Equivalency condition of symmetric inequalities

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Abstract: This paper researches on the judgment theorem and proof of the equivalency condition of a class of symmetric inequalities. By controlling two elementary symmetric polynomials and using the monotonicity of functions and Jensen inequality, it finds the necessary and sufficient condition of the equivalency a class of three-variable and n-variables symmetric inequalities. And we illustrate the application of this method in proof of these inequalities. Then we obtain several judgment theorems on symmetric and cyclic inequalities.

Key words: inequality; symmetric; cyclic; three-variables; n-variables; Judgment theorem of the equivalency condition

完全对称不等式的取等判定

摘要:本文探讨了一类完全对称不等式的取等判定及其证明. 通过控制两个初等对称多项式，利用函数的单调性及 Jensen 不等式，说明了一类三元完全对称不等式取等的充要条件，继而推广得到关于一类 n 元完全对称不等式取等的充要条件的若干定理，并举例说明此方法在证明完全对称不等式中的应用. 由此推出有关完全对称不等式与轮换对称不等式的判定定理.

关键词：不等式；完全对称；轮换对称；三元； n 元；取等判定

1 Introduction

The inequality has the wildly application in mathematics and other sciences, but to prove a inequality, there is no general method and fixed way, especially for difficult inequalities. Usually it doesn't work by enlarging or reducing directly. The full symmetry inequality, because of its especial property, has become an active branch of this field. Some researchers have used derivative method, the increment method, variable controlled method or local revision method to deal with inequalities of this type[1-3]. However, there may be a lot of computations, and often do not work successfully. Academician Yang Lu, Mr. Chen Shengli, Mr. Yao Yong, Mr. Liu Baoqian etc. have done many works in this field by using the computer as a tool. In 1985, in a conference held in Shanghai, Academician Wu Wenjun had point out that the automated proving for inequalities is a difficult problem[4]. In 1982, Choi etc. obtained the judgment of the necessary and sufficient condition for the semi-positive definiteness of a symmetric form of degree 3 with n variables[5]. In 1999 William Harris gave a necessary and sufficient condition for the semi-positive definiteness of a symmetric form of degree 4 and 5 with 3 variables[6]. Notice that the degrees of these results no more than 5. In 2001 Vlad Timofte considered the necessary and sufficient condition for the semi-positive definiteness for symmetric forms of degree d with n variable in R_+^n. But his result is difficult to be judged when d>5[7]. In 1993 Chen Shengli deeply discussed the semi-positive definiteness for more general symmetric forms with 3 variables[8]. Now it is still an unsolved problem to judge the semi-positive definiteness of the symmetric form of degree 6(or higher degree) with n variables[9]. So far, there is no report on exploring the equivalency condition of symmetric inequalities and proving an inequality using the equivalency condition in China. The aim of our research is to explore the equivalency condition of symmetric inequalities and give a theorem of the judgment of the equivalency condition for 3 and n variables, then we get a judgment theorem for homogeneous fully symmetric forms of degree 6 with 3 variables and for homogeneous cyclic symmetric inequalities of degree 4 with 3 variables and try to explore the judgment for inequalities of higher degree, which can be used for the exploration of the method to prove inequalities by hand and supply a basis for the automated proving of inequalities.

2 A judgment theorem of equivalency condition for some fully symmetric inequalities with 3 variables

2.1 The judgment theorem of equivalency condition and its proof

We firstly introduce the properties for fully symmetric inequalities with 3 variables.

Lemma 1 A polynomial $f(x,y,z)$ with 3 variables is fully symmetric if and only if $f(x,y,z)$ can be expressed uniquely by basic polynomials $\sigma_1=\sum x=x+y+z$, $\sigma_2=\sum xy=xy+yz+zx$, $\sigma_3=\prod x=xyz$ ($\sum$,$\prod$ denote the sum and times) (from the fundamental theorem for symmetric polynomials in linear algebra). Set $f(x,y,z)=g(\sum x,\sum xy,xyz)$.

Lemma 2 A fully symmetric polynomial $f(x,y,z)$ with 3 variables can be expressed uniquely by $\sigma_1=\sum x$, $\sigma_2=\sum x^2$, $\sigma_3=\sum x^3$.

Lemma 3 A fully symmetric polynomial $f(x,y,z,t)$ with 4 variables can be expressed uniquely by $t^n g(\sum\frac{x}{t},\sum\frac{xy}{t^2},\frac{xyz}{t^3})$.

Next we give the judgment theorem of equivalency condition for fully symmetric inequalities with 3 variables and its proof.

Theorem 1 For any real number a,b,c, we have

$$\frac{\sum ab\sum a+(6\sum ab-2(\sum a)^2)x_1}{9}\le abc\le\frac{\sum ab\sum a+(6\sum ab-2(\sum a)^2)x_2}{9}$$

where $x_1=\frac{\sum a+\sqrt{(\sum a)^2-3\sum ab}}{3}$, $x_2=\frac{\sum a-\sqrt{(\sum a)^2-3\sum ab}}{3}$.

The equalities hold if and only if $(a-b)(b-c)(c-a)=0$.

Proof: Suppose real number a,b,c satisfy $c\ge b\ge a$.

Consider the function $f(x)=(x-a)(x-b)(x-c)=x^3-\sum ax^2+\sum abx-abc$.

Then $f'(x)=3x^2-2\sum ax+\sum ab$.

Let x_1,x_2 be two roots of $f'(x)=0$ with $x_1\ge x_2$. Then it is easy to get

$$x_1=\frac{\sum a+\sqrt{(\sum a)^2-3\sum ab}}{3},\ x_2=\frac{\sum a-\sqrt{(\sum a)^2-3\sum ab}}{3}\qquad(1)$$

If $x_1 > x_2$, then $f(x)$ is monotone increasing on $(-\infty, x_2]$, monotone decreasing in $(x_2, x_1]$ and monotone increasing $(x_1, +\infty)$.

Meanwhile $f(x)$ has three zeros, that is, $f(a)=0$, $f(b)=0$, $f(c)=0$.

Hence $a \le x_2 \le b \le x_1 \le c$. Then $f(x_2) \ge 0, f(x_1) \le 0$, that is,

$$x_2^{\ 3} - \sum a x_2^{\ 2} + \sum ab x_2 - abc \ge 0 , \quad x_1^{\ 3} - \sum a x_1^{\ 2} + \sum ab x_1 - abc \le 0 .$$

By substituting (1) we obtain

$$\frac{\sum ab \sum a + (6\sum ab - 2(\sum a)^2) x_1}{9} \le abc \le \frac{\sum ab \sum a + (6\sum ab - 2(\sum a)^2) x_2}{9},$$

and any equality holds if and only if $(a-b)(b-c)(c-a)=0$. This completes the proof of Theorem 1.

Corollary 1 For any real number a,b,c, we have

$$\max\left(\frac{\sum ab \sum a + (6\sum ab - 2(\sum a)^2) x_1}{9}, 0\right) \le abc \le \frac{\sum ab \sum a + (6\sum ab - 2(\sum a)^2) x_2}{9} .$$

Proof. If $\sum a^2 \le 2\sum ab$, then

$$\frac{\sum ab \sum a + (6\sum ab - 2(\sum a)^2) x_1}{9} \le abc \le \frac{\sum ab \sum a + (6\sum ab - 2(\sum a)^2) x_2}{9} .$$

If $\sum a^2 \ge 2\sum ab$, then $0 \le abc \le \frac{\sum ab \sum a + (6\sum ab - 2(\sum a)^2) x_2}{9}$.

Combining these two cases we have

$$\max\left(\frac{\sum ab \sum a + (6\sum ab - 2(\sum a)^2) x_1}{9}, 0\right) \le abc \le \frac{\sum ab \sum a + (6\sum ab - 2(\sum a)^2) x_2}{9} .$$

This completes the proof of Corollary 1.

Inequalities in Theorem 1 and Corollary are very strong and have many applications in prove inequalities.

Theorem 2 For a fully symmetric inequality $f(\sum a, \sum ab, abc) \ge 0$ about real umbers a,b,c (From Lemma 1 we know that it can be denotes in this form).

Supposed $\sum a$, $\sum ab$ are fixed, and consider $f(\sum a, \sum ab, abc)$ as a function of abc .

(i) If $f'(abc) \geq 0$, then function $f(\sum a, \sum ab, abc)$ attain its maximum when two of numbers are equal which are no greater than the third, attain its minimum when two of numbers which are no less than the third.

（ii） If $f'(abc) = 0$, then function $f(\sum a, \sum ab, abc)$ attains its extremum when two of numbers are equal.

（iii） If $f'(abc) \leq 0$, then function $f(\sum a, \sum ab, abc)$ attains its maximum when two of numbers are equal which are no less than the third, attains its minimum two of numbers are equal which are no bigger than the third.

Proof: At first we prove Theorem 2 (i). Consider (x_1, x_1, y_1) , (x_2, x_2, y_2)

where $x_1 = \dfrac{\sum a + \sqrt{(\sum a)^2 - 3\sum ab}}{3}$, $y_1 = \dfrac{\sum a - 2\sqrt{(\sum a)^2 - 3\sum ab}}{3}$,

$$x_2 = \frac{\sum a - \sqrt{(\sum a)^2 - 3\sum ab}}{3} , \quad y_2 = \frac{\sum a + 2\sqrt{(\sum a)^2 - 3\sum ab}}{3} .$$

(x_1, y_1, x_2, y_2 are the same in the following.)

Then $x_1 + x_1 + y_1 = x_2 + x_2 + y_2 = a + b + c$,

$$x_1^2 + x_1 y_1 + x_1 y_1 = x_2^2 + x_2 y_2 + x_2 y_2 = ab + bc + ca .$$

From theorem 1 we know

$$\frac{\sum ab \sum a + (6\sum ab - 2(\sum a)^2)x_1}{9} \leq abc \leq \frac{\sum ab \sum a + (6\sum ab - 2(\sum a)^2)x_2}{9} .$$

In fact :

$$\frac{\sum ab \sum a + (6\sum ab - 2(\sum a)^2)x_1}{9} = x_1^2 y_1 , \frac{\sum ab \sum a + (6\sum ab - 2(\sum a)^2)x_2}{9} = x_2^2 y_2 ,$$

Then $x_1^2 y_1 \leq abc \leq x_2^2 y_2$.

Meanwhile $f'(abc) \geq 0$, i.e., function $f(abc)$ is monotone increasing with respect to

abc and $\sum a$, $\sum ab$ are fixed. Let (x_1, x_1, y_1) , (x_2, x_2, y_2) take the place of (a,b,c). Then

$$f(x_1+x_1+y_1, x_1^2+x_1y_1+x_1y_1, x_1^2y_1) = f(\sum a, \sum ab, x_1^2y_1) \le f(\sum a, \sum ab, abc),$$

$$f(\sum a, \sum ab, abc) = f(x_2+x_2+y_2, x_2^2+x_2y_2+x_2y_2, abc) \le f(x_2+x_2+y_2, x_2^2+x_2y_2+x_2y_2, x_2^2y_2)$$

Obviously $x_1 \ge y_1$, $x_2 \le y_2$, hence the function $f(\sum a, \sum ab, abc)$ attain its maximum when two of numbers are equal which are no greater than the third, attain its minimum when two of numbers which are no less than the third. This proves Theorem 2 (ⅰ).

