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Research into the Order 3 Magic Hexagon: Its Properties, Construction and Extensions

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Abstract

An order 3 magic hexagon resembles the shape of a 19-cell honeycomb, arranged in a 3 4 5 4 3 manner. The requirement is to fill the numbers 1-19 in the grids so that each row (15 in total) adds up to 38.

Previously invented methods aimed at solving this problem and proving its uniqueness were either not rigorous enough or too intricate. So by analyzing its properties, I wanted to find a combinatorial solution to its construction, prove its uniqueness, and investigate whether its mathematical principles can be used in real-world applications.

The difficulty depends on the viewpoint, so the first step was to label each grid in a convenient way. I chose to look at the magic hexagon as a network composed of a center and rings. Then the connections and restrictions of each number set could be found by formula derivation. In a similar fashion, symmetrical properties were also found. The next step was to analyze possible distributions of odd and even numbers. Out of the 9 configurations, only 1 proved to be usable. The final step was construction. With all the properties known, the few impossibilities were easily eliminated, and only one solution remained, thus proving its uniqueness.

The procedures used on the order 3 magic hexagon may be extended to those of higher orders, providing more ease in their construction. The unique properties of magic hexagons may be used in some fields of application, such as in password systems, large-scale roof structure, composite material, national security systems and many other fields.

Contents

Introduction

Everything started from a math corner in a small weekly newspaper published in 1910. It was a recreational mathematics problem; the requirement was to fill the numbers 1–19 into the cells in figure 1, so that the numbers in each line (15 in total) add up to 38.

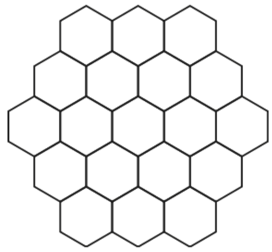


Figure 1

Ordinary people would laugh at it, skip it over, and never think of it again. But Mr. Adams, who had great love for mathematics, took it seriously. He wanted to find an answer.

To reduce labor and protect the earth, Adams obtained a numbered set of hexagonal ceramic tiles and tried to arrange them on a board randomly. He worked hard and 47 years later, in 1957, he finally got a solution, which is shown in figure 2.

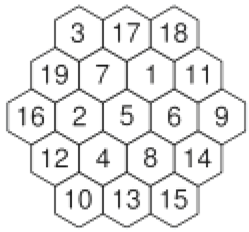


Figure 2

Adams's solution was a real big success at that time. However, mathematicians did not stop there; they proposed two critical questions, as listed below:

1. Is Adams's solution unique or are there other arrangements that also work? (Ignore rotations and reflections of the original arrangement.)

2. Is there a systematic approach through which all such solutions could be found mathematically?

I personally want to add a little bit more: the questions being the same, while subjected to higher order magic hexagons.

Answer to the first question:

It is unique, proved by computers. A program which can analyze 196729 configurations was ran on an IBM 1620. The calculation took 42 minutes.

Comment:

You may think, that is good, we have got the answer, and it really did not take that much time. However, here is a problem: ironically, mathematicians usually do not like computer-based proofs, as they lack mathematical methods and logic, and sometimes aesthetic aspects. This method will also experience great difficulty when used on higher order magic hexagons.

For now, the best approach to the second question:

This method was proposed by Charles W. Trigg, the first step is to list all the possibilities of the outer sides (since each of them is made up of three numbers only), and then eliminate. In this way 1896 configurations have to be examined manually, out of which 121 proved usable. Then, list all the possibilities of the inside. 120 of them were again eliminated manually; only one is left, thus constructing the magic hexagon successfully.

Comment:

While he did answer the second question, I believe there should be better and simpler methods. Examining 1896 configurations sound extremely intricate, at least to me.

My Approach: An Overview

In the combinatorial method I have proposed, the order 3 magic hexagon can be constructed easily, and its uniqueness can be proved at the same time.

Step One: Formula Derivation

To make the analysis and drawing easier, I have simplified the diagram to the form illustrated in figure 3; the numbers then goes to each intersection. The next important step is labeling. Trying to label along each line gives weird symbols and the analysis could not continue, so I chose to look at the magic hexagon as a system composed of a center and rings.

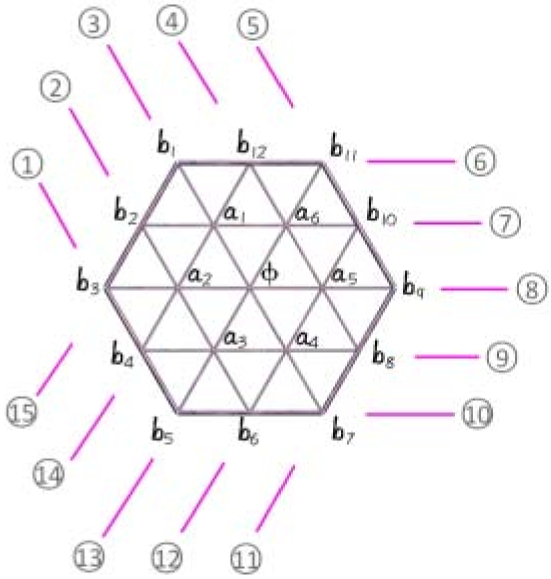


Figure 3

Here $A = a_1 + a_2 + a_3 + a_4 + a_5 + a_6$

$$B = b_1 + b_2 + b_3 + b_4 + \cdots + b_{12}$$

$$b' = b_1 + b_3 + b_5 + b_7 + b_9 + b_{11}$$

$$\Delta b_1 = b_2 + b_6 + b_{10}$$

$$\Delta b_2 = b_4 + b_8 + b_{12}$$

We can now try to combine the 15 initial restrictions and seek for reasonable results. In this way, more restrictions (which are good and helpful in solving such a system) could be found.

$$\begin{cases} ③+⑧+⑬ = 3\Phi + A + b' = 114 \Rightarrow \Phi = 38 - \dfrac{A+b'}{3} \\ \begin{cases} ④+⑨+⑭ = 2\Delta b_1 + A = 114 \\ ②+⑦+⑫ = 2\Delta b_2 + A = 114 \end{cases} \Rightarrow \begin{cases} \Delta b_1 = \Delta b_2 \text{ (set as } \Delta b) \\ A + 2\Delta b = 114 \end{cases} \Rightarrow \\ ①+⑤+⑥+⑩+⑪+⑮ = 2b' + 2\Delta b = 228 \Rightarrow b' + \Delta b = 114 \end{cases}$$

$$\begin{cases} 3\Phi + A = 114 - b' = \Delta b = \frac{114-A}{2} \Rightarrow A + 2\Phi = 38 \\ \Delta b = 114 - b' = A + 3\Phi = 38 + \Phi \\ b' = 114 - \Delta b = A + \Delta b = \begin{cases} A + 38 + \Phi = 2A + 3\Phi = 76 - \Phi \\ A + 114 - b' \Rightarrow A = 2b' - 114 \end{cases} \end{cases}$$

Some important formulae:

(1) $\Phi = 38 - \frac{A+b'}{3}$

(2) $\Delta b_1 = \Delta b_2$ (to simplify, we can call each of them the Δb)

(3) $A + 2\Delta b = 114$

(4) $\Delta b + b' = 114$

(5) $A + 2\Phi = 38$

(6) $\Delta b = A + 3\Phi = 38 + \Phi$

(7) $b' = A + \Delta b = 2A + 3\Phi = 76 - \Phi$

(8) $A = 2b' - 114$

Some significance:

<1> From $A + 2\Phi = 38$ we know A must be even.

<2> From $b' = A + \Delta b = 76 - \Phi$ we know b', Δb and Φ are of the same odd-even property.

<3> From Formula (3)–(8) we can see that Φ, A, Δb and b' have exact mutual relationships, that is, if we knew one of them, we can work out all the other three values.

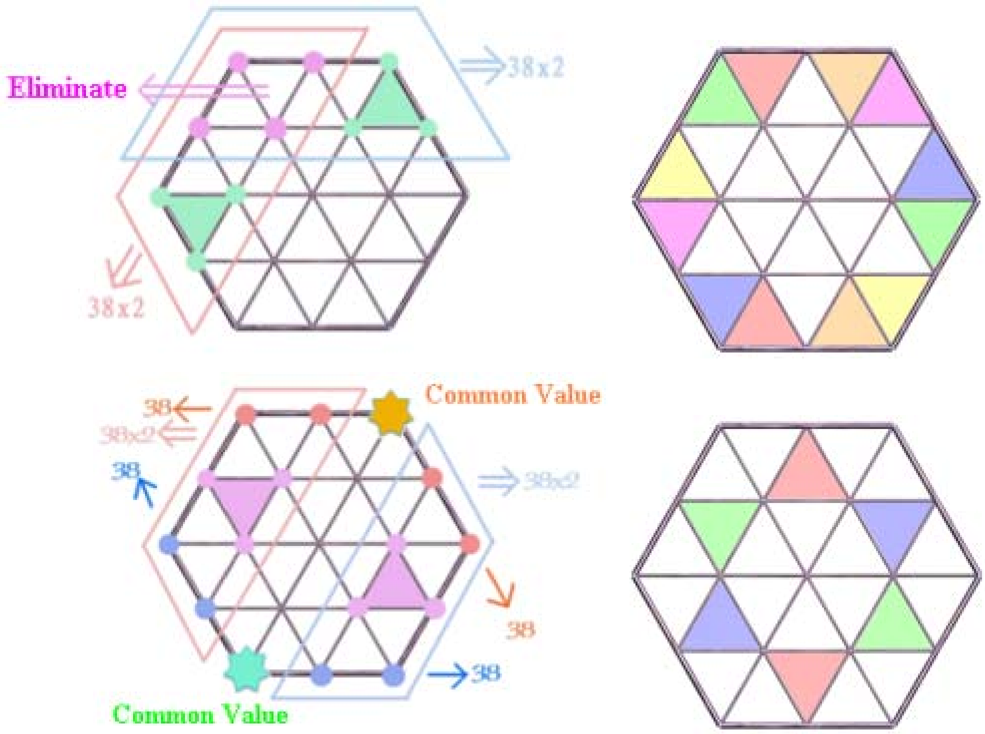
<4> $A_{min} = 1 + 2 + 3 + 4 + 5 + 7 = 22$, then $\Phi_{max} = 8$. We can now work out the ranges of A, Δb and b':

$$\begin{cases} \Phi \le 8 \\ 22 \le A \le 36 \\ 39 \le \Delta b \le 46 \\ 68 \le b' \le 75 \end{cases}$$

