

Study on the Problem of the Number Ring Transformation

Nanjing JinLing High School

Jiang Su, China

Yan Meng-yuan Rong Xin-yun Zheng Xin

Director Teacher: *Feng Hui-yu Wang Bing-feng*

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ABSTRACT

A topic for study derives from an interesting problem on the internet. Some significative conclusions about the problem of number ring transformation are made in the paper. Firstly, an assumption that a given starting number ring could be transformed to the end ring in finite degree is proved. Secondly, transformation degree is proved to be unconcerned with the order of the transformation. Furthermore, a function of computing transformation degree is worked out. Finally, the same end ring is gained with a given starting number ring transformed in different sequence.

Keywords: Number Ring, End Ring, Transformation Sequence,
Transformation Degree

INTRODUCTION

This topic derives from an interesting problem on the internet^[1]. We make an effort to develop the researches on this problem. A series of interesting conclusions were found. Among them, the problem in the 27th IMO is a specific instance of our first conclusion.

The problem is described as follows.

Each vertex of regular n -gon ($n \in \mathbb{N}^*, n \geq 3$) corresponds to an integer. The sum of the n integers is equal to 1. Three adjacent vertexes in turn correspond to (x, y, z) , and $y < 0$. The action that (x, y, z) is replaced by $(x+y, -y, z+y)$ in turn is considered to be a transformation. **If only** there is a negative at least in the n integers, the transformation will be carried out. **At last**, the end ring would be gained, which is composed by $(1, 0, \dots, 0)$, and number of 0 is $n-1$.

With regard to each given n -membered ring ($n \in \mathbb{N}^*, n \geq 3$), We will prove that

- (1) Transformation degree is finite.
- (2) Transformation degree is definite and computable.
- (3) End ring of n -membered ring is fixed.

1 Finiteness and Definiteness of Transformation Degree

1.1 $n=3$ (n is equal to 3)

When $n=3$, we suppose that the figures of vertexes of 3-membered ring are respectively a_1, a_2, a_3 , **and** $a_1+a_2+a_3=1$, $a_2<0$. After transformed once, the three figures (a_1, a_2, a_3) are replaced respectively by $(a_1+a_2, -a_2, a_3+a_2)$.

Design objective function f

$$f(a_1, a_2, a_3) = |a_1| + |a_2| + |a_3| + |a_1+a_2| + |a_2+a_3| + |a_3+a_1| + |a_1+a_2+a_3|$$

Obviously $f(a_1, a_2, a_3) \geq 0$.

$$\begin{aligned} \text{Then } f(a_1+a_2, -a_2, a_3+a_2) - f(a_1, a_2, a_3) &= |a_1+a_2+a_3+a_2| - |a_1+a_3| \\ &= |1+a_2| - |1-a_2| = -2 \end{aligned}$$

Because $f(a_1, a_2, a_3)$ is finite positive number, such process couldn't be performed infinitely. As regards $n=3$, transformation degree is finite.

The ring in the last stage is called as end ring. Then the value of the three-membered end ring is denoted as $f(1, 0, 0)$ which is equal to 4.

Thereby, the transformation degree that the three-membered ring (a_1, a_2, a_3) is transformed to the end ring $(1, 0, 0)$ is indicated as the function g as follows.

$$g(a_1, a_2, a_3) = \frac{1}{2} f(a_1, a_2, a_3) - 2$$

1.2 $n \in \mathbb{N}^*, n \geq 3$

For $n \in \mathbb{N}^*, n \geq 3$, we suppose that the negative of each transformation is a_2 .

We construct the objective function

$$\begin{aligned} f(a_1, a_2, a_3, \dots, a_n) &= |a_1| + |a_2| + \dots + |a_n| + |a_1+a_2| + |a_2+a_3| + \dots + \\ &|a_n+a_1| + |a_1+a_2+a_3| + |a_2+a_3+a_4| + \dots + |a_n+a_1+a_2| + \dots + |a_1+a_2+a_3| \dots \end{aligned}$$

$$+a_{n-1}|+|a_2+a_3+a_4+\cdots+a_n|+\cdots+|a_n+a_1+\cdots+a_{n-2}|+|a_1+a_2+\cdots+a_n|$$

