.6389	112	42.6243	144	5.01462	175	2.50731
20	17.5512	51	37.6097	82	47.6389	113	42.6243	145	5.01462	176	2.50731
21	17.5512	52	40.117	83	47.6389	114	42.6243	146	5.01462	177	2.50731
22	20.0585	53	40.117	84	47.6389	115	40.117	147	5.01462	178	2.50731
23	20.0585	54	40.117	85	47.6389	116	40.117	148	5.01462	179	2.50731
24	20.0585	55	40.117	86	47.6389	117	37.6097	149	5.01462	180	2.50731
25	22.5658	56	40.117	87	47.6389	118	37.6097	150	5.01462		
26	22.5658	57	42.6243	88	47.6389	119	37.6097	151	5.01462		
27	22.5658	58	42.6243	89	47.6389	120	37.6097	152	5.01462		
28	25.0731	59	42.6243	90	47.6389	121	35.1024	153	2.50731		
29	25.0731	60	42.6243	91	47.6389	122	35.1024	154	2.50731		
30	25.0731	61	42.6243	92	47.6389	123	35.1024	155	2.50731		

[B]

```
#include <cstdlib>
#include <fstream>
using namespace std;
ifstream fin("v20-thea.in");
ofstream fout("Cp.out");
int main(int argc, char *argv[])
{
    double a[185][2];
    int i,j;
```

```
    for(i=0;i<=180;i++)
      {
      for(j=0;j<2;j++) fin>>a[i][j];
      }
   for(i=0;i<180;i++)
      {
          a[i][1]=1-(a[i][1]/20)*(a[i][1]/20);
      }
     for(i=0;i<180;i++)
      {
      for(j=0;j<2;j++) fout<<a[i][j]<<" ";
      fout <<endl;
      }
    return 0;
}
```

[C]

0	0.858551	45	-2.08044	90	-4.67367	135	0.22989
1	0.858551	46	-2.53622	91	-4.67367	136	0.22989
2	0.858551	47	-2.53622	92	-4.67367	137	0.434205
3	0.858551	48	-2.53622	93	-4.67367	138	0.434205
4	0.858551	49	-2.53622	94	-4.67367	140	0.937134
5	0.858551	50	-2.53622	95	-4.67367	141	0.937134
6	0.858551	51	-2.53622	96	-4.67367	142	0.937134
7	0.858551	52	-3.02343	97	-4.67367	143	0.937134
8	0.748536	53	-3.02343	98	-4.67367	144	0.937134
9	0.748536	54	-3.02343	99	-4.67367	145	0.937134
10	0.748536	55	-3.02343	100	-4.67367	146	0.937134
11	0.607087	56	-3.02343	101	-4.67367	147	0.937134
12	0.607087	57	-3.54208	102	-4.67367	148	0.937134
13	0.434205	58	-3.54208	103	-4.67367	149	0.937134
14	0.434205	59	-3.54208	104	-4.67367	150	0.937134
15	0.434205	60	-3.54208	105	-4.67367	151	0.937134
16	0.22989	61	-3.54208	106	-4.67367	152	0.937134
17	0.22989	62	-4.09216	107	-4.67367	153	0.984283
18	0.22989	63	-4.09216	108	-4.67367	154	0.984283
19	0.22989	64	-4.09216	109	-4.67367	155	0.984283
20	0.22989	65	-4.09216	110	-4.09216	156	0.984283
21	0.22989	66	-4.09216	111	-3.54208	157	0.984283

22	-0.00586	67	-4.09216	112	-3.54208	158	0.984283
23	-0.00586	68	-4.09216	113	-3.54208	159	0.984283
24	-0.00586	69	-4.67367	114	-3.54208	160	0.984283
25	-0.27304	70	-4.67367	115	-3.02343	161	0.984283
26	-0.27304	71	-4.67367	116	-3.02343	162	0.984283
27	-0.27304	72	-4.67367	117	-2.53622	163	0.984283
28	-0.57165	73	-4.67367	118	-2.53622	164	0.984283
29	-0.57165	74	-4.67367	119	-2.53622	165	0.984283
30	-0.57165	75	-4.67367	120	-2.53622	166	0.984283
31	-1.26318	76	-4.67367	121	-2.08044	167	0.984283
32	-1.26318	77	-4.67367	122	-2.08044	168	0.984283
33	-1.26318	78	-4.67367	123	-2.08044	169	0.984283
34	-0.9017	79	-4.67367	124	-1.65609	170	0.984283
35	-1.26318	80	-4.67367	125	-1.26318	171	0.984283
36	-1.26318	81	-4.67367	126	-1.26318	172	0.984283
37	-1.26318	82	-4.67367	127	-0.9017	173	0.984283
38	-1.65609	83	-4.67367	128	-0.9017	174	0.984283
39	-1.65609	84	-4.67367	129	-0.57165	175	0.984283
40	-1.65609	85	-4.67367	130	-0.57165	176	0.984283
41	-1.65609	86	-4.67367	131	-0.27304	177	0.984283
42	-2.08044	87	-4.67367	132	-0.27304	178	0.984283
43	-2.08044	88	-4.67367	133	-0.00586	179	0.984283
44	-2.08044	89	-4.67367	134	-0.00586	180	0.984283

