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Title: Decision over the irrationality of the roots of the simple indicator with forms as $a^x + b^x = c^x$

Abstract

Starting from the roots of the equation $3^x + 4^x = 5^x$ discussed in the Tenth Grade, we utilized the method, the algebraic extension of the rational number field, to produce the ways to judge whether the roots of the basic exponential equation with a form as $a^x + b^x = c^x$ are rational or not. For equations with more than two terms on the left side, as is the equation $a_1^x + a_2^x + \cdots + a_n^x = d^x$, the determination of whether the root was irrational was comparatively difficult. Therefore, we provided a prevalent method for the examination of the root of a three-term equation as well as a conclusion that if the equation doesn't have integer roots, the roots won't be a rational number with a denominator of two. Finally, based on the method, the algebraic extension of the rational number field, we concluded that under special occasions, the root of the equation can't be some rational numbers with certain denominators.

Keywords:exponential equation, algebraic extension, unit root

Chapter 1 Introduction

During the Maths class in the sophomore year, we used the monotone of the indicator function to analyze and prove that the only integral root of function $3^x + 4^x = 5^x$ is $x = 2$ and that of function $3^x + 4^x + 5^x = 6^x$ is $x = 3$. Applying the same method can show that function $3^x + 4^x = 6^x$ doesn't have integral roots. After class, we made a further assumption that such equation doesn't have rational roots. Therefore, we did a investigation into simple indicator functions as $a^x + b^x = c^x$ along with the similar ones, proving that this function doesn't have rational root and acquire some consequences over the ones with more terms.

Chapter 2 Investigation into simple indicator functions with forms as$a^x + b^x = c^x$

Preliminaries

Lemma 1. The necessary and sufficient condition of the rational factorization of the polynomial$y^x - d$ is the existence of $m(m \in \mathbb{Z}^+ \backslash \{1\})$, allowing $d = d_1^m$ and $x = x_1 m$. Both d_1,x_1 are positive integers.

Proof: Adequacy: When such m exits, $({y_1}^{x_1})^m = d_1^m$ can be derived from the original function. Obviously, when$y_1^{x_1} = d_1$, the original function is true. Therefore, there at least exists a factor as ${y_1}^{x_1} - d_1$ that can be factorized.

Necessity:Considering its x factors that are factorized within the range of complex number.

$$y^x - d = (y - \sqrt[x]{d}\xi_1)(y - \sqrt[x]{d}\xi_2) \cdots (y - \sqrt[x]{d}\xi_x)$$

$\xi_1, \xi_2 \cdots \xi_x$are xth roots of unity of 1. If $y^x - d$ has the factor $h(y)$, with the degree of a, then the numerical size of its absolute term is $\sqrt[x]{d^a}$, a rational number. Due to the fact that d is an integer, $\sqrt[x]{d^a}$ must also be an integer.Let$d =$

d_0^p and p be the largest integer allowing d_0 to also be an integer. $\sqrt[x]{d^a}$ is an integer,then $x \mid ap.a < x$,x doesn't divide a exactly, so we can let $x = hq$, with $h \mid a$ and $q \mid p$,also$a = a_1 h$,$p = p_1 q$. Suppose $m = q$. We now turn to the demonstration of the claim that when $d = d_1^m x = m x_1$,d_1,x_1 are both integers.

$$d = d_0^m = d_0^{m_1 q} = ({d_0}^{m_1})^q = {d_1}^q,$$

$$d_1 = d_0^{m_1} \in \mathbb{Z}, x_1 = \frac{x}{q} = h \in \mathbb{Z}.$$

This completes the proof of Lemma 1.

Lemma 2. Suppose K/F is a random extension and $\alpha \in K$, then the following statements are equivalent:

(1)$F(\alpha)/F$ is a algebraic expension

(2)α is algebraical in F

(3)$F(\alpha)/F$ is a finite extension.

When one of the conditions is matched,$[F(\alpha) : F]$ equals the degree of α.

For a specific proof of this Lemma the reader is referred to

[1]P211Theorom3.