Step Two: Symmetrical Properties

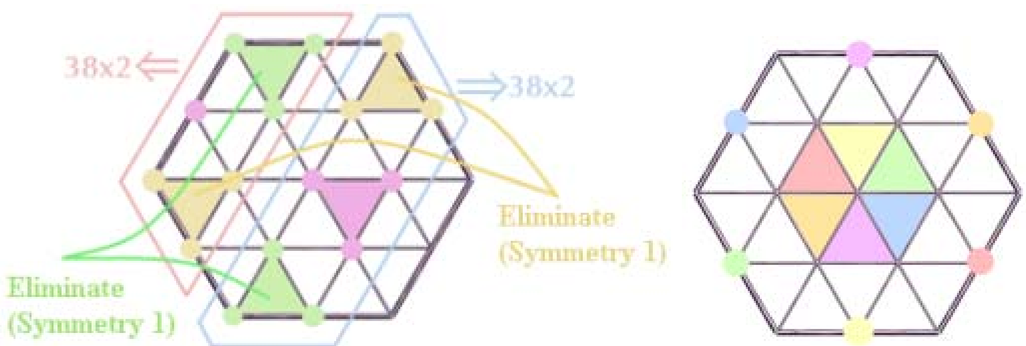
Beside the "group" view shown above, another view, featuring mostly triangular constituents, can also be considered. They are especially useful in eliminating odd-even distributions, since only 4 or 6 numbers are involved in each equation.

Type 1 and 2:



The sum of the three numbers on the vertices of a colored triangle equals to that of the other same-colored triangle.

Type 3:



The sum of the three numbers on the vertices of a colored triangle equals to the other same-colored one on the opposite side of the B ring.

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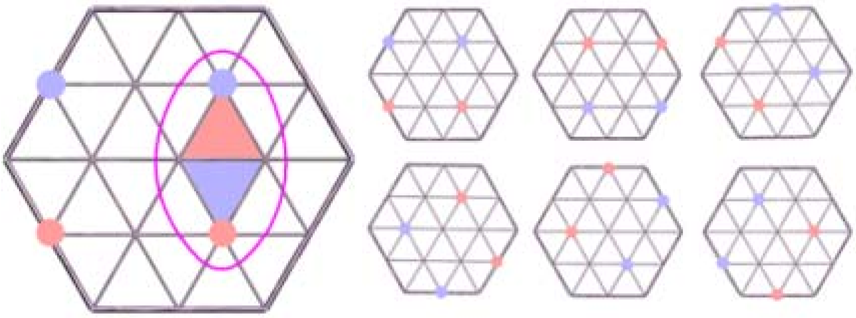


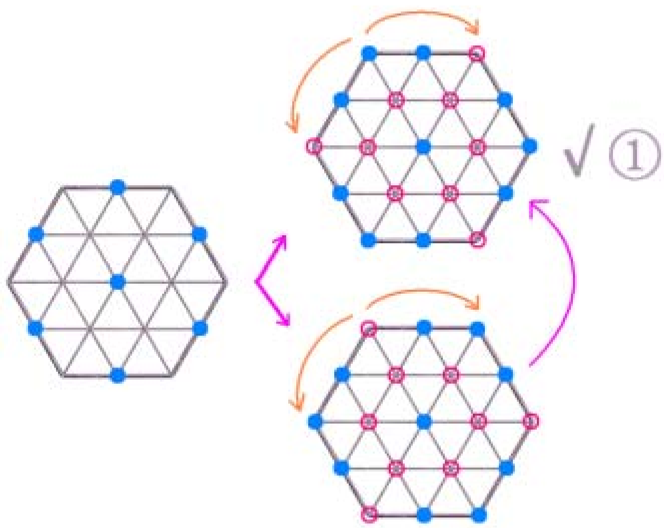
The sum of the numbers on the two blue spots equals to the sum of the red ones.

Step Three: Odd-Even Distribution (● stands for an odd number, ○ stands for an even number)

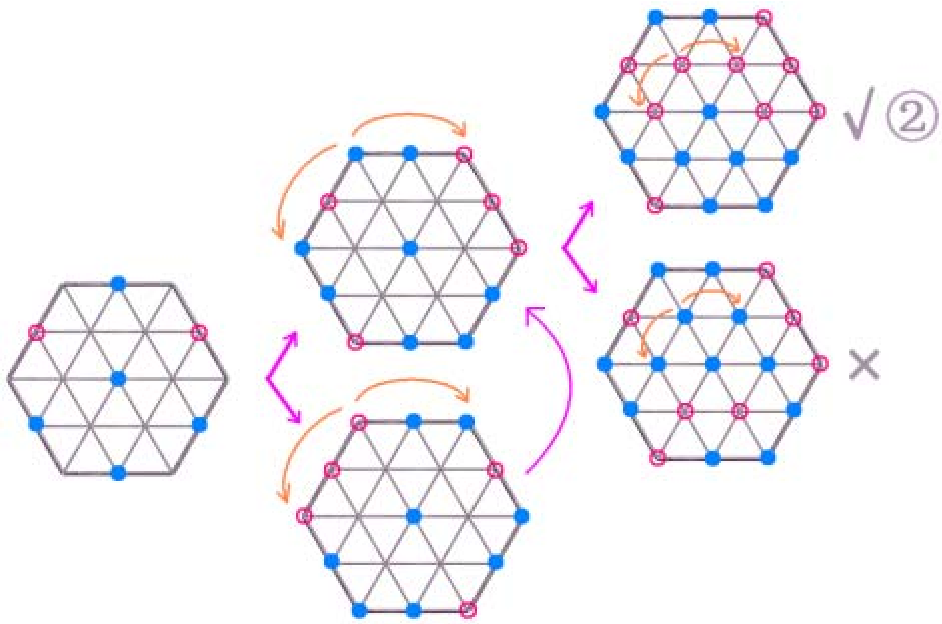
This is a major step; since each line should add up to 38, which is an even number, we can try to analyze whether it is a good restriction or not. From the formulae in step one, we know that A is even, b', Δb and Φ are of the same odd-even property. Out of these sets, Δb and Φ are the easiest to analyze. In the procedure illustrated below, only nine possible distributions can be found.

If Φ is odd, then Δb is odd, possible matches of the two Δb's are 3 odd + 3 odd, 3 odd + 1 odd 2 even, 1 odd 2 even + 1 odd 2 even.

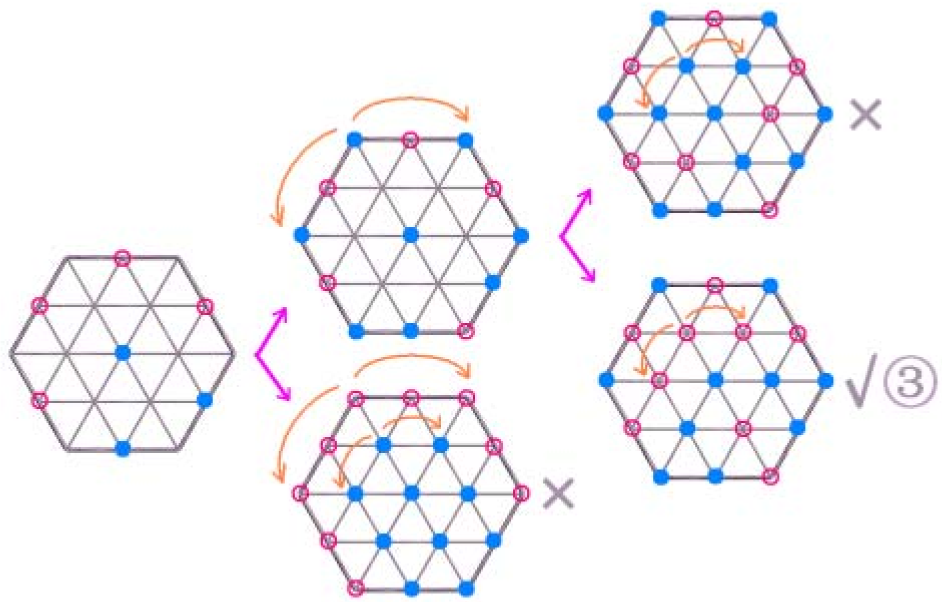
3 odd + 3 odd:



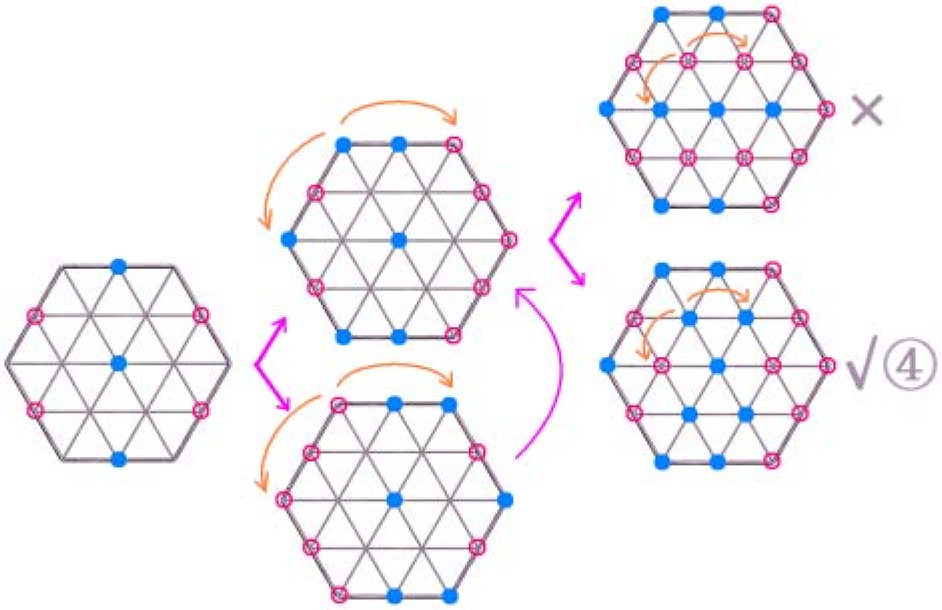
3 odd + 1 odd 2 even:



1 odd 2 even + 1 odd 2 even (ortho-):

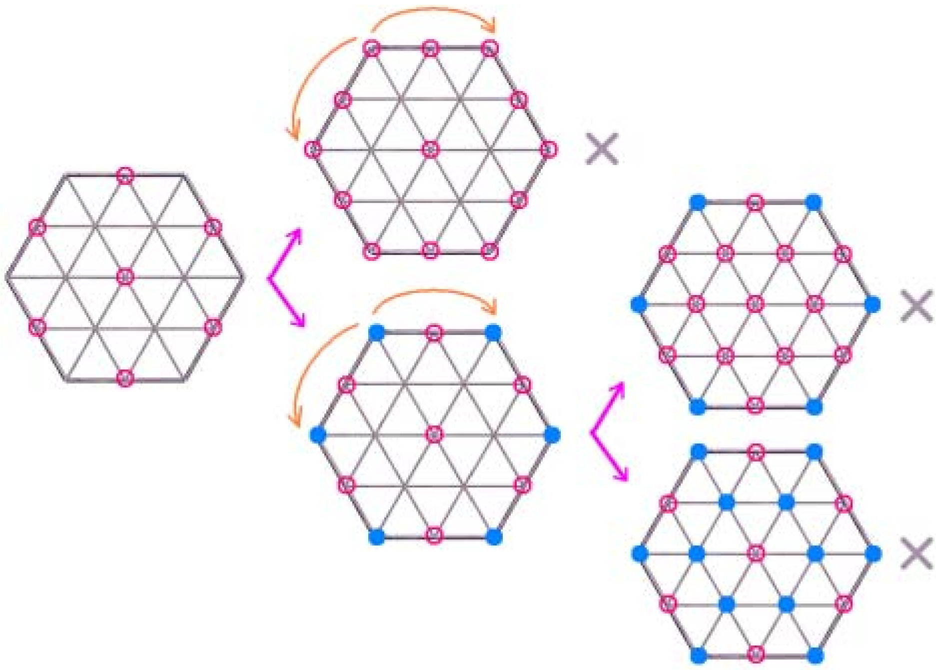


1 odd 2 even + 1 odd 2 even (para-):

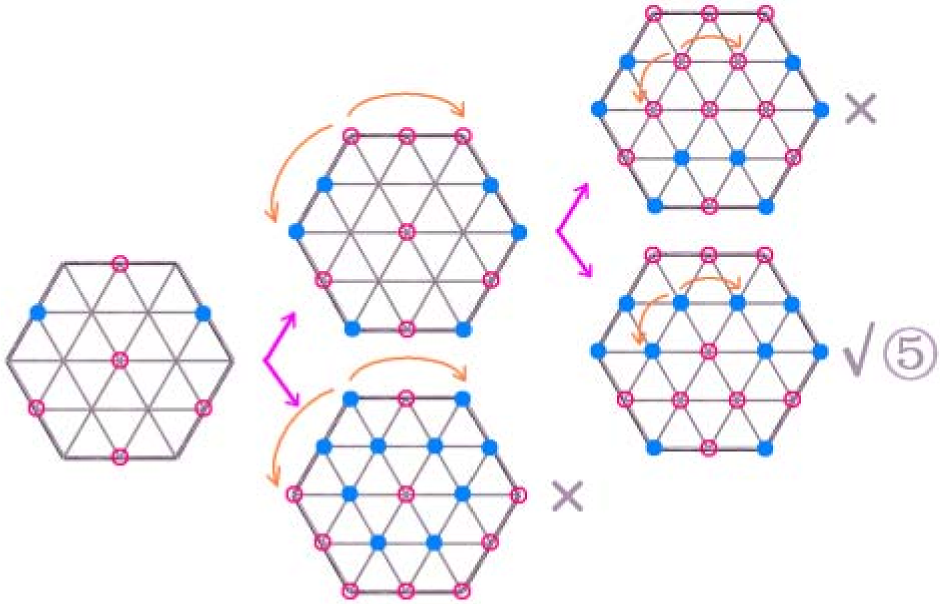


If Φ is even, then Δb is even, possible matches of the two Δb's are 3 even + 3 even, 3 even + 1 even 2 odd, 1 even 2 odd + 1 even 2 odd.

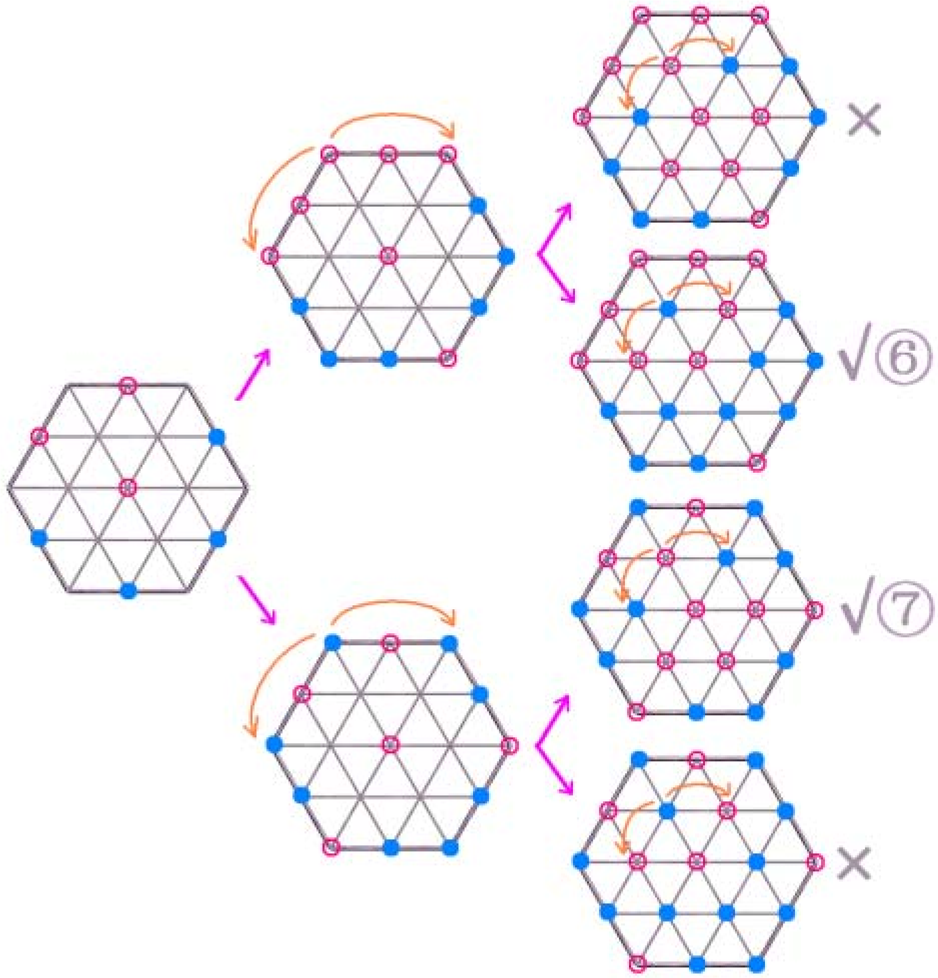
3 even + 3 even:



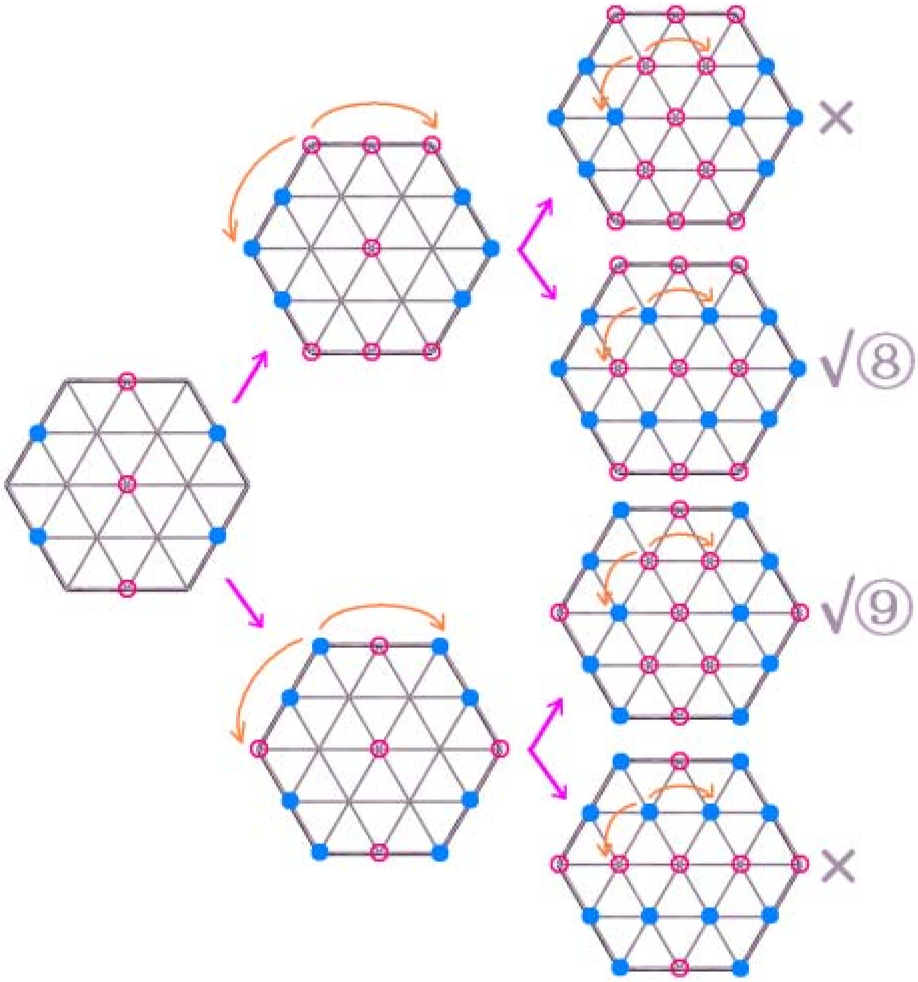
3 even + 1 even 2 odd:

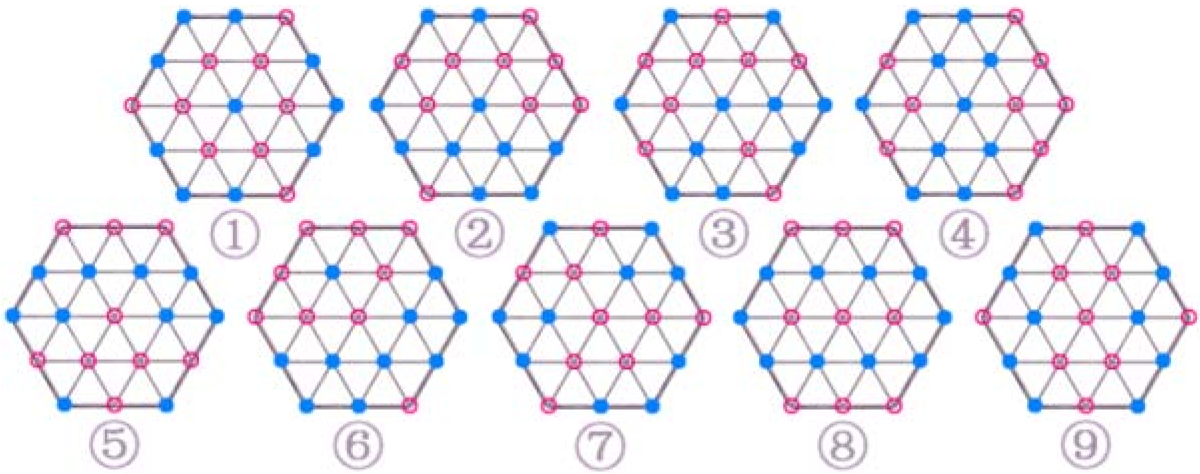


1 even 2 odd + 1 even 2 odd (ortho-):



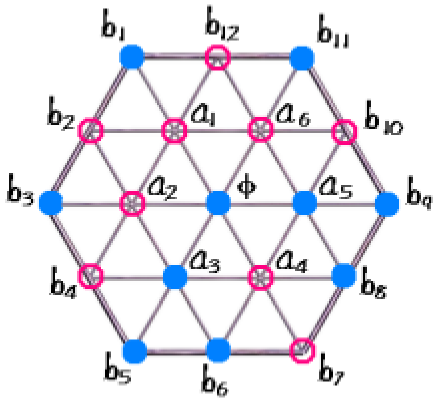
1 even 2 odd + 1 even 2 odd (para-):





Let us first take a look at graph ①. Consider the smallest possible sum of A ring, it is the sum of the first six even numbers $2+4+6+8+10+12=42$. However, the formula $A+2\Phi=38$ tells us A must be smaller than or equal to 36. The outcome of this distribution violates our formula, so it must be eliminated. Similarly, graph ③—⑨ can also be proved to be unusable, as shown below:

Elimination of graph ③:



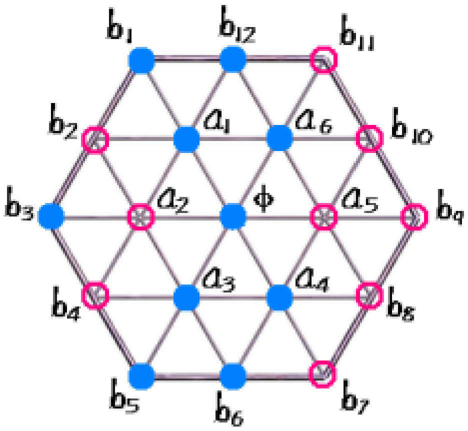
a_1 is the key spot here, so we may try setting a_1 as any even number and see the consequences. Let us first see what happens if $a_1 = 18$.

Since $b_2 + a_1 + a_6 + b_{10} = 38 = b_4 + a_2 + a_1 + b_{12}$, we can set $b_2 + a_6 + b_{10} = X_a$, $b_4 + a_2 + b_{12} = X_b$, then $X_a = X_b = 38 - 18 = 20$.

All even numbers sum to 90, and $90 - (2 \times 20) - 18 = 32$.

We have already used 7 even numbers including 18, and now, as we can see, there are no even-number pair that can sum to 32. This is equivalent to saying that there can be no two sets of triple even numbers both with a sum of 20. So this graph does not work if $a_1 = 18$. If we then set a_1 as 16, 14, 12, 10, 8, 6, 4 or 2, similar outcome results, proving that this graph does not work.

Elimination of graph ④:

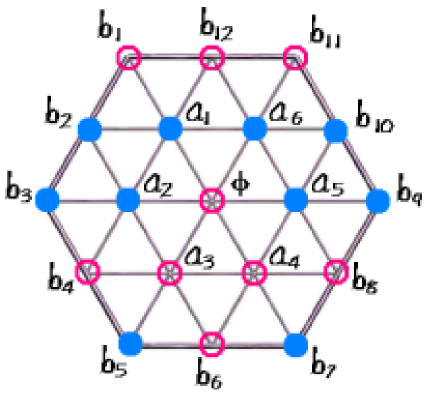


In this graph Φ can be 1, 3, 5 or 7. We may first set $\Phi = 7$, then $A = 24 = 1 + 3 + 5 + 9 + 2 + 4$.

According to symmetrical property No. 3, the number 6 cannot be at any of the Δb positions and should only be at b_9 or b_{11} (which is equivalent to b_7). If 6 is at b_9, we must find two sets of 32s beside it, which is impossible since only $14 + 18 = 32$.

If 6 is at b_{11}, then b_9 and b_{10} require the use of 14 and 18; accordingly b_1 and b_{12} will use 13 and 19, or 15 and 17. We then try to put these four numbers to b_{12} respectively. When $b_{12} = 13$, $b_1 = 19$, we get $a_3 + a_4 = 6 = 1 + 5$, 3 and 9 are left for a_1 and a_6, from symmetrical property we get $b_6 = 19$, which is a repetition of b_1. Letting $b_{12} = 13$, 15 or 17 results the same. Similarly, letting $\Phi = 1$, 3 or 5 does not make the situation any better, so this graph should be eliminated.

Elimination of graph ⑤:



<1>18, 16, 4 and 14, 12, 10, 2, leaving 6, 8;

<2>18, 16, 4 and 14, 10, 8, 6, leaving 2, 12;

<3>18, 14, 6 and 16, 10, 8, 4, leaving 2, 12;

<4>18, 12, 8 and 16, 14, 6, 2, leaving 4, 10;

<5>16, 14, 8 and 18, 12, 6, 2, leaving 4, 10;

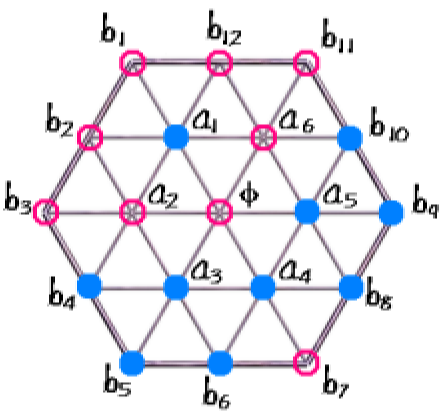
<6>16, 14, 8 and 18, 10, 6, 4, leaving 2, 12;

<7>16, 12, 10 and 18, 14, 4, 2, leaving 6, 8;

There are only 7 combinations that satisfies both row 1 and row 4:

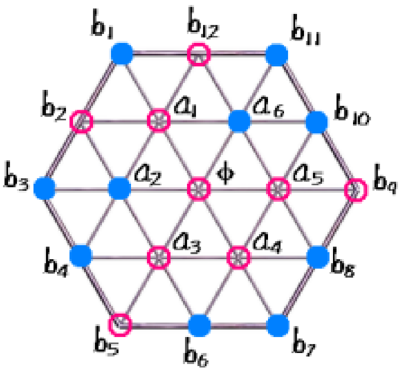
We can first take situation <1> as an example. The minimum value of $a_3 + a_4$ will be $10 + 2 = 12$, which implies that Φ cannot be 8, otherwise $b_{12} = 20$. So $\Phi = 6$, $b_6 = 8$, which requires $a_1 + a_6 = 2$ and is impossible. This possibility is thus excluded. Situations <2>～<7> produce similar results, proving that this graph is unusable.