[D]

```
#include <cstdlib>
#include <fstream>
#include <math.h>
using namespace std;
ifstream fin("Cp.in");
ofstream fout("sigema.out");
int main(int argc, char *argv[])
{   long double a[181][2];
    int i,j;
    double r;
    long double s=0;
    for(i=0;i<180;i++)
       {
       for(j=0;j<2;j++) fin>>a[i][j];
```

```
      }
      for(r=0;r<0.15;r=r+0.0001)
        {
        for(i=0;i<180;i++)
          {
s=s+sin(i*3.1415926/180)*sqrt(1-a[i][1])*(3.1415926/180)*r*0.0001;            }
        fout<<s<<endl;
        }
   return 0;
}
```

[E]

```
#include <cstdlib>
#include <fstream>
#include <math.h>
using namespace std;
ofstream fout("x.out");
int main(int argc, char *argv[])
{
   int i,m;
   double a,b;
   double v[500][3];
   double a1[500][3];
   double a2[500][3];
   double x[500][3];
   double t=0.01;

   v[0][0]=0;
   v[0][1]=17.32;
   v[0][2]=10;

   x[0][0]=0;
```

```
    x[0][1]=0;
    x[0][2]=0;
    for(i=0;i<500;i++)
      {
      a=0.020*(v[i][0]*v[i][0]+v[i][1]*v[i][1]+v[i][2]*v[i][2]);
      a1[i][0]=a*(-v[i][0])/sqrt(v[i][0]*v[i][0]+v[i][1]*v[i][1]+v[i][2]*v[i][2]);
      a1[i][1]=a*(-v[i][1])/sqrt(v[i][0]*v[i][0]+v[i][1]*v[i][1]+v[i][2]*v[i][2]);
      a1[i][2]=a*(-v[i][2])/sqrt(v[i][0]*v[i][0]+v[i][1]*v[i][1]+v[i][2]*v[i][2]);

      b=0.632*sqrt(v[i][0]*v[i][0]+v[i][1]*v[i][1]+v[i][2]*v[i][2]);
      a2[i][0]=b*(v[i][1])/sqrt(v[i][0]*v[i][0]+v[i][1]*v[i][1]);
      a2[i][1]=b*(-v[i][0])/sqrt(v[i][0]*v[i][0]+v[i][1]*v[i][1]);
      a2[i][2]=0;

      v[i+1][0]=v[i][0]+(a2[i][0]+a1[i][0])*t;
      v[i+1][1]=v[i][1]+(a2[i][1]+a1[i][1])*t;
      v[i+1][2]=v[i][2]+(a2[i][2]+a1[i][2]-9.8)*t;

      x[i+1][0]=x[i][0]+v[i][0]*t;
      x[i+1][1]=x[i][1]+v[i][1]*t;
      x[i+1][2]=x[i][2]+v[i][2]*t;
      if(x[i+1][2]<0)//ensure z>0
        {
        m=i;
        break;
        }
      }
   for(i=0;i<m;i++) fout<<x[i][0]<<" "<<x[i][1]<<" "<<x[i][2]<<" "<<endl;
   return 0;
}
```

[F]