Compare the value of $f(a_1, a_2, a_3, \cdots, a_n)$ and $f(a_1', a_2', a_3', \cdots, a_n')$ after one transformation, **then**

$$|a_1'|=|a_1+a_2|, |a_2'|=|a_2|, |a_3'|=|a_2+a_3|, |a_i'|=|a_i|, i \geq 4, |a_n'|=|a_n|$$

$$|a_1'+a_2'|=|a_1|, |a_2'+a_3'|=|a_3|, |a_3'+a_4'|=|a_2+a_3+a_4|,$$

$$|a_i'+a_{i+1}'|=|a_i+a_{i+1}|, \cdots, |a_n'+a_1'|=|a_n+a_1+a_2|,$$

.....

$$|a_1'+a_2'+\cdots+a_k'|=|a_1+a_2+\cdots+a_k|, |a_2'+a_3'+\cdots+a_{k+1}'|=|a_3+a_4+\cdots$$

$$+a_{k+1}|, |a_3'+a_4'+\cdots+a_{k+2}'|=|a_2+a_3+\cdots+a_{k+2}|, \cdots, ,$$

$$|a_i'+a_{i+1}'+\cdots+a_{i+k-1}'|=|a_i+a_{i+1}+\cdots+a_{i+k-1}|, 4 \leq i \leq n+1-k \cdots$$

$$|a_{n-k+2}'+a_{n-k+3}'+\cdots+a_n'+a_1'|=|a_{n-k+2}+a_{n-k+3}+\cdots+a_n+a_1+a_2|,$$

$$|a_{n-k+3}'+a_{n-k+4}'+\cdots+a_n'+a_1'+a_2'|=|a_{n-k+3}+a_{n-k+4}+\cdots+a_n+a_1|,$$

$$|a_{n-k+i+1}'+a_{n-k+i+2}'+\cdots+a_n'+a_1'+\cdots+a_i'|=|a_{n-k+i+1}+a_{n-k+i+2}+\cdots+a_n+a_1+\cdots+a_i|, \quad 3$$

$$\leq i \leq k-1$$

.....

$$|a_1'+a_2'+\cdots+a_{n-2}'|=|a_1+a_2+\cdots+a_{n-2}|, |a_2'+a_3'+\cdots+a_{n-1}'|=|a_3+a_4+\cdots$$

$$a_{n-1}|, |a_3'+a_4'+\cdots+a_n'|=|a_2+a_3+\cdots+a_n|, \cdots$$

$$|a_i'+a_{i+1}'+\cdots+a_n'+a_1'+\cdots+a_{i-3}'|=|a_i+a_{i+1}+\cdots+a_n+a_1+\cdots+a_{i-3}|, 4 \leq i \leq n$$

$$|a_1'+a_2'+\cdots+a_{n-1}'|=|a_1+a_2+\cdots+a_{n-1}|, |a_2'+a_3'+\cdots+a_n'|=|a_3+a_4+\cdots$$

$$+a_n|, |a_3'+a_4'+\cdots+a_n'+a_1'|=|a_2+a_3+\cdots+a_n+a_1+a_2|, \cdots$$

$$|a_i'+a_{i+1}'+\cdots+a_n'+a_1'+\cdots+a_{i-2}'|=|a_i+a_{i+1}+\cdots+a_1+\cdots+a_{i-2}|, 4 \leq i \leq n$$

$$|a_1'+a_2'+\cdots+a_n'|=|a_1+a_2+\cdots+a_n|$$

(i, k is the number of the node of the number ring, $1 \leq i, k \leq n$)

So we can make a conclusion:

$$f(a_1', a_2', a_3', \cdots, a_n') - f(a_1, a_2, a_3, \cdots, a_n) = |1+a_2| - |1-a_2| = -2 < 0$$

Thus, we find that the value is decreased by 2 after one transformation.

As the value of $f(a_1, a_2, a_3, \cdots, a_n)$ is finite positive number, such process

couldn't be performed infinitely. For $n \geq 3$, transformation degree is finite.