Elimination of graph ⑥:



In this graph Φ can be 2, 4, 6 or 8. First take $\Phi = 8$ as an example. Now $A = 22 = 1 + 3 + 5 + 7 + 2 + 4$; in the left over even numbers, only two sets of triple numbers which sum to 38 can be found: $18 + 14 + 6$ and $16 + 12 + 10$; they have no common number, thus does not satisfy the requirement of b_1, so $\Phi = 8$ does not work. $\Phi = 2$, 4 or 6 produces similar result, showing that this graph should be eliminated.

Elimination of graph ⑦:



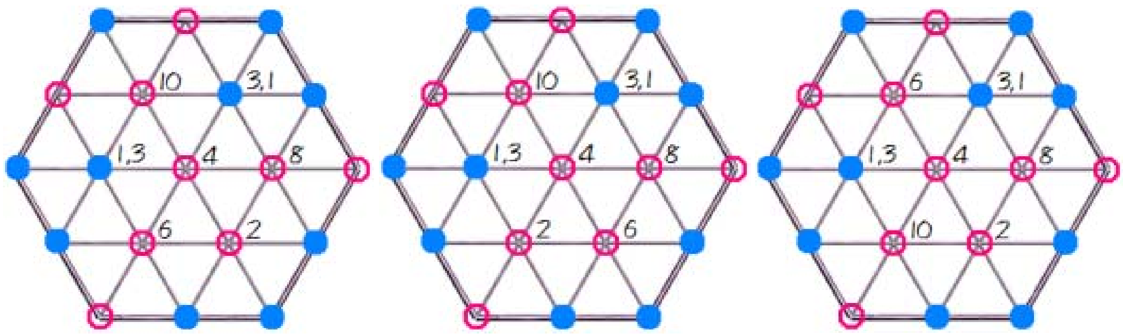
Since $A + 2\Phi = 38$, and that there are 5 even numbers in Φ and A, we can see that only three combinations are possible:

$$\Phi = 2, A = 4 + 6 + 8 + 10 + 1 + 5$$

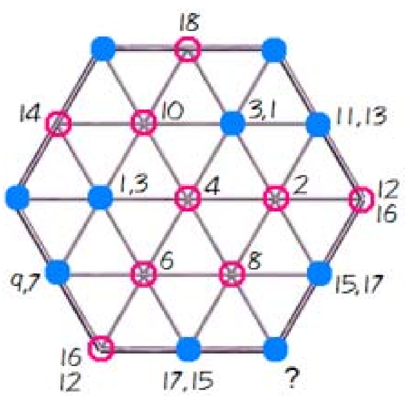
$$\Phi = 2, A = 4 + 6 + 8 + 12 + 1 + 3$$

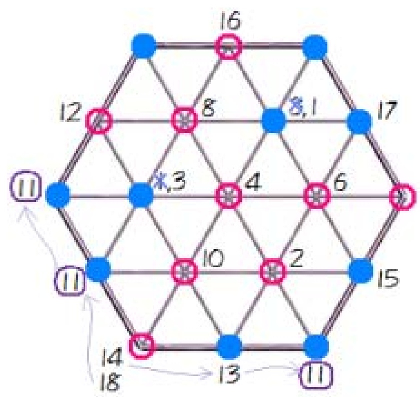
$$\Phi = 4, A = 2 + 6 + 8 + 10 + 1 + 3$$

We may take $\Phi = 4$ as an example. If the number 10 is at a_4, then either b_1 or b_{12} will be larger than 19, so 10 can only be at a_1 or a_3 (in this circumstance a_4 must be 2):



No matter how we arrange 1 and 3, repetition of numbers will occur on Δb.

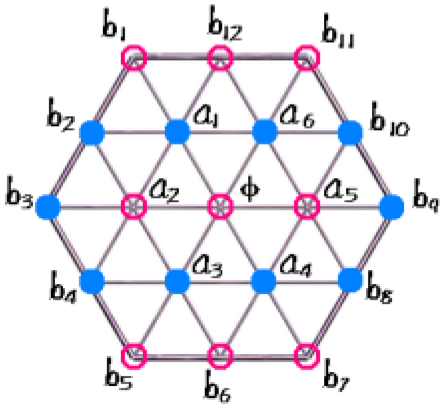




There does not exist a possible b_7. Repetition of numbers will occur.

Letting $\Phi = 2$ leads to similar results, so this graph is unsuitable.

Elimination of graph ⑧:



To satisfy the requirements of row 1 and row 5, there is only one possible combination: 6, 14, 18 with 10, 12, 16. Then 2, 4, 8 should be at Φ, a_2, a_5. First set $\Phi = 8$, then $A = 38 - 16 = 22 = 1 + 3 + 5 + 7 + 2 + 4$, the four odd numbers have three ways to combine:

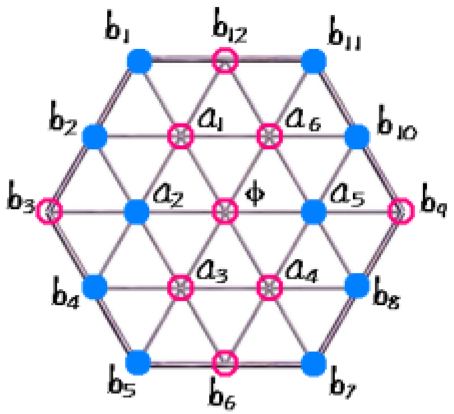
a_1, a_6: 1, 3 $\Rightarrow b_6 = 12$ a_3, a_4: 5, 7 $\Rightarrow b_{12} = 20$;

a_1, a_6: 1, 5 $\Rightarrow b_6 = 14$ a_3, a_4: 3, 7 $\Rightarrow b_{12} = 18$, but 14 and 18 are already in the same row;

a_1, a_6: 1, 7 $\Rightarrow b_6 = 16$ a_3, a_4: 3, 5 $\Rightarrow b_{12} = 16$, 16 are used twice;

So the possibility of $\Phi = 8$ is excluded. Similar contradiction occurs if we set $\Phi = 2$ or 4. This graph is thus eliminated.

Elimination of graph ⑨:



Since $A + 2\Phi = 38$, and that there are 5 even numbers in Φ and A, we can see that only three combinations are possible:

$\Phi = 2, A = 4 + 6 + 8 + 10 + 1 + 5 \Rightarrow b_3 + b_9 = 30 = 12 + 18$ or $14 + 16$

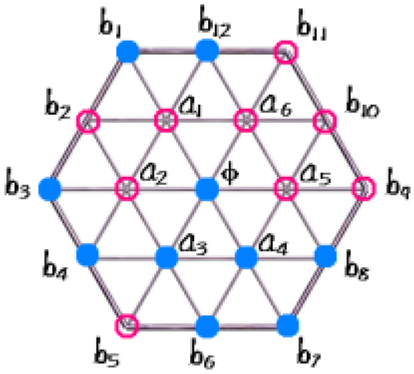
$\Phi = 2, A = 4 + 6 + 8 + 12 + 1 + 3 \Rightarrow b_3 + b_9 = 32 = 14 + 18$

$\Phi = 4, A = 2 + 6 + 8 + 10 + 1 + 3 \Rightarrow b_3 + b_9 = 30 = 12 + 18$ or $14 + 16$

For each case, no matter how the four even numbers in A ring arrange, repetition of numbers among b_6, b_{12} and b_3, b_9 results, implying that this distribution is unusable.

Step Four: Construction

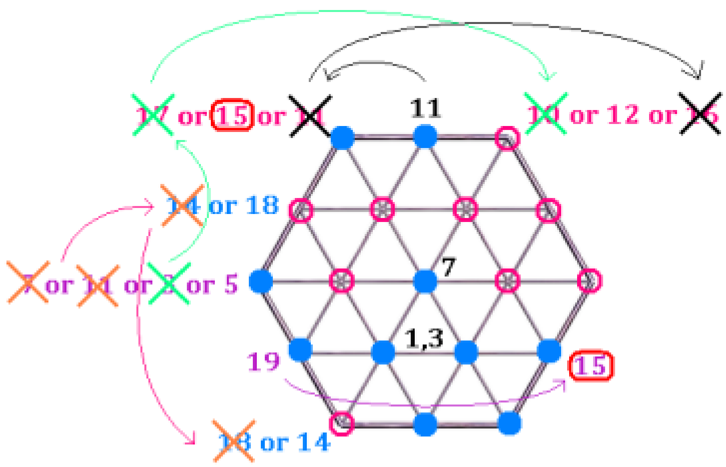
So here is the only possible distribution. Since the smallest sum of the A ring in this distribution is $2+4+6+8+1+3=24$, the largest possible Φ is 7. Now we can write down a table of the possible values of each number set.



Φ	A	Δb	b'
1	36	39	75
3	32	41	73
5	28	43	71
7	24	45	69

Only four configurations are left for us to examine.

Let us first take $\Phi=7$ and see if everything can work out. Now $A=38-2\Phi=24=2+4+6+8+1+3$, 1 and 3 must be at a_3 and a_4, then $b_{12}=7+1+3=11$. The left-over even numbers are 10, 12, 14, 16, 18; within them only $10+12+16=38$, now set b_{11} as any of the three numbers and then calculate anti-clockwise. When we get to b_2, only 14 and 18 are left for us to choose from. Continue processing, we find that b_1 and b_8 both require 15 to be there, which violates the basic requirement of the magic hexagon, thus $\Phi=7$ does not satisfy.