In the same way we can prove Theorem 2（iii）.

If $f^{'}(abc)=0$, the degree of abc in $f(\sum a, \sum ab, abc)$ is zero, then we can let any one of (x_1, x_1, y_1) , (x_2, x_2, y_2) take the place of (a,b,c), and the value of $f(\sum a, \sum ab, abc)$ is unchanged. Hence for a function $f(\sum a, \sum ab, abc)$ there is a corresponding (x_1, x_1, y_1) and (x_2, x_2, y_2). So when $f(\sum a, \sum ab, abc)$ attains its extremum, there is a corresponding (x_1, x_1, y_1) and (x_2, x_2, y_2), i.e., the function attains its extremum when two of the numbers are equal. This proves Theorem 2（iii）.

Hence Theorem 2 is proved.

Corollary 2 For a fully symmetric inequality about non-negative real umbers a,b,c

$$f(\sum a, \sum ab, abc) \ge 0,$$

(i) If $f^{'}(abc) \ge 0$, then $f(\sum a, \sum ab, abc)$ attain its maximum when two of numbers are equal which are no greater than the third, attain its minimum when two of numbers which are no less than the third.

(ii) If $f^{'}(abc)=0$, then function $f(\sum a, \sum ab, abc)$ attains its extremum when two of numbers are equal.

（iii） If $f^{'}(abc) \le 0$, then function $f(\sum a, \sum ab, abc)$ attains its maximum when two of numbers are equal which are no less than the third, attains its minimum two of numbers are equal which are no bigger than the third.

Proof: At first we prove Corollary 2 (i).

When x_1, y_1, x_2, y_2 are nonnegative, the proof is similar to the proof of Theorem 2.

However, $x_1 = \dfrac{\sum a + \sqrt{(\sum a)^2 - 3\sum ab}}{3} \ge 0$, $x_2 = \dfrac{\sum a - \sqrt{(\sum a)^2 - 3\sum ab}}{3} \ge 0$

$y_2 = \dfrac{\sum a + 2\sqrt{(\sum a)^2 - 3\sum ab}}{3} \ge 0$, then we only have to consider the case that $y_1 = \dfrac{\sum a - 2\sqrt{(\sum a)^2 - 3\sum ab}}{3} \le 0$.

We consider $(x_3, y_3, 0)$, where

$$x_3 = \frac{\sum a + \sqrt{(\sum a)^2 - 4\sum ab}}{2}, \quad y_3 = \frac{\sum a - \sqrt{(\sum a)^2 - 4\sum ab}}{2}.$$

From $\sum a^2 \ge 2\sum ab$ we know x_3, y_3 are real numbers. Obviously $x_3 \ge 0$, $y_3 \ge 0$, so x_3, y_3 are nonnegative real numbers. Then

$$x_3 + y_3 + 0 = \sum a ,$$

$$x_3 y_3 + 0 \cdot x_3 + 0 \cdot y_3 = \sum ab .$$

Since $f^{'}(abc) \ge 0$, let $(x_3, y_3, 0)$ takes the lace of (a,b,c), then

$$f(x_3 + y_3 + 0, x_3 y_3 + 0 \cdot x_3 + 0 \cdot y_3, 0 \cdot x_3 \cdot y_3) = f(\sum a, \sum ab, 0) \le f(\sum a, \sum ab, abc) .$$

Hence the function attains its maximum when two of numbers are equal which are no greater than the third. The function attains its minimum when two of the numbers are equal (and these two numbers are no less than the third.) or one of the numbers is equal to zero. This proves Corollary 2 (i).

In the same way we can prove Corollary 2 (iii).

When $f^{'}(abc) = 0$, let (x_2, x_2, y_2) take the place of (a,b,c), then using similar method to proving Theorem 2(ii), we can prove Corollary 2(ii).

This completes the proof of Corollary 2.

Theorem 3 For a fully symmetric inequality about real umbers a,b,c

$$f(\sum a, \sum ab, abc) \geq 0,$$

(i) If $f''(abc) \geq 0$, then maximum of the function $f(\sum a, \sum ab, abc)$ is attained when two of numbers are equal.

(ii) If $f''(abc) = 0$, then it can be reduced to one of the cases in Theorem 2.

(iii) If $f''(abc) \leq 0$, the minimum of the function $f(\sum a, \sum ab, abc)$ is attained when two of numbers are equal.

Proof: At first we prove Theorem 3(i).

Since $f''(abc) \geq 0$, $f(abc)$ is convex to the downwards. So the maximum of $f(abc)$ is attained in the end points. Hence

$$f(\sum a, \sum ab, abc) = f(x_1 + x_1 + y_1, x_1^2 + x_1y_1 + x_1y_1, abc) = f(x_2 + x_2 + y_2, x_2^2 + x_2y_2 + x_2y_2, abc)$$
$$\leq \max\left\{ f(x_1 + x_1 + y_1, x_1^2 + x_1y_1 + x_1y_1, x_1^2y_1), f(x_2 + x_2 + y_2, x_2^2 + x_2y_2 + x_2y_2, x_2^2y_2) \right\}$$

i.e., the maximum of $f(\sum a, \sum ab, abc)$ must be attained only when two of numbers are equal. This proves Theorem 3 (i).

In the same way we can prove Theorem 3(iii).

If $f''(abc) = 0$, i.e., the degree of abc in $f(\sum a, \sum ab, abc)$ is less than 2, the sign of $f'(abc)$ is invariant, hence it can be reduced to one of cases in Theorem 2. Theorem 3(ii) is proved. This completes the proof of Theorem 3.

Corollary 3 For a fully symmetric inequality about non-negative real umbers a,b,c

$$f(\sum a, \sum ab, abc) \geq 0,$$

(i) If $f''(abc) \geq 0$, then the maximum of the function $f(\sum a, \sum ab, abc)$ is attained when two of numbers are equal or one of numbers is equal to zero.

(ii) If $f^{''}(abc)=0$, then it is reduced to one of cases in Corollary 2, and the extremum of $f(\sum a,\sum ab,abc)$ is attained when two of numbers are equal or one of numbers is equal to zero.

(iii) If $f^{''}(abc)\le 0$, then the minimum of the function $f(\sum a,\sum ab,abc)$ is attained when two of numbers are equal or one of numbers is equal to zero.

The proof of Corollary is similar to Theorem 3 and Corollary and we omit it here.

The main idea of theorems and corollaries above is that the fully symmetric polynomial about real numbers a,b,c can be expressed in the form $f(\sum a,\sum ab,abc)$, and we can control two quantities $\sum a$, $\sum ab$, and adjust abc.

Corollary 4 A fully symmetric inequality of degree n with 3 variables $f(a,b,c)\ge 0$ if and only if $f(x,1,0)\ge 0$ and $f(x,1,1)\ge 0$ （$n\le 5$）.

Proof: The inequality to be proved can be rewritten as $f(\sum a,\sum ab,abc)\ge 0$. Since the degree of the inequality is less than, the degree of abc is less than one and $f^{''}(abc)=0$. Then from Corollary 3 we know that the extremum of the function is attained when two of numbers are equal or some of numbers is zero. Hence if $f(x,1,0)\ge 0$ and $f(x,1,1)\ge 0$, then the inequality holds. Obviously it is the sufficient condition of the inequality. This completes the proof of Corollary 4.

Remark: Corollary 4 is stronger than the result of Mr. Chen Shengli about the nonnegative homogeneous fully symmetric inequality of degree 4 with 3 variables[10].

With respect theorems and corollaries above we may make some further extensions.

Extension 1: For fully symmetric inequalities with 3 variables, $f(\sum a,\sum ab,abc)$ can be rewritten in the form $g(\sum a,\sum a^2,\sum a^3)$. We can make $\sum a$, $\sum a^2$ unchanged and adjust $\sum a^3$, similar results also hold. This is because that for such polynomials, quantities such as $\sum a, \sum a^2$ may determinate $\sum a, \sum ab$ uniquely. Since $\sum a^3 = (\sum a)^3 + 3abc - 3\sum a \sum ab$, quantities such as $\sum a^3$ is abc in fact., hence similar results hold.

If theorems and corollaries above can not be applied to the primary function, we can make some replacement of a, b, c such that theorems and corollaries above may work for now function. (see example 2)

Extension 2: an extension for homogeneous fully symmetric inequalities of degree n with 4 variables defined on real number field or nonnegative real number field:

Every homogeneous fully symmetric inequality of degree n with 4 variables defined on real number field or nonnegative real number field, except 0, is equivalent to a fully symmetric inequality with 3 variables.

Proof: For a fully symmetric in equality of degree n with 4 variables $f(x, y, z, t) \geq 0$, from Lemma 3 we know that every fully symmetric in equality of degree n with 4 variables $f(x, y, z, t)$ can be expressed uniquely in the form $t^n g(\sum \frac{x}{t}, \sum \frac{xy}{t^2}, \frac{xyz}{t^3})$.

Hence $f(x, y, z, t) \geq 0 \Leftrightarrow t^n g(\sum \frac{x}{t}, \sum \frac{xy}{t^2}, \frac{xyz}{t^3}) \geq 0$, that is. $g(\sum \frac{x}{t}, \sum \frac{xy}{t^2}, \frac{xyz}{t^3}) \geq 0$.

Then it is equivalent to a fully symmetric inequality with 3 variables about $\frac{x}{t}, \frac{y}{t}, \frac{z}{t}$.

Furthermore we can apply theorems and corollaries above to fully symmetric inequalities of degree n with 4 variables defined on real number field or nonnegative real number field.

For fully symmetric inequalities of degree n with 4 variables to which theorems and corollaries can be applied, we may adjust two of variables to be equal. Furthermore, because of the homogeneity we may suppose these two variables are 1, hence we cam reduce it to be a fully symmetric inequality with 2 variables.

2.2 Application

We explain application of the equivalency condition in proving inequalities.

Example 1. Verify that if a, b, c are nonnegative real numbers, then

$$a^3 + b^3 + c^3 + 3abc - a^2b - ab^2 - a^2c - ac^2 - b^2c - bc^2 \geq 0.$$

Proof 1: From Corollary 1 we know that

$$\max\left(\frac{\sum ab\sum a+(6\sum ab-2(\sum a)^2)x_1}{9},0\right)\le abc \text{ ,(here } x_1=\frac{\sum a+\sqrt{(\sum a)^2-3\sum ab}}{3}\text{)}$$

Set $\sum a=p$, $(\sum a)^2-3\sum ab=t^2$, $abc=r$. Obviously $p\ge t$. Hence we obtain

$$\max\left(\frac{(p+t)^2(p-2t)}{27},0\right)\le r . \qquad (2)$$

The original problem is equivalent to $27r+4pt^2-p^3\ge 0$.

If $t\ge \frac{p}{2}$, then obviously the inequality holds.

If $t\le \frac{p}{2}$, by using (2), we have $27r\ge (p+t)^2(p-2t)=p^3-3t^2p-2t^3\ge p^3-4pt^2$,

This completes the proof.