X	Y	Z	X	Y	Z
0	0	0	4.03179	13.0121	4.16217
0	0.1732	0.1	4.10715	13.1173	4.15677
0.001264	0.345707	0.19862	4.18299	13.2218	4.1504
0.003779	0.51752	0.295868	4.25929	13.3255	4.14306
0.007531	0.688635	0.391752	4.33604	13.4285	4.13477
0.012509	0.859052	0.48628	4.41326	13.5307	4.12552
0.018698	1.02877	0.579459	4.49092	13.6322	4.11531
0.026087	1.19778	0.671298	4.56902	13.7329	4.10414
0.034663	1.36609	0.761802	4.64757	13.8328	4.09203
0.044414	1.5337	0.850981	4.72655	13.932	4.07897
0.055327	1.7006	0.93884	4.80596	14.0305	4.06496
0.06739	1.86679	1.02539	4.88579	14.1282	4.05
0.080591	2.03227	1.11063	4.96604	14.2251	4.0341
0.094919	2.19704	1.19457	5.0467	14.3213	4.01727
0.110361	2.36109	1.27723	5.12778	14.4167	3.99949
0.126907	2.52443	1.3586	5.20926	14.5114	3.98079
0.144544	2.68706	1.43869	5.29113	14.6053	3.96114
0.163261	2.84897	1.5175	5.3734	14.6984	3.94057
0.183047	3.01016	1.59506	5.45606	14.7908	3.91907
0.20389	3.17063	1.67135	5.53911	14.8824	3.89665
0.22578	3.33039	1.74639	5.62253	14.9733	3.8733
0.248707	3.48942	1.82018	5.70633	15.0634	3.84903
0.272658	3.64773	1.89274	5.7905	15.1527	3.82384
0.297623	3.80532	1.96405	5.87504	15.2413	3.79773
0.323592	3.96218	2.03414	5.95994	15.329	3.77071
0.350554	4.11832	2.10301	6.04519	15.4161	3.74277
0.378499	4.27374	2.17066	6.13079	15.5023	3.71393
0.407416	4.42843	2.23709	6.21674	15.5878	3.68418
0.437296	4.58239	2.30232	6.30303	15.6725	3.65352
0.468128	4.73562	2.36634	6.38966	15.7564	3.62196
0.499903	4.88813	2.42917	6.47662	15.8396	3.58949
0.53261	5.03991	2.49081	6.56391	15.922	3.55613
0.56624	5.19095	2.55126	6.65152	16.0036	3.52187
0.600783	5.34127	2.61054	6.73945	16.0844	3.48672
0.636229	5.49086	2.66863	6.82769	16.1644	3.45068
0.67257	5.63971	2.72556	6.91624	16.2437	3.41374
0.709796	5.78784	2.78132	7.0051	16.3222	3.37591
0.747897	5.93523	2.83591	7.09425	16.3999	3.3372
0.786864	6.08189	2.88935	7.1837	16.4768	3.29761
0.826689	6.22781	2.94164	7.27344	16.5529	3.25714
0.867362	6.37301	2.99278	7.36347	16.6282	3.21578

0.908874	6.51747	3.04278	7.45378	16.7028	3.17355
0.951218	6.66119	3.09164	7.54436	16.7765	3.13044
0.994383	6.80418	3.13937	7.63521	16.8495	3.08646
1.03836	6.94644	3.18597	7.72633	16.9216	3.04162
1.08314	7.08796	3.23144	7.81771	16.993	2.9959
1.12872	7.22874	3.27579	7.90935	17.0636	2.94931
1.17509	7.36879	3.31902	8.00124	17.1333	2.90187
1.22224	7.50811	3.36114	8.09338	17.2023	2.85356
1.27016	7.64668	3.40215	8.18576	17.2704	2.80439
1.31884	7.78452	3.44206	8.27838	17.3378	2.75436
1.36828	7.92163	3.48086	8.37123	17.4043	2.70348
1.41846	8.058	3.51857	8.46431	17.4701	2.65175
1.46939	8.19363	3.55519	8.55762	17.535	2.59916
1.52105	8.32852	3.59071	8.65114	17.5991	2.54573
1.57343	8.46267	3.62515	8.74487	17.6624	2.49145
1.62653	8.59609	3.65851	8.83882	17.7249	2.43633
1.68033	8.72877	3.69079	8.93296	17.7865	2.38037
1.73484	8.86071	3.722	9.02731	17.8474	2.32357
1.79005	8.99192	3.75213	9.12185	17.9074	2.26593
1.84594	9.12238	3.7812	9.21658	17.9666	2.20745
1.90251	9.25211	3.8092	9.31149	18.025	2.14814
1.95975	9.38109	3.83614	9.40658	18.0825	2.08801
2.01766	9.50934	3.86202	9.50184	18.1393	2.02704
2.07622	9.63685	3.88685	9.59727	18.1951	1.96525
2.13544	9.76362	3.91063	9.69287	18.2502	1.90263
2.1953	9.88965	3.93336	9.78862	18.3044	1.8392
2.2558	10.0149	3.95505	9.88452	18.3578	1.77494
2.31693	10.1395	3.9757	9.98057	18.4104	1.70987
2.37869	10.2633	3.9953	10.0768	18.4621	1.64399
2.44106	10.3864	4.01388	10.1731	18.5129	1.57729
2.50405	10.5087	4.03142	10.2696	18.5629	1.50978
2.56763	10.6303	4.04793	10.3661	18.6121	1.44147
2.63182	10.7511	4.06342	10.4628	18.6605	1.37235
2.6966	10.8713	4.07788	10.5597	18.7079	1.30242
2.76197	10.9906	4.09133	10.6566	18.7546	1.2317
2.82791	11.1092	4.10376	10.7536	18.8003	1.16018
2.89443	11.2271	4.11517	10.8507	18.8453	1.08787
2.96151	11.3443	4.12557	10.948	18.8893	1.01476
3.02915	11.4607	4.13497	11.0453	18.9325	0.940857
3.09735	11.5763	4.14336	11.1427	18.9749	0.866168
3.1661	11.6912	4.15075	11.2401	19.0163	0.790694
3.23539	11.8054	4.15713	11.3377	19.057	0.714436
3.30522	11.9188	4.16252	11.4352	19.0967	0.637396

