Title

Sharp Polynomial Estimate of Integral Points in Real-Angled Simplices

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ABSTRACT: Characterization of homogeneous polynomials with isolated critical point at the origin follows from a study of complex geometry. Yau previously proposed a Numerical Characterization Conjecture. A step forward in solving this Conjecture, the Granville-Lin-Yau Conjecture was formulated, with a sharp estimate that counts the number of positive integral points in n-dimensional ($n \geq 3$) real right-angled simplices with vertices whose distance to the origin are at least $n-1$. The estimate was proven for $n \leq 6$ but has a counterexample for $n = 7$. In this project we come up with an idea of forming a new sharp estimate conjecture where we need the distances of the vertices to be n. We have proved this new sharp estimate conjecture for $n \leq 7$ and are in the process of proving the general n case.

1. INTRODUCTION

Let $\Delta(a_1, a_2, ..., a_n)$ be an n-dimensional simplex described by

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + ... + \frac{x_n}{a_n} \leq 1, \;\; x_1, x_2, ..., x_n \geq 0 \tag{1.1}$$

where $a_1 \geq a_2 \geq ... \geq a_n \geq 1$ are positive real numbers. Let $P_{(a_1,a_2,...,a_n)}$ be defined as the number of positive integral solutions of (1.1) and $Q_{(a_1,a_2,...,a_n)}$ be defined as the number of nonnegative integral solutions of (1.1). It is known that the study of $P_{(a_1,a_2,...,a_n)}$ and $Q_{(a_1,a_2,...,a_n)}$ are equivalent. If we let $a = \frac{1}{a_1} + \frac{1}{a_2} + .. + \frac{1}{a_n}$, the relation is given by the following formulas:

$$Q_{(a_1,a_2,...,a_n)} = P_{(a_1(1+a),a_2(1+a),...,a_n(1+a))} \tag{1.2}$$

$$P_{(a_1,a_2,...,a_n)} = Q_{(a_1(1-a),a_2(1-a),...,a_n(1-a))} \tag{1.3}$$

The computation of $Q_{(a_1,a_2,...,a_n)}$ has generated interest among leading mathematicians for decades. Hardy and Littlewood wrote several papers that have applications to problems of Diophantine approximation ([9], [10], [11]). The effort was carried on by D. C. Spencer who subsequently wrote on the problem of estimating $Q_{(a_1,a_2,...,a_n)}$ ([23],[24]). The general problem of counting $P_{(a_1,a_2,...,a_n)}$ and $Q_{(a_1,a_2,...,a_n)}$ where $a_1, a_2, ..., a_n$ are positive integers continues to be a challenge in recent years, and tremendous research is being put into developing an exact formula (see [4],[3],[6],[12]). Mordell gave a formula for $Q_{(a_1,a_2,a_3)}$, expressed in terms of three Dedekind sums, in the case that a_1, a_2, and a_3 are pairwise relatively prime [19]. Pommersheim extended the formula for $Q_{(a_1,a_2,a_3)}$ to arbitrary a_1, a_2, and a_3 using toric varieties [20].

The earliest results to approximate $P_{(a_1,a_2,...,a_n)}$ or $Q_{(a_1,a_2,...,a_n)}$ were asymptotic in nature. Because of this, they are short of practical applications in number theory and geometry. Recent efforts are also restricted in application as they are limited to integral simplices. Furthermore, the involvement of generalized Dedekind sums or other complicated terms [1] makes it difficult to determine the order of magnitude of $P_{(a_1,a_2,...,a_n)}$.

Although we do not know if any such formula exists, ideally $P_{(a_1,a_2,...,a_n)}$ could be counted in terms of a polynomial in $a_1, a_2, ..., a_n$, where $a_1, a_2, ..., a_n$

are not limited to integers, but can be any positive real numbers. However for the applications in number theory and singularity theory, a relatively sharp upper estimate should be more than sufficient. The research of lattice points in simplices is currently a very active area. An excellent article relating to lattice points in rational tetrahedra was written by Barvinok and Pommersheim [2]. For more information, please refer to the collection "Integer Points in Polyhedra- Geometry, Number Theory, Algebra, Optimization," a Snowbird Conference Proceedings published by the AMS (Contemporary Mathematics, vol. 374, 2005).

An upper polynomial estimate of $P_{(a_1,a_2,...,a_n)}$ would have many applications. According to Granville [8], it is a key topic in number theory. Such an estimate could be applied to finding large gaps between primes, to Waring's problem, to primality testing and factoring algorithms, and to bounds for the least prime k-th power residues and non-residues (mod n). Given a set P of primes $p_1 < p_2 < ... < p_n < y$, number theorists are interested in counting the number of integers $m \leq y^u$ where $m = p_1^{l_1} p_2^{l_2} ... p_n^{l_n}$ for all $u \geq 2$. This is equivalent to counting the number of $(l_1, l_2, ... l_n) \in \mathbb{Z}^n_{\geq 0}$ such that $l_1 p_1 + l_2 p_2 + ... + l_n p_n \leq \log y^u$, which is also equivalent to counting the number of $(l_1, l_2, ..., l_n) \in \mathbb{Z}^n_{\geq 0}$ such that

$$\frac{l_1}{a_1} + \frac{l_2}{a_2} + ... + \frac{l_n}{a_n} \leq 1, \text{ where } a_i = \frac{\log y^u}{\log p_i} \tag{1.4}$$

Observe that the a_i's are not integral in general. Please see Carl Pomerance's ICM 1994 lecture at Zürich [21] and his lecture notes [22] for more information about applications of $P_{(a_1,a_2,...,a_n)}$ and $Q_{(a_1,a_2,...,a_n)}$.

The current method for counting $P_{(a_1,a_2,...,a_n)}$ is the polynomial estimate (1.6) provided by number theorists. Attach a unit cube to the right of and above each lattice point of $\Delta(a_1, a_2, ..., a_n)$. Then

$$\begin{aligned} Q_{(a_1,a_2,...,a_n)} &\leq \sum \text{ volume of the unit cube attached to each lattice point} \\ &\leq \text{ volume of } (x_1, x_2, ..., x_n) \in \mathbb{R}^n_+ : \sum_{i=1}^{n} \frac{x_1 - 1}{a_i} \leq 1 \\ &= \frac{1}{n!}(a_1 a_2 ... a_n)(\sum_{i=1}^{n} \frac{1}{a_i})^n \qquad (1.5) \end{aligned}$$

In view of (1.2), (1.5) can be rewritten as

$$P_{(a_1,a_2,...,a_n)} \leq \frac{1}{n!} a_1 a_2 ... a_n \tag{1.6}$$

The estimate of $P_{(a_1,a_2,...,a_n)}$ given by (1.6) is interesting. However, it is not strong enough to be useful, particularly when many of the a_i's are small [8].

In geometry and singularity theory, estimating P_n for real right-angled simplices is connected with the Durfee Conjecture. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of a complex analytic function with an isolated critical point at the origin. Let $V = \{(z_1, ..., z_n) \in \mathbb{C}^n : \ f(z_1, ..., z_n) = 0\}$. The Milnor number of the singularity (V,0) is defined as

$$\mu = \dim\, \mathbb{C}\{z_1, ..., z_n\}/(f_{z_1}, ..., f_{z_n})$$

the geometric genus p_g of (V,0) is defined as

$$p_g = \dim\, H^{n-2}(M, \mathcal{O}),$$

where M is a resolution of V and $\mathcal{O}$ is the sheaf of germs of holomorphic functions on M. In 1978, Durfee [7] made the following conjecture:

Durfee Conjecture. $n!p_g \leq \mu$ with equality only when $\mu = 0$.

If $f(z_1, ..., z_n)$ is a weighted homogeneous polynomial of type $(a_1, a_2, ..., a_n)$ with an isolated singularity at the origin, Milnor and Orlik [18] proved that $\mu = (a_1 - 1)...(a_n - 1)$. On the other hand, Merle and Teissier [17] showed that $p_g = P_n$, where P_n is the number of positive integral solutions of (1.1). Finding an estimate of P_n eventually led to a resolution of the Durfee Conjecture [29].

Starting from early 90's, Yau, Xu and Lin ([15],[26],[28]) tried to get sharp upper estimates of P_n where a_i are just positive real numbers. They were able to obtain it under certain conditions, specifically when n =3,4, and 5. Surprisingly enough, these sharp estimates are all polynomials of a_i:

$3!P_3 \ \leq \ f_3 = a_1a_2a_3 - (a_1a_2 + a_1a_3 + a_2a_3) + a_1 + a_2$

$4!P_4 \ \leq \ f_4 = a_1a_2a_3a_4 - \frac{3}{2}(a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4) + \frac{11}{3}(a_1a_2 + a_1a_3 + a_2a_3) - 2(a_1 + a_2)$

$5!P_5 \ \leq \ f_5 = a_1a_2a_3a_4a_5 - 2(a_1a_2a_3a_4 + a_1a_2a_3a_5 + a_1a_2a_4a_5 + a_1a_3a_4a_5 + a_2a_3a_4a_5) + 354(a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4) - \frac{50}{6}(a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_2a_5) + 6(a_1 + a_2 + a_3 + a_4)$

These estimates are considered sharp because the equality holds true if and only if all a_i take the same integer. Inspired by the similarity of these estimates, the general form of the upper estimate was conjectured.

Granville-Lin-Yau (GLY) conjecture Let P_n = number of element of set $\left\{(x_1, x_2, ...x_n) \in \mathbf{Z}_+^n; \frac{x_1}{a_1} + \frac{x_2}{a_2} + ... + \frac{x_n}{a_n} \leq 1\right\}$. Let $n \geq 3$,

(1) Sharp Estimate: if $a_1 \geq a_2 \geq ... \geq a_n \geq n-1$, then

$$(1.7) \qquad n!P_n \le f_n := A_0^n + \frac{s(n,n-1)}{n} A_1^n + \sum_{l=1}^{n-2} \frac{s(n,n-1-l)}{\binom{n-1}{l}} A_l^{n-1}$$

$s(n,k)$ is the Stirling number of the first kind defined by (3.2) and A_k^n be defined as in (3.1). Equality holds if and only if $a_1 = a_2 = ... = a_n =$ integer.

(2) Rough Estimate: If $a_1 \ge a_2 \ge ... \ge a_n > 1$

$$(1.8) \qquad n!P_n < q_n := \prod_{i=1}^{n} (a_i - 1)$$

The rough estimate in (1.8) has recently been proven true by Yau and Zhang [29]. When n=3,4, and 5, this conjecture is true ([14],[15],[26],[28]). The sharp estimate conjecture was first formulated in [16]. In private communication to Yau, Granville formulated this sharp estimated conjecture independently after reading [14].

The importance of this Upper Estimate Conjecture is twofold. First the Durfee Conjecture in singularity theory becomes a special case. And second, more importantly, it is the first main step to prove the following conjecture made by Yau in 1995:

Conjecture 1 Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of a weighted homogeneous polynomial with isolated critical points at the origin. let μ, P_g and v be respectively the Milnor number, geometric genus and multiplicity of the singularity $V = \{z \in \mathbb{C}^{n+1} : f(z) = 0\}$. Then
$\mu - h(v) \ge (n+1)!p_g$ where $h(v) = (v-1)^{n+1} - v(v-1)...(v-n)$, and the equality holds if and only if f is a homogeneous polynomial.

The above conjecture was proven for the case n=3 in [27] and for the case n=4 in [13]. it leads to the following numerical characterization of an affine variety in $\mathbb{C}^{n+1}$ as a cone over nonsingular projective variety $\mathbb{C}P^n$.

Conjecture 2 Let V be an affine hyperspace in $\mathbb{C}^{n+1}$. Then V is a cone over nonsingular hypersurface in $\mathbb{C}P^n$ if and only if V has only isolated singularity at the origin, $\mu = \tau$ and $\mu - (v-1)^{n+1} + v(v-1)...(v-n) = (n+1)!p_g$, where $\tau = \dim\ \mathbb{C}\{z_1, ..., z_{n+1}\}(f, f_{z_1}, ..., f_{z_{n+1}})$

The GLY Conjecture has been proved individually for $n = 3, 4, 5$ and generally for $n \le 6$. However, for the case $n = 7$, a counterexample to the conjecture has been given by [25].

Counter-example to GLY Conjecture for $n = 7$ Let $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 2000$ and $a_7 = 6.09$. Then consider the following 7-dimensional tetrahedron: $x_i > 0, 1 \le i \le 7$.

$$\frac{x_1}{2000}+\frac{x_2}{2000}+\frac{x_3}{2000}+\frac{x_4}{2000}+\frac{x_5}{2000}+\frac{x_6}{2000}+\frac{x_7}{6.09}\leq 1$$

P_7 has been computed to be $0.39656226290532420 \times 10^{17}$
Now we compute the sharp estimate f_7 when $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 2000$ and $a_7 = 6.09$.

$$\begin{aligned}f_7 &= A_0^7 + A_1^7\frac{s(7,6)}{7} + \sum_{l=1}^{5} A_l^6 \frac{s(7,6-l)}{\binom{6}{l}} \\ &= .199840413 \times 10^{21}\end{aligned} \tag{1.9}$$

So we have

$$f_7 - 7!P_7 = -.269675 \times 10^{17}$$

This shows that the sharp estimate of GLY Conjecture fails in the case $n = 7$. After discovering this counter-example, Wang and Yau modified the GLY Conjecture.

Modified GLY Conjecture. There exists an integer α which depends only on n such that the sharp estimate (1.7) holds when $a_1 \geq a_2 \geq ... \geq a_n \geq \alpha$

In order to get the estimate of P_n for the general n, Wang and Yau [25] drew upon ideas from $n = 4, 5$ and proposed a uniform method of partitioning the n-dimensional right-angled simplex into several $(n-1)$-dimensional right-angled simplices. Since the conjecture is true for $n = 3$, the proof of the general theorem would follow inductively. However, since α is not known, when induction is applied to prove the sharp estimate conjecture by dissecting the n-dimensional right-angled simplex along the x_n-axis into $(n-1)$-dimensional right-angled simplices, we cannot apply the lower-dimensional sharp estimate conjecture.

In our new conjecture, we modify (1.7) to give a larger estimate. We decrease what is subtracted in the second term to give a new estimate as follows:

$$Y_7 := A_0^7 - \frac{7}{2}(a_7)A_1^6 + \sum_{l=1}^{5} \frac{s(7,6-l)}{\binom{6}{l}} A_l^6 \tag{1.10}$$

(1.10) is very similar to (1.7) for $n = 7$ because only the second term is changed. It is also considered sharp because the homogenous case is not affected by the change. With this modification $Y_7 \leq 7!P_7$ can be proven for

$a_7 \geq 7$. Furthermore, the counterexample is no longer a counterexample to our estimate. Extending this from $n = 7$ to the general n, we get:

New Sharp Estimate Conjecture
Let $P_n =$ number of element of set $\left\{(x_1, x_2, ...x_n) \in \mathbf{Z}_+^n; \frac{x_1}{a_1} + \frac{x_2}{a_2} + ... + \frac{x_n}{a_n} \leq 1\right\}$.
Let $n \geq 3$,

If $a_1 \geq a_2 \geq ... \geq a_n \geq n$, then

$$n!P_n \leq Y_n := A_0^n - \frac{n}{2}(a_n)A_1^{n-1} + \sum_{l=1}^{n-2} \frac{s(n, n-1-l)}{\binom{n-1}{l}} A_l^{n-1} \tag{1.11}$$

Equality holds if and only if $a_1 = a_2 = ... = a_n =$ integer.