Proof 2: Denote by $f(\sum a,\sum ab,abc)$ the left hand side of the inequality. Since the highest degree of the function if less than 3, the degree of abc is less than 3. Hence $f''(abc)=0$. By using Corollary 3 we know that the minimum of $f(\sum a,\sum ab,abc)$ is attained when two of numbers are equal. We may assume that $a=c$, hence it is sufficient to prove

$$a^3+b^3+a^3+3a^2b-a^2b-ab^2-a^3-a^3-b^2a-a^2b\ge 0 \Leftrightarrow b(a-b)^2\ge 0 .$$

Hence original inequality holds.

Remark: This example is a part of the well-known Schur inequality. The whole Schur inequality can be found in [11].

Example 2. Suppose x,y,z are three nonnegative real number satisfying $x^2+y^2+z^2=1$. Verify that $\sum\frac{x}{x^3+yz}\ge 3$.

Proof : Let $a=\frac{yz}{x},b=\frac{xz}{y},c=\frac{xy}{z}$. Then it sufficient to prove that if a,b,c are nonnegative real number satisfying $\sum ab=1$, then $\sum\frac{1}{ab+c}\ge 3$

$$\Leftrightarrow \sum(ab+c)(ac+b)-3(ab+c)(ac+b)(bc+a)\ge 0 .$$

Suppose $f(\sum a, \sum ab, abc) = \sum(ab+c)(ac+b) - 3(ab+c)(ac+b)(bc+a)$, then $f^{"}(abc) \geq 0$. Hence form Theorem 3(i) we know that the minimum of $f(\sum a, \sum ab, abc)$ is attained when two of numbers are equal or one of numbers is equal to 0.

If one is zero, we may assume $c=0$. Then we only have to show $\frac{1}{ab}+\frac{1}{a}+\frac{1}{b} \geq 3$ under the condition $ab=1$. Since $\frac{1}{a}+\frac{1}{b}+\frac{1}{ab} \geq \frac{2}{\sqrt{ab}}+\frac{1}{ab}=3$, this prove the case that one of numbers is zero.

If two of numbers are equal, we may assume $a=c$, then we only have to show $\frac{2}{ab+a}+\frac{1}{a^2+b} \geq 3$ under conditions that a,b are nonnegative real numbers and $2ab+a^2=1$. Substitute $b=\frac{1-a^2}{2a}$, $\Leftrightarrow \frac{2}{\frac{1-a^2}{2}+a}+\frac{1}{a^2+\frac{1-a^2}{2a}} \geq 3$, when $a \in [0,1]$.

$$\Leftrightarrow \frac{1-(a^2+\frac{1-a^2}{2a})}{a^2+\frac{1-a^2}{2a}} \geq 2\frac{\frac{1-a^2}{2}+a-1}{\frac{1-a^2}{2}+a},$$

$$\Leftrightarrow \frac{(1-a^2)(1+\frac{1}{2a})}{a^2+\frac{1-a^2}{2a}}+\frac{(1-a)^2}{\frac{1-a^2}{2}+a} \geq 0.$$

$\because a \in [0,1]$, hence inequality above holds, i.e., the original inequality holds when two of numbers are equal.

Hence the inequality holds.

Remark: By a skillful substitution the degree is reduced form 9 to 6, which makes it is impossible to use the theorems. The substitution is in fact consider the function in the form $f(\sum \frac{yz}{x}, \sum y^2, xyz)$, which is the extension of fully symmetric inequality with 3 variables as we said.

Example 3 Suppose a,b,c are nonnegative real numbers satisfying $a+b+c=1$.

Verify that $\sum a^2+\frac{\sqrt{3}}{2}(abc)^{\frac{1}{2}}\geq\frac{1}{2}$.

Proof. Original inequality is equivalent to $\frac{\sqrt{3}}{2}(abc)^{\frac{1}{2}}\sqrt{\sum a}\geq\frac{1}{2}(\sum a)^2-\sum a^2$

$\Leftrightarrow\sqrt{3}(abc)^{\frac{1}{2}}\sqrt{\sum a}\geq 2\sum ab-\sum a^2$,

If $\sum a^2\geq 2\sum ab$, the inequality holds obviously.

If $\sum a^2\leq 2\sum ab$, suppose $f(\sum a,\sum ab,abc)=\sqrt{3}(abc)^{\frac{1}{2}}\sqrt{\sum a}-2\sum ab+\sum a^2$,

So $f^{'}(abc)=\frac{1}{2}(abc)^{-\frac{1}{2}}\geq 0$. Hence from Theorem 2(i) we know the minimum of the function when two numbers are equal, and they are bigger than the third. We may suppose $a=c$. Then $\Leftrightarrow\sqrt{3}\sqrt{a^2b}\sqrt{2a+b}\geq 2(a^2+2ab)-2a^2-b^2$ and $a\geq b$,

$\Leftrightarrow 3a^2(2a+b)\geq b(4a-b)^2$

$\Leftrightarrow 6a^3+3a^2b\geq 16a^2b+b^3-8ab^2$,

$\Leftrightarrow 6a^2(a-b)+7ab(b-a)+b^2(a-b)\geq 0$,

$\Leftrightarrow (a-b)(6a^2-7ab+b^2)\geq 0$,

$\Leftrightarrow (a-b)^2(6a-b)\geq 0$,

Notice that $a\geq b$, hence inequality above holds. Combining these two cases the original inequality is proved.

There is a detail which may be ignored easily: if $\sum a^2\leq 2\sum ab$, then the minimum is attained when two of numbers are equal(and these two numbers are no less than third.). Hut it is essential in dealing with this example.

In this section we choose three typical examples about proving inequalities by using judgment of equivalency condition. In fact the judgment of equivalency condition has wide application in proving fully symmetric inequalities with 3 variables. Here we don't list one by one.

3 Judgment of equivalency condition a class of fully symmetric inequalities with n variables.

3.1 Judgment theorem f equivalency condition a class of fully symmetric inequalities with n variables.

Theorem 4 (i) Given nonnegative real numbers $a \geq b \geq c$, real number $m \leq 0$ which are not same at the same time, for variables $x \leq y \leq z$ satisfying $x+y+z=a+b+c$, $x^m+y^m+z^m=a^m+b^m+c^m$.(in particular $xyz=abc$ if $m=0$).

Then there exist nonnegative real numbers x_1, x_2, if $x=x_1$, then $x=x_1 \leq y=z$; if $x=x_2$ then $x=x_2=y \leq z$,; if $x \in (x_1, x_2)$, then $x<y<z$.

(ii) Given nonnegative real numbers $a \geq b \geq c$, real number $m>0$ in which at most two numbers are equal, at most one is equal to zero and $m \neq 1$, for variables $x \leq y \leq z$ satisfying $x+y+z=a+b+c$, $x^m+y^m+z^m=a^m+b^m+c^m$, then there exist nonnegative real numbers x_1, x_2, if $x=x_1$, then $x=x_1 \leq y=z$; if $x \in (x_1, x_2)$, then $x<y<z$; if $x=x_1$, then $0=x_1=x<y \leq z$ or $x_1=x \leq y=z$.

Proof: We make some preparation. Consider y, z as functions of x, then form $x+y+z=a+b+c$, $x^m+y^m+z^m=a^m+b^m+c^m$ we know

$$1+y'+z'=0, \quad mx^{m-1}+my'y^{m-1}+mz'z^{m-1}=0.$$

We obtain $y'=\dfrac{x^{m-1}-z^{m-1}}{z^{m-1}-y^{m-1}}$, $z'=\dfrac{y^{m-1}-x^{m-1}}{z^{m-1}-y^{m-1}}$.(it is easy to see that equality holds when m=0). Form $x<z, z>y$, we get $y'<0$, $z'>0$. Hence when x increases, y is monotone decreasing and z is monotony increasing.

Firstly we prove Theorem 4(i). If $m=0$, consider function

$$f(x)=x^3-2\sum ax^2+(\sum a)^2 x-4abc$$

$$f(0)=-4abc \leq 0, \quad f(\frac{\sum a}{3})=(\frac{\sum a}{3})^3-2\sum a(\frac{\sum a}{3})^2+(\sum a)^2\frac{\sum a}{3}-4abc>0.$$

Then there is nonnegative real number x_1 such that $f(x_1)=0$ and $x_1 \in \left[0, \frac{\sum a}{3}\right)$.

Let $x = x_1$, then $(a+b+c-x_1)^2 x_1$, i.e., $(y+z)^2 = 4yz$. Hence $y = z$, which implies $x = x_1 < y = z$. Since y is monotony decreasing, when $x = x_1$, $x < y$. Hence there is x_2 such that when $x = x_2$, $x = x_2 = y \le z$. From the monotony of z we know when $x \in (x_1, x_2)$, $x < y < z$. Hence when m=0 theorem is proved.

When $m < 0$, consider function

$$f(x) = 2(abcx)^{-m} - (x^{-m}((ab)^{-m} + (ac)^{-m} + (bc)^{-m}) - (abc)^{-m})(\frac{a+b+c-x}{2})^{-m}.$$

If $x = 0$, $f(0) = (abc)^{-m}(\frac{a+b+c}{2})^{-m} \ge 0$.

If $x = c$, $f(c) = 2(abc^2)^{-m} - c^{-2m}(a^{-m} + b^{-m})(\frac{a+b}{2})^{-m} \le 0$. Mean while $c < \frac{a+b+c}{3}$, hence there is x_1 such that $0 \le x_1 < \frac{a+b+c}{3}$, $f(x_1) = 0$. Then we can get that when $x = x_1$, $y = z$, hence $x = x_1 < y = z$ by using similar argument as the case $m = 0$. From the monotony of y and z, there is x_2 such that when $x = x_2$, $x = x_2 = y \le z$, when $x \in (x_1, x_2)$, $x < y < z$. Hence when $m < 0$ theorem is proved.

Secondly we prove Theorem 4(ii). When $m > 1$, consider function

$$f(x) = a^m + b^m + c^m - x^m - 2\left(\frac{a+b+c-x}{2}\right)^m.$$

Obviously $f'(x) = -mx^{m-1} + m\left(\frac{a+b+c-x}{2}\right)^{m-1} > 0$ when $x \in \left(0, \frac{\sum a}{3}\right)$.

If $f(0) \ge 0$, then there are y, z such that $a^m + b^m + c^m = y^m + z^m$ and $y + z = a + b + c$ 且 $y > 0$.

Let $x_1 = 0$, then $x < y \le z$.

If $f(0) < 0$, since $f(\frac{a+b+c}{3}) = \sum a^m - 3\left(\frac{a+b+c}{3}\right)^m > 0$, there is x_1 such that

$0 < x_1 < \frac{a+b+c}{3}$, and $f(x_1) = 0$. Using similar argument as the case $m = 0$ we can get that $y = z$, hence $x = x_1 < y = z$. Since y is monotone decreasing and z is monotone increasing strictly, in any one of two cases above there is x_2 such that when $x = x_2$, $x = x_2 = y \le z$, when $x \in (x_1, x_2)$, $x < y < z$. The case $m > 1$ is proved.

When $0 < m < 1$, consider function $f(x) = a^m + b^m + c^m - x^m - 2\left(\frac{a+b+c-x}{2}\right)^m$.