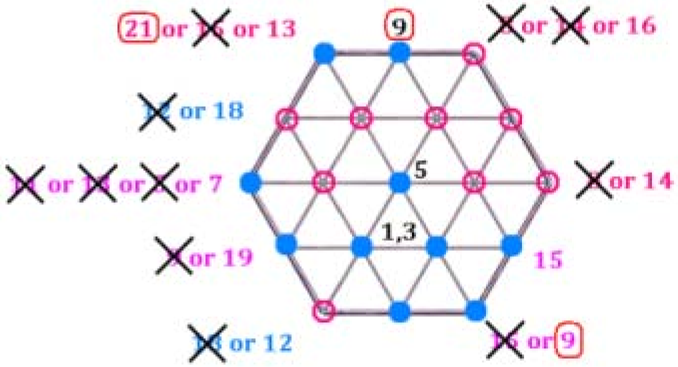


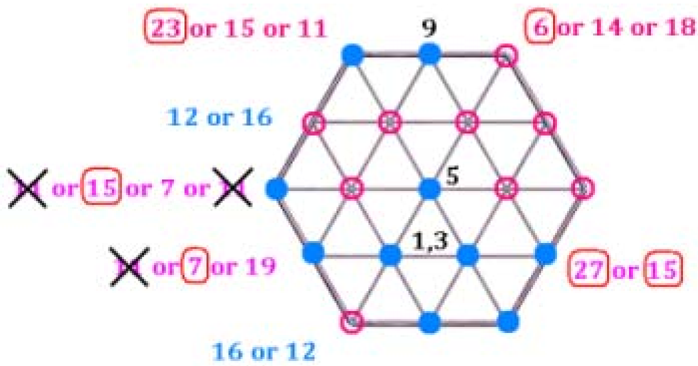
Similarly, $\Phi=1$ or 3 are also unsatisfactory, 5 is our last chance. When $\Phi=5$, there are 3 possible sets of A ring:

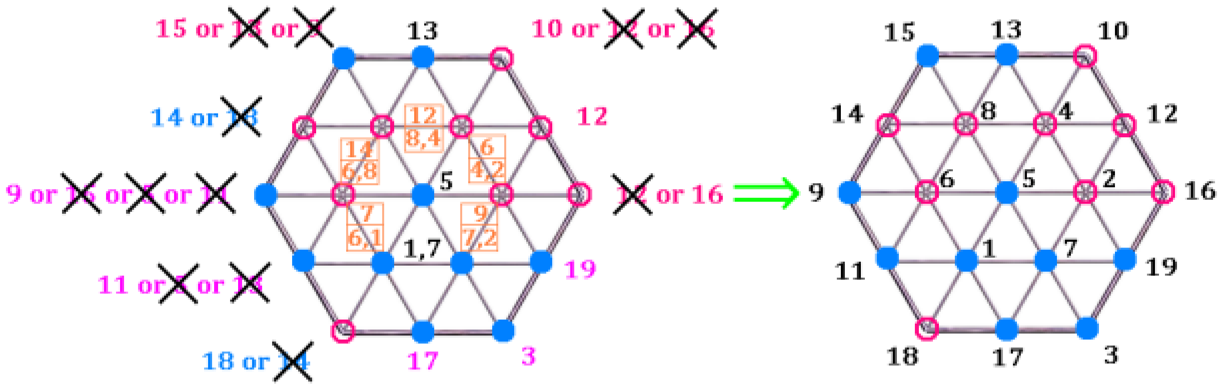
<1> $A=2+4+6+12+1+3$, the three even numbers are 8, 14, 16, leaving 12 and 18;

<2> A = 2 + 4 + 8 + 10 + 1 + 3, the three even numbers are 6, 14, 18, leaving 12 and 16;

<3> A = 2 + 4 + 6 + 8 + 1 + 7, the three even numbers are 10, 12, 16, leaving 14 and 18.







We have got to the correct solution! A more significant achievement is that, through all our steps, all the possible configurations have been examined, which means this result is essentially unique. We have solved the two problems simultaneously and easily.

Order 4 Magic Hexagons

Looking for the solutions to higher order magic hexagons is a real difficult problem for computers. For now, the best result is an order 7 magic hexagon using 2–128, created by

Zahray Arsen in 2007, using multiple computers (parallel computing) with a traversal algorithm. This suggests that any further achievement using this methodology will be constrained by the speed of computers. This is not good news for mathematics, which is considered to be logical, structural and insightful. Currently, I do not have a complete solution to these higher order magic hexagons, but some progress on order 4 magic hexagons has been made. For example, I have found 8 formulae;

Here:

$A = a_1 + a_2 + a_3 + a_4 + a_5 + a_6$

$b' = b_1 + b_3 + b_5 + b_7 + b_9 + b_{11}$

$b'' = b_2 + b_4 + b_6 + b_8 + b_{10} + b_{12}$

$\Delta b_1 = b_2 + b_6 + b_{10}$

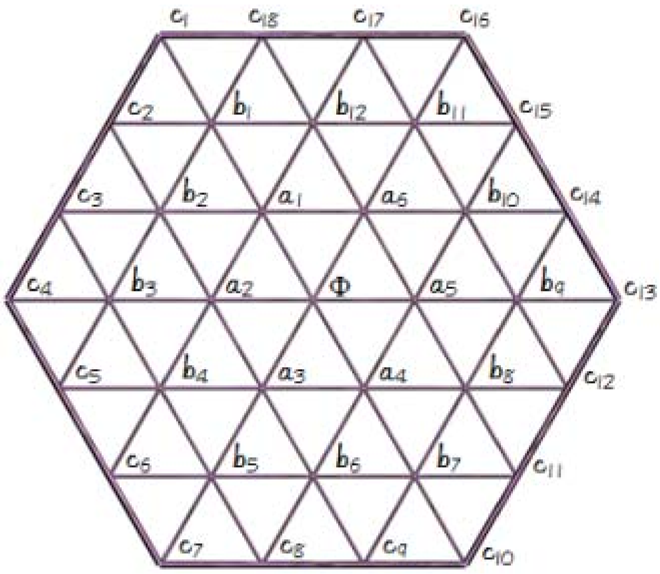
$\Delta b_2 = b_4 + b_8 + b_{12}$

$c' = c_1 + c_4 + c_7 + c_{10} + c_{13} + c_{16}$

$c'' = c_2 + c_3 + c_5 + c_6 + c_8 + c_9 + c_{11}$
$+ c_{12} + c_{14} + c_{15} + c_{17} + c_{18}$

$\Delta c_1 = c_2 + c_3 + c_8 + c_9 + c_{14} + c_{15}$

$\Delta c_2 = c_5 + c_6 + c_{11} + c_{12} + c_{17} + c_{18}$



We can get:

(1) $\Phi + b' = 111$

(2) $\Delta b_1 = \Delta b_2 = \Delta b$

(3) $A + b'' = c'$

(4) $A + c' = 2b' = 222 - 2\Phi$

(5) $b'' + c'' = 444 + 2\Phi$

(6) $2c' + c'' = 666$

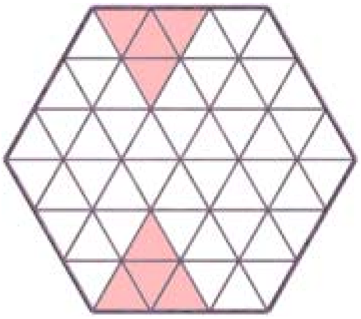
(7) $\Phi + A + \Delta b = 111$

(8) $\Delta c_1 = \Delta c_2 = \Delta c$

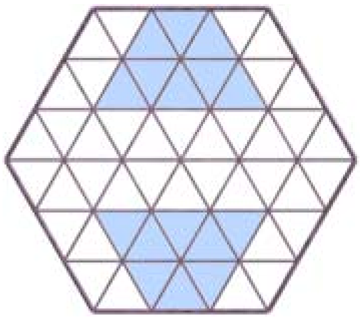
Some restrictions of each number set:

$$\begin{cases} \Phi \geq 3 \\ A \geq 33 \\ \Delta b \leq 75 \\ b' \leq 108 \\ c' \leq 183 \\ \Delta c \geq 150 \end{cases}$$

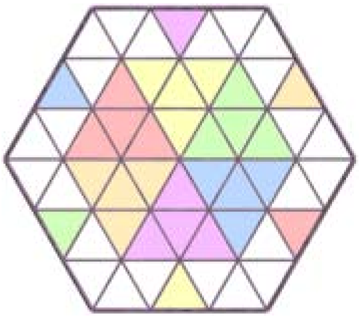
And 7 kinds of symmetry:



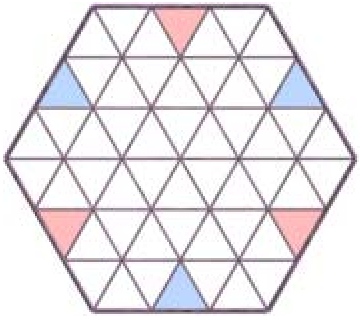
The sum of the 6 numbers on the vertices of a colored triangle equals to that of the other colored triangle (6 pairs in total).



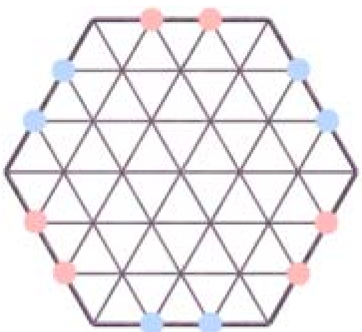
The sum of the 9 numbers on the vertices of a colored trapezoid equals to that of the other colored trapezoid (3 pairs in total).