The above conjecture is sharp enough for application to Conjecture 1. Here, we only need the distances of the vertices to the origin to be at least n. The above conjecture given by Y_n can be easily proved for $n = 3, 4, 5, 6$ by a direct comparison with f_n because it can be proven for all n that $Y_n \geq f_n$. We have also proved this new conjecture for $n \leq 7$ and are in the process of proving the general n case.

2. MAIN THEOREM

Let $P_7 =$ the number of element of set

$$\left\{(x_1, x_2, ...x_7) \in \mathbf{Z}_+^7 : \frac{x_1}{a_1} + \frac{x_2}{a_2} + ... + \frac{x_7}{a_7} \leq 1\right\}$$

.
When $a_1 \geq a_2 \geq ... \geq a_7 \geq 7$, the sharp estimate is given by

$$7!P_7 \leq Y_7 := A_0^7 - \frac{7}{2}a_7 A_1^6 + \sum_{l=1}^{5} \frac{s(7, 6-l)}{\binom{6}{l}} A_l^6$$

and the equality holds if and only if $a_1 = a_2 = ...a_7 = integer$.

This can also be expressed as
$7!P_7 \leq Y_7 = a_1a_2a_3a_4a_5a_6a_7 - \frac{7}{2}a_7(a_1a_2a_3a_6 + a_1a_2a_4a_6 + a_1a_3a_4a_6 + a_2a_3a_4a_6 + a_1a_2a_5a_6 + a_1a_3a_5a_6 + a_1a_4a_5a_6 + a_2a_3a_5a_6 + a_2a_4a_5a_6 + a_3a_4a_5a_6 + a_1a_2a_3a_4 + a_1a_2a_3a_5 + a_1a_2a_4a_5 + a_1a_3a_4a_5 + a_2a_3a_4a_5) - \frac{735}{15}(a_1a_2a_3a_6 + a_1a_2a_4a_6 + a_1a_3a_4a_6 + a_2a_3a_4a_6 + a_1a_2a_5a_6 + a_1a_3a_5a_6 + a_1a_4a_5a_6 + a_2a_3a_5a_6 + a_2a_4a_5a_6 + a_3a_4a_5a_6 + a_1a_2a_3a_4 + a_1a_2a_3a_5 + a_1a_2a_4a_5 + a_1a_3a_4a_5 + a_2a63a_4a_5) + \frac{1624}{20}(a_1a_2a_6 + a_1a_3a_6 + a_1a_4a_6 + a_2a_3a_6 + a_2a_4a_6 + a_3a_4a_6 + a_1a_5a_6 + a_2a_5a_6 + a_3a_5a_6 + a_4a_5a_6 + a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4 + a_1a_2a_5 + a_1a_3a_5 + a_1a_4a_5 + a_2a_3a_5 + a_2a_4a_5 + a_3a_4a_5) - \frac{1764}{15}(a_1a_6 + a_2a_6 + a_3a_6 + a_4a_6 + a_5a_6 + a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4 + a_1a_5 + a_2a_5 + a_3a_5 + a_4a_5) + 120(a_1 + a_2 + a_3 + a_4 + a_5 + a_6)$

3. NOTATION

Notation 1: Polynomial of a_i : A_k^n

$$A_k^n = (\prod_{i=1}^{n} a_i)(\sum_{1\leq i_1\leq i_2\leq ...\leq i_k\leq n} \frac{1}{a_{i_1}a_{i_2}...a_{i_k}}) \tag{3.1}$$

Defined recursively we have,

$$A_k^n = a_n A_k^{n-1} + A_{k-1}^{n-1}$$

Notation 2: s(n,k)

s(n,k) is the Stirling number of the first kind defined by the generating function

$$x(x-1)...(x-n+1) = \sum_{k=0}^{n} s(n,k)x^k \tag{3.2}$$

Notation 3: Bernoulli Number and Polynomial

We will also use Bernoulli number B_k, which is definied by recursion formula

$$B_k = \sum_{i=0}^{k} \binom{k}{i} B_i, \text{ with } B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}$$

The most important property of the Bernoulli number is

$$B_{2k+1} = 0, \ for \ k \geq 1 \tag{3.3}$$

Bernoulli polynomial $B_k[x]$ is defined as

$$B_k[x] = \sum_{i=0}^{k} \binom{k}{i} B_i x^{k-i}$$

We will use the following equality about the Bernoulli number and Bernoulli polynomial:

$$\sum_{k=1}^{m} k^n = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} (m+1)^{n+1-k} B_k, \ for \ n \geq 1 \tag{3.4}$$

4. SHARP ESTIMATE ANALYSIS

Since the conjecture has been proved true for case $n = 6$, for $a_6 \geq 5$, we can use induction method to prove $n = 7$. The basic approach is to partition the higher dimension tetrahedron into several lower dimension tetrahedra [25]. We have:

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + ... + \frac{k}{a_7} \leq 1 \tag{4.1}$$

Using a simple computation, we can change the above form into:

$$\frac{x_1}{a_1(1-\frac{k}{a_7})} + \frac{x_2}{a_2(1-\frac{k}{a_7})} + ... + \frac{x_6}{a_6(1-\frac{k}{a_7})} \leq 1 \tag{4.2}$$

Let $P_6(k)$ be the number of positive integral solutions. Then

$$P_7 = \sum_{k=1}^{[a_7]} P_6(k) \tag{4.3}$$

Assume $a_6(1-\frac{k}{a_7}) \geq 5$ for all $1 \leq k \leq [a_7]$. We can apply sharp estimate from the induction assumption on the 6 dimensional tetrahedron.

We have sharp estimate $Y_6(k)$:

$$\begin{aligned} 6!P_6(k) \leq Y_6(k) &= A_0^6(1-\frac{k}{a_7})^6 \\ &- \frac{6}{2}A_1^5 a_6(1-\frac{k}{a_7})^5 + \sum_{l=1}^{4} \frac{s(6,5-l)}{\binom{5}{l}} A_l^5 (1-\frac{k}{a_6})^{5-l} \end{aligned}$$

By (4.3), we have

$$7!P_7 = 7\sum_{k=1}^{[a_7]} (6)!P_6(k)$$

$$\leq 7\sum_{k=1}^{[a_7]} Y_{(6)}(k)$$

In order to prove $7!P_7 \leq Y_7$, it is sufficient to prove

$$a_7^6 Y_7 - 7\sum_{k=1}^{[a_7]} a_7^6 Y_6(k) \geq 0 \tag{4.4}$$

The difficulties arise when $a_6(1-\frac{k}{a_7}) \geq 5$ is not satisfied and the sharp estimate can not be used.
However, if $k = m'$ satisfy the conditions, then all $1 \leq k < m'$ must statisfy this conditions. This is true since

$$a_6(1-\frac{k}{a_7}) > a_6(1-\frac{m'}{a_7}) \geq 5,\ for\ 1 \leq k < m'$$

So we can sum up k from 1 to m'.

The left hand side of (4.4) is a polynomial of $a_1, a_2, ..., a_7$. It is a difficult and lengthy computation to work out this polynomial manually so Python is used for our computations. Also, to compute this polynomial we transform the left hand to satisfy the following two requirements:

1. The lower and upper limits of the summation is determined by the degree of the polynomial.
2. The number of summation symbols in one term is minimized.

Lemma 3 transforms the polynomial to fufill the first requirement.
The proofs of the following two lemmas are given in [25]. Our proofs of Lemmas 3,4, and 5 are given in the Appendix.

5. **FIVE LEMMAS**

Lemma 1*(Coefficient Criteria)* [25] *Let $f(\beta)$ be a polynomial defined by*

$$f(\beta) = \sum_{i=0}^{n} c_i \beta^i, where \beta \in (0,1)$$

If for any $k = 0, 1, ..., n$

$$\sum_{i=0}^{k} c_i \geq 0$$

then $f(\beta) \geq 0$ for $\beta \in (0,1)$. If "$\geq$" is replaced by "$>$", the Lemma is still true.

The second Lemma allows us to use the initial value of all partial derivatives to determine the sign of the polynomial. For $i \le i_1 \le i_2 \le ... \le i_k \le n-1, k = 1, 2, ...n-1$, we use the following notation:

$$f^{(k)}(i_1, i_2, ...i_k) = \frac{\partial^k f}{\partial a_{i_1} \partial a_{i_2} ... \partial a_{i_k}}$$

Lemma 2 *(Initial Value Criteria)* [25] *Let $f(a_1, a_2, ..., a_n, \beta)$ be a polynomial of $a_i, 1 \le i \le n$ and β, where the degree of variable $a_i, i = 1, 2, ..., n-1$ is 1, and $\beta \in (0, 1)$. If*

(1) $f(a_n, a_n, ...a_n, \beta) \ge 0$, for $a_n \ge \alpha$ and $\beta \in (0, 1)$

(2) $f^{(k)}(i_1, i_2, ...i_k)|_{(a_n, a_n, ..., a_n, \beta)} \ge 0$, for $a_n \ge \alpha$ and $\beta \in (0, 1)$, and for all

$1 \le i_1 \le i_2 \le ... \le i_k \le n-1, k = 1, 2, ...n-1$;

Then $f(a_1, a_2, ..., a_n, \beta) \ge 0$ for $a_1 \ge a_2 \ge ... \ge a_n \ge \alpha$ and $\beta \in (0, 1)$.

If " $\ge$ " in $f(a_1, a_2, ..., a_n, \beta) \ge 0$ of condition (1)
and $f^{(k)}(i_1, i_2, ...i_k)|_{(a_n, a_n, ..., a_n, \beta)} \ge 0$ of condition (2) are replaced by " $>$ ", then $f(a_1, a_2, ..., a_n, \beta) > 0$

Corollary 1 *For $1 \le t \le n$, let $f(a, a_{t+1}, ..., a_n, \beta)$ be a polynomial of $a, a_i, t+1 \le i \le n$ and β, where the degree of variable a is t and the degree of $a_i, i = t+1, ...n-1$ is 1 and $\beta \in (0, 1)$. If*

(1)$f(a_n, a_n, ..., a_n, \beta) \ge 0$, for $a_n \ge \alpha$ and $\beta \in (0, 1)$;

(2)$\frac{\partial^s f}{\partial a^s}|_(a_n, a_n, ..., a_n, \beta) \ge 0$ for $1 \le s \le t$ and $a_n \ge \alpha$ and $\beta \in (0, 1)$

(3) For $1 \le s \le t$ and $1 \le k \le n-1-t$
where $t+1 \le i_{t+1} \le i_{t+2} \le ... \le i_{t+k} \le n-1$.

Then $f(a, a_{t+1}, ..., a_n, \beta) \ge 0$ for $a \ge a_{t+1} \ge ... \ge a_n \ge \alpha$ and $\beta \in (0, 1)$

When $t = 1$, this collary is the same as Lemma 2. When $t = n$, condition (3) is not needed.

Lemma 3 (For details and proof of Lemma 3 please see Appendix)
Let

$$G(m') = \sum_{k=1}^{m'} Y_{n-1}(k) \tag{5.1}$$

Then $G(m')$ can be expressed by a summation with limits determined by n alone

$$G(m') = \frac{1}{n}A_0^{n-1}\sum_{i=0}^{n-2}\binom{n}{i}(-\frac{1}{a_n})^{n-1-i}\sum_{k=0}^{n-1-i}\binom{n-i}{k}(m'+1)^{n-i-k}B_k$$
$$-\frac{1}{2}a_{n-1}A_1^{n-2}\sum_{i=0}^{n-3}\binom{n-1}{i}(-\frac{1}{a_n})^{n-2-i}\sum_{k=0}^{n-2-i}\binom{n-1-i}{k}(m'+1)^{n-1-i-k}B_k$$
$$+\sum_{l=1}^{n-3}\frac{1}{n-1-l}\frac{s(n-1,n-2-l)}{\binom{n-2}{l}}A_l^{n-2}\sum_{i=0}^{n-3-l}\binom{n-1-l}{i}(-\frac{1}{a_n})^{n-2-l-i}$$
$$\times\sum_{k=0}^{n-2-l-i}\binom{n-1-l-i}{k}(m'+1)^{n-1-l-i-k}B_k + m'Y_{n-1} \tag{5.2}$$

where

$$Y_{n-1} = [A_0^{n-1} - \frac{n-1}{2}a_{n-1}A_1^{n-2} + \sum_{l=1}^{n-3}\frac{s(n-1,n-2-l)}{\binom{n-2}{l}}A_l^{n-2}] \tag{5.3}$$

Lemma 4 (For proof, please see appendix)
Let

$$g(m') = na_n^{n-1}G(m') \tag{5.4}$$

then,

$$g(a_n-\beta-m) = na_n^{n-1}\sum_{h=m}^{a_n-\beta-1}Y_{(n-1)}(a_n-\beta-h) \tag{5.5}$$

$$= A_0^{n-1}\left\{-\sum_{s=0}^{n}(-1)^{s-1}B_s\binom{n}{s}a_n^{n-s} + (-1)^{n-1}B_n[1-\beta-m]\right\}$$
$$-\frac{n}{2}a_{n-1}A_1^{n-2}\left\{-(n-1)a_n^{n-1} - \sum_{s=0}^{n-1}(-1)^{s-1}B_s\binom{n-1}{s}a_n^{n-s} + (-1)^{n-2}a_nB_{n-1}[1-\beta-m]\right\}$$
$$+\sum_{l=1}^{n-3}\frac{n}{n-1-l}\frac{s(n-1,n-2-l)}{\binom{n-2}{l}}A_1^{n-2}$$
$$\times\left\{-(n-1-l)a_n^{n-1} - \sum_{s=0}^{n-1-l}(-1)^{s-l}B_s\binom{n-1-l}{s}a_n^{n-s} + (-1)^{n-2-l}a_n^{l+1}B_{n-l}[1-\beta-m]\right\}$$

(5.6)

Lemma 5 (For proof, please see appendix)