The value of the end ring $(1, 0, 0, \dots, 0)$ is gained as follows:

$$f(1, 0, 0, \dots, 0) = 1 + 2 + 3 + \dots + (n-1) + 1 = \frac{n^2 - n + 2}{2}$$

Thereby, the function g , which is used to gain the transformation degree of n -membered ring, is denoted as follows:

$$g(a_1, a_2, a_3, \dots, a_n) = \frac{1}{2} f(a_1, a_2, a_3, \dots, a_n) - \frac{n^2 - n + 2}{4}$$

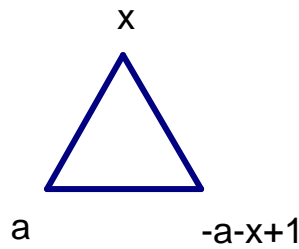
After a lot of experiments, we find that for a given n -membered ring, the position of figure 1 in the end ring is unique no matter the transformation sequence. We will make further researches on this theme.

2 Fixity of the Position of figure 1 in the End Ring

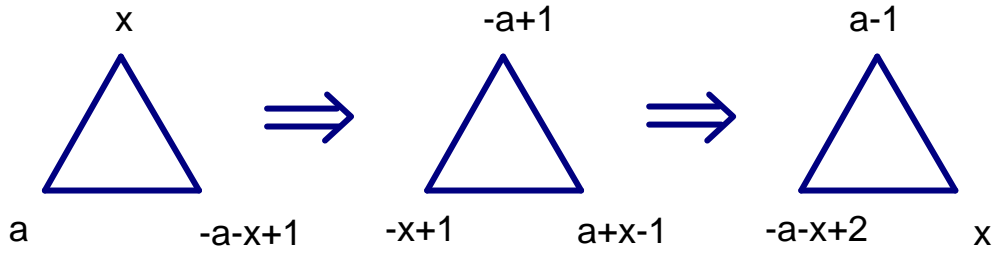
2.1 $n=3$ (n is equal to 3)

Positions of the three-membered ring are marked as follows, top vertex is marked as position one, bottom-left vertex is marked as position two, and bottom-right vertex is marked as position three. We suppose position $3n+1 (n \in \mathbb{Z})$ as position one, position $3n+2 (n \in \mathbb{Z})$ as position two, and position $3n (n \in \mathbb{Z})$ as position three. Meanwhile, we suppose that figure of the position one is x , figure of the position two is a , and figure of the position three is $-a-x+1$, in addition $a \geq x \geq 0$.

As shown in the figure.

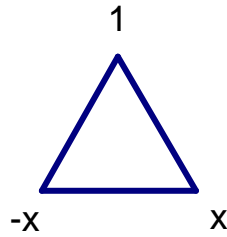


The following transformation is proposed.



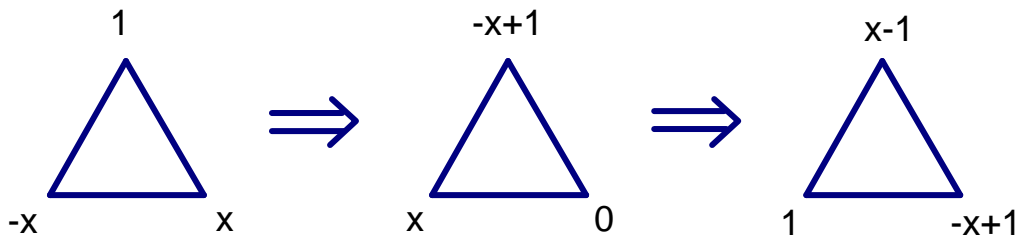
It's that, the transformation could change a to be $a-1$, and $-a-x+1$ to be $-a-x+2$. Absolute value of them is reduced by 1. Besides, position of x is moved to position three from position one, that is, the position is moved clockwise once. If $-a-x+k+1 < 0$ and $-a+k < 0$ (that is $k < a$, the mean of k is transformation degree) then the transformation will be carried out.

After transformed $a-1$ times, a new three-membered ring is gained.



Among it, the position of x is moved clockwise $a-1$ times. It will move to the position $-a+2$, that is, $1-(a-1)$. So figure 1 will lie in the position $-a+3$.

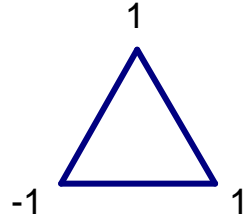
Then, the three-membered ring $(1, -x, x)$ is transformed as follows.



Among it, x is replaced by $x-1$ and $-x$ is replaced by $-x+1$. The position of figure 1 will be moved to position two from position one, that is, it is moved counterclockwise once.