Main Results:

By representing functions similar to $3^x + 4^x = 6^x$ with $a^x + b^x = c^x (a, b, c \in \mathbb{Q}^+)$, we reached a few simple conclusions. Assuming $a < b$, then

(1)If x exists, then :

when x < 0, a, b are both bigger than c;

when x > 0, a, b are both smaller than c

when x $= 0$, no such a, b, c exist

(2)If the function have integral root, the root is unique.

(3)If $a + b < c$, x doesn't exist.

(4)If $a < b < c$, the range of x is

$$\max\{\log_{(\frac{a}{c})} \frac{1}{2}, 0\} < x < \log_{(\frac{b}{c})} \frac{1}{2}.$$

If $c < a < b$, the range of x is

$$-\log_{(\frac{c}{b})} \frac{1}{2} < x < \min\{-\log_{(\frac{c}{a})} \frac{1}{2}, 0\}.$$

Proof: // Utilizing the monotony of the indicator function, the proof of (1),(2) and (3) is trivial. By dividing both sides with c^x and transforming the function into $(\frac{a}{c})^x + (\frac{b}{c})^x = 1$, some tedious manipulation yields to conclusions (1) to (3). Applying the same method, a system of inequalities can be provided. Based on the later form , and due to the fact that $(\frac{a}{c})^x < (\frac{b}{c})^x$ when $a < b < c$, therefore

$$\begin{cases} (\frac{a}{c})^x < \frac{1}{2}, \\ (\frac{b}{c})^x > \frac{1}{2}. \end{cases}$$

By solving the system of inequalities, the first part of conclusion (4) can be proved. As the remainder of the argument is analogous to that of the first part, it is left to the reader. In order to simplify the problem, we started with the situation when $b = 1$.

If the function has rational root, then there must exists p, q (p and q are mutually prime, p,q$\in \mathbb{Z}^+$, $p \neq 1$) that allows

$$c^{\frac{q}{p}} - a^{\frac{q}{p}} = 1,$$

Let $m = c^q = (1 + a^{\frac{q}{p}})^p, n = a^q = (c^{\frac{q}{p}} - 1)^p$, then the function is represented by $\sqrt[x]{m} = 1 + \sqrt[x]{n}$. By studying the simplified equation, we attained Proposition1.

Proposition 1. Functions with forms as $\sqrt[x]{m} - \sqrt[x]{n} = 1 (m, n \in \mathbb{Z}^+)$ doesn't have positive integral roots which satisfies $\sqrt[x]{m} \in \mathbb{Q}$.

Proof: Assume that there exists such root x, then $\sqrt[x]{m} = 1 + \sqrt[x]{n}$.

$$m = (1+\sqrt[x]{n})^x = 1+\mathrm{C}_x^1\sqrt[x]{n}+\mathrm{C}_x^2(\sqrt[x]{n})^2+\cdots+\mathrm{C}_x^{x-1}(\sqrt[x]{n})^{x-1}+n,$$

So $\sqrt[x]{n}$ is the root of a $x-1$-degree polynomial with integral coefficient.As it is also the root of function $y^x - n = 0$, its minimal polynomial is the common factor of the mentioned polynomials.

The following argument is split into two parts:

1° If $y^x - n$ is the minimal polynomial of $\sqrt[x]{n}$: the fact that $y^x - n$ is also the root of a $x - 1$-degree polynomial with

integral coefficient leads to a contradiction. 2° If $y^x - n$ isn't the minimal polynomial of $\sqrt[x]{n}$:
According to Lemma 1, there exists such $d > 1$ that allows $n = n_1^d$ and $x = x_1 d$. By taking the largest d, d_{max}, function (1) can be transformed into

$$\sqrt[x]{m} = 1 + \sqrt[x_1]{n_1},$$

so $\mathbb{Q}(\sqrt[x]{m}) = \mathbb{Q}(\sqrt[x_1]{n_1})$

According to Lemma 2, $\mathbb{Q}(\sqrt[x]{m})$'s degree of extension is also x_1. Therefore, the degree of the minimal polynomial of $\mathbb{Q}(\sqrt[x]{m})$ is x_1. Because $\sqrt[x]{m}$ is the root of function $y^x - m$, its minimal polynomial is a factor of $y^x - m$. Consulting the proof of Lemma 1, we factorized $y^x - m$ within the range of the complex number as follows:

$$y^x - m = (y - \sqrt[x]{m}\xi_1)(y - \sqrt[x]{m}\xi_2)\cdots(y - \sqrt[x]{m}\xi_x),$$