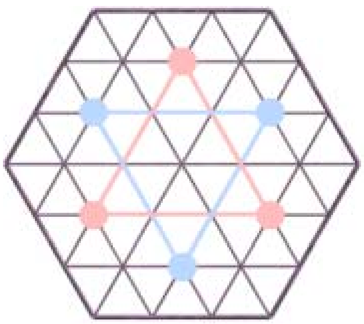


The sum of the 6 numbers on the vertices of an inside colored triangle equals to that of the outer same-colored triangle.

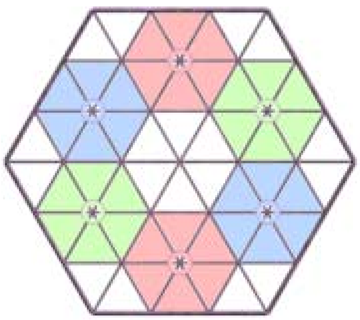


The sum of all the numbers on the vertices of the red-colored triangles equals to that of the blue-colored triangles.



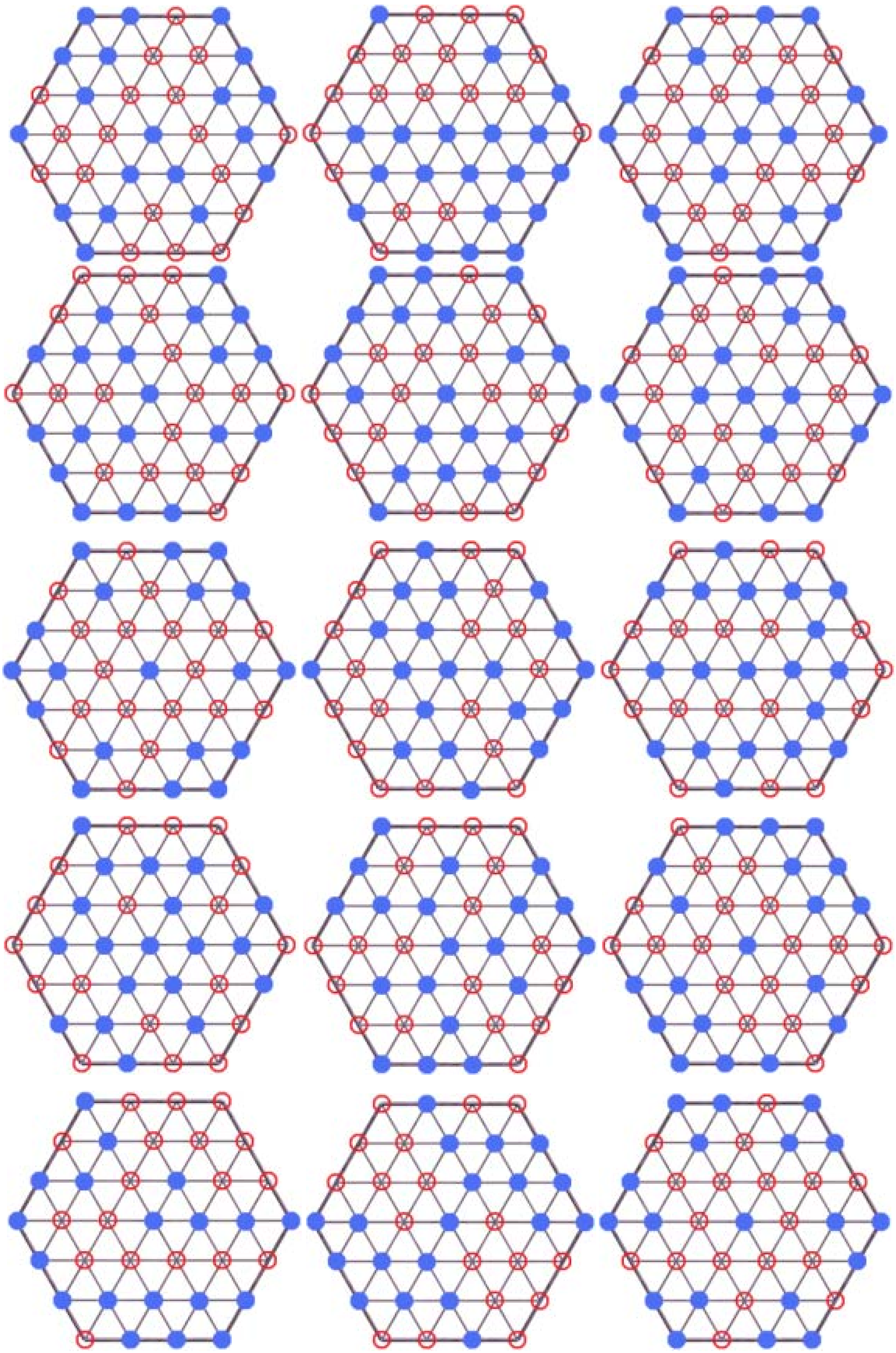


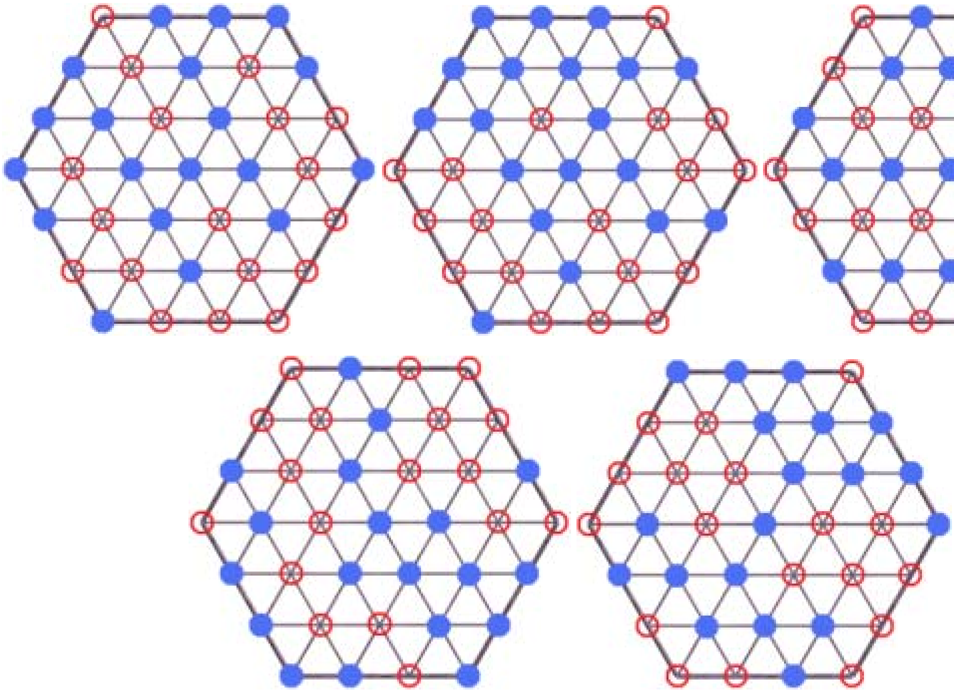
The sum of all the numbers on the red colored places equals to that of the blue colored ones (which means the same as formula 2 and 8).



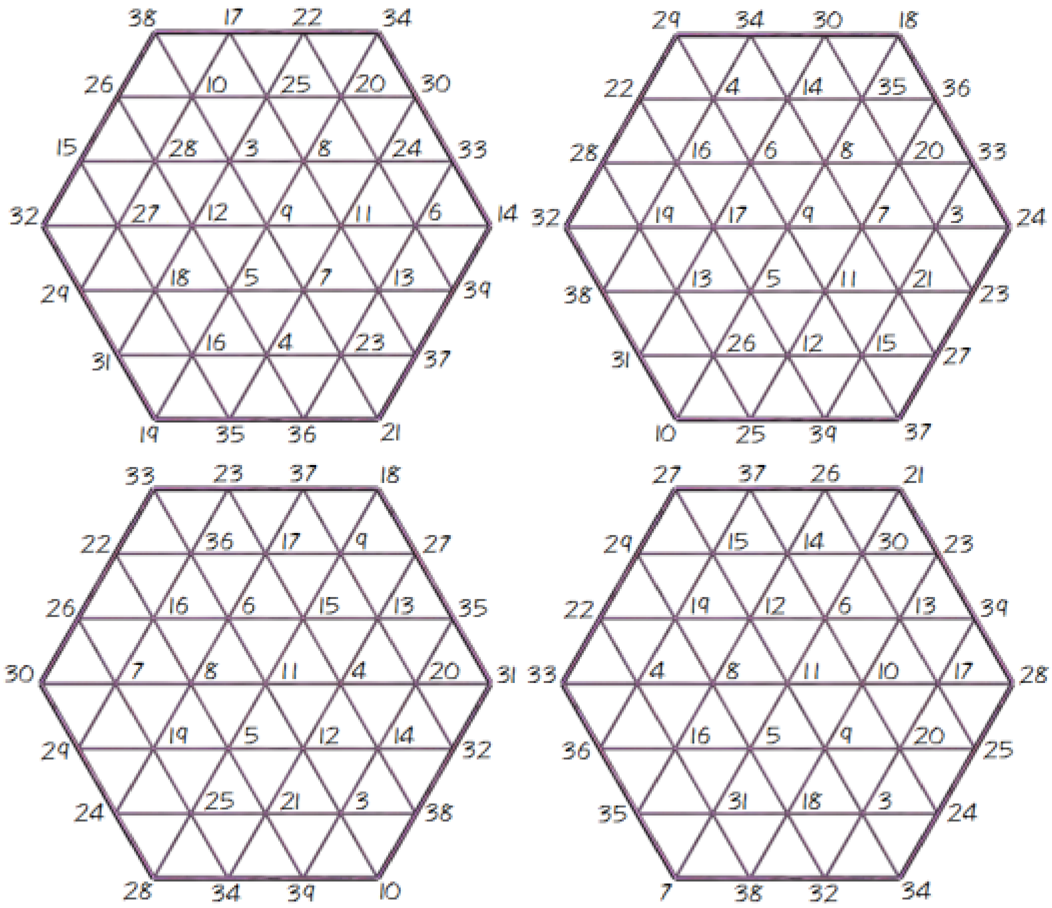
Within each colored hexagon, the sum of the 6 vertices plus twice the value of the inner dot equals that of the other same-colored hexagon.