Let

$$\Delta_0(a_n-\beta-m)=a_n^{n-1}Y_n-g(a_n-\beta-m) \tag{5.7}$$

then,

$$\Delta_0(a_n-\beta-m)=\sum_{i=n}^{2n-2}T_i+T_{n-1}(m)+\Phi(m,\beta) \tag{5.8}$$

where $T_i, n\le i\le 2n-2$ are polynomials of $a_1,a_2,...,a_n$ with coefficients that do not depend on β or m. Each term in T_i has degree of i. The expression of T_i are:

$$\begin{aligned}
T_{2n-2} &= \frac{n}{2}A_0^{n-1}a_n^{n-1}+\frac{n}{2}a_{n-1}a_n^nA_1^{n-2}-\frac{n}{2}A_1^{n-1}a_n^n\\
T_{2n-3} &= -\frac{n(n-1)}{4}a_{n-1}a_n^{n-1}A_1^{n-2}+\frac{s(n,n-2)}{\binom{n-1}{1}}A_1^{n-1}a_n^{n-1}\\
&\quad -\binom{n}{2}B_2A_0^{n-1}a_n^{n-2}-\frac{n}{n-2}\frac{s(n-1,n-3)}{\binom{n-2}{1}}A_1^{n-2}a_n^n\\
T_i &= \sum_{i=n}^{2n-4}\frac{s(n,i-(n-1))}{\binom{n-1}{2n-2-i}}a_n^{n-1}A_{2n-2-i}^{n-1}\\
&\quad +\sum_{i=n}^{2n-4}(-1)^i\binom{n}{2n-1-i}B_{2n-1-i}A_0^{n-1}a_n^{i-(n-1)}\\
&\quad +\sum_{i=n}^{2n-4}\frac{n}{2}(-1)^i\binom{n-1}{2n-2-i}B_{2n-2-i}A_1^{n-2}a_{n-1}a_n^{i-(n-2)}\\
&\quad -\sum_{i=n}^{2n-4}\frac{n}{i-n+1}\frac{s(n-1,i-n)}{\binom{n-2}{2n-2-i}}A_{2n-2-i}^{n-2}a_n^n\\
&\quad +\sum_{i=n}^{2n-4}\frac{n}{2}\frac{s(n-1,i-n+1)}{\binom{n-2}{2n-3-i}}A_{2n-3-i}^{n-2}a_n^{n-1}\\
&\quad +\sum_{i=n}^{2n-4}(-1)^i\sum_{s=1}^{2n-4-i}\frac{(-1)^{1+s}n}{n-1-s}\binom{n-1-s}{2n-2-i-s}\\
&\quad \times\frac{s(n-1,n-2-s)}{\binom{n-2}{s}}B_{2n-2-i-s}A_s^{n-2}a_n^{i+s-(n-2)}
\end{aligned}$$

$T_{n-1}(m)$ is a polynomial of $a_1, a_2, ..., a_n$ with coefficients depending only on n and m. Each term in $T_{n-1}(m)$ has degree of n-1. The expression of $T_{n-1}(m)$ is

$$\begin{aligned} T_{n-1}(m) &= (-1)^{n-2}\{B_n[1-m] - B_n\}A_0^{n-1} \\ &+ (-1)^{n-2}\frac{n}{2}\{B_{n-1}[1-m] - B_{n-1}\}A_1^{n-2}a_{n-1}a_n \\ &+ (-1)^{n-2}\sum_{l=1}^{n-3}\frac{n(-1)^{l+1}}{n-1-l}\frac{s(n-1,n-2-l)}{\binom{n-2}{l}} \\ &\times \{B_{n-1-l}[1-m] - B_{n-1-l}\}A_l^{n-2}a_n^{l+1} \end{aligned}$$

$\Phi(m,\beta)$ is the polynomial of $a_1, a_2, ..., a_n$ with coefficients depending on m and β. $\Phi(m,\beta) = 0$ if $\beta = 0$. Each term in $\Phi(m,\beta)$ has degree of n-1 and
$\Phi(m,\beta) = (-1)^{n-2}A_0^{n-1}\Psi(n,m,\beta) + (-1)^{n-2}\frac{n}{2}A_1^{n-2}a_{n-1}a_n\Psi(n-1,m,\beta)$
$+(-1)^{n-2}\sum_{l=1}^{n-3}\frac{n(-1)^{l+1}}{n-1-l}\frac{s(n-1,n-2-l)}{\binom{n-2}{l}}A_l^{n-2}a_n^{l+1}\Psi(n-1-l,m,\beta)$
where
$\Psi(n,m,\beta) = (-1)^n\sum_{s=0}^{n-1}\binom{n}{s}B_s[m]\beta^{n-s}$

6. PROOF OF SHARP UPPER ESTIMATE

We know that the Y_6 estimate is true. In order to get the proof of 7, we partition the 7th dimensional tetrahedron into the lower dimensional case. The inequality $\frac{x_1}{a_1}+\frac{x_2}{a_2}+...+\frac{k}{a_7} \le 1$ is the partition of the 7 dimensional tetrahedron into 6 dimensional tetrahedron. Using the notation $k = a_7 - \beta - h$ in (8.2), we can then transform the kth partition in (4.1) into the following form:

$$\frac{x_1}{\frac{a_1}{a_7}(\beta+h)} + \frac{x_2}{\frac{a_2}{a_7}(\beta+h)} + ... + \frac{x_7}{\frac{a_6}{a_7}(\beta+h)} \le 1 \qquad (6.1)$$

where $h = 0, 1, 2, ..., a_7 - \beta - 1$
Let $P_6(h)$ be the number of positive integer solution of (6.1). Then we have

$$P_7 = \sum_{h=0}^{[a_7]-1} P_6(h)$$

We also use the notation $q_6(a_7 - \beta - h)$ and $Y_6(a_7 - \beta - h)$ to denote the rough and sharp estimate defined for (6.1).

For each $P_6(h)$, there are three cases regarding its upper estimate:

(a). $P_6(h) = 0$, then we do not need to count this partition.

Also if $P_6(h') = 0$, then $P_6(h) = 0$ for all $h \leq h'$. Let h_0 be the smallest number such that $P_6(h_0) > 0$.

(b). $P_6(h) > 0$, and $\frac{a_6}{a_7}(\beta + h) < 5$. We know $\frac{a_6}{a_7}(\beta + h) > 1$. Then we can apply the rough estimate:

$$(6)!P_6(h) \leq q_6(a_7 - \beta - h)$$

(c). $P_6(h) > 0$, and $\frac{a_6}{a_7}(\beta+h) \geq (6)$. Then we can apply the sharp stimate:

$$(6)!P_6(h) \leq Y_6(a_7 - \beta - h)$$

So we have

$$7!P_7 = 7 \sum_{h=h_0}^{[a_7]-1} (6)!P_6(h) \tag{6.2}$$

$$\leq 7 \sum_{h=h_0}^{m-1} q_6(a_7 - \beta - h) + 7 \sum_{h=m}^{[a_7]-1} Y_6(a_7 - \beta - h)$$

where m is the smallest integer for which the sharp estimate condition $\frac{a_6}{a_7}(\beta+m) \geq 5$ is true. In order to show $7!P_7 \leq Y_7$, we only need to show that

$$Y_7 \geq 7 \sum_{h=h_0}^{m-1} q_6(a_6 - \beta - h) + 7 \sum_{h=m}^{[a_7]-1} Y_6(a_7 - \beta - h)$$

Now define

$$\Delta = a_7^6 Y_7 - 7a_7^6 \sum_{h=m}^{[a_7-1]} Y_6(a_7 - \beta - h) - 7a_7^6 \sum_{h=h_0}^{m-1} q_6(a_7 - \beta - h) \tag{6.3}$$

Using the definitions given by Lemmas 3 and 4, we have

$$\Delta = a_7^6 Y_7 - g(a_7 - \beta - m) - 7a_7^6 \sum_{h=h_0}^{m-1} q_6(a_7 - \beta - h)$$

$$= \Delta_0(a_7 - \beta - m) - 7a_7^6 \sum_{h=h_0}^{m-1} q_6(a_7 - \beta - h) \tag{6.4}$$

(The expression for $\Delta_0(a_7-\beta-m)$ is given in Lemma 5)

Prove that $\Delta \geq 0$
Notice that Δ is the polynomial of $a_1, a_2, ..., a_7, \beta$. Our task here is to show that $\Delta \geq 0$ for $a_1 \geq a_2 \geq ... \geq a_7 \geq 7$ and the equality holds when $a_1 = a_2 = ... = integer$. If we can determine m and h_0, then we can use Lemmas 1 and 2 to determine the sign of Δ. For this reason, we will study Δ in 6×7 subcases determined by

$$a_1 = a_2 = ... = a_{7-i} \geq a_{7-i+1} \geq ... \geq a_6, \text{ where } 1 \leq i \leq 6$$

$$P_6(5-j) = 0, P_6(6-j) > 0, \text{ where } 0 \leq j \leq 6$$

Also $P_6(h) = 0$ implies that $P_6(1) = P_6(2) = ... = P_6(h-1) = 0$

Proof for 7th dimension

$$\Delta_0(a_7-\beta-m)=\sum_{i=7}^{12}T_i+T_6(m)+\Phi(m,\beta) \text{ where,}$$

$$\begin{aligned}
T_{12} &= \frac{7}{2}A_0^6a_7^6+\frac{7}{2}a_6a_7^7A_1^5-\frac{7}{2}A_1^6a_7^7\\
T_{11} &= -\frac{21}{2}a_6a_7^6A_1^5+\frac{s(7,5)}{\binom{6}{1}}A_1^6a_7^6\\
&\quad -\binom{7}{2}B_2A_0^6a_7^5-\frac{7}{5}\frac{s(6,4)}{\binom{5}{1}}A_1^5a_7^7\\
T_i &= \sum_{i=7}^{10}\frac{s(7,i-6)}{\binom{6}{12-i}}a_7^6A_{12-i}^6\\
&\quad +\sum_{i=7}^{10}(-1)^i\binom{7}{13-i}B_{13-i}A_0^6a_7^{i-6}\\
&\quad +\sum_{i=7}^{10}\frac{7}{2}(-1)^i\binom{6}{12-i}B_{12-i}A_1^5a_6a_7^{i-5}\\
&\quad +\sum_{i=7}^{10}(-1)^i\sum_{s=1}^{12-i}\frac{(-1)^{1+s}7}{6-s}\binom{6-s}{12-i-s}\\
&\quad \times\frac{s(6,5-s)}{\binom{5}{s}}B_{12-i-s}A_s^5a_7^{i+s-5}\\
T_6(m) &= (-1)^5\{B_7[1-m]-B_7\}A_0^6\\
&\quad +(-1)^5\frac{n}{2}\{B_6[1-m]-B_6\}A_1^5a_6a_7\\
&\quad +(-1)^5\sum_{l=1}^{4}\frac{7(-1)^{l+1}}{6-l}\frac{s(6,5-l)}{\binom{5}{l}}\\
&\quad \times\{B_{6-l}[1-m]-B_{6-l}\}A_l^5a_n^{l+1}\\
\Phi(m,\beta) &= (-1)^5A_0^6\Psi(7,m,\beta)+(-1)^5\frac{7}{2}A_1^5a_6a_7\Psi(6,m,\beta)\\
&\quad +(-1)^5\sum_{l=1}^{4}\frac{7(-1)^{l+1}}{6-l}\frac{s(6,5-l)}{\binom{5}{l}}A_l^5a_7^{l+1}\Psi(6-l,m,\beta)
\end{aligned}$$

$$\Delta_0(a_7-\beta-m)=\sum_{i=7}^{12}T_i+T_6(m)+\Phi(m,\beta) \text{ where,}$$

$$
\begin{aligned}
T_{12} &= \frac{7}{2}a_1a_2a_3a_4a_5a_6a_7^6 + \frac{7}{2}(a_1a_2a_4a_5 + a_1a_3a_4a_5 + a_2a_3a_4a_5 \\
&\quad +a_1a_2a_3a_5 + a_1a_2a_3a_4)a_6a_7^7 - \frac{7}{2}(a_1a_2a_4a_5a_6 \\
&\quad +a_1a_3a_4a_5a_6 + a_2a_3a_4a_5a_6 + a_1a_2a_3a_5a_6 + a_1a_2a_3a_4a_6 \\
&\quad +a_1a_2a_3a_4a_5)a_7^7 \\
T_{11} &= -\frac{21}{2}(a_1a_2a_4a_5 + a_1a_3a_4a_5 + a_2a_3a_4a_5 + a_1a_2a_3a_5 \\
&\quad +a_1a_2a_3a_4)a_6a_7^6 + \frac{175}{6}(a_1a_2a_4a_5a_6 + a_1a_3a_4a_5a_6 \\
&\quad +a_2a_3a_4a_5a_6 + a_1a_2a_3a_5a_6 + a_1a_2a_3a_4a_6 \\
&\quad +a_1a_2a_3a_4a_5)a_7^6 - \frac{7}{2}a_1a_2a_3a_4a_5a_6a_7^5 - \frac{119}{5}(a_1a_2a_4a_5 \\
&\quad +a_1a_3a_4a_5 + a_2a_3a_4a_5 + a_1a_2a_3a_5 + a_1a_2a_3a_4)a_7^7 \\
T_{10} &= -49(a_1a_4a_5a_6 + a_2a_4a_5a_6 + a_3a_4a_5a_6 + a_1a_2a_5a_6 \\
&\quad +a_1a_3a_5a_6 + a_2a_3a_5a_6 + a_1a_2a_4a_6 + a_1a_3a_4a_6 \\
&\quad +a_2a_3a_4a_6 + a_1a_2a_3a_6 + a_1a_2a_4a_5 + a_1a_3a_4a_5 \\
&\quad +a_2a_3a_4a_5 + a_1a_2a_3a_5 + a_1a_2a_3a_4)a_7^6 + \frac{35}{4}(a_1a_2a_4a_5 \\
&\quad +a_1a_3a_4a_5 + a_2a_3a_4a_5 + a_1a_2a_3a_5 + a_1a_2a_3a_4)a_6a_7^5 \\
&\quad +\frac{119}{2}(a_1a_2a_4a_5 + a_1a_3a_4a_5 + a_2a_3a_4a_5 + a_1a_2a_3a_5 \\
&\quad +a_1a_2a_3a_4)a_7^6 + \frac{315}{8}(a_1a_4a_5 + a_2a_4a_5 + a_3a_4a_5 + a_1a_2a_5 \\
&\quad +a_1a_3a_5 + a_2a_3a_5 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4 + a_1a_2a_3)a_7^7 \\
T_9 &= \frac{1624}{20}(a_1a_5a_6 + a_3a_5a_6 + a_4a_5a_6 + a_1a_4a_6 + a_2a_4a_6 \\
&\quad +a_3a_4a_6 + a_1a_2a_6 + a_1a_3a_6 + a_2a_3a_6 + a_1a_4a_5 + a_2a_4a_5 \\
&\quad +a_2a_5a_6 + a_3a_4a_5 + a_1a_2a_5 + a_1a_3a_5 + a_2a_3a_5 + a_1a_2a_4 \\
&\quad +a_1a_3a_4 + a_2a_3a_4 + a_1a_2a_3)a_7^6 + \frac{7}{6}a_1a_2a_3a_4a_5a_6a_7^3 \\
&\quad -\frac{119}{3}(a_1a_2a_4a_5 + a_1a_3a_4a_5 + a_2a_3a_4a_5 + a_1a_2a_3a_5 \\
&\quad +a_1a_2a_3a_4)a_7^5 - \frac{315}{4}(a_1a_4a_5 + a_2a_4a_5 + a_3a_4a_5 + a_1a_2a_5 \\
&\quad +a_1a_3a_5 + a_2a_3a_5 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4 \\
&\quad +a_1a_2a_3)a_7^6 - \frac{959}{15}(a_1a_5 + a_2a_5 + a_3a_5 + a_4a_5 \\
&\quad +a_1a_4 + a_2a_4 + a_3a_4 + a_1a_2 + a_1a_3 + a_2a_3)a_7^7
\end{aligned}
$$