$\xi_1, \xi_2 \cdots \xi_x$ are the xth roots of unity of 1. The minimal polynomial of $\sqrt[x]{m}$ is the product of x_1 terms selected from above. The numerical size of its absolute term is $\sqrt[x]{m^{x_1}}$, a rational number. So $\sqrt[x]{m} = \sqrt[x_1]{m_1}$. As d is the largest number available, the minimal polynomial of $\sqrt[x_1]{n_1}$ is $y^{x_1} - n_1$. The repetition of the argument from 1° can lead to a contradiction. The proof is completed.

Extension 1.1. Functions with forms as $\sqrt[x]{m} - \sqrt[x]{n} = k(m, n \in \mathbb{Z}^+, k \in \mathbb{Q})$ doesn't have positive integral root which satisfies $\sqrt[x]{m} \in \mathbb{Q}$.

Proof: By replacing the 1 in the proving process of Proposition1 with $k= \frac{q}{p}$, an argument similar to the one used in Proposition1 shows that Extension1.1 is true.

Extension 1.2. Functions with forms as $\sqrt[x]{m} - \sqrt[y]{n} = k(m, n \in \mathbb{Z}^+, k \in \mathbb{Q})$ doesn't have positive integral root which satisfies $\sqrt[x]{m} \in \mathbb{Q}$.

Proof:This function can be transformed into $\sqrt[xy]{m^y} - \sqrt[xy]{n^x} = k$, and thus been proved by Extension1.1.

By consulting the proving skills of Proposition1, we tried to discuss about the irrationality of the root under general conditions. First, we handle the original function $a^x + b^x = c^x$ as follows:
Represent a,b,c in such forms that have the smallest integer as base number with a integer as exponent.

$$a = a_1^{k_a}, b = b_1^{k_b}, c = c_1^{k_c}.$$

Let $k = (k_a, k_b, k_c)$ and $y = kx$.
So the final form of function is $a_0^y + b_0^y = c_0^y$. If $k \neq 1$, we only discussed the final form and acquired Proposition2.

Proposition 2. When a,b,c are changed into forms that have the smallest base numbers and integral exponents that have no common factor other than 1, if functions as $a^x + b^x = c^x$,$(a, b, c \in \mathbb{Q})$ don't have integral roots, they don't have rational roots.
Proof:Both sides of the function divide c^x,

$$(\frac{a}{c})^x - (\frac{b}{c})^x = 1.$$

Applying the reduction to absurdity:

If the original function have rational root, let it be $\frac{q}{p}(p, q$ are mutually prime, p,q$\in \mathbb{Z}^+$). Marka^q, b^q, c^q respectively as m, n, k.So,

$$\sqrt[p]{\frac{m}{k}} - \sqrt[p]{\frac{n}{k}} = 1$$

As a,b,c are changed into forms that have the smallest base numbers and integral exponents that have no common factor other than 1, the minimal polynomial of $\sqrt[p]{\frac{n}{k}}$ is $y^p - \frac{n}{k} = 0$. Using the same argument from 1° in Proposition1 can lead to contradictory.

Using Proposition2 and the monotony of indicator function, we can quickly tell whether the root of the mentioned indicator function is rational or not.

Example 1.Function$27^x + 64^x = 216^x$.

(1)Changing every term of the function into forms with the smallest base number and a integral exponent

$$3^{3x} + 4^{3x} = 6^{3x}.$$

Let $y = 3x, 3^y + 4^y = 6^y$.
(2)Replacing the original function with $(\frac{1}{2})^y + (\frac{2}{3})^y = 1$.
Because the function on the left side is monotone decreasing, the root is unique.
(3)When $y = 1$, the left side of the function is larger than the right side and when $y = 2$,it is the opposite. Therefore, there is no integral root.
The root of the original function is irrational.