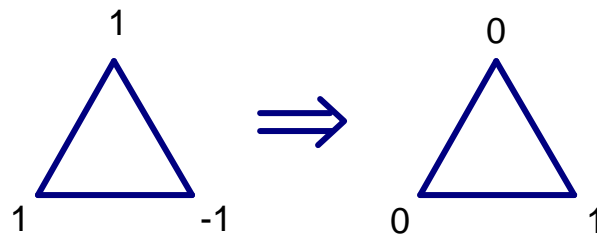
If $x-k > 0$ **then** the transformation will be carried out.

After transformed $x-1$ times, a new three-membered ring is gained.



At this time, figure 1 lies in the position $-a+x+2$, that is, $-a+3+(x-1)$.

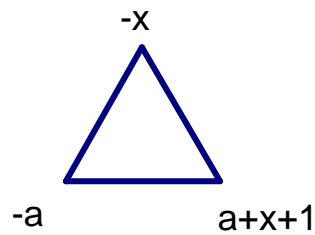
To follow, the last transformation is carried out to make the end ring brought out.



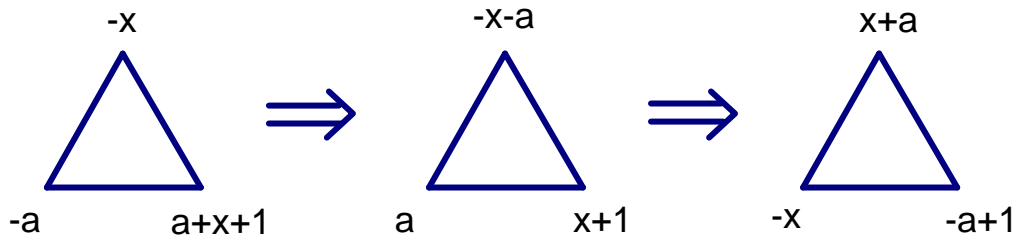
At last, figure 1 will lie in the position $-a+x+3$ with position $-a+x+2$ moved counterclockwise once.

A three-membered ring with a couple of negatives could be transformed in like manner. We suppose that $-x$ lies in the position one, $-a$ lies in the position two, and $a+x+1$ lies in the position three. Among it, $a \geq x > 0$.

As shown in the figure.

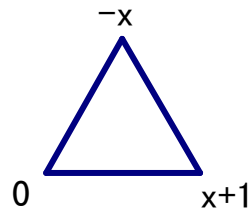


The following transformation is proposed.

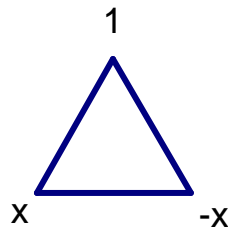


It's that, the transformation could change $-a$ to be $-a+1$ and $a+x+1$ to be $a+x$. Absolute value of them is reduced by 1. Besides, position of $-x$ is moved to position two from position one, that is, the position is moved counterclockwise once.

After transformed a times, a new three-membered ring is gained.

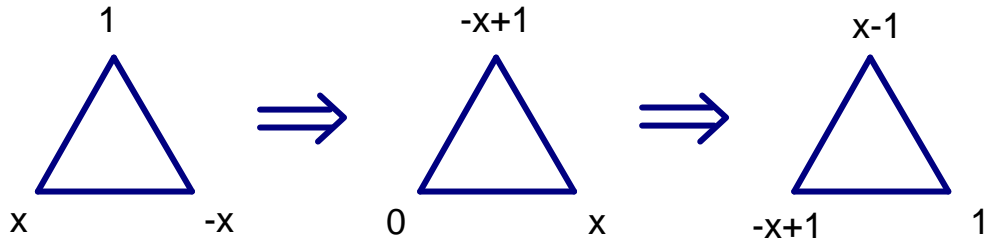


The three-membered ring $(1, x, -x)$ is gained after transformed once again.



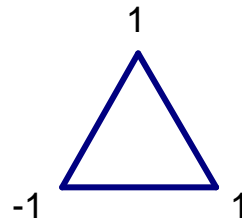
Among it, the position of $-x$ is moved clockwise $a-1$ times. It will move to the position $a+2$, that is, $1+(a+1)$. So figure 1 will lie in the position $a+3$.