With 20 possible odd-even distributions:





Here are four completed order 4 magic hexagons that can be found:



From these four configurations, we can find two other properties of the order 4 magic hexagon. The first one is that there exist 4 adjacent numbers in A ring whose sum is 31, for example $5+7+11+8=31$， $5+11+7+8=31$， $6+8+5+12=31$， $6+12+$

$8+5=31$. The second property is that, if the difference of A and Δb is multiples of 3 (including 0), then the difference divide by 3 plus any one number on Δb equals to the sum of the two numbers on the opposite side of A ring. This is actually an extension of symmetrical property type 3 of order 3 magic hexagon. Because in order 3 we have $\Delta b = A + 3\Phi$, which means the difference between A and Δb is 3Φ, so a number on Δb equals to the sum of Φ and the two numbers on the opposite side of A ring. For example, in the last graph, $A = 50$, $\Delta b = 50$, the components of Δb are $14 = 5 + 9$, $19 = 10 + 9$, $16 = 10 + 6$, $18 = 12 + 6$, $20 = 8 + 12$, $13 = 8 + 5$.

To find all solutions, a possible approach is to compile a program that can execute the following operations:

1. Set Φ as an arbitrary value, calculate $b' = 111 - \Phi$;
2. Assign a number set $S = \{3, 4, \ldots, 38, 39\} - \{\text{the value of } \Phi\}$;
3. Within S, list all 6-number groups $\{axis_i\}$ whose sum is b';
4. Within $\{axis_i\}$, list all combinations of 3 non-overlapping groups $\{\{x_i\}, \{y_i\}, \{z_i\}\}$;
5. Within $\{\{x_i\}, \{y_i\}, \{z_i\}\}$, pick 2 numbers from each sub-group, their sum should equal to b', label them as b_{x1}, b_{x2}, b_{y1}, b_{y2}, b_{z1}, b_{z2}. If that kind of combination could not be found, then this whole group should be eliminated;
6. Again, pick 2 other numbers from each sub-group, their sum should satisfy the value of A, label them as a_{x1}, a_{x2}, a_{y1}, a_{y2}, a_{z1}, a_{z2}. The rest should be labeled as c_{x1}, c_{x2}, c_{y1}, c_{y2}, c_{z1}, c_{z2}. Similarly, if such combination could not be found, then this whole group should be eliminated;
7. Since symbols with subscripts 1 and 2 are interchangeable, so for each 18-number group, there are $2^9 \div 2 \times 2 = 256$ different configurations ($\div 2$ to cancel out mirror images, $\times 2$ to change the sequence of axis);
8. For each configuration, the program should then work out the values on other spots by using the symmetrical properties and formulae mentioned above. If a number ever appears more than once, that configuration should be eliminated, leaving us with correct solutions. Compared with the simple traversal algorithm, this method eliminated most of the ineffective calculations, and is thus more effective.

Discussion

Through the above analysis and procedure, we have investigated several problems relating to the order 3 magic hexagon. For example, the problem of listing – how many arrangements there are, the structural problem – what properties it has, how efficient the algorithm can be, whether an arrangement exists if the restrictions became more rigorous. These problems are what combinational mathematics concerns. In a word, with the above mathematical principles, the magic hexagon has now stepped into the field of combinatorial mathematics.

1. Although in the formula derivation step we started with 15 equations, they are not all linear independent; in fact, all equations and formulae are linear combinations of only 12 linear equations. This means that unless we knew seven exact numbers, there is no simple way of solving this system using methods of linear algebra.

2. As a graph symmetrical about its center, an odd number occupies this point. This also happens to other magic hexagons and magic squares, each of which has a center and whose numerical constituents have different quantities of odd and even numbers. The larger group ones get to be at the center. Is there a name for this principle?

3. There are 10 odd numbers and 9 even numbers, all the odd numbers sum to 100, all the even numbers sum to 90. These two number sets are not symmetrical.

I came up with an explanation why all the symmetrical distributions were eliminated; I called it the "superimposition of symmetry", that is, since the distribution describes odd-even property, they can be superimposed, the rule is:

Symmetrical + Symmetrical = Symmetrical

Symmetrical + Asymmetrical = Asymmetrical

Asymmetrical + Symmetrical = Asymmetrical

Asymmetrical + Asymmetrical = Symmetrical

(Here "+" means "superimposed to".)

Since odd and even numbers are naturally asymmetrical, and that the resultant magic hexagon is perfectly symmetrical, we can easily conclude that the distribution must be an asymmetrical one.

4. In analyzing the properties of higher order magic hexagons, we may adopt a virtual filling method. This method uses arbitrary numbers and repeated use is allowed. The only requirement is to make sure that each line adds to the same value as the real magic hexagon. This step may provide provision in deriving formulae of magic hexagons of orders higher than four. From observing the properties of the easy-to-construct virtual one, we can find some clues of the properties of the real one. We then just have to put the relevant initial equations together and actually prove the property, since it is much easier to test whether a statement is true than to find it out starting from the middle of nowhere, just as the P vs NP problem describes.

5. By analyzing the few higher order magic hexagons that can be found on the internet, I found a tricky trend among them:

Order	Number of Rows	Magic Sum	Sum of All Numbers	Qty. of All Numbers	$\frac{\textbf{Magic Sum}}{\textbf{Qty. of All Numbers}} = \frac{\textbf{Average of All Numbers}}{\textbf{Number of Rows}}$
3	5	38	190	19	**2**
4	7	111	777	37	**3**
5	9	244	2196	61	**4**
6	11	546	6006	91	**6**
7	13	635	8255	127	**5**

The last column shows 5 consecutive integers, which leads to some other questions: Is there a “highest possible order”? Is there a systematic solution that can work it out? Or that magic hexagons of any order is possible? If then, how to prove it?

6. If we are to optimize a numerical structure with geometrical features, such as the construction of a magic hexagon, is it impossible to have a universal, non-traversal algorithm if the scale of the structure increases without bound?

7. Although the magic hexagon is a 2-dimensional shape, in construction, it behaves more like a 3-dimensional object, as each position is restricted by three independent directions. Therefore, if we can extend the planar magic hexagon into the third dimension, we may construct a virtually higher dimensional structure in the 3D world. If that happens, we may be able to investigate some properties of higher dimensions in a more obvious way.

8. Magic hexagons are unique in many aspects, such as in their complexity (high order ones), isotropy, and geometry. Based on these properties, and through thought experiments, I came up with some possible applications.

(1) Password Systems

Since none of the magic hexagons of orders higher than 7 has been found, we can use the diagrams of still higher order magic hexagons as the “lock”. The owner will first fill in the numbers 1 to $3n^2 - 3n + 1$ randomly, then calculate the sum of each line, and input them to the confirmation mechanism. In this way, a cracker, restrained by the speed of the computers on earth, cannot reach to the solution in sufficiently short time. What is more, we can always leave a gap between the math achievement and the password system.

(2) Architecture: Roof Structure for Large-scale Buildings

Both diagrams can be bent in the third dimension, generating domes made up of rods and joints. Similar to Fuller’s dome structure, these two also have the properties of being light and strong. Another kind of structure mimics a bird’s nest. In the modified graph, the Δb is an obvious component. If we single it out, rotate it and stack it many times, we can get a bird’s nest structure. In this configuration most of the material is distributed on the outside where the largest load is to be supported. On the inside, however, they support less material, and they are supported by what is underneath; both require less weight and less strength, so minimum amount of material is used there.

(3) Composite Material

We can put hexagonal disc springs between two steel boards, making a sandwich-type board. The disc springs can have different stiffness, and can be arranged in a magic hexagon fashion. In this way, the composite board will not only absorb strong impact resulting from heavy storms in the ocean or cannon shots on the battlefield, but also prevent resonance. This combination is not a simple one including only materials, but an integration of mechanical elements into ordinary material.

(4) National Security Systems

Most weapons fire along a straight line. Therefore, it is natural for us to assume that the power of most weapons, within their range, can be added linearly. If we can rate the power of a weapon with a number, and then arrange the weapons of different power in the form of a magic hexagon, with numbers matching power, we can create a fire net of equal strength, which has no weak spots; terrorists will not succeed in breaking the defense system. Higher order magic hexagons will make the system more realistic.

Where else can magic hexagons be used? A professor from Shantou University once said that ordinary magic squares can be used in fields such as artificial intelligence, graph theory, game theory, experiment design, electric circuit theory, feedback control, analysis situs, algorithm improvement, parallel processing, economic dispatch decision, security processing of graphics and possibility estimation of engineering works. But unfortunately, no further information was provided and no relevant information could be found on the internet. If the statement is indeed true, then I think magic hexagons will have more advantages over magic squares, because as mentioned above, they contain one more dimension, which may provide convenience in processing real world (3-dimensional world) problems.

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