Example 2.Function$1^x + 4^x = 9^x$.
(1)Changing every term of the function into forms with the smallest base number and a integral exponent

$$1^{2x} + 2^{2x} = 3^{2x}.$$

(2)Let $y = 2x$, yielding function $1^y + 2^y = 3^y$ with the

integral root $y = 1$.
Thus original function have the rational root $\frac{1}{2}$.

By summarizing the mentioned propositions and examples, we yield the following theorem:

Theorem 1 When discuss functions with forms as $a^x + b^x = c^x$ （$a, b, c \in \mathbb{Q}^+$）, find the largest positive integer k that allows

$$a^x = a_1^{kx}, b^x = b_1^{kx}, c^x = c_1^{kx}, a_1, b_1, c_1 \in \mathbb{Z}$$

Let$y = kx$, and simplify the equation into forms as $a_1^y + b_1^y = c_1^y$.
If $a_1^y + b_1^y = c_1^y$ doesn't have integral roots, then the original function doesn't have rational root;
If $a_1^y + b_1^y = c_1^y$ have integral roots but not satisfying $k \mid y$, the original function have rational yet non-integral root
If $a_1^y + b_1^y = c_1^y$ have integral roots and $k \mid y$, the original function have integral root.

Chapter 3 Study of Three-term Equations

Repeating the simplification and change of variable introduced in chapter one, the equation turns into $\sqrt[x]{a} + \sqrt[x]{b} = 1 - \sqrt[x]{c}$.To find contradictions in degree of extensions, we have to prove $[\mathbb{Q}(\sqrt[x]{a} + \sqrt[x]{b}) : \mathbb{Q}] \neq [\mathbb{Q}(\sqrt[x]{c}) : \mathbb{Q}]$
(All the" $\frac{q}{p}$" s stand for rational numbers are fractions in lowest terms, and the equations have all been simplified.)
Lemma 3 Let $K \supset E \supset F$ be extension fields over F, and $[K : F]$ is finite. Then

$$[K : F] = [K : E][E : F].$$

For a rigid proof of this Lemma reader is referred to [1]P211 Theorem4.

Based on those conclusions,we can prove proposition 3:
Proposition 3 Simquations like $a_1^x + a_2^x + a_3^x = d^x, (a, b, c, d \in \mathbb{Q})$ without integer solutions don't have radical solutions, either.

Proof: Let $\dfrac{q}{2}=x(\text{q}\in\mathbb{Z},(q,2)=1)$,$m_i=(\dfrac{a_i}{d})^q(i=1,2,3)$,

the equation turns into $\sqrt{m_1}+\sqrt{m_2}=1-\sqrt{m_3}, so\mathbb{Q}(\sqrt{m_1}+\sqrt{m_2})=\mathbb{Q}(\sqrt{m_3})$.

Consider$\mathbb{Q}(\sqrt{m_1},\sqrt{m_2})$. It contains$\sqrt{m_1}+\sqrt{m_2}$,

so$\mathbb{Q}(\sqrt{m_1},\sqrt{m_2})\supseteq\mathbb{Q}(\sqrt{m_1}+\sqrt{m_2})$, Also,

$$\frac{m_1-m_2}{\sqrt{m_1}+\sqrt{m_2}}=\sqrt{m_1}-\sqrt{m_2},$$

$$\frac{1}{2}[(\sqrt{m_1}+\sqrt{m_2})+(\sqrt{m_1}-\sqrt{m_2})]=\sqrt{m_1},$$

$$\frac{1}{2}[(\sqrt{m_1}+\sqrt{m_2})-(\sqrt{m_1}-\sqrt{m_2})]=\sqrt{m_2},$$

$\mathbb{Q}(\sqrt{m_1},\sqrt{m_2})\subseteq\mathbb{Q}(\sqrt{m_1}+\sqrt{m_2})$, $\mathbb{Q}(\sqrt{m_1},\sqrt{m_2})=\mathbb{Q}(\sqrt{m_1}+\sqrt{m_2})$ Then we will turn to prove $[\mathbb{Q}(\sqrt{m_1},\sqrt{m_2}):\mathbb{Q}]\neq[\mathbb{Q}(\sqrt{m_3}):\mathbb{Q}]$.