Then, the three-membered ring $(1, x, -x)$ is transformed as follows.

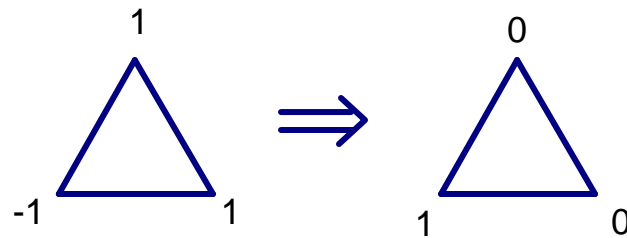


Among it, x is replaced by $x-1$ and $-x$ is replaced by $-x+1$. The position of figure 1 will be moved to position three from position one, that is, it is moved clockwise once.

After transformed $x-1$ times, a three-membered ring is gained.



The end ring will be gained after the last transformation.



At last, figure 1 will lie in the position $a-x+3$ with position $a+3$ moved clockwise x times, that is, $x-1+1$.

If the given 3-membered ring satisfies the condition $a > x > 0$, we can conclude the position of the figure 1 in the end ring. **If** the given 3-membered ring satisfies the condition $x > a > 0$, we can conclude the position of figure 1 in the end ring by turning over the position one and two.

Summing up the above, we can make the following conclusions:

We suppose the given 3-membered ring is $(x, y, 1-x-y)$, where $xy \geq 0$, meanwhile, x lies in the position one, y lies in the position two, and $1-x-y$ lies in the position two $x-y$. We assume $h(x, y, 1-x-y) = x-y$ ($xy \geq 0$).

We prove that, for 3-membered ring $(x, y, 1-x-y)$, the value of function h remains unchanged after each transformation .

For $x > 0$ and $x > 0$,

$$h(x, y, 1-x-y) = x-y.$$

$$h(1-y, 1-x, x+y-1) = x-y = h(x, y, 1-x-y)$$

For $x < 0$ and $y < 0$,

$$h(x, y, 1-x-y) = x-y.$$

$$h(-x, y+x, 1-y) = h(1-y, -x, y+x) - 1 = 1-y+x-1 = x-y$$

$$h(x+y, -y, 1-x) = h(-y, 1-x, x+y) + 1 = -y-(1-x)+1 = x-y$$

For $x > y = 0$,

$$h(x, 0, -x+1) = x-0 = x$$

$$h(1, -x+1, x-1) = h(x-1, 1, -x+1) - 1 = x-1-1-1 = x-3 \equiv h(x, 0, -x+1)$$

So it has been prove that the value doesn't change after each transformation when $n=3$. **That is**, the end ring is always the same no matter the transformation sequence.

2.2 $n=4$ (n is equal to 4)

For a four-membered ring $(a, b, c, 1-a-b-c)$, we suppose that a lies in position one, b in position two, c in position three and $1-a-b-c$ in position four, whose sequence is anti-clockwise. Meanwhile, we assume $4n+1$ is the position one, $4n+2$ is the position two, $4n+3$ is the position three and $4n$ the position four ($n \in \mathbb{Z}$).

Assuming $h(a, b, c, 1-a-b-c) = a-2b+c$,

When the parameter a is changed

$$h(-a, a+b, c, 1-b-c) = -a-c+2(a+b) = a+2b-c = h(a, b, c, 1-a-b-c);$$

When the parameter b is changed

$$\begin{aligned} h(a+b, -b, b+c, 1-a-b-c) &= (a+b) - (b+c) + 2(-b) = a-2b-c \\ &\equiv a+2b-c = h(a, b, c, 1-a-b-c); \end{aligned}$$

When the parameter c is changed

$$\begin{aligned} h(a, b+c, -c, 1-a-b) &= a - (-c) + 2(b+c) = a+2b+3c \\ &\equiv a+2b-c = h(a, b, c, 1-a-b-c); \end{aligned}$$

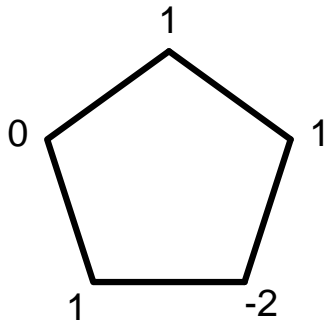
When the parameter $1-a-b-c$ is changed

$$h(1-b-c, b, 1-a-b, a+b+c-1) = (1-b-c) - (1-a-b) + 2b = a+2b-c = h(a, b, c, 1-a-b-c).$$