$$\begin{aligned}
T_8 &= -\frac{588}{5}(a_1a_6+a_2a_6+a_3a_6+a_4a_6+a_5a_6+a_1a_5 \\
&+a_2a_5+a_3a_5+a_4a_5+a_1a_4+a_2a_4+a_3a_4+a_1a_2 \\
&+a_1a_3+a_2a_3)a_7^6-\frac{7}{4}(a_1a_2a_4a_5+a_1a_3a_4a_5 \\
&+a_2a_3a_4a_5+a_1a_2a_3a_5+a_1a_2a_3a_4)a_6a_7^3 \\
&+\frac{315}{8}(a_1a_4a_5+a_2a_4a_5+a_3a_4a_5+a_1a_2a_5+a_1a_3a_5 \\
&+a_2a_3a_5+a_1a_2a_4+a_1a_3a_4+a_2a_3a_4+a_1a_2a_3)a_7^5 \\
&+\frac{959}{10}(a_1a_5+a_2a_5+a_3a_5+a_4a_5+a_1a_4+a_2a_4 \\
&+a_3a_4+a_1a_2+a_1a_3+a_2a_3)a_7^6+84(a_1+a_2+a_3+a_4+a_5)a_7^7 \\
T_7 &= 120(a_1+a_2+a_3+a_4+a_5+a_6)a_7^6 \\
&-\frac{1}{6}(a_1a_2a_3a_4a_5a_6)a_7+\frac{119}{30}(a_1a_2a_4a_5+a_1a_3a_4a_5 \\
&+a_2a_3a_4a_5+a_1a_2a_3a_5+a_1a_2a_3a_4)a_7^3-\frac{959}{30}(a_1a_5 \\
&+a_2a_5+a_3a_5+a_4a_5+a_1a_4+a_2a_4+a_3a_4 \\
&+a_1a_2+a_1a_3+a_2a_3)a_7^5-84(a_1+a_2+a_3+a_4+a_5)a_7^6
\end{aligned}$$

$$\begin{aligned}
T_6(m) &= -\{B_7[1-m]\}a_1a_2a_3a_4a_5a_6-\frac{7}{2}\{B_6[1-m]-B_6\}(a_1a_2a_4a_5 \\
&+a_1a_3a_4a_5+a_2a_3a_4a_5+a_1a_2a_3a_5+a_1a_2a_3a_4)a_6a_7 \\
&-\frac{119}{5}\{B_5[1-m]\}(a_1a_2a_4a_5+a_1a_3a_4a_5+a_2a_3a_4a_5+a_1a_2a_3a_5 \\
&+a_1a_2a_3a_4)a_7^2-\frac{315}{8}\{B_4[1-m]-B_4\}(a_1a_4a_5+a_2a_4a_5 \\
&+a_3a_4a_5+a_1a_2a_5+a_1a_3a_5+a_2a_3a_5+a_1a_2a_4+a_1a_3a_4+a_2a_3a_4 \\
&+a_1a_2a_3)a_7^3-\frac{959}{15}\{B_3[1-m]\}(a_1a_5+a_2a_5+a_3a_5+a_4a_5 \\
&+a_1a_4+a_2a_4+a_3a_4+a_1a_2+a_1a_3+a_2a_3)a_7^4 \\
&-84\{B_2[1-m]-B_2\}(a_1+a_2+a_3+a_4+a_5)a_7^5
\end{aligned}$$

$$\begin{aligned}
\Phi(m,\beta) &= -A_0^6(-1)^7\sum_{s=0}^{6}\binom{7}{s}B_s[m]\beta^{7-s} \\
&+(-1)^5\frac{7}{2}A_1^5a_6a_7(-1)^6\sum_{s=0}^{5}\binom{6}{s}B_s[m]\beta^{6-s}
\end{aligned}$$

$$-\frac{7}{5}\frac{s(6,4)}{\binom{5}{1}}A_1^5a_7^2(-1)^5\sum_{s=0}^{4}\binom{5}{s}B_s[m]\beta^{5-s}$$
$$+\frac{7}{4}\frac{s(6,3)}{\binom{5}{2}}A_2^5a_7^3(-1)^4\sum_{s=0}^{3}\binom{4}{s}B_s[m]\beta^{4-s}$$
$$-\frac{7}{3}\frac{s(6,2)}{\binom{5}{3}}A_3^5a_7^4(-1)^3\sum_{s=0}^{2}\binom{3}{s}B_s[m]\beta^{3-s}$$
$$+\frac{7}{2}\frac{s(6,1)}{\binom{5}{4}}A_4^5a_7^5(-1)^2\sum_{s=0}^{1}\binom{2}{s}B_s[m]\beta^{2-s}$$

CASE 1 $i = 1$ implies that $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a$

Subcase 1.0 $P_6(0) = P_6(1) = ... = P_6(4) = P_6(5) = 0$

$$\begin{aligned}
T_{12} &= \frac{7}{2}(a^6)a_7^6 + \frac{35}{2}(a^5)a_7^7 - 21(a^5)a_7^7 \\
T_{11} &= -\frac{105}{2}(a^5)a_7^6 + 175(a^5)a_7^6 - \frac{7}{2}(a^6)a_7^5 - 119(a^4)a_7^7 \\
T_{10} &= -735(a^4)a_7^6 + \frac{175}{4}(a^5)a_7^5 + \frac{595}{2}(a^4)a_7^6 + \frac{1575}{4}(a^3)a_7^7 \\
T_9 &= 1624(a^3)a_7^6 + \frac{7}{6}(a^6)a_7^3 - \frac{595}{3}(a^4)a_7^5 - \frac{1575}{2}(a^3)a_7^6 - \frac{1918}{3}(a^2)a_7^7 \\
T_8 &= -1764(a^2)a_7^6 - \frac{35}{4}(a^5)a_7^3 + \frac{1575}{4}(a^3)a_7^5 + 959(a^2)a_7^6 + 420(a)a_7^7 \\
T_7 &= 720(a)a_7^6 - \frac{1}{6}(a^6)a_7 + \frac{119}{6}(a^4)a_7^3 - \frac{959}{3}(a^2)a_7^5 - 420(a)a_7^6
\end{aligned}$$

Taking $m = 6$, we have

$T_6(6) = 143605a^6 - 464625a^5a_7 + 582505a^4a_7^2 - 354375a^3a_7^3 + 105490a^2a_7^4 - 12600aa_7^5$
and
$\Phi(6,\beta) = a^6[\beta^7 + \frac{77}{2}\beta^6 + \frac{1267}{2}\beta^5 + 5775\beta^4 + \frac{188993}{6}\beta^3 + 102795\beta^2 + \frac{1115101}{6}\beta]$
$-\frac{35}{2}a^5a_7[\beta^6 + 33\beta^5 + \frac{905}{2}\beta^4 + 3300\beta^3 + \frac{26999}{2}\beta^2 + 29370\beta]$
$+119a^4a_7^2[\beta^5 + \frac{55}{2}\beta^4 + \frac{905}{3}\beta^3 + 1650\beta^2 + \frac{26999}{6}\beta]$
$-\frac{1575}{4}a^3a_7^3[\beta^4 + 22\beta^3 + 181\beta^2 + 660\beta]$
$+\frac{1918}{3}a^2a_7^4[\beta^3 + \frac{33}{2}\beta^2 + \frac{181}{2}\beta]$
$-420aa_7^5[\beta^2 + 11\beta]$

So, $\Delta 1.0 = \frac{7}{2}a^6a_7^6 - \frac{7}{2}a^5a_7^7 - \frac{105}{2}a^5a_7^6 + 175a^5a_7^6 - \frac{7}{2}a^6a_7^5 - 119a^4a_7^7 + \frac{175}{4}a^5a_7^5 - 735a^4a_7^6 + \frac{1575}{4}a^3a_7^7 + \frac{595}{2}a^4a_7^6 + 1624a^3a_7^6 + \frac{7}{6}a^6a_7^3 - \frac{1918}{3}a^2a_7^7 - \frac{1575}{2}a^3a_7^6 - \frac{595}{3}a^4a_7^5 - 1764a^2a_7^6 - \frac{35}{4}a^5a_7^3 + 420aa_7^7 + 959a^2a_7^6 + \frac{1575}{4}a^3a_7^5 + 720aa_7^6 - \frac{1}{6}a^6a_7 -$

$420aa_7^6 + \frac{119}{6}a^4a_7^3 - \frac{959}{3}a^2a_7^5$
$+143605a^6 - 464625a^5a_7 + 582505a^4a_7^2 - 354375a^3a_7^3 + 105490a^2a_7^4 - 12600aa_7^5$
$+a^6[\beta^7 + \frac{77}{2}\beta^6 + \frac{1267}{2}\beta^5 + 5775\beta^4 + \frac{188993}{6}\beta^3 + 102795\beta^2 + \frac{1115101}{6}\beta]$
$-\frac{35}{2}a^5a_7[\beta^6 + 33\beta^5 + \frac{905}{2}\beta^4 + 3300\beta^3 + \frac{26999}{2}\beta^2 + 29370\beta]$
$+119a^4a_7^2[\beta^5 + \frac{55}{2}\beta^4 + \frac{905}{3}\beta^3 + 1650\beta^2 + \frac{26999}{6}\beta]$
$-\frac{1575}{4}a^3a_7^3[\beta^4 + 22\beta^3 + 181\beta^2 + 660\beta]$
$+\frac{1918}{3}a^2a_7^4[\beta^3 + \frac{33}{2}\beta^2 + \frac{181}{2}\beta]$
$-420aa_7^5[\beta^2 + 11\beta]$

$$\Delta_{1.0} = \sum_{i=7}^{12} T_i + T_6(6) + \Phi(6, \beta)$$

$\Delta_{1.0}|_{a=a_7} = a_7^6\beta^7 + 21a_7^6\beta^6 + 175a_7^6\beta^5 + 735a_7^6\beta^4 + 1624a_7^6\beta^3 + 1764a_7^6\beta^2 + 720a_7^6\beta$

$\frac{\partial\Delta_{1.0}}{\partial a}|_{a=a_7} = \frac{7}{2}a_7^{11} + 231a_7^{10} - 350a_7^9 + \frac{809}{2}a_7^8 - 45a_7^7 - 261a_7^6 + 3780a_7^5 + 6a_7^5\beta^7 + \frac{287}{2}a_7^5\beta^6 + \frac{2779}{2}a_7^5\beta^5 + 6965a_7^5\beta^4 + \frac{38255}{2}a_7^5\beta^3 + \frac{55671}{2}a_7^5\beta^2 + 18621a_7^5\beta$

$\frac{\partial^2\Delta_{1.0}}{\partial a^2}|_{a=a_7} = 35a_7^{10} + 917a_7^9 - 2012.5a_7^8 + \frac{4183}{3}a_7^7 + 577.5_7a^6 - \frac{1219}{3}a_7^5 - 904407a_7^4 + 30a_7^4\beta^7 + 805a_7^4\beta^6 + 8883a_7^4\beta^5 + \frac{103565}{2}a_7^4\beta^4 + \frac{510146}{3}a_7^4\beta^3 + \frac{617421}{2}a_7^4\beta^2 + \frac{13686233}{3}a_7^4\beta$

$\frac{\partial^3\Delta_{1.0}}{\partial a^3}|_{a=a_7} = 210a^{10} + 4074a_7^8 - 5512.5a_7^7 + 399a_7^6 + 1837.5a_7^5 + 456a_7^4 + 1208970a_7^3 + 120a_7^3\beta^7 + 1050a_7^3\beta^6 + 44226a_7^3\beta^5 + 294052.5a_7^3\beta^4 + 1124445a_7^3\beta^3 + 15203212.5a_7^3\beta^2 + 2755794a_7^3\beta$

$\frac{\partial^4\Delta_{1.0}}{\partial a^4}|_{a=a_7} = 840a_7^8 + 10584a_7^7 - 15750a_7^6 - 4340a_7^5 - 1050a_7^4 + 416a_7^3 + 9922920a_7^2 + 360a_7^2\beta^7 + 11760a_7^2\beta^6 + 161616a_7^2\beta^5 + 1524040a_7^2\beta^4 + 5271140a_7^2\beta^3 + 13369650a_7^2\beta^2 + 18080584a_7^2\beta$

$\frac{\partial^5\Delta_{1.0}}{\partial a^5}|_{a=a_7} = 2100a_7^7 + 12180a_7^6 - 5250a_7^5 + 840a_7^4 - 1050a_7^3 + 120a_7^2 + 47640600a_7 + 720a_7\beta^7 + 25620a_7\beta^6 + 386820a_7\beta^5 + 3207750a_7\beta^4 + 15749160a_7\beta^3 + 73776158.75a_7\beta^2 + 72135120a_7\beta$

$\frac{\partial^6\Delta_{1.0}}{\partial a^6}|_{a=a_7} = 2520a_7^6 - 2520a_7^5 + 840a_7^3 - 120a_7 + 103395600 + 720\beta^7 + 27720\beta^6 + 483840\beta^5 + 4641840\beta^4 + 22679160\beta^3 + 9669156 - \beta^2 + 235303680\beta$

We regard these as polynomials in β with coefficients in a_7. Under the condition of $a_7 \geq 7$ and $0 < \beta < 1$, the first polynomial is positive since all its coefficients are postitive. For the other polynomials, we need to check the summation of the coefficients as described in Lemma 1.