Obviously $f'(x) = -mx^{m-1} + m\left(\frac{a+b+c-x}{2}\right)^{m-1} < 0$ when $x \in \left(0, \frac{\sum a}{3}\right)$.

If $f(0) \le 0$, there are y, z, such that $a^m + b^m + c^m = y^m + z^m$, $y + z = a + b + c$ and $y > 0$. Let $x_1 = 0$, then $x < y \le z$.

If $f(0) > 0$, since $f(\frac{a+b+c}{3}) = \sum a^m - 3\left(\frac{a+b+c}{3}\right)^m < 0$, there is x_1, such that $0 < x_1 < \frac{a+b+c}{3}$, $f(x_1) = 0$. Using similar argument as the case $m = 0$ we get that $y = z$, hence $x = x_1 < y = z$. Since y is monotone decreasing and z is monotone increasing strictly, in any one of two cases above there is x_2 such that when $x = x_2$, $x = x_2 = y \le z$, when $x \in (x_1, x_2)$, $x < y < z$. The case $m < 1$ is proved.

This completes the proof of the theorem.

Remark: When $m > 0$, if two of a, b, c are equal to 0, then the only possibility is $x = y < z$.

Theorem 5 For nonnegative real numbers x, y, z, and a function in the form $F(x, y, z) = f(x) + f(y) + f(z)$, let $g(x^{m-1}) = f'(x)$.

(i) If $g(x)$ is convex to the downwards,

When hen $m > 0$ the minimum of $F(x, y, z)$ is attained when $x \le y = z$ or $0 = x < y \le z$, the maximum is attained when $x = y \le z$.

When $m \le 0$, the minimum of $F(x,y,z)$ is attained when $x \le y = z$, the maximum is attained when $x = y \le z$.

(ii) If $g(x)$ is convex to the upwards,

When $m > 0$ the maximum of $F(x,y,z)$ is attained when $x \le y = z$ or $0 = x < y \le z$, the minimum is attained when $x = y \le z$.

When $m \le 0$, the maximum of $F(x,y,z)$ is attained when $x \le y = z$, the minimum is attained when $x = y \le z$.

If $f(x)$, $g(x)$ are continuous functions, then $f(x)$, $g(x)$ are continuous between the minimum and the maximum, i.e., any value between the minimum and the maximum can be attained.

Proof: Firstly consider the case that $g(x)$ is convex to the downwards

Let $x+y+z$, $x^m + y^m + z^m$ fixed(m is a real number non-equal to 1), then there are $a \ge b \ge c$ in which at most two are the same, (when $m > 0$, there is at most one is zero among such a, b, c.) satisfying $x+y+z = a+b+c$, $x^m + y^m + z^m = a^m + b^m + c^m$.

Consider y,z as functions of x, then from $x+y+z = a+b+c$, $x^m + y^m + z^m = a^m + b^m + c^m$ we know

$1 + y' + z' = 0$, $mx^{m-1} + my'y^{m-1} + mz'z^{m-1} = 0$.

Hence $y' = \dfrac{x^{m-1} - z^{m-1}}{z^{m-1} - y^{m-1}}$, , $z' = \dfrac{y^{m-1} - x^{m-1}}{z^{m-1} - y^{m-1}}$ (obviously when $m = 0$ the equality holds either.)

Then $F'(x) = f'(x) + y'f'(y) + z'f'(z) = f'(x) + \dfrac{x^{m-1} - z^{m-1}}{z^{m-1} - y^{m-1}} f'(y) + \dfrac{y^{m-1} - x^{m-1}}{z^{m-1} - y^{m-1}} f'(z)$.

Hence

$$\frac{F'(x)}{(x^{m-1} - z^{m-1})(y^{m-1} - x^{m-1})} = \frac{f'(x)}{(x^{m-1} - z^{m-1})(x^{m-1} - y^{m-1})} + \frac{f'(y)}{(y^{m-1} - z^{m-1})(y^{m-1} - x^{m-1})} + \frac{f'(z)}{(z^{m-1} - x^{m-1})(z^{m-1} - y^{m-1})}$$

Since $g(x^{m-1}) = f^{'}(x)$, $g(y^{m-1}) = f^{'}(y)$, $g(z^{m-1}) = f^{'}(z)$, then

$$\frac{F^{'}(x)}{(x^{m-1}-z^{m-1})(y^{m-1}-x^{m-1})} = \frac{g(x^{m-1})}{(x^{m-1}-z^{m-1})(x^{m-1}-y^{m-1})} + \frac{g(y^{m-1})}{(y^{m-1}-z^{m-1})(y^{m-1}-x^{m-1})} + \frac{g(z^{m-1})}{(z^{m-1}-x^{m-1})(z^{m-1}-y^{m-1})}$$

From $x \le y \le z$ we know

$$(x^{m-1}-z^{m-1})(x^{m-1}-y^{m-1}) \ge 0, \quad (z^{m-1}-x^{m-1})(z^{m-1}-y^{m-1}) \ge 0,$$

$$(z^{m-1}-y^{m-1})(y^{m-1}-x^{m-1}) \ge 0.$$

Since $g(x)$ is convex to the downwards, form Jensen inequality we obtain

$$\frac{g(x^{m-1})}{(x^{m-1}-z^{m-1})(x^{m-1}-y^{m-1})} + \frac{g(z^{m-1})}{(z^{m-1}-x^{m-1})(z^{m-1}-y^{m-1})} \ge$$

$$\left(\frac{1}{(x^{m-1}-z^{m-1})(x^{m-1}-y^{m-1})} + \frac{1}{(z^{m-1}-x^{m-1})(z^{m-1}-y^{m-1})}\right) g\left(\frac{\frac{x^{m-1}}{(x^{m-1}-z^{m-1})(x^{m-1}-y^{m-1})} + \frac{z^{m-1}}{(z^{m-1}-x^{m-1})(z^{m-1}-y^{m-1})}}{\frac{1}{(x^{m-1}-z^{m-1})(x^{m-1}-y^{m-1})} + \frac{1}{(z^{m-1}-x^{m-1})(z^{m-1}-y^{m-1})}}\right)$$

$$= \frac{g(y^{m-1})}{(z^{m-1}-y^{m-1})(y^{m-1}-x^{m-1})}.$$

i.e., $F^{'}(x)$ is monotone increasing, form Theorem 4 we know when $m > 0$, the minimum of $F(x, y, z)$ is attained when $x \le y = z$ or $0 = x < y \le z$, the maximum is attained when $x = y \le z$; when $m \le 0$, the minimum of $F(x, y, z)$ is attained when $x \le y = z$, the maximum is attained when $x = y \le z$.

Similarly if $g(x)$ is convex to the upwards on $[0, +\infty)$, when $m > 0$, the maximum of $F(x, y, z)$ is attained when $x \le y = z$ or $0 = x < y \le z$, the minimum is attained when $x = y \le z$, when $m \le 0$, the maximum of $F(x, y, z)$ is attained when $x \le y = z$, the minimum is attained when $x = y \le z$.

This completes the proof of Theorem.

Remark 1 : For functions in the form $F(x, y, z) = f(x, y) + f(y, z) + f(z, x)$ ($f(x, y)$ is a symmetric function with respect to x, y), we may fix $\sum x$, $\sum x^2$ or

$\sum x$, xyz, only notice that $xy=-\frac{1}{2}\sum x^2+\frac{1}{2}\left(\sum x\right)^2+z^2-z(x+y+z)$, $xy=\frac{xyz}{z}$, $x+y=x+y+z-z$, and a symmetric function of x,y can be considered as a function of xy , $x+y$.

Remark 2: The extremum of $F(x,y,z)$ is assumed to exist.

Theorem 6 For nonnegative real numbers x, y, z, and a function in the form $F(x,y,z)=f(x)+f(y)+f(z)$, let $h(x^{m-1})=\frac{f''(x)}{(x^{m-1}-z^{m-1})(x^{m-1}-y^{m-1})}$

(i) If $m>0$, for nonnegative real number x,

If $f'(x)\le 0$, $f''(x)\ge 0$ or $h(x)$ is convex to downwards, the maximum of $F(x,y,z)$ is attained when two of numbers are equal or some of numbers is equal to 0.

If $f'(x)\ge 0$, $f''(x)\le 0$ or $h(x)$ is convex to upwards, the minimum of $F(x,y,z)$ is attained when two of numbers are equal or some of numbers is equal to 0.

(ii) If $m\le 0$, for nonnegative real number x,

If $f'(x)\le 0$, $f''(x)\ge 0$ or $h(x)$ is convex to downwards, the maximum of $F(x,y,z)$ is attained when two of numbers are equal.

If $f'(x)\ge 0$, $f''(x)\le 0$ or $h(x)$ is convex to upwards, the minimum of $F(x,y,z)$ is attained when two of numbers are equal and $f(x)$ is continuous between the minimum and the maximum, i.e., any value between the minimum and the maximum can be attained.

Since the proof is similar to Theorem 5, here we only give a sketch of the proof.

Sketch of the proof.: Firstly we prove Theorem 6(i). Similar to theorem 5, we know that $y'=\frac{x^{m-1}-z^{m-1}}{z^{m-1}-y^{m-1}}$, $,z'=\frac{y^{m-1}-x^{m-1}}{z^{m-1}-y^{m-1}}$ （it is easy to see that it holds when $m=0$）

then $F'(x)=f'(x)+y'f'(y)+z'f'(z)=f'(x)+\frac{x^{m-1}-z^{m-1}}{z^{m-1}-y^{m-1}}f'(y)+\frac{y^{m-1}-x^{m-1}}{z^{m-1}-y^{m-1}}f'(z)$,.

Hence

$$F^{"}(x)=f^{"}(x)+\frac{(m-1)x^{m-2}}{z^{m-1}-y^{m-1}}f^{'}(y)-\frac{(m-1)x^{m-2}}{z^{m-1}-y^{m-1}}f^{'}(z)+(\frac{x^{m-1}-z^{m-1}}{z^{m-1}-y^{m-1}})^2 f^{"}(y)+(\frac{y^{m-1}-x^{m-1}}{z^{m-1}-y^{m-1}})^2 f^{"}(z)$$

If $f^{'}(x)\le 0$, then $\frac{(m-1)x^{m-2}}{z^{m-1}-y^{m-1}}f^{'}(y)-\frac{(m-1)x^{m-2}}{z^{m-1}-y^{m-1}}f^{'}(z)\ge 0$. Hence if $f^{"}(x)\ge 0$,

$F^{"}(x)\ge 0$.

If $h(x)$ is convex to downwards, then

$$f^{"}(x)+(\frac{x^{m-1}-z^{m-1}}{z^{m-1}-y^{m-1}})^2 f^{"}(y)+(\frac{y^{m-1}-x^{m-1}}{z^{m-1}-y^{m-1}})^2 f^{"}(z)=$$

$$(x^{m-1}-z^{m-1})^2(y^{m-1}-x^{m-1})^2(\frac{h(x^{m-1})}{(x^{m-1}-z^{m-1})(x^{m-1}-y^{m-1})}+\frac{h(y^{m-1})}{(y^{m-1}-z^{m-1})(y^{m-1}-x^{m-1})}+\frac{h(z^{m-1})}{(z^{m-1}-x^{m-1})(z^{m-1}-y^{m-1})})\ge 0$$

(by Jensen inequality). Hence $F^{"}(x)\ge 0$.