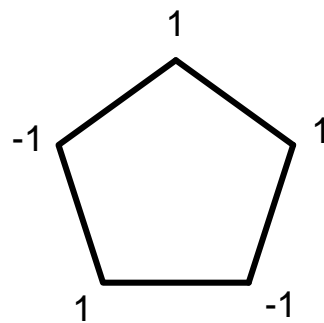
So we have proved that the value of function is fixed in any case of transformation when $n=4$. **That is**, the end ring is the same, which is unconcerned with the transformation sequence.

2.3 $n \in \mathbb{N}^*, n \geq 3$

Based on the data of the program ^[2], we gained **the theorem of arrowhead**. Here we give an example for demonstration. According to the program, we know that figure 1 in the first five-membered ring lies in position one and in the second five-membered ring lies in position three.



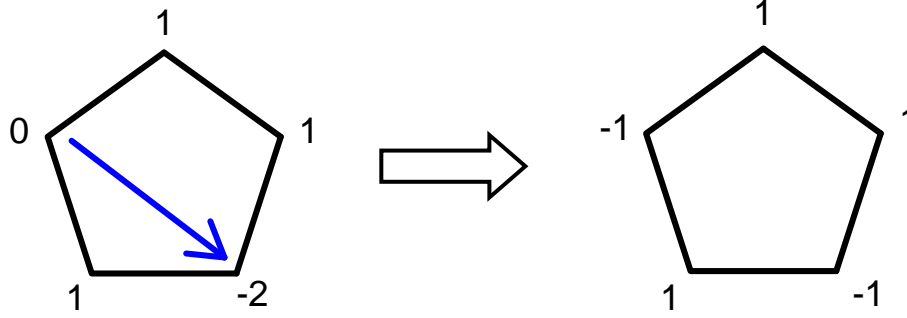
the first five-membered ring



the second five-membered ring

The second five-membered ring is achieved by remotion of the arrowhead based on the first five-membered ring. The course of transformation is

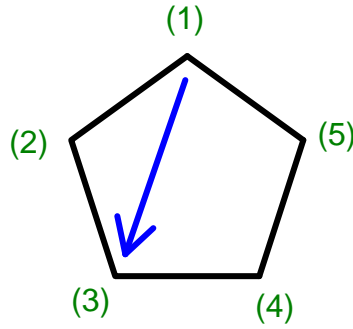
showed in the nether figure.



the first five-membered ring

the second five-membered ring

We circumvolve the tail of arrowhead to the position one (the original position of figure 1). At this time, the head of arrowhead points to the position marked 3(the new position of figure 1), which is showed in the nether figure.



According to **the theorem of arrowhead**, we design a right function, which can prove that the position of figure 1 in n-membered ring is fixed.

For a n-membered ring $(a_1, a_2, a_3 \cdots, a_n)$,

$$h(a_1, a_2, a_3 \cdots, a_n) = a_1 + 2a_2 + 3a_3 + \cdots + ka_k + \cdots + (n-1)a_{n-1}$$

Supposing the changed negative is a_k , Assuming $a_{n+1} = a_1, a_0 = a_n$, **then**

$$\begin{aligned} & h(a_1, a_2, a_3, \cdots, a_{k-1} + a_k, -a_k, a_{k+1} + a_k, \cdots, a_n) \\ & \equiv (a_1 + 2a_2 + 3a_3 + \cdots + ka_k + \cdots + (n-1)a_{n-1}) - [(k-1)a_{k-1} + ka_k + (k+1)a_{k+1}] \\ & \quad + [(k-1)(a_{k-1} + a_k) + k(-a_k) + (k+1)(a_{k+1} + a_k)] = h(a_1, a_2, a_3 \cdots, a_n) \end{aligned}$$

So we have proved that the value of function is fixed for any n ($n \in \mathbb{N}^*, n \geq 3$). **That is**, the end ring is the same, which is unconcerned with the

transformation sequence. The value of the end ring is $h=1 \times 0+2 \times 0+3 \times 0+4 \times 0+\cdots+r \times 1+\cdots+(n-1) \times 0=r \equiv h(a_1,a_2,a_3,\cdots,a_n)(\text{mod } n)$. So the arithmetical compliment r of $h(a_1,a_2,a_3,\cdots,a_n) \text{ mod } n$, is the position of figure 1 in the end ring. (If $r=1$,figure 1 in the end ring lies in position one ; \cdots ; if $r=k$, in position k ; if $r=0$, in position n)

CONCLUSION

After a lot of experiments, we draw the following conclusions by constructing a function.