Now that$[\mathbb{Q}(\sqrt{m_1}):\mathbb{Q}]=[\mathbb{Q}(\sqrt{m_3}):\mathbb{Q}]=2$,if$[\mathbb{Q}(\sqrt{m_1},\sqrt{m_2}):\mathbb{Q}]=[\mathbb{Q}(\sqrt{m_3}):\mathbb{Q}]$, then according to Lemma 5, we have:

$$[\mathbb{Q}(\sqrt{m_1},\sqrt{m_2}):\mathbb{Q}]=[\mathbb{Q}(\sqrt{m_1}):\mathbb{Q}][\mathbb{Q}(\sqrt{m_2},\sqrt{m_1}):\mathbb{Q}(\sqrt{m_1})]$$

$\therefore[\mathbb{Q}(\sqrt{m_2}):\mathbb{Q}(\sqrt{m_1})]=1$, $\sqrt{m_2}\in\mathbb{Q}(\sqrt{m_1})$.

$$\because \mathbb{Q}(\sqrt[2]{m_1}) = a_0 + a_1\sqrt[2]{m_1}(a_0, a_1 \in \mathbb{Q}).$$

$\sqrt{m_2} = a_0 + a_1\sqrt{m_1}$ $m_2 = a_0^2 + a_1^2 m_1 + 2a_0a_1\sqrt{m_1}$ $\because$ $m_2\mathbb{Q}, a_0^2+a_1^2m_1+2a_0a_1\sqrt[2]{m_1}$ is irrational $\therefore$ $[\mathbb{Q}(\sqrt{m_1}, \sqrt{m_2}) : \mathbb{Q}] \neq [\mathbb{Q}(\sqrt{m_3}) : \mathbb{Q}]$

,which is contradictory with the assume $\mathbb{Q}(\sqrt{m_1}+\sqrt{m_2}) = \mathbb{Q}(\sqrt{m_3})$.

The proof is completed.

Lemma 4. If a,b$\in \mathbb{Q}^+$, $\frac{\sqrt[x]{a}}{\sqrt[x]{b}} \notin \mathbb{Q}$, then$\mathbb{Q}(\sqrt[x]{a}) \neq \mathbb{Q}(\sqrt[x]{b})$.

The proof of this Lemma is postponed to the Appendix.

Lemma 5.Suppose α,β are n-degree algebraic number,all zero points of their minimum polynomials are

$$\alpha_1, \alpha_2, \cdots, \alpha_n; \beta_1, \beta_2, \cdots, \beta_n,$$

respectively. Then for each $h \neq \frac{\beta_i-\beta_j}{\alpha_k-\alpha_l}(1 \leqslant k, l, i, j \leqslant n)$,we have $\mathbb{Q}(h\alpha + \beta) = \mathbb{Q}(\alpha, \beta)$

For a specific proof of this Lemma the reader is referred to [2]P9Theorem15

According to Lemma 4,if$\frac{\beta_i-\beta_j}{\alpha_k-\alpha_l} \neq 1$as k,l,i,j values from 1 to n, let h be 1,we get $\mathbb{Q}(\alpha+\beta) = \mathbb{Q}(\alpha,\beta)$.
According to Lemma3,$[\mathbb{Q}(\sqrt[x]{a},\sqrt[x]{b}):\mathbb{Q}] \neq 1$, so$[\mathbb{Q}(\sqrt[x]{a},\sqrt[x]{b}):\mathbb{Q}] > [\mathbb{Q}(\sqrt[x]{c}):\mathbb{Q}] = x$, The degrees of extensions of each side of the equation are different, which is contradictory.