So the maximum is attained at the end, i.e., the maximum of $F(x,y,z)$ is attained when two of numbers are equal or some of numbers is zero.

Using similar method we can prove the case that $f^{'}(x)\ge 0$, $f^{"}(x)\le 0$ or $h(x)$ is convex to upwards. Hence Theorem 6{i} holds.

Similarly we can prove Theorem 6(ii). This completes the proof of Theorem 6.

Theorem 7 For nonnegative real numbers $x_1, x_2 \dots, x_n$ and a function in the form $F(x_1,x_2,\dots,x_n)=f(x_1)+f(x_2)+\dots+f(x_n)$, let $g(x^{m-1})=f^{'}(x)$.($n\ge 3$, n is integral numbers.)

(i) If $g(x)$ is convex to downwards,

When $m>0$, the minimum of $F(x_1,x_2,\dots,x_n)$ is attained when $x_1\le x_2=x_3=\dots=x_n$ or the d of numbers are zero and n-d-1 of positive numbers are equal, the maximum is attained when $x_1=x_2=\dots x_{n-1}\le x_n$ (here $1\le d\le n-1$, $d\in N$, the following is the same.)

When $m\le 0$, the minimum of $F(x_1,x_2,\dots,x_n)$ is attained when $x_1\le x_2=x_3=\dots=x_n$,, the maximum is attained when $x_1=x_2=\dots x_{n-1}\le x_n$..

(ii) If $g(x)$ is convex upwards,

When $m>0$, the maximum of $F(x_1,x_2,\ldots,x_n)$ are attained when $x_1 \le x_2 = x_3 = \ldots = x_n$, the minimum is attained when $x_1 = x_2 = \ldots x_{n-1} \le x_n$.

If $f(x)$, $g(x)$ are continuous then $F(x,y,z)$ is continuous between the minimum and the maximum, i.e., every value between the minimum and the maximum can be attained.

Proof: Firstly we prove the case that $g(x)$ is convex to downwards.

When $m>0$, we first prove that when the minimum of $F(x_1,x_2,\ldots,x_n)$ then $x_1 \le x_2 = x_3 = \ldots = x_n$, or d of numbers are zero and n-d-1 of the positive numbers are equal. We adjust $x_1,x_2,\ldots,x_n$ such that three of them x_i,x_j,x_k $\left(1 \le i < j < k \le n\right)$ (at most one of x_i,x_j,x_k is zero), fix other variables and $x_i+x_j+x_k$, $x_i^{\ m}+x_j^{\ m}+x_k^{\ m}$ $(m \in R, m \ne 1)$ such that $F(x_i,x_j,x_k)$ attains its minimum. From Theorem 5 we know that when $F(x_i,x_j,x_k)$ attains its minimum, $x_i \le x_j = x_k$ or $0 = x_i < x_j \le x_k$.

If the adjustment is taken as far as $(x_1,x_2,\ldots,x_n)$ and $x_1 \le x_2 = x_3 = \ldots = x_n$ or d of numbers are zero and n-d-1 of positive numbers are equal, the adjustment will stop, we call it the end of adjustment. So we only have to there are only two cases as stated when the adjustment can not carry over.

Suppose that the adjustment carries over until

$F(x_1,x_2,\ldots,x_n) = f(x_1)+f(x_2)+\ldots+f(x_n)$, we may assume $x_1 \le x_2 \le \ldots \le x_n$.

If there is zero among $x_i (1 \le i \le n)$, we may suppose that x_{d+1} is the smallest one which are not zero. If $d = n-1, n-2$, then the proposition is proved(hence it is proved when $n=3$). If $d \le n-3$ ($n \ge 4$), consider $(x_{d+1},x_{d+2},x_n),(x_{d+1},x_{d+3},x_n),\ldots,(x_{d+1},x_{n-1},x_n)$. Since $g(x)$ is convex to downwards, hence when $F(x_{d+1}, x_i, x_n)$, $(d+2 \le i \le n-1)$ attains its minimum, there holds $x_{d+1} \le x_i = x_n$. Combining these $n-d-2$ formulas we know d of numbers are zero and at least $n-d-1$ of positive numbers are equal when the adjustment is ended.

When $x_i (1 \le i \le n)$ are all positive, similarly we can prove $x_1 \le x_2 = x_3 = \ldots = x_n$.

Hence if $m > 0$, when $F(x_1, x_2, \ldots, x_n)$ attains the minimum, there holds $x_1 \le x_2 = x_3 = \ldots = x_n$ or d of numbers are zero and at least $n-d-1$ of positive numbers are equal.

Similarly $x_1 = x_2 = \ldots x_{n-1} \le x_n$ when $F(x_1, x_2, \ldots, x_n)$ attains the maximum.

So when m>0 the proposition is proved.

Hence the case that $g(x)$ is convex to downwards is proved.

In the similar way if $g(x)$ is convex to upwards, when $m > 0$, the maximum of $F(x_1, x_2, \ldots, x_n)$ is attained when $x_1 \le x_2 = x_3 = \ldots = x_n$ or d of numbers are zero and at least $n-d-1$ of positive numbers are equal., the minimum is attained when $x_1 = x_2 = \ldots x_{n-1} \le x_n$; when $m \le 0$, the maximum of $F(x_1, x_2, \ldots, x_n)$ is attained when $x_1 \le x_2 = x_3 = \ldots = x_n$, the minimum is attained when $x_1 = x_2 = \ldots x_{n-1} \le x_n$.

This proves Theorem 7.

Using similar argument we can get a generalization of Theorem 6 with 6 variables. We omit it here.

Corollary 5(i) If $(x_1, x_2, \ldots, x_n)$ are nonnegative real numbers, $x_1 + x_2 + \ldots + x_n$, $x_1^m + x_2^m + \ldots + x_n^m$ $(m \in R, m \ne 1)$ are fixed, function $F(x_1, x_2, \ldots, x_n) = x_1^p + x_2^p + \ldots + x_n^p$ $(p \in R, p \ne 1, m)$. In particular when p=0, $F(x_1, x_2, \ldots, x_n) = x_1 x_2 \ldots x_n$.$(n \ge 3, n$ is integer); when $p(p-1)(p-m) \ge 0$ and $p \ne 0$ or $p = 0$ and $m > 0$: if $m > 0$, the minimum of $F(x_1, x_2, \ldots, x_n)$ is attained when $x_1 \le x_2 = x_3 = \ldots = x_n$ or d of numbers are zero and at least $n-d-1$ of positive numbers are equal, the maximum is attained when $x_1 = x_2 = \ldots x_{n-1} \le x_n$, if $m \le 0$, the minimum of $F(x_1, x_2, \ldots, x_n)$ is attained when $x_1 \le x_2 = x_3 = \ldots = x_n$, the maximum is attained when $x_1 = x_2 = \ldots x_{n-1} \le x_n$.

(ii) If $(x_1, x_2, \ldots, x_n)$, and $x_1 + x_2 + \ldots + x_n$, $x_1^m + x_2^m + \ldots + x_n^m$ $(m \in R, m \ne 1)$ are fixed , function $F(x_1, x_2, \ldots, x_n) = x_1^p + x_2^p + \ldots + x_n^p$ $(p \in R, p \ne 1, m)$, in particular when p=0 $F(x_1, x_2, \ldots, x_n) = x_1 x_2 \ldots x_n$ ($n \ge 3, n$ is integer), when $p(p-1)(p-m) \le 0$ and $p \ne 0$ or $p = 0$ 及 $m < 0$, if $m > 0$, then the maximum of $F(x_1, x_2, \ldots, x_n)$ is attained when $x_1 \le x_2 = x_3 = \ldots = x_n$ or d of numbers are zero and at least $n-d-1$ of positive numbers are equal, the minimum is attained when $x_1 = x_2 = \ldots x_{n-1} \le x_n$, if $m \le 0$, then the maximum of $F(x_1, x_2, \ldots, x_n)$ is attained when $x_1 \le x_2 = x_3 = \ldots = x_n$, the minimum is attained when $x_1 = x_2 = \ldots x_{n-1} \le x_n$,

If $f(x)$, $g(x)$ are continuous then $F(x, y, z)$ is continuous between the minimum and the maximum, i.e., every value between the minimum and the maximum can be attained.

Proof : Firstly we prove Corollary (i). If $p \ne 0$, set $g(x^{m-1}) = f'(x)$. Since

$g''(x) = p\dfrac{(p-1)(p-m)}{(m-1)^2} x^{\frac{p-2m+1}{m-1}} \ge 0$, we know from Theorem 6 that when $m > 0$,

the minimum f $F(x_1, x_2, \ldots, x_n)$ is attained when $x_1 \le x_2 = x_3 = \ldots = x_n$ or d of numbers are zero and at least $n-d-1$ of positive numbers are equal, the maximum is attained when $x_1 = x_2 = \ldots x_{n-1} \le x_n$, if $m \le 0$, then the minimum of $F(x_1, x_2, \ldots, x_n)$ is attained when $x_1 \le x_2 = x_3 = \ldots = x_n$, the maximum is attained when $x_1 = x_2 = \ldots x_{n-1} \le x_n$,.

If $p = 0$, set $f(x) = \ln(x)$, $g(x^{m-1}) = f'(x)$, since $g''(x) = \dfrac{m}{(m-1)^2} x^{\frac{-2m+1}{m-1}} \ge 0$, and

$\ln(x)$ is continuous, the minimum of $\sum_{i=1}^{n}\ln(x_i)$ when $x_1 \le x_2 = x_3 = \ldots = x_n$ or d of numbers tend to zero and n-d-1 of numbers are equal, the maximum is attained when $x_1 = x_2 = \ldots x_{n-1} \le x_n$, i.e., the minimum of $\prod_{i=1}^{n} x_i$ when $x_1 \le x_2 = x_3 = \ldots = x_n$ or d of numbers are zero and n-d-1 of positive umbers are equal, the maximum is attained when $x_1 = x_2 = \ldots x_{n-1} \le x_n$,.

Hence Corollary 5(i) is proved. Similarly we can prove Corollary 5(ii). This completes the proof of Corollary.

Remark: In corollary 5(i) if n=3,m=2,p=0，then when $x+y+z$, $x^2+y^2+z^2$ are fixed(i.e., $x+y+z$, $xy+yz+zx$ since $x^2+y^2+z^2+2(xy+yz+zx)=(x+y+z)^2$) , the minimum of xyz is attained when $x \le y = z$ or there is a zero, the maximum is attained when $x = y \le z$. This is corollary 1, also the core to prove Theorem 2,3, and Corollary 2,3.

In corollary 5(i) if n=3,m=0,p=2 when $x+y+z$, xyz are fixed, the minimum of xyz is attained when $x \le y = z$, the maximum is attained when $x = y \le z$ i.e., the maximum of $xy+yz+zx$ is attained when $x \le y = z$, the minimum is attained when $x = y \le z$.