3.37558	12.0315	4.16692	11.5329	19.1356	0.559578
3.44647	12.1435	4.17032	11.6306	19.1736	0.480983
3.51787	12.2546	4.17274	11.7283	19.2108	0.401615
3.58979	12.3651	4.17417	11.8261	19.247	0.321474
3.66222	12.4748	4.17461	11.9239	19.2824	0.240565
3.73515	12.5838	4.17408	12.0218	19.3169	0.158888
3.80858	12.692	4.17256			
3.8825	12.7994	4.17007			
3.9569	12.9061	4.16661			

[G]

17.0	52.4	17.4	44.8
16.6	53.6	17.2	44.5
15.8	55.8	17.5	44.3
14.7	57.1	17.1	44.2
17.1	51.7	17.5	43.9
17.4	51.5	17.2	43.8
17.4	50.9	17.1	43.4
17.6	49.9	17.4	43.5
17.6	48.8	17.5	43.2
14.7	46.9	17.4	42.7
15.6	46.3	17.1	42.7
15.3	45.3	17.3	41.9
14.6	44.5	17.3	41.6
14.7	43.7	17.7	39.4
14.5	42.7	17.6	38.0
14.8	41.7	17.5	36.4
14.8	40.4	17.2	35.6
14.5	39.2	17.1	34.3
15.4	40.1	16.7	33.8
15.7	41.9	16.5	32.8
16.3	42.7	16.4	32.2
15.2	43.3	16.0	32.0
15.7	43.8	15.9	31.3
16.3	44.3	15.6	31.1
17.2	45.0	17.8	47.6

[H]

```
#include <cstdlib>
```

```
#include <fstream>
#include <math.h>
using namespace std;
ifstream fin("02.in");
ofstream fout("02.out");
int main(int argc, char *argv[])
{
    double a[50],b[50];
    int c[60][90];
    int i,j,k,n;
    for(i=0;i<50;i++)
      {
      fin>>a[i];
      fin>>b[i];
      }
       for(i=0;i<60;i++)
        {
        for(j=0;j<90;j++)
          {
         n=0;
          for(k=0;k<50;k++)
             {
             if(a[k]>i&&a[k]<=(i+1)&&b[k]>j&&b[k]<=(j+1)) n++;
             }
          c[i][j]=n;
          }
        }
           for(i=0;i<60;i++)  fout<<i+0.5<<" ";
     fout<<endl<<endl;
     for(j=0;j<90;j++)  fout<<j+0.5<<" ";
```

```
 fout<<endl<<endl;
 for(j=0;j<90;j++)
  {
  for(i=0;i<60;i++)
    {
    fout<<c[i][j]/50.0<<" ";
    }
  fout<<";";
  }
 return EXIT_SUCCESS;
}
```