Inspect the possible values of h. All zero points of the minimum polynomial of $\sqrt[x]{a},\sqrt[x]{b}$ can be projected onto the complex plane as apexes of the inscribed regular polygons of two circles centering around origin. Their radiuses are α,β respectively and their ratio of similitude is $\sqrt[n]{a},\sqrt[n]{b}$If there exists k,l,i,j satisfying$\beta_i - \beta_j = \alpha_k - \alpha_l$,there exists two diagonal of these two polygons which are both parallel and equal in length to each other.
Since we can calculate the length of each diagonal with the

formula

$$L_m = L\frac{\sin\frac{m\pi}{n}}{\sin\frac{\pi}{n}}$$

m stands for the number of edges between the end points of the diagonal, the equation

$$\frac{\sin\frac{p\pi}{n}}{\sin\frac{q\pi}{n} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}}(p, q = 1, 2, \cdots, b)$$

can be used to give out impossible values of n. Take n=5 as an example(as shown in Picture 3.1).

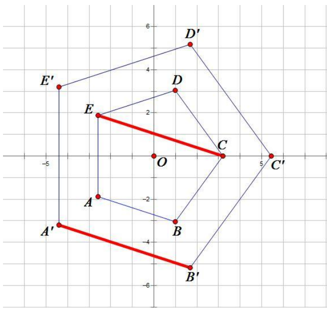

图 1: Picture 3.1

If EC=A'B', $\frac{AB}{EC} = \frac{AB}{A'B'}, \frac{\sin\frac{\pi}{5}}{\sin\frac{2\pi}{5}} = \frac{\sqrt{5}+1}{2}$. For a,b $\in \mathbb{Q}$, and $(\frac{\sqrt{5}+1}{2})^5$ is not in Q, we can conclude :If equation $a^x + b^x + c^x = d^x$ doesn't have integer solution ,it doesn't have rational solution with 5 as denominator, either.

When n=6,$\frac{\sin\frac{\pi}{6}}{\sin\frac{3\pi}{6}}=\frac{1}{2}\in\mathbb{Q}$. If $\frac{a}{b}=\frac{1}{64}$, consider$\frac{b}{c}$, $\frac{c}{a}$.At least one of them can't be 64 or $\frac{1}{64}$,we suggest it be$\frac{b}{c}$. Then we have $[Q(\sqrt[n]{c}+\sqrt[n]{b}):Q]\neq[Q(\sqrt[n]{a}):Q]$, so fraction in lowest terms with 6 as denominator also can't be solution of equation $a^x+b^x+c^x=d^x$.

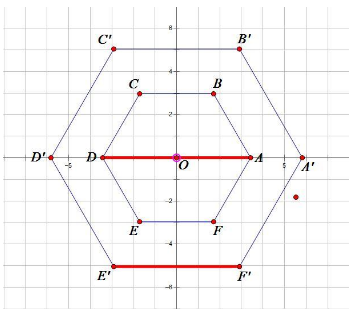

图 2: pentagon

The ratio of sines of different angles and their integer powers are mostly irrational, and the method also works when there's only one rational ratio. Even when there are more than one rational ratio,we can still get contradiction by calculating the concrete value of ratio of similitude, so this method can be widely and effectively used in proving the impossibility of a 3-term exponential equation having solution with appointed denominator.

Chapter 4 Discussion on equations with multiple terms

We tried to popularize Theorem 1 in Chapter 2 to more terms.Namely,we conjecture that equations without integer solutions like

$$a_1^x + a_2^x + \ldots + a_n^x = d^x, (a_1, a_2, \cdots, a_n \in \mathbb{Q})$$

also doesn't have rational solutions. In this chapter, we give out several situations that we can prove the impossibility of solution with appointed denominator.

Lemma 6 Ifα is an n-degree algebraic element of $\mathbb{Q}$, β is an m-degree algebraic element of $\mathbb{Q}$, and $(m, n) = 1$, then $\alpha + \beta$ is an (n+m)-degree algebraic element of $\mathbb{Q}$.