In corollary 5(ii) let n=3, m=0, p= $\frac{1}{2}$ and $x_1 = x^{\frac{1}{2}}, y_1 = y^{\frac{1}{2}}, z_1 = z^{\frac{1}{2}}$, ,then when ${x_1}^2+{y_1}^2+{z_1}^2$, $x_1y_1z_1$ are fixed, the maximum of $x_1+y_1+z_1$ is attained when $x_1 \le y_1 = z_1$ or there is a zero, the minimum is attained when $x_1 = y_1 \le z_1$.

Corollary 6 For nonnegative real numbers x, y, z, if $\sum xy$, xyz are fixed, the minimum of $\sum x$ is attained when $x \le y = z$, and its maximum is attained when $x = y \le z$.

Proof: If some of x,y,z is zero, we may assume $x=0$, then it is sufficient to get the range of $y+z$ when yz is fixed. Obviously $y+z$ attains its minimum when $y=z$, while $y+z$ tends to infinity as y tends to 0, i.e.., the maximum is attained when $x=y\le z$. If x,y,z are all positive, we may assume 设 $xy=c$， $yz=a$， $xz=b$，, then $x=\sqrt{\frac{bc}{a}}$, $y=\sqrt{\frac{ac}{b}}$, $z=\sqrt{\frac{ab}{c}}$. Then it is sufficient to get the range of $\sum x=(abc)^{0.5}\sum\frac{1}{a}$ when $\sum xy=\sum a$， $xyz=(abc)^{0.5}$ are fixed, or the range of $\sum\frac{1}{a}$.

From Corollary 5(ii) we know that $\sum\frac{1}{a}$ attains the minimal when $a=b\le c$, attains the maximum when $a\le b=c$. Hence $\sum x$ attains the minimal when $z\le x=y$, attains the maximum when $z=y\le x$.

When handling fully symmetric inequalities with 3 variables, we may fix two of the three fundamental polynomials and change the value of the third polynomial to attain the extremum .

Theorem 8 For a fully symmetric inequality about nonnegative real numbers a,b,c $f(\sum a,\sum ab,abc)\ge 0$. If $f^{'}(abc)f^{'}(\sum ab)\ge 0$ or $f^{'}(\sum a)f^{'}(\sum ab)\ge 0$, equalities hold if and only if three numbers are the same.

Proof: We prove the case that $f^{'}(\sum a)f^{'}(\sum ab)\ge 0$, the case that $f^{'}(abc)f^{'}(\sum ab)\ge 0$ can be proved in the same way.

We only have to prove the case when $f^{'}(\sum a)\ge 0$, the case $f^{'}(\sum a)\le 0$ can be proved in the similar way.

Form Corollary 6 we know that we may control $\sum ab$， abc and adjust a,b,c such that $\sum a$ attains the minimum when $a\le b=c$, i.e., $f\left(a,b,c\right)$ attains the minimum and $\sum a$ attains the maximum when $a=b\le c$, i.e., $f\left(a,b,c\right)$ attains the maximum. Since $\sum ab$, abc are fixed, we have $f^{'}(\sum a) \geq 0$, hence $f^{'}(\sum ab) \geq 0$. From Corollary 5(ii) we know that we may control $\sum a$, abc and adjust a,b,c such that $\sum ab$ attains the maximum when $a \leq b = c$, i.e., $f(a,b,c)$ attains the maximum and $\sum ab$ attains the minimum when $a = b \leq c$, i.e., $f(a,b,c)$ attains the minimum. Since $\sum a$, abc are fixed, we have $f^{'}(\sum ab) \geq 0$, hence $f^{'}(\sum a) \geq 0$.Hence if we continue the adjustment until when (a,b,c) and at this time the adjustment of (a,b,c) is still (a,b,c), then we call it the end of the adjustment(it is easy to see when $a = b = c$ the end of the adjustment cones, hence there is always a end of adjustment.)

Suppose that the end comes when (a,b,c) , then $a \leq b = c$ and $a = b \leq c$ or $a = b \leq c$ 及 $a \leq b = c$. In any case we have $a = b = c$. This completes the proof of Theorem 8.

Remark: From Theorem 8 we obtain a judgment of some fully symmetric inequality with 3 variables when three numbers are equal to each other.

For Theorems and Corollary, using similar method we can obtain similar results as result in this section, when variables are defined on $[\alpha, \beta]$ or (α, β) or $[\alpha, \beta)$ or $(\alpha, \beta]$, $0 \leq \alpha < \beta$ which is omitted here. For variables defined on the real number field, as long as the degree m such that variables are well-defined we also have similar result.

3.2 Application

By through following examples, we show the application of the equivalency condition in proving inequalities.

Using same replacement argument as in Corollary 6 we cam prove following conjecture proposed by Mr. Yang Xuzhi[12].

Example 1. Conjecture of Yang Xuzhi: Suppose $x_1, \ldots, x_n$ are real numbers such that

$\sum_{i=1}^{n} x_i^2 \le n$, then $2+(n-2)\prod_{i=1}^{n} x_i \ge \prod_{i=1}^{n} x_i \sum_{i=1}^{n} \frac{1}{x_i}$.

Proof: The condition $\sum_{i=1}^{n} x_i^2 \le n$ when n=1 is in fact the inequality to be proved.

When n=2, from $x_1^2 + x_2^2 \le 2$ we know $2x_1x_2 \le 2$, this is the the inequality to be proved.

Now we prove the case $n \ge 3$.

Let $\prod_{i=1}^{n} x_i \cdot \frac{1}{x_i} = \sqrt{\frac{1}{y_i}}$, $(i=1,2,\ldots, n)$ $\therefore \prod_{i=1}^{n} x_i = (\frac{1}{\prod_{i=1}^{n} y_i})^{\frac{1}{2(n-1)}} \Rightarrow x_i = \frac{y_i^{\frac{1}{2}}}{\prod_{i=1}^{n} y_i^{\frac{1}{2(n-1)}}}$,

Hence $\sum_{i=1}^{n} y_i \le n(\prod_{i=1}^{n} y_i)^{\frac{1}{n-1}}$.

We want to prove $2+(n-2)(\frac{1}{\prod_{i=1}^{n} y_i})^{\frac{1}{2(n-1)}} \ge \sum_{i=1}^{n} \sqrt{\frac{1}{y_i}}$.

Fixed $\sum_{i=1}^{n} y_i$, $\prod_{i=1}^{n} y_i$, then from Corollary 6 we know the maximum of $\sum_{i=1}^{n} \sqrt{\frac{1}{y_i}}$ when $y_1 \le y_2 = y_3 = \ldots = y_n$, or when $x_2 = x_3 = \ldots = x_n \ge x_1$.

So we only have to prove when $(n-1)x^2 + y^2 \le n$ we have

$$2+(n-2)x^{n-1}y \ge x^{n-1} + (n-1)x^{n-2}y \Leftrightarrow (2-x^{n-1}) \ge x^{n-2}y\left((n-1)-x(n-2)\right).$$

Hence we only have to prove the case when $(n-1)x^2 + y^2 = n$.

$$\Leftrightarrow 2+(n-2)x^{n-1}\sqrt{n-(n-1)x^2} \ge x^{n-1} + (n-1)x^{n-2}\sqrt{n-(n-1)x^2} \quad (1 \le x^2 \le \frac{n}{n-1})$$

Set $f(x) = (n-2)x^{n-1}\sqrt{n-(n-1)x^2} - x^{n-1} + (n-1)x^{n-2}\sqrt{n-(n-1)x^2}$

$$f'(x) = (n-2)(n-1)x^{n-2}\sqrt{n-(n-1)x^2} - (n-1)x^{n-2} - (n-2)(n-1)x^{n-3}\sqrt{n-(n-1)x^2} - (n-2)(n-1)x^n(n-(n-1)x^2)^{-\frac{1}{2}} + (n-1)^2x^{n-1}(n-(n-1)x^2)^{-\frac{1}{2}}$$

$$f'(x) \ge 0 \Leftrightarrow (n-2)(x-1)(n-(n-1)x^2) - x\sqrt{n-(n-1)x^2} + x^2((n-1)-(n-2)x) \ge 0$$

$$\Leftrightarrow (n-2)(x-1)(n-(n-1)x^2) + (n-2)x^2(1-x) + x^2 - x\sqrt{n-(n-1)x^2} \ge 0$$

$$\Leftrightarrow -n(n-2)(x-1)^2(x+1)+nx\frac{(x^2-1)}{x+\sqrt{n-(n-1)x^2}}\geq 0$$

$$\Leftrightarrow \frac{x}{x+\sqrt{n-(n-1)x^2}}\geq (n-2)(x-1)，记\ g(x)=x+\sqrt{n-(n-1)x^2}$$

then ： $g'(x)=1-\frac{(n-1)x}{\sqrt{n-(n-1)x^2}}\leq 1-(n-1)x<0$ ， hence $g(x)$ is monotone decreasing in the field of definition.

Hence : $\frac{x}{x+\sqrt{n-(n-1)x^2}}\geq \frac{x}{2}$， $\Leftarrow \frac{x}{2}\geq (n-2)x-n+2$

$$\Leftrightarrow 2n-4\geq (2n-5)x，\ \Leftrightarrow x\leq \frac{2n-4}{2n-5}，\ \Leftarrow (\frac{2n-4}{2n-5})^2\geq \frac{n}{n-1}，\ \Leftrightarrow \frac{4n-9}{(2n-5)^2}\geq \frac{1}{n-1}$$

$$\Leftrightarrow (4n-9)(n-1)\geq (2n-5)^2，\ \Leftrightarrow 4n^2-13n+9\geq 4n^2-20n+25，\ \Leftrightarrow 7n\geq 16$$

The last inequality holds obviously when $n\geq 3$。So $f'(x)\geq 0$， hence $f(x)$ is monotone increasing.

Thus $f(x)\geq f(1)=0$.

This proves the proposition. !

Example 2 : Suppose $x_1,\ldots,x_n$ are real numbers such that $\sum_{i=1}^{n}x_i=1$. Try to compute the maximum of $\sum_{i=1}^{n}\prod_{j\neq i}x_j^{\,x}$ (x is a nonnegative real number.)

Solution: When $x\leq 1$, $\frac{\sum_{i=1}^{n}\prod_{j\neq i}x_j^{\,x}}{n}\leq (\frac{\sum_{i=1}^{n}\prod_{j\neq i}x_j}{n})^x\leq 1$, that is, $\sum_{i=1}^{n}\prod_{j\neq i}x_j^{\,x}\leq n$.