Notice the degree of β in these polynomials is 7. Let c_i, $0 \leq i \leq 7$ be the coefficient of β^i. For $\frac{\partial \Delta_{1.0}}{\partial a}|_{a=a_7}$ We have

$$\begin{aligned}
c_0 &= \tfrac{7}{2}a_7^{11} + 231a_7^{10} - 350a_7^9 + \tfrac{809}{2}a_7^8 - 45a_7^7 - 261a_7^6 + 3780a_7^5 \\
&= a_7^7(\tfrac{7}{2}a_7^4 - 45) + a_7^9(231a_7 - 350) + a_7^6(\tfrac{809}{2}a_7^2 - 261) + 3780a_7^5 \\
&> 0 \text{ when } a_7 > 6 \text{ and } 0 < \beta < 1 \\
c_0 + c_1 &= \tfrac{7}{2}a_7^{11} + 231a_7^{10} - 350a_7^9 + \tfrac{809}{2}a_7^8 - 45a_7^7 - 261a_7^6 + 3780a_7^5 \\
&\quad +18621a_7^5 \\
&> 0 \text{ when } a_7 > 6 \text{ and } 0 < \beta < 1 \\
c_0 + c_1 + c_2 &= \tfrac{7}{2}a_7^{11} + 231a_7^{10} - 350a_7^9 + \tfrac{809}{2}a_7^8 - 45a_7^7 - 261a_7^6 + 3780a_7^5 \\
&\quad +18621a_7^5 + \tfrac{55671}{2}a_7^5 \\
&> 0 \text{ when } a_7 > 6 \text{ and } 0 < \beta < 1
\end{aligned}$$

It is obvious that $\sum_{i=0}^{k} c_i > 0 \; for \; 0 \leq k \leq 7$
By Lemma 1, it follows that

$$\frac{\partial \Delta_{1.0}}{\partial a}|_{a=a_7} \geq 0$$

$$\begin{aligned}
c_0 &= 35a_7^{10} + 917a_7^9 - 2012.5a_7^8 + \tfrac{4183}{3}a_7^7 + 577.5_7a^6 - \tfrac{1219}{3}a_7^5 \\
&\quad -90440a_7^4 \\
&= 35a_7^{10} + 917a_7^8(a_7 - \tfrac{4025}{1834}) + \tfrac{4183}{3}a_7^5(a_7^2 + \tfrac{1219}{4183}) \\
&\quad + \tfrac{11550}{2}a_7^4(a^2 - \tfrac{180880}{11550}) \\
&> 0 \text{ when } a_7 > 6 \text{ and } 0 < \beta < 1 \\
c_0 + c_1 &= 35a_7^{10} + 917a_7^8(a_7 - \tfrac{4025}{1834}) + \tfrac{4183}{3}a_7^5(a_7^2 + \tfrac{1219}{4183}) \\
&\quad + \tfrac{11550}{2}a_7^4(a^2 - \tfrac{180880}{11550}) + \tfrac{13686233}{3}a_7^4 \\
&> 0 \text{ when } a_7 > 6 \text{ and } 0 < \beta < 1 \\
c_0 + c_1 + c_2 &= 35a_7^{10} + 917a_7^8(a_7 - \tfrac{4025}{1834}) + \tfrac{4183}{3}a_7^5(a_7^2 + \tfrac{1219}{4183}) \\
&\quad + \tfrac{11550}{2}a_7^4(a^2 - \tfrac{180880}{11550}) + \tfrac{13686233}{3}a_7^4 + \tfrac{617421}{2}a_7^4 \\
&> 0 \text{ when } a_7 > 6 \text{ and } 0 < \beta < 1
\end{aligned}$$

It is obvious that $\sum_{i=0}^{k} c_i > 0 \; for \; 0 \leq k \leq 7$
By Lemma 1, it follows that

$$\frac{\partial^2 \Delta_{1.0}}{\partial a^2}|_{a=a_7} \geq 0$$

In a similar way, we can show that $\frac{\partial^k \Delta_{1.0}}{\partial a^k}|_{a=a_n} \geq 0$ is true for all $0 \leq k \leq n$. Then by Corollary 1, we have $\Delta = \Delta_{1.0} > 0$ for all $a \geq a_n \geq n-1$ and $0 < \beta < 1$

Also we can check in this case that for Δ to be equal to zero, a_i must be the same number and this number must be an integer. In other words, $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7$ and $\beta = 0$. Looking at $\Delta_{1.0}$, equality for the estimate occurs only at $\Delta_{1.0}|_{a=a_7}$, when $\beta = 0$.

Subcase 1.1: $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a$
$P_6(0) = P_6(1) = ... = P_6(4) = 0, P_6(5) > 0$

Since $\frac{a_6}{a_7}(\beta + h) \geq 5$ for all $h \geq 5$, we can apply sharp estimate to $P_6(h)$. So $m = 5$ and $h_0 = 5$. We have

Since $P_6(5) > 0$, we have

$$\frac{1}{\frac{a_1}{a_7}(\beta + (5))} + \frac{1}{\frac{a_2}{a_7}(\beta + (5))} + ... + \frac{1}{\frac{a_6}{a_7}(\beta + (5))} \leq 1$$

This is equivalent to

$$\frac{a}{a_7}(\beta + (5)) \geq 6$$

We have the minimum value for a:

$$a \geq a_0 := \frac{6}{\beta + 5} a_6$$

As stated, $a_0 > a_6$. Since $\Delta_{1.1}$ does not have value for $a \in [a_7, a_0)$, we extend the definition of $\Delta_{1.1}$ to interval $[a_7, a_0]$ by assigning
$\Delta_{1.1}(a, a_7, \beta) = \Delta_{1.1}(a_0, a_7, \beta)$, for $a \in [a_7, a_0)$

So we can check the derivative of $\Delta_{1.1}$ at $a = a_0$ instead of $a = a_7$. For $k = 0, 1, 2, ...6$, we need to verify that $\frac{\partial^k \Delta_{1.1}}{\partial a^k}|_{a=a_0} \geq 0$ for $a_7 \geq 7$ and $0 < \beta < 1$

$T_6(5) = 34230a^6 - 136500a^5 a_7 + 210630a^4 a_7^2 - 157500a^3 a_7^3 + 57540a^2 a_7^4 - 8400a a_7^5$
and
$\Phi(5, \beta) = a^6[\beta^7 + 31.5\beta^6 + 423.5\beta^5 + 3150\beta^4 + \frac{83993}{6}\beta^3 + 37170\beta^2 + \frac{327601}{6}\beta] - \frac{35}{2}a^5 a_7[\beta^6 + 27\beta^5 + 302.5\beta^4 + 1800\beta^3 + 5999.5\beta^2 + 10620\beta] + 119a^4 a_7^2[\beta^5 + 22.5\beta^4 + \frac{605}{3}\beta^3 + 900\beta^2 + \frac{11999}{6}\beta] - \frac{1575}{4}a^3 a_7^3[\beta^4 + 18\beta^3 + 121\beta^2 + 360\beta] + \frac{1918}{3}a^2 a_7^4[\beta^3 + 13.5\beta^2 + 60.5\beta] - 420a a_7^5[\beta^2 + 9\beta]$

$$\Delta_{1.1} = \sum_{i=7}^{12} T_i + T_6(5) + \Phi(5, \beta)$$

Here we check the derivatives at $a_0 = \frac{6a_7}{\beta+5}$

$\Delta_{1.1}|_{a=a_0} = \frac{7}{2}a^6a_7^6 - \frac{7}{2}a^5a_7^7 - \frac{105}{2}a^5a_7^6 + 175a^5a_7^6 - \frac{7}{2}a^6a_7^5 - 119a^4a_7^7 + \frac{175}{4}a^5a_7^5 - 735a^4a_7^6 + \frac{1575}{4}a^3a_7^7 + \frac{595}{2}a^4a_7^6 + 1624a^3a_7^6 + \frac{7}{6}a^6a_7^3 - \frac{1918}{3}a^2a_7^7 - \frac{1575}{2}a^3a_7^6 - \frac{595}{3}a^4a_7^5 - 1764a^2a_7^6 - \frac{35}{4}a^5a_7^3 + 420aa_7^7 + 959a^2a_7^6 + \frac{1575}{4}a^3a_7^5 + 720aa_7^6 - \frac{1}{6}a^6a_7 - 420aa_7^6 + \frac{119}{6}a^4a_7^3 - \frac{959}{3}a^2a_7^5 + 34230a^6 - 136500a^5a_7 + 210630a^4a_7^2 - 157500a^3a_7^3 + 57540a^2a_7^4 - 8400aa_7^5 + a^6[\beta^7 + 31.5\beta^6 + 423.5\beta^5 + 3151\beta^4 + \frac{83993}{6}\beta^3 + 37170\beta^2 + \frac{327370}{6}\beta] - \frac{35}{2}a^5a_7[\beta^6 + 27\beta^5 + 302.5\beta^4 + 1800\beta^3 + 5999.5\beta^2 + 10620\beta] + 119a^4a_7^2[\beta^5 + 22.5\beta^4 + \frac{605}{3}\beta^3 + 900\beta^2 + \frac{11999}{6}\beta] - \frac{1575}{4}a^3a_7^3[\beta^4 + 18\beta^3 + 115\beta^2 + 360\beta] + \frac{1918}{3}a^2a_7^4[\beta^3 + 13.5\beta^2 + 60.5\beta] - 420aa_7^5[\beta^2 + 11\beta]$

For $n = 7$

$\Delta_{1.1}|_{a=a_0} = \frac{a_7^6}{(\beta+5)^6}[136080a_7^6 + 635040a_7^5 - 141750a_7^4 + 44940a_7^3 - 9450a_7^2 + 8220a_7 + 776635056 + 46656\beta^7 + \beta^6(146944 - 136080a_7) + \beta^5(19758816 - 3674160a_7 + 154224a_7^2) + \beta^4(147013056 - 41164200a_7^2 + 3470040a_7^3) + \beta^3(653129568 - 244944000a_7 + 31101840a_7^2 - 1530900a_7^3 + 23016a_7^4) + \beta^2(1734203520 + 138801600a_7^2 - 9780750a_7^3 + 310716a_7^4 - 2520a_7^5) + \beta(2545629123 - 1445169600a_7 + 3084377184a_7^2 - 30618000a_7^3 + 1392468a_7^4 - 27720a_7^5)]$

$\frac{\partial\Delta_{1.1}}{\partial a}|_{a=a_0} = \frac{a_7^5}{(\beta+5)^5}[140616a_7^6 + 1208088a_7^5 - 51975a_7^4 - 34258a_7^3 - 23415a_7^2 + 5824a_7 + 878171280 + 46656\beta^7\beta^6(146944 - 113400a_7) + \beta^5(19758816 - 3061800a_7 + 102816a_7^2) + \beta^4(147013056 - 34303560a_7 + 2313360a_7^2 - 42525a_7^3) + \beta^3(653219568 - 204120000a_7 + 20734560a_7^2 - 765450a_7^3 + 7672a_7^2) + \beta^2(1734203520 - 680343300a_7 + 11823840a_7^2 - 765450a_7^3 + 103572a_7^4) + \beta(2545629120 - 1204308000a_7 + 205614864a_7^2 - 15309000a_7^3 + 464156a_7^4]$

$\frac{\partial^2\Delta_{1.1}}{\partial a^2}|_{a=a_0} = \frac{a_7^4}{(\beta+5)^4}[120960a_7^6 + 341712a_7^5 + 1215a_7^4 - 12124a_7^3 + 577.5a_7^2 + 6265a_7 + \frac{4345}{3} + 38880\beta^7 + \beta^6(1224720 - 75600a_7) + \beta^5(16465680 - 2041200a_7 + 51408a_7^2) + \beta^4(122510880 - 22869000a_7 + 1156680a_7^2 - 14175a_7^3) + \beta^3(5942740 - 22680000a_7 + 10367280a_7^2 - 255150a_7^3 + 7672a_7^4) + \beta^2(1445169600 - 453562200a_7 - 407295000a_7^2 - 1630125a_7^3 + 17262a_7^4) + \beta(2121357600 - 802872000a_7 + 102807432a_7^2 - 5103000a_7^3 + 77350a_7^4)]$

$\frac{\partial^3\Delta_{1.1}}{\partial a^3}|_{a=a_0} = \frac{a_7^3}{(\beta+5)^3}[83160a_7^6 + 156744a_7^5 - 9161.25a_7^4 + 27139a_7^3 + +1837.5a_7^2 898.5a_7 + 621787320 + 25920\beta^7 + \beta^6(816480 - 37800a_7) + \beta^5(1097712 - 1020600a_7 + 17136) + \beta^4(81673920 - 11434500a_7 + 38560a_7^2 - 2362.5a_7^3) + \beta^3(362849760 - 68040000a_7 + 3455760a_7^2 - 42525a_7^3) + \beta^2(96446400 - 226781100a_7 + 3455760a_7^2 - 271687.5a_7^3) + \beta(1414288400 - 401436000a_7 + 34269144a_7^2 - 850500a_7^3)]$

$\frac{\partial^4\Delta_{1.1}}{\partial a^4}|_{a=a_0} = \frac{a_7^2}{(\beta+5)^2}[42840a_7^6 + 39984a_7^5 - 5250a_7^4 + 10360a_7^3 - 1050a_7^2 - 1684a_7 + 350395920 + 12960\beta^7 + \beta^6(408240 - 12600a_7) + \beta^5(5488560 - 340200a_7 + 2865) + \beta^4(40836960 - 3811500a_7 + 64260a_7^2) + \beta^3(181425880 - 22680000a_7 + 575960a_7^2) + \beta^2(481723200 - 75593700a_7 + 2570400a_7^2) + \beta(707119200 -$

$133812000a_7 + 5711524a_7^2)]$

$\frac{\partial^5 \Delta_{1.1}}{\partial a^5}|_{a=a_0} = \frac{a_7}{(\beta+5)}[14700a_7^6 - 420a_7^5 + 5250a_7^4 + 5040a_7^3 - 1050a_7^2 - 720a_7 + 131493600 + 4320\beta^7 + \beta^6(136080 - 2100a_7) + \beta^5(1829520 - 56700a_7) + \beta^4(13612320 - 635250a_7) + \beta^3(60474960 - 3780000a_7) + \beta^2(160574400 - 12598950a_7) + \beta(235716400 + 2230200a_7)]$

$\frac{\partial^6 \Delta_{1.1}}{\partial a^6}|_{a=a_0} = [720a_7^6 - 22680a_7^5 + 30490a_7^3 - 120a_7 + 24645600]$

$\frac{\partial^7 \Delta_{1.1}}{\partial a^7}|_{a=a_0} = 0$

$$
\begin{aligned}
c_0 &= 136080a_7^6 + 635040a_7^5 - 141750a_7^4 + 44940a_7^3 - 9450a_7^2 \\
&\quad +8220a_7 + 776635056 \\
&= 136080a_7^4(a_7^2 - \tfrac{14175}{136080}) + 635040a_7^3(a_7^2 - \tfrac{9450}{635040}) \\
&\quad +44940a_7(a_7^2 - \tfrac{9450}{44940}) \\
&> 0 \text{ when } a_7 > 6 \text{ and } 0 < \beta < 1 \\
c_0 + c_1 &= 136080a_7^6 + (635040 - 27720)a_7^5 + (1392458 - 141750)a_7^4 \\
&\quad +(44940 - 30618000)a_7^3 - (3084377184 - 9450)a_7^2 \\
&\quad +(8220 - 27720)a_7 + 776635056 + 2545629120 \\
&> 0 \text{ when } a_7 > 6 \text{ and } 0 < \beta < 1 \\
c_0 + c_1 + c_2 &= 136080a_7^6 + (-2520 + 635040 - 27720)a_7^5 \\
&\quad +(310716 + 1392458 - 141750)a_7^4 \\
&\quad +(44940 - 30618000 - 9780750)a_7^3 \\
&\quad -(138801600 + 3084377184 - 9450)a_7^2 \\
&\quad +(8220 - 27720 - 816411960)a_7 \\
&\quad +776635056 + 2545629120 + 1734203520 \\
&> 0 \text{ when } a_7 > 6 \text{ and } 0 < \beta < 1
\end{aligned}
$$

It is obvious that $\sum_{i=0}^{k} c_i > 0 \ for \ 0 \le k \le 7$
By Lemma 1, it follows that

$$\frac{\partial \Delta_{1.1}}{\partial a}|_{a=a_0} \ge 0$$

In a similar way, we can show that $\frac{\partial^k \Delta_{1.1}}{\partial a^k}|_{a=a_0} \ge 0$ is true for all $0 \le k \le 7$. Then by Corollary 1, we have $\Delta = \Delta_{1.1} > 0$ for all $a \ge a_7 \ge 7$ and $0 < \beta < 1$

Again, as in the rest of the cases from here on, we can check that for Δ to be equal to zero, a_i must equal the same integer ($a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7$ and $\beta = 0$).