Proof:Consider the field $\mathbb{Q}(\alpha) \cap \mathbb{Q}(\beta)$,

$[\mathbb{Q}(\alpha) \cap \mathbb{Q}(\beta) : \mathbb{Q}]\Big|[\mathbb{Q}(\alpha) : \mathbb{Q}], [\mathbb{Q}(\alpha) \cap \mathbb{Q}(\beta) : \mathbb{Q}]\Big|[\mathbb{Q}(\beta) : \mathbb{Q}]$,

$\therefore [\mathbb{Q}(\alpha) \cap \mathbb{Q}(\beta) : \mathbb{Q}]\Big|m, [\mathbb{Q}(\alpha) \cap \mathbb{Q}(\beta) : Q]\Big|n$,

$\because (m,n)=1, \therefore [\mathbb{Q}(\alpha)\cap\mathbb{Q}(\beta):\mathbb{Q}]=1, \mathbb{Q}(\alpha)\cap\mathbb{Q}(\beta)=\mathbb{Q}$.
Let all zero points of α's minimum polynomials be $\alpha_1 = \alpha, \alpha_2, \alpha_3, \ldots, \alpha_n$; all zero points of β's minimum polynomials be $\beta_1=\beta, \beta_2, \beta_3, \ldots, \beta_n$;
If $\alpha_i+\beta_j=\alpha_k+\beta_l$, then $\alpha_i-\alpha_k=\beta_l-\beta_j$,
For $\mathbb{Q}(\alpha)\cap\mathbb{Q}(\beta)=\mathbb{Q}$, we can infer $i=k, l=j$.
$f(\alpha_1+\beta_1-x)$ and $g(x)$ have the only common factor $x-\beta_1$, and the coefficients of $f(\alpha_1+\beta_1-x)$, $g(x)$ are all in $\mathbb{Q}(\alpha_1+\beta_1)$.
There exist polynomials u(x), v(x) in $\mathbb{Q}(\alpha_1+\beta_1)[x]$ that satisfy $f(\alpha_1+\beta_1-x)u(x)+g(x)v(x)=x-\beta_1$.
$\therefore \beta_1\in\mathbb{Q}(\alpha_1+\beta_1), \alpha_1\in\mathbb{Q}(\alpha_1+\beta_1)$.
$\mathbb{Q}(\alpha_1,\beta_1)\subseteq\mathbb{Q}(\alpha_1+\beta_1)$, $\mathbb{Q}(\alpha_1+\beta_1)\subseteq\mathbb{Q}(\alpha_1,\beta_1)$,
$\therefore \mathbb{Q}(\alpha_1,\beta_1)=\mathbb{Q}(\alpha_1+\beta_1)$,
$\because [\mathbb{Q}(\alpha_1):\mathbb{Q}]\big|[\mathbb{Q}(\alpha_1,\beta_1):\mathbb{Q}], [\mathbb{Q}(\beta_1):\mathbb{Q}]\big|[\mathbb{Q}(\alpha_1,\beta_1):\mathbb{Q}]$,
$m\big|[\mathbb{Q}(\alpha_1,\beta_1):\mathbb{Q}], n\big|[\mathbb{Q}(\alpha_1,\beta_1):\mathbb{Q}]$, $\because (m,n)=1$,
$\therefore mn\big|[\mathbb{Q}(\alpha_1,\beta_1):\mathbb{Q}]$,
$\because [\mathbb{Q}(\alpha_1,\beta_1):\mathbb{Q}]=mn$,

$\therefore [\mathbb{Q}(\alpha_1 + \beta_1) : \mathbb{Q}] = mn.$

$\therefore \alpha_1 + \beta_1 (\alpha + \beta)$ is an $n + m$-degree algebraic element of$\mathbb{Q}$.

Using the conclusions above, we found several situations in which we can get impossible values of denominators of the solution.

For equations like $a_1^x + a_2^x + \ldots + a_n^x = d^x$, let x =$\frac{q}{p}$. Using the method mentioned in Proposition 2, we turn it to $m_1^{\frac{qx_1}{p}} + m_2^{\frac{qx_2}{p}} + \cdots + m_n^{\frac{qx_n}{p}} = 1$ ($m_i^{x_1} = a_i^x, x_1$ is the biggest integer when$m_i \in \mathbb{Z}$. By studying the relationship among $x_i (i = 1, 2, \ldots, n)$, we can get some impossible values of p. Two examples below:

Example 3.$16^x + 25^x + 27^x = 64^x$.