When $x>1$, note that $\sum_{i=1}^{n}\prod_{j\neq i}x_j^{\,x}=\prod_{i=1}^{n}x_i^{\,x}\sum_{i=1}^{n}\frac{1}{x_i^{\,x}}$. Fix $\sum_{i=1}^{n}x_i$ 、 $\prod_{i=1}^{n}x_i$, from Corollary 5(II) we know that the maximum of $\sum_{i=1}^{n}\frac{1}{x_i^{\,x}}$ is attained when $x_1\leq x_2=x_3=\cdots\cdots=x_n=a$. Hence we only have to compute the maximum of

$$f(a)=a^{(n-1)x}+(n-1)a^{(n-2)x}(n-(n-1)a)^x\ (1\le a\le \frac{n}{n-1}).$$

When the maximum of $f(a)$ is attained when $a=\frac{n}{n-1}$, $f(a)\le f(\frac{n}{n-1})$,

When the maximum of $f(a)$ is not attained when $a=\frac{n}{n-1}$, that is the maximum is attained when $n-(n-1)a\ne 0$,

$$f'(a)=(n-1)x(a^{(n-1)x-1}+(n-2)a^{(n-2)x-1}(n-(n-1)a)^x-(n-1)a^{(n-2)x}(n-(n-1)a)^{x-1})$$

$$=(n-1)xa^{(n-2)x-1}(n-(n-1)a)^x\left(\left(\frac{a}{n-(n-1)a}\right)^x+(n-2)-(n-1)\frac{a}{n-(n-1)a}\right)$$

Set $\frac{a}{n-(n-1)a}=k$, $g(k)=k^x+(n-2)-(n-1)k$, then $k\ge 1$, when $x\ge n-1$,

$$g(k)=k^x+(n-2)-(n-1)k\ge (n-1)k^{\frac{x}{n-1}}-(n-1)k\ge 0.$$

When $x<n-1$, $g'(k)=xk^{x-1}-(n-1)$. When k is a positive real number, $g(k)$ only have one stationary point $k=\left(\frac{n-1}{x}\right)^{\frac{1}{x-1}}$, hence there are two zeros at most. Meanwhile $k=1$ is a stationary point of $g(k)$, and $\left(\frac{n-1}{x}\right)^{\frac{1}{x-1}}>1$. Hence the maximum of $f(a)$ is attained when $k=1$ or k tends to infinite. However $n-(n-1)a=0$ when k tends to infinity, a contradiction.

Hence the maximum of $f(a)$ is attained when k=1 or a=1.

Thus $f(a)\le \max(f(1),f(\frac{n}{n-1}))$.

When $1\le x\le \frac{\lg n}{\lg n-\lg(n-1)}$, $f(1)\ge f(\frac{n}{n-1})$, while when $x>\frac{\lg n}{\lg n-\lg(n-1)}$, $f(1)<f(\frac{n}{n-1})$. Hence

$$\sum_{i=1}^{n}\prod_{j\ne i}x_j^{\,x}\le\begin{cases} n, when 0<x\le \frac{\lg n}{\lg n-\lg(n-1)} \\ \left(\frac{n}{n-1}\right)^{(n-1)x}, when x>\frac{\lg n}{\lg n-\lg(n-1)}\end{cases}.$$

When $n\ge 3$, $(1+\frac{1}{n-1})^{\frac{1}{n}}<1+\frac{1}{n(n-1)}<n-1\Leftrightarrow \frac{n+1}{n}<\frac{\lg n}{\lg n-\lg(n-1)}$.

Hence we have when $x_1,\ldots,x_n \in R^+, \sum_{i=1}^{n} x_i^n = n$, $\sum (x_2 x_3 \cdots\cdots x_n)^{n+1} \le n$. This is another unsolved conjecture of Mr. Yang Xuezhi[13].

This example solved a problem of optimal exponent, while the Bottema developed by Academician Yang Lu can not dual with this problem[14].

4 Judgment of un-normal equivalency condition for some fully symmetric or cyclic inequalities.

In the beginning of this section we emphasize following fact: when σ_1, σ_2 are fixes, σ_3 is continuous between the minimum and the maximum(from Corollary 6), i.e., if the maximum of $f(\sigma_3)$ is bigger than 0, and the minimum of $f(\sigma_3)$ is less than 0, and $f(\sigma_3)$ is a continuous then there is a σ_3 such that σ_3 =0.

4.1 Judgment for homogeneous fully symmetric inequality of degree n with 3 variables $(n >= 6, n \in N)$

We have proved that the equality of a fully symmetric inequality of degree n with 3 variables ($n \le 5, n \in N$) holds if and only if two variables are equal or some of variables is zero. We call it the normal equivalency condition. However when the degree is 6, the equivalency condition is not like this, so what is the equivalency condition for a fully symmetric inequality of degree 6 with 3 variables? Many scholars have studied this problem[3]、[15]. But there is no result on the whole judgment for fully symmetric inequalities of degree 6 with 3 variables. In the following we obtained a general result, and extend it to higher degrees.

A fully symmetric inequality of degree 6 with 3 variables can be written in the form

$$f(a,b,c) = A\sigma_3^{\ 2} + (B\sigma_1^{\ 3} + C\sigma_1 t^2)\sigma_3 + g(\sigma_1, t) \ge 0$$

Where $g(\sigma_1, t) = D\sigma_1^{\ 6} + E\sigma_1^{\ 4} t^2 + F\sigma_1^{\ 2} t^4 + Gt^6$, $\sigma_1 = \sum a, \sum ab = \dfrac{\sigma_1^{\ 2} - t^2}{3}, abc = \sigma_3^{\ 2}$.

Hence $0 \le t \le \sigma_1$. From the homogeneity we may assume that $\sigma_1 = 1$. Hence we obtain following theorem.

Theorem 9 Judgment theorem for fully symmetric inequalities of degree 6 with 3 variables defined on nonnegative real field.

For a fully symmetric inequality of degree 6 with 3 variables defined on nonnegative real field, it holds if and only if:

$D(x+1)^6+E(x+1)^4(x^2-x+1)+F(x+1)^2(x^2-x+1)^2+G(x^2-x+1)^3\geq 0$ i.e., $f(x,1,0)\geq 0$,

$Ax^2+(B(2x+1)^3+C(2+x)(x-1)^2)x+D(x+2)^6+E(x+2)^4(x-1)^2+G(x-1)^6\geq 0$ i.e., $f(x,1,1)\geq 0$. Two inequalities holds when $x\in[0,+\infty)$.

When $A\geq 0$,

$4At^3+(27C-6A)t^2+2A+27B\geq 0, 4At^3+(6A-27C)t^2-2A-27B\geq 0$,

$B+Ct^2\leq 0$ Have solutions when $t\in[0,1]$, and their intersection of solutions is non-empty. Let this intersection by (3). Then elements in (3) satisfy

$4AGt^6+(4AF-C^2)t^4+(4AE-2BC)t^2+4AD-B^2\geq 0$.

Proof: We fix σ_1, t and change the value of σ_3.

When $A\leq 0$, then $f''(\sigma_3)\leq 0$. From corollary 3 w3 only have to prove $f(x,1,0)\geq 0$ or $f(x,1,1)\geq 0$.

When $A\geq 0$, then $f'(\sigma_3)=0$ has no root. Since σ_3 is continuous between the minimum and the maximum we know that the sigh of $f(\sigma_3)$ is fixed. Hence from Corollary 2 we only have to prove $f(x,1,0)\geq 0$ or $f(x,1,1)\geq 0$.

If $A\geq 0$ and $f'(\sigma_3)=0$ has roots, i.e., $f'(\sigma_3)=2A\sigma_3+B\sigma_1^3+C\sigma_1 t^2=0$ has root $\sigma_3=-\frac{C\sigma_1 t^2+B\sigma_1^3}{2A}$. From Corollary 1 we know that

$$\max(0,\frac{(\sigma_1+t)^2(\sigma_1-2t)}{27})\leq\sigma_3\leq\frac{(\sigma_1-t)^2(\sigma_1+2t)}{27}.$$

Hence $\max(0,\frac{(\sigma_1+t)^2(\sigma_1-2t)}{27})\le\frac{-C\sigma_1 t^2-B\sigma_1^3}{2A}\le\frac{(\sigma_1-t)^2(\sigma_1+2t)}{27}$.

From formula above we obtain

$4At^3+(27C-6A)t^2+2A+27B\ge 0, 4At^3+(6A-27C)t^2-2A-27B\ge 0$,

$B+Ct^2\le 0$ （$0\le t\le 1$）

The intersection of the solutions of three inequalities above is the range of t (3), and the original inequality holds if and only if

$$\min f(\sigma_3)=f(-\frac{C\sigma_1 t^2+B\sigma_1^3}{2A})\ge 0$$

$$\Leftrightarrow\frac{4Ag(\sigma_1,t)-(B+Ct^2)^2}{4A}\ge 0$$

$$\Leftrightarrow 4A(D+Et^2+Ft^4+Gt^6)-(B^2+C^2t^4+2BCt^2)\ge 0$$

$$\Leftrightarrow 4AGt^6+(4AF-C^2)t^4+(4AE-2BC)t^2+4AD-B^2\ge 0.$$

If for t satisfying (3) formula above holds then the original inequality holds, otherwise the original inequality doesn’t hold.

Then Theorem 9 holds.

Hence the judgment for fully symmetric inequalities of degree 6 with 3 variables is solved in theory. Since $f(x,1,0)\ge 0$, $f(x,1,1)\ge 0$ are inequalities of degree 6 with one variable and there are 23 cases in discussing the solution of a inequality of degree 6 with one variable, hence the judgment for fully symmetric inequalities of degree 6 with 3 variables is very difficult.

Further for variables defined on any interval (in real number filed), we all obtain perfect judgment.

In theory we know easily that if we can get all real roots of $f'(\sigma_3)$ or $f'(\sigma_2)$ of $f'(\sigma_1)$, then we can using Theorem 9 to judge the homogeneous fully symmetric inequalities of degree n with 3 variables. Hence as long as the degree of one among $\sigma_3,\sigma_2,\sigma_1$ is less than 5(the derivation has degree less than or equal to 4). These inequalities can be solved whenever the equivalency condition is. For some special problems of higher dimensions can be solved either. Using similar method we may get judgment theorem for fully symmetric inequalities of degree n with 3 variables $(n=7,8,9,10,11)$ and we omit it here. But when the degree is 12-14，15-17, we have to solve a equation with 3 variables or 4 variables, and there are a large quantity computations. It is hard to give a full judgment theorem.

4.2 Judgment of equivalency condition for homogeneous fully symmetric inequality with 3 variables

A cyclic symmetric form can be expressed by fundamental polynomials; hence theorems above work for cyclic symmetric inequalities. Since all cyclic symmetric inequalities can be written in the form $f(a,b,c)=g(\sigma_1,\sigma_2,\sigma_3)+h(\sigma_1,\sigma_2,\sigma_3)\sum_{cyc}a^2b\ge 0$. The quantity $\sum_{cyc}a^2b$ is very important; hence it is essential to estimate $\sum_{cyc}a^2b$. Firstly we fix σ_1,σ_2 and get the upper and lower bounds of $\sum_{cyc}a^2b$. Since

$$\sum_{cyc}a^2b+\sum_{cyc}ab^2=\sigma_1\sigma_2-3\sigma_3,(\sum_{cyc}a^2b-\sum_{cyc}ab^2)^2=\sigma_1^2\sigma_2^2-4\sigma_2^3+2\sigma_1(9\sigma_2-2\sigma_1^2)\sigma_3-27\sigma_3^2$$

Then $\sum_{cyc}a^2b=\dfrac{\sigma_1\sigma_2-3\sigma_3\pm\sqrt{\sigma_1^2\sigma_2^2-4\sigma_2^3+2\sigma_1(9\sigma_2-2\sigma_1^2)\sigma_3-27\sigma_3^2}}{2}$

Set $f(\sigma_3)=\sigma_1\sigma_3-3\sigma_3+\sqrt{\sigma_1^2\sigma_2^2-4\sigma_2^3+2\sigma_1(9\sigma_2-2\sigma_1^2)\sigma_3-27\sigma_3^2}$

$$f'(\sigma_3)=-3+\frac{1}{2}(-54\sigma_3+2\sigma_1(9\sigma_2-2\sigma_1^2))\left|(a-b)(b-c)(c-a)\right|^{-1}$$

When $f'(\sigma_3)=0$，$\sum_{cyc}a^2b$ attains the maximum.