Subcase 1.j: $P_6(0) = P_6(1) = ... = P_6(5 - j) = 0$, $P_6(6 - j) > 0$.
In this case, $m = 5$ and $h_0 = 6 - j$. Then

$$\Delta = \Delta_0(a_7 - \beta - 5) - 7a_7^6 \sum_{h=6-j}^{4} q_6(a_7 - \beta - h)$$

Let

$$\Delta_{1.j} = \Delta|_{a_1 = a_2 ... = a_6 = a}$$

Since $\Delta_{1.j}$ is the polynomial of a, a_7, β, we also use $\Delta_{a,a_7,\beta}$ for $\Delta_{1.j}$. From $P_6(6-j) > 0$, we can get

$$a \geq a_0 := \begin{cases} a_7 & \text{for j=0} \\ \frac{6}{\beta+5} & \text{for j =1} \\ \frac{6}{7-j} & \text{for } j \geq 2 \end{cases}$$

We extend Δ and use the same method as Subcase 1.1 to check the derivatives of $\Delta_{1.1}$ at $a = a_0$. Lemmas 1 and 2 are then applied.

The other 6 cases, where
$P_6(0) = P_6(1) = ... = P_6(5-j) = 0, P_6(6-j) > 0$ are verified in a similar manner using the computer program along with lemmas 1 and 2. As in Case 1, recall Δ as given in (6.4). In order to apply Lemma 2, there is one difference in what we compute. If we define

$$\Delta_{i.j} = \Delta|_{a_1 = a_2 = ... = a_{7-i} = a}$$

$\Delta_{i.j}$ is a polynomial in $a, a_{7-i+1}, ..., a_7, \beta$
For each case i, we have already shown in case i-1 that

$$\Delta_{i-1.j} \geq 0 \; for \; a \geq a_{7-i+2} \geq ... \geq a_7 \geq 7 \; and \; 0 < \beta < 1$$

So, $\Delta_{i.j}|_{a=a_{7-i-1}} \geq 0$. We are left to check for $0 \leq s \leq 7-i$ that,

$$\frac{\partial^s \Delta_{i.j}}{\partial a^s}|_{a=a_{7-i+1}=...=a_7} \geq 0$$

$$\frac{\partial^k}{\partial a_{i_{7-i+1}} \partial a_{i_{7-i+2}} ... \partial a_{i_{7-i+k}}} \left(\frac{\partial^s \Delta_{i.j}}{\partial a^s}\right)|_{a=a_{7-i+1}=...=a_7} \geq 0$$

Q.E.D

7. DISCUSSION

The sharp estimate analysis in our paper can be extended to the general n. We also generalized the computer program used in our proof to check the sign of Δ to verify the sign for an arbitrary input of n. As with the $n = 7$ case, we have recently succeeded in proving the conjecture for $n = 8$ and $n = 9$. However, for $n = 10$ the sign of Δ is not positive, indicating that the conjecture fails to be true in the case of $n = 10$. In order for the estimate to be applied to $n = 10$, we must have $a_{10} > 11$ instead of $a_{10} > 10$. It is possible that the only way this particular estimate can be applied to the general n is if we have $a_n > \alpha$, where α is a function of n. Unfortunately, this approach makes induction very difficult. A way around this problem is to change the estimate again and make all terms of the estimate larger. In this paper, we only modified the second term. There is still room to make our estimate even larger and have it remain a sharp estimate. With this in mind we come up with the following conjecture, given in (7.1).

SHARP ESTIMATE CONJECTURE

Let $P_n =$ number of element of set $\left\{(x_1, x_2, ...x_n) \in \mathbf{Z}^n_+; \frac{x_1}{a_1} + \frac{x_2}{a_2} + ... + \frac{x_n}{a_n} \leq 1\right\}$.
Let $n \geq 3$,
We define $A^n_k = a_1 a_2 ... a_{n-k}$ for k is even or zero, $A^n_k = a_{k+1} a_{k+2} ... a_n$ for k is odd.

If $a_1 \geq a_2 \geq ... \geq a_n \geq n - 1$, then

$$n! P_n \leq Y_n := \sum_{g=0}^{n-1} s(n, n-g) A^n_g \tag{7.1}$$

Equality holds if and only if $a_1 = a_2 = ... = a_n =$ integer.

8. APPENDIX

The proofs of Lemmas 3, 4, and 5 are similar to proofs of lemmas given in [25].

Proof of Lemma 3: Plugging in the expression of $Y_{n-1}(k)$, we have

$$\begin{aligned} G(m') &= A_0^{n-1}\sum_{k=1}^{m'}(1-\frac{k}{a_n})^{n-1} - \frac{n-1}{2}a_{n-1}A_1^{n-2}\sum_{k=1}^{m'}(1-\frac{k}{a_n})^{n-2} \\ &+ \sum_{l=1}^{n-3}\frac{s(n-1,n-2-l)}{\binom{n-2}{l}}A_l^{n-2}\sum_{k=1}^{m'}(1-\frac{k}{a_n})^{n-2-l} \end{aligned}$$

Applying the binomial theorem, $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$

$$\begin{aligned} G(m') &= A_0^{n-1}\sum_{k=1}^{m'}\sum_{i=0}^{n-1}\binom{n-1}{i}(-\frac{k}{a_n})^{n-1-i} \\ &- \frac{n-1}{2}a_{n-1}A_1^{n-2}\sum_{k=1}^{m'}\sum_{i=0}^{n-2}\binom{n-2}{i}(-\frac{k}{a_n})^{n-2-i} \\ &+ \sum_{l=1}^{n-3}\frac{s(n-1,n-2-l)}{\binom{n-2}{l}}A_l^{n-2}\sum_{k=1}^{m'}\sum_{i=0}^{n-2-l}\binom{n-2-l}{i}(-\frac{k}{a_n})^{n-2-l-i} \end{aligned}$$

$$\begin{aligned} &= A_0^{n-1}\sum_{i=0}^{n-1}\binom{n-1}{i}(-\frac{1}{a_n})^{n-1-i}\sum_{k=1}^{m'}k^{n-1-i} \\ &- \frac{n-1}{2}a_{n-1}A_1^{n-2}\sum_{i=0}^{n-2}\binom{n-2}{i}(-\frac{1}{a_n})^{n-2-i}\sum_{k=1}^{m'}k^{n-2-i} \\ &+ \sum_{l=1}^{n-3}\frac{s(n-1,n-2-l)}{\binom{n-2}{l}}A_l^{n-2}\sum_{i=0}^{n-2-l}\binom{n-2-l}{i}(-\frac{1}{a_n})^{n-2-l-i}\sum_{k=1}^{m'}k^{n-2-l-i} \end{aligned}$$

Let B_k be the Bernoulli number. This number has the property
$\sum_{k=1}^{m}k^n = \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}(m+1)^{n+1-k}B_k$, for $n \geq 1$

$$\begin{aligned}
G(m') &= A_0^{n-1}\sum_{i=0}^{n-2}\binom{n-1}{i}(-\frac{1}{a_n})^{n-1-i}\frac{1}{n-i}\sum_{k=0}^{n-1-i}\binom{n-i}{k}(m'+1)^{n-i-k}B_k \\
&- \frac{n-1}{2}a_{n-1}A_1^{n-2}\sum_{i=0}^{n-3}\binom{n-2}{i}(-\frac{1}{a_n})^{n-2-i}\frac{1}{n-1-i} \\
&\times \sum_{k=0}^{n-2-i}\binom{n-1-i}{k}(m'+1)^{n-1-i-k}B_k
\end{aligned}$$

$$\begin{aligned}
+ &\sum_{l=1}^{n-3}\frac{s(n-1,n-2-l)}{\binom{n-2}{l}}A_l^{n-2}\sum_{i=0}^{n-3-l}\binom{n-2-l}{i}(-\frac{1}{a_n})^{n-2-l-i}\frac{1}{n-1-l-i} \\
\times &\sum_{k=0}^{n-2-l-i}\binom{n-1-l-i}{k}(m'+1)^{n-1-l-i-k}B_k \\
+ &A_0^{n-1}\binom{n-1}{n-1}(-\frac{k}{a_n})^{n-1-(n-1)}\sum_{k=1}^{m'}k^{n-1-(n-1)} \\
- &\frac{n-1}{2}a_{n-1}A_1^{n-2}\binom{n-2}{n-2}(-\frac{1}{a_n})^{n-2-(n-2)}\sum_{k=1}^{m'}k^{n-2-(n-2)} \\
+ &\sum_{l=1}^{n-3}\frac{s(n-1,n-2-l)}{\binom{n-2}{l}}A_l^{n-2}\binom{n-2-l}{n-2-l}(-\frac{1}{a_n})^{n-2-l-(n-2-l)}\sum_{k=1}^{m'}k^{n-2-l-(n-2-l)}
\end{aligned}$$

Let Y_{n-1} be defined as in (5.3)

$$\begin{aligned}
= &A_0^{n-1}\sum_{i=0}^{n-2}\binom{n-1}{i}(-\frac{1}{a_n})^{n-1-i}\frac{1}{n-i}\sum_{k=0}^{n-1-i}\binom{n-i}{k}(m'+1)^{n-i-k}B_k \\
- &\frac{n-1}{2}a_{n-1}A_1^{n-2}\sum_{i=0}^{n-3}\binom{n-2}{i}(-\frac{1}{a_n})^{n-2-i}\frac{1}{n-1-i}\sum_{k=0}^{n-2-i}\binom{n-1-i}{k}(m'+1)^{n-1-i-k}B_k \\
+ &\sum_{l=1}^{n-3}\frac{s(n-1,n-2-l)}{\binom{n-2}{l}}A_l^{n-2}\sum_{i=0}^{n-3-l}\binom{n-2-l}{i}(-\frac{1}{a_n})^{n-2-l-i}\frac{1}{n-1-l-i} \\
\times &\sum_{k=0}^{n-2-l-i}\binom{n-1-l-i}{k}(m'+1)^{n-1-l-i-k}B_k + m'Y_{n-1}
\end{aligned}$$

(5.2) follows easily from above.

Q.E.D

Notice that the maximum number of summation symbols in one term in the expression of $G(m')$ is three. Let

$$\beta = a_n - [a_n] \tag{8.1}$$

$$k = [a_n] - h = a_n - \beta - h \text{ where } h = 0, 1, 2, ..., [a_n] - 1 \tag{8.2}$$

$$m' = a_n - \beta - m \tag{8.3}$$

Then

$$G(a_n - \beta - m) = \sum_{h=m}^{a_n-\beta-1} Y_{n-1}(a_n - \beta - h) \tag{8.4}$$

Using this notation, we can further simplify $G(m')$ by reducing the number of summation symbols.

m' will be used in the proof of the main theorem inductively starting with the largest integer that satisfies the inequality $a_6(1 - \frac{m'}{a_7}) \geq 5$.

Proof of Lemma 4 From (8.4), (5.4) is obvious. Plugging into Lemma 3 we have, $g(m') = na_n^{n-1}G(m') = A_0^{n-1} \sum_{i=0}^{n-2} \binom{n}{i} (-1)^{n-1-i} a_n^i \sum_{k=0}^{n-1-i} \binom{n-i}{k} (m'+1)^{n-i-k} B_k$
$-\frac{n}{2} a_{n-1} A_1^{n-2} \sum_{i=0}^{n-3} \binom{n-1}{i} (-1)^{n-2-i} a_n^{1+i} \sum_{k=0}^{n-2-i} \binom{n-1-i}{k} (m'+1)^{n-1-i-k} B_k$
$+ \sum_{l=1}^{n-3} \frac{n}{n-1-l} \frac{s(n-1, n-2-l)}{\binom{n-2}{l}} A_l^{n-2} \sum_{i=0}^{n-3-l} \binom{n-1-l}{i} (-1)^{n-2-l-i} a_n^{1+l+i} \sum_{k=0}^{n-2-l-i} \binom{n-1-l-i}{k}$
$(m'+1)^{n-1-l-i-k} B_k + na_n^{n-1} m' Y_{n-1}$

Replacing m' by $a_n - \beta - m$, we have

$g(a_n - \beta - m) =$
$A_0^{n-1} \sum_{i=0}^{n-2} \binom{n}{i} (-1)^{n-1-i} (a_n)^i \sum_{k=0}^{n-1-i} \binom{n-i}{k} (a_n - \beta - m + 1)^{n-i-k} B_k$
$-\frac{n}{2} a_{n-1} A_1^{n-2} \sum_{i=0}^{n-3} \binom{n-1}{i} (-1)^{n-2-i} (a_n)^{1+i} \sum_{k=0}^{n-2-i} \binom{n-1-i}{k} (a_n - \beta - m + 1)^{n-1-i-k} B_k$
$+ \sum_{l=1}^{n-3} \frac{n}{n-1-l} \frac{s(n-1, n-2-l)}{\binom{n-2}{l}} A_l^{n-2} \sum_{i=0}^{n-3-l} \binom{n-1-l}{i} (-1)^{n-2-l-i} a_n^{1+l+i}$

$$\times \sum_{k=0}^{n-2-l-i} \binom{n-1-l-i}{k}$$
$$(a_n-\beta-m+1)^{n-1-l-i-k}B_k + na_n^{n-1}(a_n-\beta-m)Y_{n-1}$$

Applying the binomial theorem again,

$$= A_0^{n-1}\sum_{i=0}^{n-2}\binom{n}{i}(-1)^{n-1-i}(a_n)^i\sum_{k=0}^{n-1-i}\binom{n-i}{k}B_k\sum_{t=0}^{n-i-k}\binom{n-i-k}{t}$$
$$\times a_n^{n-i-k-t}(1-\beta-m)^t$$
$$-\tfrac{n}{2}a_{n-1}A_1^{n-2}\sum_{i=0}^{n-3}\binom{n-1}{i}(-1)^{n-2-i}(a_n)^{1+i}\sum_{k=0}^{n-2-i}\binom{n-1-i}{k}B_k$$
$$\times \sum_{t=0}^{n-1-i-k}\binom{n-1-i-k}{t}a_n^{n-1-i-k-t}(1-\beta-m)^t$$
$$+\sum_{l=1}^{n-3}\frac{n}{n-1-l}\frac{s(n-1,n-2-l)}{\binom{n-2}{l}}A_l^{n-2}\sum_{i=0}^{n-3-l}\binom{n-1-l}{i}(-1)^{n-2-l-i}(a_n)^{1+l+i}$$
$$\times \sum_{k=0}^{n-2-l-i}\binom{n-1-l-i}{k}B_k\sum_{t=0}^{n-1-l-i-k}\binom{n-1-l-i-k}{t}a_n^{n-1-l-i-k-t}(1-$$
$$\beta-m)^t + na_n^{n-1}(a_n-\beta-m)Y_{n-1}$$

$$= A_0^{n-1}\sum_{i=0}^{n-2}\binom{n}{i}(-1)^{n-1-i}\sum_{k=0}^{n-1-i}\binom{n-i}{k}B_k\sum_{t=0}^{n-i-k}\binom{n-i-k}{t}$$
$$\times a_n^{n-k-t}(1-\beta-m)^t$$
$$-\tfrac{n}{2}a_{n-1}A_1^{n-2}\sum_{i=0}^{n-3}\binom{n-1}{i}(-1)^{n-2-i}\sum_{k=0}^{n-2-i}\binom{n-1-i}{k}B_k$$
$$\times \sum_{t=0}^{n-1-i-k}\binom{n-1-i-k}{t}a_n^{n-k-t}(1-\beta-m)^t$$
$$+\sum_{l=1}^{n-3}\frac{n}{n-1-l}\frac{s(n-1,n-2-l)}{\binom{n-2}{l}}A_l^{n-2}\sum_{i=0}^{n-3-l}\binom{n-1-l}{i}(-1)^{n-2-l-i}\sum_{k=0}^{n-2-l-i}\binom{n-1-l-i}{k}$$
$$B_k\sum_{t=0}^{n-1-l-i-k}\binom{n-1-l-i-k}{t}a_n^{n-k-t}(1-\beta-m)^t+na_n^{n-1}(a_n-\beta-m)Y_{n-1}$$