Let x be $\frac{q}{p}$, we can get that $p \neq 6$. If p=6,$\sqrt[3]{\left(\frac{1}{2}\right)^q} + \sqrt[3]{\left(\frac{5}{8}\right)^q} + \sqrt[2]{\left(\frac{3}{4}\right)^q} = 1.$

$[\mathbb{Q}(\sqrt[3]{\left(\frac{1}{2}\right)^q} + \sqrt[3]{\left(\frac{5}{8}\right)^q}) : \mathbb{Q}]$ mod 9, while $[\mathbb{Q}(\sqrt[2]{\left(\frac{3}{4}\right)^q}) : \mathbb{Q}] = 2$ does not. So $p \neq 6$. As well, we can infer that

$p \neq 30$: $[\mathbb{Q}(\sqrt[15]{\left(\frac{1}{2}\right)^q} + \sqrt[15]{\left(\frac{5}{8}\right)^q}) : \mathbb{Q}]$ mod 225, while $[\mathbb{Q}(\sqrt[10]{\left(\frac{3}{4}\right)^q}) : \mathbb{Q}] = 10$ does not. Namely, $p \neq 6k, (k,6) = 1$

Example 4.$2^x + 54^x + 100^x = 65536^x$.
Lex x be $\frac{q}{30}$,

$$\sqrt[2]{\left(\frac{1}{2}\right)^q} + \sqrt[10]{\left(\frac{3}{32}\right)^q} + \sqrt[15]{\left(\frac{5}{128}\right)^q} = 1$$

$$[\mathbb{Q}(\sqrt[2]{\left(\frac{1}{2}\right)^q} + \sqrt[10]{\left(\frac{3}{32}\right)^q}) : \mathbb{Q}]$$

so p$\neq$ 30.

Now we come to conclusion:

For each term in the equation

$$m_1^{\frac{qx_1}{p}} + m_2^{\frac{qx_2}{p}} + \cdots + m_n^{\frac{qx_n}{p}} = 1((x_1, x_2, \cdots, x_i) = 1),$$

Pay attention to $\beta_i = \frac{p}{(p,x_i)}$,which stands for the degree of extension of each term. If $(\beta_{k1}, \beta_{k2}, \cdots, \beta_{kj}) = 1$ （put into Group A）,and they have at least one factor not had by any of the remaining $\beta_{k_{j+1}}$ to $\beta_{k_n}(k_1, \ldots, k_j, k_{j+1}, \ldots, k_n$ is a permutation of $1, 2, \cdots, n$)(put into Group B), then the equation doesn't have a fractional solution with p as its denominator.

Show the equation with β_i as $\sqrt[\beta_1]{m_1} + \sqrt[\beta_2]{m_2} + \cdots + \sqrt[\beta_k]{m_k} = 1$.If its terms can be put into two groups according to the law above, the extension degree of Group A is $I_A = \prod_{i=1}^{k} \beta_{k_i}$, and that of Group B must be a factor of l_A can,t exactly divide $\prod_{i=j+1}^{n} \beta_{k_i}$,so $l_A \neq l_B$. With Proposition 5, we can infer a series of impossible values of p from one:

Proposition 5 For equations like $a_1^x + a_2^x + , + a_n^x = d^x$, If it doesn't have a fractional solution with p as its denominator, it also can't have a fractional solution with hp as its denominator(（$h, i = 1$,$i = 1, 2$,$n \neq k$） .

Sketch of the proof: Continue the discussion above.

$\because (I_A, q) = 1, \therefore I_A' = I_A q, I_B' \mod I_B q^{n-j}$. Let $\frac{I_A}{(I_A, I_B)}$ be c,c can't divide $I_B q^2$ exactly,so $I_A q$ can't divide $I_B q^{n-j}$ exactly.We get $l_A' \neq l_B'$,which is contradictory to our assume.

In chapter three, we have solved the problem in most situations,but can't give out the proof when the ratios of the base numbers are integer powers of integers. Discussion in this chapter is complement to it in some ways. So far, our discussion on exponential equations with multiple terms remains tentative and initial. We mainly aim to provide a train of thought, and we hope our brick will attract a jadestone.

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