While $f'(\sigma_3)=0\Leftrightarrow 3|a-b|\ |b-c|c-a|=(9\sigma_1\sigma_2-2\sigma_1^3-27\sigma_3)$

$$\Leftrightarrow\sum a^3+6abc=3\sum_{cyc}ab^2$$

$$\Leftrightarrow 243\sigma_3^2+\left(36\sigma_1^3-162\sigma_1\sigma_2\right)\sigma_3+\sigma_1^6+18\sigma_1^2\sigma_2^2+9\sigma_2^3-9\sigma_1^4\sigma_2=0$$

$$\Leftrightarrow\sigma_3=\frac{9\sigma_1\sigma_2-2\sigma_1^3\pm\sqrt{\sigma_1^6+27\sigma_1^2\sigma_2^2-9\sigma_1^4\sigma_2-27\sigma_2^3}}{27},$$

It is easy to see that when σ_3 is the bigger one, $\sum_{cyc} a^2b$ attains its maximum. Hence

$$\max\sum_{cyc} a^2b = \sigma_1\sigma_2 - 3\sigma_3 - \frac{\sum a^3 + 6abc}{3} = \sigma_1\sigma_2 - 3\sigma_3 - \frac{\sigma_1^3 + 9\sigma_3 - 3\sigma_1\sigma_2}{3} = \frac{6\sigma_1\sigma_2 - \sigma_1^3}{3} - 6\sigma_3$$

$$= \frac{6\sigma_1\sigma_2 - \sigma_1^3}{3} - \frac{2}{9}(9\sigma_1\sigma_2 - 2\sigma_1^3 + \sqrt{\sigma_1^6 + 27\sigma_1^2\sigma_2^2 - 9\sigma_1^4\sigma_2 - 27\sigma_2^3})$$

$$= \frac{\sigma_1^3}{9} + \frac{2}{9}\sqrt{\sigma_1^6 + 27\sigma_1^2\sigma_2^2 - 9\sigma_1^4\sigma_2 - 27\sigma_2^3} = \frac{\sigma_1^3}{9} + \frac{2}{9}(\sigma_1^2 - 3\sigma_2)^{\frac{3}{2}}$$

Similarly we get $\min\sum_{cyc} a^2b = \frac{\sigma_1^3}{9} - \frac{2}{9}(\sigma_1^2 - 3\sigma_2)^{\frac{3}{2}}$

Hence $\min\sum_{cyc} a^2b = \frac{\sigma_1^3}{9} - \frac{2}{9}(\sigma_1^2 - 3\sigma_2)^{\frac{3}{2}}, \max\sum_{cyc} a^2b = \frac{\sigma_1^3}{9} + \frac{2}{9}(\sigma_1^2 - 3\sigma_2)^{\frac{3}{2}}$.

Using similar method we can prove some other cyclic symmetric inequalities with 3 variables.

Now we consider the judgment for cyclic symmetric inequalities. Chen Shengli has obtained a sufficient and necessary condition for cyclic symmetric inequalities of degree 3 with 3 variables and considered cyclic symmetric inequalities of degree 4 with 3 variables and obtained some results[17]. However there is no result on the judgment for cyclic symmetric inequalities of degree 4 with 3 variables. We will begin with cyclic symmetric inequalities of degree 4 with 3 variables, get some general results and make some extensions to higher degrees.

A cyclic symmetric inequality of degree 4 with 3 variables can be written as

$$F(a,b,c) = k_1\sigma_1^4 + k_2\sigma_1^2\sigma_2 + k_3\sigma_2^2 + k_4\sigma_1\sigma_3 + k_0\sigma_1\sum_{cyc} a^2b$$

and $\sum_{cyc} a^2b \geq \sum_{cyc} ab^2$, that is

$$\sum_{cyc} a^2b = \frac{\sigma_1\sigma_2 - 3\sigma_3 + \sqrt{\sigma_1^2\sigma_2^2 - 4\sigma_2^3 + 2\sigma_1(9\sigma_2 - 2\sigma_1^2)\sigma_3 - 27\sigma_3^2}}{2}.$$

From the homogeneity of the inequality we may assume $\sigma_1 = 1$.

Theorem 10 Judgment theorem for cyclic symmetric inequalities of degree 4 with 3 variables defined on nonnegative real field:

A cyclic symmetric inequality of degree 4 with 3 variables holds if and only if

$f(x,1,1)\ge 0, f(x,1,0)\ge 0$ ， $x\in[0,+\infty)$;

If the equation $A\sigma_3^2+B\sigma_3+C=0$ about σ_3 has real roots σ_{3_i} （$i=1,2$）

Here $A=729k_0^2+27(2k_4-3k_0)^2$,

$B=-486k_0^2\sigma_1\sigma_2+108\sigma_1^3k_0^2-2(2k_4-3k_0)^2\sigma_1(9\sigma_2-2\sigma_1^2)$,

$C=(81\sigma_1^2\sigma_2^2+4\sigma_1^6-36\sigma_1^4\sigma_2)k_0^2-(2k_4-3k_0)^2(\sigma_1^2\sigma_2^2-4\sigma_2^3)$

and there are σ_{3_i} such that $\max(0,\frac{(1+t)^2(1-2t)}{27})\le\sigma_{3_i}\le\frac{(1-t)^2(1+2t)}{27}$ (4)

Then for t,σ_{3_i} satisfying (4) also satisfy $f(\sigma_{3_i})_{\min}\ge 0$.

Proof: we fix σ_1, σ_2 and change σ_3.

$$F'(\sigma_3)=k_4\sigma_1+k_0\sigma_1\frac{-3+\frac{1}{2}(18\sigma_1\sigma_2-4\sigma_1^3-54\sigma_3)(\sigma_1^2\sigma_2^2-4\sigma_2^3+2\sigma_1(9\sigma_2-2\sigma_1^2)\sigma_3-27\sigma_3^2)^{-\frac{1}{2}}}{2}$$

$$\frac{2F'(\sigma_3)}{\sigma_1}=(2k_4-3k_0)(\sigma_1^2\sigma_2^2-4\sigma_2^3+2\sigma_1(9\sigma_2-2\sigma_1^2)\sigma_3-27\sigma_3^2)^{\frac{1}{2}}+k_0(9\sigma_1\sigma_2-2\sigma_1^3-27\sigma_3)$$

$$F'(\sigma_3)=0\Leftrightarrow(2k_4-3k_0)^2(\sigma_1^2\sigma_2^2-4\sigma_2^3+2\sigma_1(9\sigma_2-2\sigma_1^2)\sigma_3-27\sigma_3^2)=k_0^2(9\sigma_1\sigma_2-2\sigma_1^3-27\sigma_3)^2$$

$$\Leftrightarrow A\sigma_3^2+B\sigma_3+C=0 \qquad (5)$$

where $A=729k_0^2+27(2k_4-3k_0)^2$,

$B=-486k_0^2\sigma_1\sigma_2+108\sigma_1^3k_0^2-2(2k_4-3k_0)^2\sigma_1(9\sigma_2-2\sigma_1^2)$,

$C=(81\sigma_1^2\sigma_2^2+4\sigma_1^6-36\sigma_1^4\sigma_2)k_0^2-(2k_4-3k_0)^2(\sigma_1^2\sigma_2^2-4\sigma_2^3)$

$\sigma_{3_{1,2}}=\frac{-B\pm\sqrt{B^2-4AC}}{2A}$, and $\max(0,\frac{(\sigma_1+t)^2(\sigma_1-2t)}{27})\le\sigma_3\le\frac{(\sigma_1-t)^2(\sigma_1+2t)}{27}$

If (5) has no solution or there is no σ_{3_i} satisfying (4), then inequality holds if and only if $f(x,1,1)\ge 0, f(x,1,0)\ge 0$ ， $x\in[0,+\infty)$. If not then $f(\sigma_3)_{\min}=f(\sigma_{3_i})_{\min}$.

Hence it is sufficient to prove that $f(\sigma_{3_i})_{\min}\ge 0$ holds on the range of σ_{3_i} such that （4） holds. This completes the proof.

Further more for variables defined on any interval (in the real number field), we can also get perfect ways of judgment.

Like before, in theory, as long as we get all roots of $f'(\sigma_3)$ or $f'(\sigma_2)$ or $f'(\sigma_1)$, then we can judge cyclic symmetric inequalities of degree n with 3 variables using Theorem 10. But when the degree is 5 or 6, we have to solve a equation of degree 3 or 4, and it becomes more complicated since there are a large quantities of computations. Hence it is hard to give a complete judgment theorem.

5 Conclusions

This research, by controlling two fundamental symmetric polynomials and using the monotony of functions and Jensen inequality, gives some sufficient and necessary conditions for the equivalency of some fully symmetric inequalities with 3 variables. Then we obtain some theorem on the equivalency conditions for some fully symmetric inequalities of degree 6 with 3 variables, and give some applications ny using some examples. At last we obtain judgment theorems for homogeneous fully symmetric inequalities of degree 6 with 3 variables and homogeneous fully cyclic inequalities of degree 4 with 3 variables and consider the possibility of the judgment for higher degrees in theory. Our research has widely applications in the proof of inequalities.

6 Problems and prospects

In the proof there are some coincidences that can not be explained, and there are many difficulties in the extension. We list them here and make some prospects.

6.1 Can we extend inequalities in Theorem 1 to degree 4, even degree n?

This requires that $f'(\sum a_1, \sum a_1a_2, \sum a_1a_2a_3, \cdots\cdots, \prod_{i=1}^{n} a_i)$ has at least two real roots.

For such function of degree n, its derivation is of degree n-1. Yet we don't know the discriminant of equations with real coefficient of degree $n \geq 5$. Hence it is almost impossible to make extensions. To extend it to 4 or 5 variables we have to solve a equation of higher degree which contains a large quantity of computations.

6.2 Jensen inequality can be considered as fixing $\sum x_i$, and by using properties of convex or concave functions to adjust n variables to be equal. The core of the third section is by fixing two polynomials $\sum x_i$, $\sum x_i^m$, and using properties of convex or concave functions, to adjust the n variables to make n-1 of them equal or some variable touching the boundary. So can we adjust the n variables to make n-p-1 of them equal or some variable touching the boundary by fixing p polynomials such as $\sum x_i^m$, and using properties of convex or concave functions,?

6.3 Academician Yang Lu pointed that for judgment of the number of roots when the Coefficients are constant or the text Coefficients are in given range can be solved by using computers. Hence we conjecture that using the method in section 4, and making some improvements, it can be used in the proof of inequalities using computers, in the judgment of more general fully symmetric or cyclic inequalities with 3 variables.

These problems need further and in-depth study.

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