Notice

$$\binom{n}{i}\binom{n-i}{k}\binom{n-i-k}{t} = \binom{n-k-t}{i}\binom{n}{k,t}$$

$$\binom{n-1}{i}\binom{n-1-i}{k}\binom{n-1-i-k}{t} = \binom{n-1-k-t}{i}\binom{n-1}{k,t}$$

$$\binom{n-1-l}{i}\binom{n-1-l-i}{k}\binom{n-1-l-i-k}{t} = \binom{n-1-l-k-t}{i}\binom{n-1-l}{k,t}$$

where

$$\binom{n}{k,t} = \frac{n!}{k!t!(n-k-t)!}$$

and

$$\binom{N}{k}\binom{N-k}{t} = \binom{N}{k,t}$$

So $g(a_n-\beta-m) =$
$A_0^{n-1}\sum_{i=0}^{n-2}(-1)^{n-1-i}\sum_{k=0}^{n-1-i}B_k\sum_{t=0}^{n-i-k}\binom{n-t-k}{i}\binom{n}{k,t}a_n^{n-k-t}(1-\beta-m)^t$
$-\frac{n}{2}a_{n-1}A_1^{n-2}\sum_{i=0}^{n-3}(-1)^{n-2-i}\sum_{k=0}^{n-2-i}B_k\sum_{t=0}^{n-1-i-k}\binom{n-1-t-k}{i}\binom{n-1}{k,t}$
$a_n^{n-k-t}(i-\beta-m)^t$
$+\sum_{l=1}^{n-3}\frac{n}{n-1-l}\frac{s(n-1,n-2-l)}{\binom{n-2}{l}}A_l^{n-2}\sum_{i=0}^{n-3-l}(-1)^{n-2-l-i}\sum_{k=0}^{n-2-l-i}$
$B_k\sum_{t=0}^{n-1-l-i-k}\binom{n-1-l-k-t}{i}\binom{n-1-l}{k,t}(1-\beta-m)^t a_n^{n-k-t}+na_n^{n-1}(a_n-$
$\beta-m)Y_{n-1}$

Define I_1, I_2, I_3 to be the first three terms respectively in the summation of $g(a_n-\beta-m)$. Then we have
$g(a_n-\beta-m) = I_1+I_2+I_3+na_n^{n-1}(a_n-\beta-m)Y_{n-1}$
For I_1, let $s=k+t$. Then $0\le s\le n-i$ Also define

$$\Phi = (-1)^{n-1-i}B_k\binom{n-s}{i}\binom{n}{k}\binom{n-k}{s-k}(1-\beta-m)^{s-k}a_n^{n-s}$$

Then

$$I_1 = A_0^{n-1}\sum_{i=0}^{n-2}\sum_{k=0}^{n-1-i}\sum_{s=k}^{n-i}\Phi(s,i,k)$$

By changing the order of the summations, we have

$$I_1 = A_0^{n-1}[\sum_{i=0}^{n-2}\sum_{s=0}^{n-1-i}\sum_{k=0}^{s}\Phi(s,i,k) + \sum_{i=0}^{n-2}\sum_{k=0}^{n-1-i}\Phi(n-i,i,k)]$$

$$= A_0^{n-1}[\sum_{s=1}^{n-1}\sum_{i=0}^{n-1-s}\sum_{k=0}^{s}\Phi(s,i,k) + \sum_{i=0}^{n-2}\Phi(0,i,0)$$

$$+\sum_{i=0}^{n-2}\sum_{k=0}^{n-1-i}\Phi(n-i,i,k)]$$

$$= A_0^{n-1}[\sum_{s=2}^{n-1}\sum_{i=0}^{n-1-s}\sum_{k=0}^{s}\Phi(s,i,k)+\sum_{i=0}^{n-2}\sum_{k=0}^{1}\Phi(1,i,k)+\sum_{i=0}^{n-2}\Phi(0,i,0)+\sum_{s=2}^{n}\sum_{k=0}^{s-1}\Phi(s,n-s,k)]$$

$$= A_0^{n-1}[\sum_{s=2}^{n-1}\sum_{i=0}^{n-1-s}\sum_{k=0}^{s}\Phi(s,i,k)+\sum_{i=0}^{n-2}\sum_{k=0}^{1}\Phi(1,i,k)+\sum_{i=0}^{n-2}\Phi(0,i,0)+\sum_{s=2}^{n-1}\sum_{k=0}^{s}\Phi(s,n-s,k) + \sum_{k=0}^{n-1}\Phi(n,0,k) - \sum_{s=2}^{n-1}\Phi(s,n-s,s)]$$

$$= A_0^{n-1}[\sum_{s=2}^{n-1}\sum_{i=0}^{n-s}\sum_{k=0}^{s}\Phi(s,i,k)+\sum_{i=0}^{n-2}\sum_{k=0}^{1}\Phi(1,i,k)+\sum_{i=0}^{n-2}\Phi(0,i,0)+\sum_{k=0}^{n}\Phi(n,0,k)- \sum_{s=2}^{n}\Phi(s,n-s,s)]$$

Here

$$\sum_{i=0}^{n-2}\Phi(0,i,0) = \sum_{i=0}^{n-2}(-1)^{n-1-i}\binom{n}{i}a_n^n$$

$$= (-1)\sum_{i=0}^{n-2}(-1)^{n-i}\binom{n}{i}a_n^n$$

$$= (-1)[\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i} - (-1)\binom{n}{n-1} - 1]a_n^n$$

By the binomial theorem, the first term in the above expression equals 0

$$\sum_{i=0}^{n-2}\Phi(0,i,0) = (1-n)a_n^n$$

$$\sum_{i=0}^{n-2}\sum_{k=0}^{1}\Phi(1,i,k) = \sum_{i=0}^{n-2}(\Phi(1,i,0)+\Phi(1,i,1))$$

$$= \sum_{i=0}^{n-2}[(-1)^{n-1-i}\binom{n-1}{i}n(1-\beta-m)a_n^{n-1}+(-1)^{n-1-i}\binom{n-1}{i}B_1na_n^{n-1}]$$

$$= \sum_{i=0}^{n-2}(-1)^{n-1-i}\binom{n-1}{i} na_n^{n-1}(1-\beta-m+B_1)$$
$$= -n(1-\beta-m+B_1)a_n^{n-1}$$

$$\sum_{s=2}^{n}\Phi(s,n-s,s) = \sum_{s=2}^{n}(-1)^{s-1}B_s\binom{n-s}{0}\binom{n}{s}\binom{n-s}{n-s}(1-\beta-m)^0 a_n^{n-s}$$
$$= \sum_{s=2}^{n}(-1)^{s-1}B_s\binom{n}{s}a_n^{n-s}$$
$$\sum_{k=0}^{n}\Phi(n,0,k) = \sum_{k=0}^{n}(-1)^{n-1}B_k\binom{0}{0}\binom{n}{k}\binom{n-k}{n-k}(1-\beta-m)^{n-k}a_n^{n-n}$$
$$= (-1)^{n-1}\sum_{k=0}^{n}B_k\binom{n}{k}(1-\beta-m)^{n-k}$$
$$= (-1)^{n-1}B_n[1-\beta-m]$$

$$\sum_{s=2}^{n-1}\sum_{i=0}^{n-s}\sum_{k=0}^{s}\Phi(s,i,k) =$$
$$\sum_{s=2}^{n-1}\sum_{i=0}^{n-s}\sum_{k=0}^{s}(-1)^{n-1-i}B_k\binom{n-s}{i}\binom{n}{s}\binom{s}{k}(1-\beta-m)^{s-k}a_n^{n-s}$$
$$= \sum_{s=2}^{n-1}\binom{n}{s}a_n^{n-s}\sum_{i=0}^{n-s}(-1)^{n-1-i}\binom{n-s}{i}\sum_{k=0}^{s}B_k\binom{s}{k}(1-\beta-m)^{s-k}$$
$$= \sum_{s=2}^{n-1}\binom{n}{s}a_n^{n-s}\sum_{i=0}^{n-s}(-1)^{n-1-i}\binom{n-s}{i}B_s[1-\beta-m]$$

$= 0$ By the binomial theorem.
where $B_s[1-\beta-m]$ is a Bernoulli polynomial.
So I_1 can be rewritten as

$$A_0^{n-1}\{(1-n)a_n^n - n(1-\beta-m+B_1)a_n^{n-1} - \sum_{s=2}^{n}(-1)^{s-1}B_s$$
$$\tbinom{n}{s}a_n^{n-s} + (-1)^{n-1}B_n[1-\beta-m]\}$$
$$= A_0^{n-1}\{(-n)a_n^n - n(1-\beta-m)a_n^{n-1} - \sum_{s=0}^{n}(-1)^{s-1}B_s$$
$$\tbinom{n}{s}a_n^{n-s} + (-1)^{n-1}B_n[1-\beta-m]\}$$

I_2 and I_3 can be rewritten similarly.

$$I_2 = (-\tfrac{n}{2}a_{n-1}A_1^{n-2})[(1-(n-1))a_n^n - (n-1)(1-\beta-m+B_1)a_n^{n-1}$$
$$-\sum_{s=2}^{n-1}(-1)^{s-1}B_s\binom{n-1}{s}a_n^{n-s} - (-1)^{n-2}a_nB_{n-1}[1-\beta-m]]$$
$$= (-\tfrac{n}{2}a_{n-1}A_1^{n-2})[(-(n-1))a_n^n - (n-1)(1-\beta-m)a_n^{n-1}$$

$-\sum_{s=0}^{n-1}(-1)^{s-1}B_s\binom{n-1}{s}a_n^{n-s} - a_nB_{n-1}[1-\beta-m]]$

$I_3 = [\sum_{l=1}^{n-3}\frac{n}{n-1-l}\frac{s(n-1,n-2-l)}{\binom{n-2}{l}}A_1^{n-2}][-(n-1-l)a_n^n-(n-1-l)(1-\beta-$

$m)a_n^{n-1}-\sum_{s=0}^{n-1-l}(-1)^{s-l}B_s\binom{n-1-l}{s}a_n^{n-s}+(-1)^{n-2-l}a_n^{l+1}(-1)^{n-2-l}B_{n-l}[1-\beta-m]$

In I_1 there is a term $d_1 = A_0^{n-1}[-na_n^{n-1} - n(-\beta-m)a_n^{n-1}]$
$= -na_n^{n-1}A_0^{n-1}(a_n-\beta-m)$
In I_2 there is a term $d_2 = -\frac{n(n-1)}{2}A_1^{n-2}a_{n-1}[-a_n^n - (-\beta-m)a_n^{n-1}]$
$= -na_n^{n-1}(\frac{n-1}{2})a_{n-1}A_1^{n-2}(a_n-\beta-m)$
In I_3 there is a term $d_3 = \sum_{l=1}^{n-3}\frac{n}{n-1-l}\frac{s(n-1,n-2-l)}{\binom{n-2}{l}}A_l^{n-2}[-(n-1-$
$l)(-\beta-m)a_n^{n-1}]$
$= -na_n^{n-1}\sum_{l=1}^{n}\frac{s(n-1,n-2-l)}{\binom{n-2}{l}}A_l^{n-2}(a_n-\beta-m)$
$d_1+d_2+d_3 = -[na_n^{n-1}m'Y_{n-1}(a_n-\beta-m)]$

Then Lemma 4 follows.

Q.E.D.

Now we can study the difference between the sharp estimate Y_n and the sum of the lower dimension sharp estimates $g(a_n-\beta-m)$. The next lemma plays a crucial role in our later computation.

Proof of Lemma 5 Notice that:
$a_n^{n-1}Y_n = A_0^na_n^{n-1} - \frac{n}{2}a_nA_1^{n-1}a_n^{n-1} + \sum_{l=1}^{n-2}\frac{s(n,n-1-l)}{\binom{n-1}{l}}A_l^{n-1}a_n^{n-1}$
and $A_0^na_n^{n-1} = A_0^{n-1}a_n^n$
$a_nA_1^{n-1}a_n^{n-1} = a_n^nA_1^{n-1}$

Plugging into Lemma 4,

$\Delta_0 = A_0^{n-1}a_n^n - \frac{n}{2}A_1^{n-1}a_n^n$
$+\sum_{l=1}^{n-2}\frac{s(n,n-1-l)}{\binom{n-1}{l}}A_l^{n-1}a_n^{n-1}$
$-A_0^{n-1}\{-na_n^{n-1} - \sum_{s=0}^{n}(-1)^{s-1}B_s\binom{n}{s}a_n^{n-s} + (-1)^{n-1}B_n[1-\beta-m]\}$
$-\{-\frac{n}{2}a_{n-1}A_1^{n-2}\}$

$$\times\left\{-(n-1)a_n^{n-1}-\sum_{s=0}^{n-1}(-1)^{s-1}B_s\binom{n-1}{s}a_n^{n-s}+a_n(-1)^{n-2}B_{n-1}[1-\beta-m]\right\}$$
$$-\{\sum_{l=1}^{n-3}\frac{n}{n-1-l}\frac{s(n-1,n-2-l)}{\binom{n-2}{l}}A_l^{n-2}\}$$
$$\times\left\{[-(n-1-l)a_n^{n-1}-\sum_{s=0}^{n-1-l}(-1)^{s-1}B_s\binom{n-1-l}{s}a_n^{n-s}+(-1)^{n-2-l}a_n^{l+1}B_{n-l}[1-\beta-m]\right\}$$

$$\Delta_0=A_0^{n-1}a_n^n-\tfrac{n}{2}A_1^{n-1}a_n^n+\sum_{l=1}^{n-2}\frac{s(n,n-1-l)}{\binom{n-1}{l}}A_l^{n-1}a_n^{n-1}-A_0^{n-1}-na_n^{n-1}-(-a_n^n-\frac{n}{2}a_n^{n-1})$$
$$-A_0^{n-1}\{\sum_{s=2}^{n}(-1)^{s-1}B_s\binom{n}{s}a_n^{n-s}+(-1)^{n-1}B_n[1-\beta-m]\}--\frac{n}{2}a_{n-1}A_1^{n-2}$$
$$\{-(n-1)a_n^{n-1}-(-a_n^n-\tfrac{n-1}{2}a_n^{n-1})-\sum_{s=2}^{n-1}(-1)^{s-1}B_s\binom{n-1}{s}a_n^{n-s}$$
$$+(-1)^{n-2}a_nB_{n-1-l}[1-\beta-m]\}$$
$$-[\sum_{l=1}^{n-3}\frac{n}{n-1-l}\frac{s(n-1,n-2-l)}{\binom{n-2}{l}}A_l^{n-2}]$$
$$\{a_n^n-\tfrac{1}{2}(n-1-l)a_n^{n-1}-\sum_{s=2}^{n-1-l}(-1)^{s-1}B_s\binom{n-1-l}{s}a_n^{n-s}$$
$$+(-1)^{n-2-l}a_n^{l+1}B_{n-l}[1-\beta-m]\}$$

$$\Delta_0=A_0^{n-1}a_n^n(1-1)-\tfrac{n}{2}A_1^{n-1}a_n^n+\tfrac{n}{2}A_0^{n-1}a_n^{n-1}-\tfrac{n(n-1)}{4}a_{n-1}a_n^{n-1}A_1^{n-2}$$
$$+\tfrac{n}{2}a_{n-1}a_n^nA_1^{n-2}+\sum_{l=1}^{n-2}\frac{s(n,n-1-l)}{\binom{n-1}{l}}A_l^{n-1}a_n^{n-1}$$
$$-A_0^{n-1}\{-\sum_{s=2}^{n}(-1)^{s-1}B_s\binom{n}{s}a_n^{n-s}+(-1)^{n-1}B_n[1-\beta-m]\}$$
$$+\tfrac{n}{2}a_{n-1}A_1^{n-2}\{-\sum_{s=2}^{n-1}(-1)^{s-1}B_s\binom{n-1}{s}a_n^{n-s}$$
$$+(-1)^{n-2}a_nB_{n-1}[1-\beta-m]\}$$
$$-[\sum_{l=1}^{n-3}\frac{n}{n-1-l}\frac{s(n-1,n-2-l)}{\binom{n-2}{l}}A_l^{n-2}]$$
$$\{a_n^n-\tfrac{1}{2}(n-1-l)a_n^{n-1}-\sum_{s=2}^{n-1-l}(-1)^{s-1}B_s\tbinom{n-1-l}{s}a_n^{n-s}$$
$$+(-1)^{n-2-l}a_n^{l+1}B_{n-1-l}[1-\beta-m]\}$$

In the last term, define

$$\overline{\Delta} = -\sum_{l=1}^{n-3} \frac{n}{n-1-l} \frac{s(n-1, n-2-l)}{\binom{n-2}{l}} A_l^{n-2}$$

$$- \sum_{s=2}^{n-1-l} (-1)^{s-1} B_s \binom{n-1-l}{s} a_n^{n-s}$$

$$= \sum_{l=1}^{n-3} \frac{n}{n-1-l} \frac{s(n-1, n-2-l)}{\binom{n-2}{l}} \sum_{s=2}^{n-1-l} (-1)^{s-1} B_s \binom{n-1-l}{s} A_l^{n-2} a_n^{n-s}$$

Define the new index $i = s + l$, we have

$$\overline{\Delta} = \sum_{l=1}^{n-3} \frac{n}{n-1-l} \frac{s(n-1, n-2-l)}{\binom{n-2}{l}} \sum_{i=2+l}^{n-1} (-1)^{i-l-1} B_{i-l} \binom{n-1-l}{i-l} A_l^{n-2} a_n^{n-i+l}$$

$$= \sum_{i=n-4}^{n-1} \sum_{l=1}^{i-2} \frac{n}{n-1-l} \frac{s(n-1, n-2-l)}{\binom{n-2}{l}} (-1)^{i-l-1} B_{i-l} \binom{n-1-l}{i-l} A_l^{n-2} a_n^{n-i+l}$$

$$= \sum_{l=1}^{n-3} (-1)^{n-2-l} \frac{n}{n-1-l} \frac{s(n-1, n-2-l)}{\binom{n-2}{l}} B_{n-1-l} A_l^{n-2} a_n^{1+l}$$

$$+ n \sum_{l=1}^{n-4} (-1)^{n-3-l} \frac{s(n-1, n-2-l)}{\binom{n-2}{l}} B_{n-2-l} A_l^{n-2} a_n^{2+l}$$

$$+ \sum_{i=3}^{n-3} (-1)^i \sum_{l=1}^{i-2} \frac{n(-1)^{1+l}}{n-1-i} \frac{s(n-1, n-2-l)}{\binom{n-2}{l}} B_{i-l} A_l^{n-2} a_n^{n-i+l}$$

Also notice among the terms of Δ,

$$\sum_{l=1}^{n-2} \frac{s(n, n-1-l)}{\binom{n-1}{l}} A_l^{n-1} a_n^{n-1}$$

$$= \frac{s(n,n-2)}{\binom{n-1}{1}} A_1^{n-1} a_n^{n-1} + \frac{s(n,n-3)}{\binom{n-1}{2}} A_2^{n-1} a_n^{n-1} + \frac{s(n,1)}{\binom{n-1}{n-2}} A_{n-2}^{n-1} a_n^{n-1}$$

$$+ \sum_{i=n+1}^{2n-5} \frac{s(n, i-(n-1))}{\binom{n-1}{2n-2-i}} a_n^{n-1} A_{2n-2-i}^{n-1}$$

$$-A_0^{n-1} \{ -\sum_{s=2}^{n} (-1)^{s-1} B_s \binom{n}{s} a_n^{n-s} + (-1)^{n-1} B_n [1 - \beta - m] \}$$

$$= -\binom{n}{2} B_2 A_0^{n-1} a_n^{n-2} + \sum_{i=n+1}^{2n-5} (-1)^i \binom{n}{2n-1-i} B_{2n-1-i} A_0^{n-1} a_n^{i-(n-1)}$$

$$+ (-1)^{n-2} \binom{n}{n-1} B_{n-1} A_0^{n-1} a_n + (-1)^{n-2} (-B_n + B_n [1 - \beta - m] A_0^{n-1})$$

$$\tfrac{n}{2}a_{n-1}A_1^{n-2}\{-\sum_{s=2}^{n-1}(-1)^{s-1}B_s\binom{n-1}{s}a_n^{n-s}+(-1)^{n-2}a_nB_{n-1}[1-\beta-m]\}$$
$$=(-1)^{n-2}\tfrac{n(n-1)}{2}B_{n-2}A_l^{n-2}a_{n-1}a_n^2$$
$$+\tfrac{n}{2}a_{n-1}A_1^{n-2}B_2\tbinom{n-1}{2}a_n^{n-2}+(-1)^{n-2}\tfrac{n}{2}\{B_{n-1}[1-\beta-m]-B_{n-1}\}A_1^{n-2}a_{n-1}a_n$$
$$+\sum_{i=n+1}^{2n-5}\frac{n}{2}(-1)^i\binom{n-1}{2n-2-i}B_{2n-2-i}A_1^{n-2}a_{n-1}a_n^{i-(n-2)}$$

$$a_n^n[\sum_{l=1}^{n-3}\frac{n}{n-1-l}\frac{s(n-1,n-2-l)}{\binom{n-2}{l}}A_l^{n-2}]$$
$$=-\tfrac{n}{n-2}\tfrac{s(n-1,n-3)}{\binom{n-2}{1}}A_1^{n-2}a_n^n-\tfrac{n}{n-3}\tfrac{s(n-1,n-4)}{\binom{n-2}{2}}A_2^{n-2}a_n^n$$
$$-\sum_{i=n+1}^{2n-5}\frac{n}{i-(n-1)}\frac{s(n-1,i-n)}{\binom{n-2}{2n-2-i}}A_{2n-2-i}^{n-2}a_n^n$$

$$-\tfrac{1}{2}(n-1-l)a_n^{n-1}[\sum_{l=1}^{n-3}\frac{n}{n-1-l}\frac{s(n-1,n-2-l)}{\binom{n-2}{l}}A_l^{n-2}]$$
$$=\tfrac{1}{2}\tfrac{n}{n-2}s(n-1,n-3)A_1^{n-2}a_n^{n-1}+\sum_{i=n+1}^{2n-5}\frac{n}{2}\frac{s(n-1,i-(n-1))}{\binom{n-2}{2n-3-i}}A_{2n-3-i}^{n-2}a_n^{n-1}$$
$$+\tfrac{1}{2}\tfrac{n}{n-2}s(n-1,1)A_{n-3}^{n-2}a_n^{n-1}$$

Collecting terms,

$$\Delta=\tfrac{n}{2}A_0^{n-1}a_n^{n-1}+\tfrac{n}{2}a_{n-1}a_n^nA_1^{n-2}-\tfrac{n}{2}A_1^{n-1}a_n^n-\tfrac{n(n-1)}{4}a_{n-1}a_n^{n-1}A_1^{n-2}$$
$$+\tfrac{s(n,n-2)}{\binom{n-1}{1}}A_1^{n-1}a_n^{n-1}+\tfrac{s(n,n-3)}{\binom{n-1}{2}}A_2^{n-1}a_n^{n-1}+\tfrac{s(n,1)}{\binom{n-1}{n-2}}A_{n-2}^{n-1}a_n^{n-1}$$
$$+\sum_{i=n+1}^{2n-5}\frac{s(n,i-(n-1))}{\binom{n-1}{2n-2-i}}a_n^{n-1}A_{2n-2-i}^{n-1}$$
$$-\tbinom{n}{2}B_2A_0^{n-1}a_n^{n-2}+\sum_{i=n+1}^{2n-5}(-1)^i\binom{n}{2n-1-i}B_{2n-1-i}A_0^{n-1}a_n^{i-(n-1)}$$
$$+(-1)^{n-2}\tbinom{n}{n-1}B_{n-1}A_0^{n-1}a_n+(-1)^{n-2}(-B_n+B_n[1-\beta-m)A_0^{n-1}]$$
$$+(-1)^{n-2}\tfrac{n(n-1)}{2}B_{n-2}A_l^{n-2}a_{n-1}a_n^2$$
$$+\tfrac{n}{2}a_{n-1}A_1^{n-2}B_2\tbinom{n-1}{2}a_n^{n-2}+(-1)^{n-2}\tfrac{n}{2}\{B_{n-1}[1-\beta-m]-B_{n-1}\}A_1^{n-2}a_{n-1}a_n$$
$$+\sum_{i=n+1}^{2n-5}\frac{n}{2}(-1)^i\binom{n-1}{2n-2-i}B_{2n-2-i}A_1^{n-2}a_{n-1}a_n^{i-(n-2)}$$

$$-\tfrac{n}{n-2}\tfrac{s(n-1,n-3)}{\binom{n-2}{1}}A_1^{n-2}a_n^n-\tfrac{n}{n-3}\tfrac{s(n-1,n-4)}{\binom{n-2}{2}}A_2^{n-2}a_n^n$$
$$-\sum_{i=n+1}^{2n-5}\frac{n}{i-(n-1)}\frac{s(n-1,i-n)}{\binom{n-2}{2n-2-i}}A_{2n-2-i}^{n-2}a_n^n$$
$$+\tfrac{1}{2}\tfrac{n}{n-2}s(n-1,n-3)A_1^{n-2}a_n^{n-1}+\sum_{i=n+1}^{2n-5}\frac{n}{2}\frac{s(n-1,i-(n-1))}{\binom{n-2}{2n-3-i}}A_{2n-3-i}^{n-2}a_n^{n-1}$$
$$+\tfrac{1}{2}\tfrac{n}{n-2}s(n-1,1)A_{n-3}^{n-2}a_n^{n-1}$$
$$+\sum_{l=1}^{n-4}(-1)^{n-1-l}\frac{n}{\binom{n-2}{l}}B_{n-2-l}A_l^{n-2}a_n^{2+l}$$
$$+(-1)^{n-2}\sum_{l=1}^{n-3}\frac{n(-1)^{l+1}}{n-1-l}\frac{s(n-1,n-2-l)}{\binom{n-2}{l}}\{B_{n-1-l}[1-m]-B_{n-1-l}\}A_l^{n-2}a_n^{l+1}$$
$$+\sum_{i=n+1}^{2n-5}\sum_{s=1}^{2n-4-i}(-1)^i\sum_{l=1}^{i-2}\frac{n(-1)^{1+l}}{n-1-i}\frac{s(n-1,n-2-l)}{\binom{n-2}{l}}B_{i-l}A_l^{n-2}a_n^{n-i+l}$$

Also notice that,

$$B_n[1-\beta-m]=\sum_{k=0}^{n}\binom{n}{k}(1-\beta-m)^{n-k}B_k$$
$$=\sum_{k=0}^{n-1}\binom{n}{k}(1-\beta-m)^{n-k}B_k+B_n$$
$$=\sum_{k=0}^{n-1}\binom{n}{k}B_k\sum_{t=0}^{n-k}\binom{n-k}{t}(1-m)^t(-\beta)^{n-k-t}+B_n$$
$$=\sum_{k=0}^{n-1}\binom{n}{k}B_k(1-m)^{n-k}+B_n+\sum_{k=0}^{n-1}\binom{n}{k}B_k\sum_{t=0}^{n-k-1}\binom{n-k}{t}(1-m)^t(-\beta)^{n-k-t}$$
$$=B_n[1-m]+\sum_{k=0}^{n-1}\binom{n}{k}B_k\sum_{t=0}^{n-k-1}\binom{n-k}{t}(1-m)^t(-\beta)^{n-k-t}$$

Let

$$\Psi(n,m,\beta)=\sum_{k=0}^{n-1}\binom{n}{k}B_k\sum_{t=0}^{n-k-1}\binom{n-k}{t}(1-m)^t(-\beta)^{n-k-t}$$

Define the new index $s=k+t$. We have

$$\Psi(n,m,\beta)=\sum_{k=0}^{n-1}\binom{n}{k}B_k\sum_{s=k}^{n-1}\binom{n-k}{s-k}(1-m)^{s-k}(-\beta)^{n-s}$$

$$= \sum_{s=0}^{n-1}\sum_{k=0}^{s}\binom{n}{k} B_k \binom{n-k}{s-k}(1-m)^{s-k}(-\beta)^{n-s}$$
$$= \sum_{s=0}^{n-1}(-\beta)^{n-s}\sum_{k=0}^{s}\binom{n}{s}\binom{s}{k}(1-m)^{s-k}B_k$$
$$= \sum_{s=0}^{n-1}(-\beta)^{n-s}\binom{n}{s}\sum_{k=0}^{s}\binom{s}{k}(1-m)^{s-k}B_k$$
$$= \sum_{s=0}^{n-1}\binom{n}{s}(-\beta)^{n-s}B_s[1-m]$$

Notice that $B_n[1-x] = (-1)^n B_n[x]$
So
$$\Psi(n,m,\beta) = (-1)^n \sum_{s=0}^{n-1}\binom{n}{s}B_s[m]\beta^{n-s}$$

Similarly,
$$B_{n-1}[1-\beta-m] = B_{n-1}[1-m] + \sum_{k=0}^{n-2}\binom{n-1}{k} B_k \sum_{t=0}^{n-k-2}\binom{n-1-k}{t}$$
$$(1-m)^t(-\beta)^{n-1-k-t}$$
$$B_{n-1-l}[1-\beta-m] = B_{n-1-l}[1-m] + \sum_{k=0}^{n-2-l}\binom{n-1-l}{k} B_k \sum_{t=0}^{n-k-2-l}\binom{n-1-k-l}{t}$$
$$(1-m)^t(-\beta)^{n-1-k-t-l}$$

Then,

$$B_n[1-\beta-m] = B_n[1-m] + \Psi(n,m,\beta)$$
$$B_{n-1}[1-\beta-m] = B_{n-1}[1-m] + \Psi(n-1,m,\beta)$$
$$B_{n-1-l}[1-\beta-m] = B_{n-1-l}[1-m] + \Psi(n-1-l,m,\beta)$$

Using the above results and collecting terms with the same degree, Lemma 5 follows.

Q.E.D

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