For a given n -membered ring ($n \in \mathbb{N}^*, n \geq 3$):

① The finiteness and definiteness of transformation degree

Transformation degree g

$$g(a_1, a_2, a_3, \dots, a_n) = \frac{1}{2} (|a_1| + |a_2| + \dots + |a_n| + |a_1 + a_2| + |a_2 + a_3| + \dots + |a_n + a_1| + |a_1 + a_2 + a_3| + |a_2 + a_3 + a_4| + \dots + |a_n + a_1 + a_2| + \dots + |a_1 + a_2 + a_3 \dots + a_{n-1}| + |a_2 + a_3 + a_4 \dots + a_n| + \dots + |a_n + a_1 + \dots + a_{n-2}| + |a_1 + a_2 + \dots + a_n|) - \frac{n^2 - n + 2}{4}$$

② The fixity of Position of figure 1 in the End Ring

Number $1, 2, 3, \dots, n$ anti-clockwise, where the value in position k is a_k . We define a function $h(a_1, a_2, a_3, \dots, a_n) = a_1 + 2a_2 + 3a_3 + \dots + ka_k + \dots + (n-1)a_{n-1}$. The remainder r of $h(a_1, a_2, a_3, \dots, a_n)$ modulo n is the position of figure 1 in the End Ring. (If $r=1$, it is in position 1. If $r=2$, it is in position 2, \dots , etc. If $r=k$, it is in position k . And if $r=0$, it is in position n).

At the time we draw the conclusion, we are conquered by the beauty and oneness of mathematics. We are more eager to explore the unknowns in mathematics. We are going to make researches on the problems in the condition that the sum expands from 1 to 2, 3, 4 and so on so that the conclusion can be commonly adapted. When sum increases, the end ring is more and more complicated. As the time is limited, for the moment, we have only solved the problem in 3-membered ring when the sum is 2.

(The method in this condition is similar when the sum is 1. The function becomes $h(a_1, a_2, a_3) = \frac{a_1 - a_2}{2}$, where a_1, a_2 have same parity).

The research in the future must be more challenging and interesting. We will keep on improving the ability of solving the mathematics problems and experiencing the essence of 'Fun with Math'.

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Thanks to JinLing High School for providing a strong research atmosphere.

Thanks to the Yau(HSM) Awards Committee, which provides a wonderful opportunity for us to communicate and learn from each other.

REFERENCES

IMO quizzery of previous sessions, Peijie-Liu, Harbin Institute of Technology Press p419~420

APPENDIX

【1】 Site: <http://tieba.baidu.com/f?kz=236481629>

【2】 Code:

```
program xx(input, output);
var i, j, n: integer;
    a, sum: array[1..10] of integer;
    q: array[1..1000000, 1..10] of integer;
    s, f, front, rear: integer;
    flag: boolean;
begin
s:=0;
readln(n);
for i:=1 to n-1 do begin readln(a[i]); s:=s+a[i]; end;
a[n]:=1-s;
front:=1; rear:=1; q[1]:=a;
while front<=rear do
begin
flag:=false;
for i:=1 to n do
if q[front, i]<0 then
begin
flag:=true;
inc(rear);
q[rear]:=q[front];
if i=1 then q[rear, n]:=q[rear, n]+q[rear, 1]
```

```

        else q[rear, i-1]:=q[rear, i-1]+q[rear, i];
    if i=n then q[rear, 1]:=q[rear, 1]+q[rear, n]
        else q[rear, i+1]:=q[rear, i+1]+q[rear, i];
    q[rear, i]:=0-q[rear, i];
end;
if not flag then break
    else inc(front);
end;
assign(output, 'qct.out');
rewrite(output);
for i:=front to rear do
    for j:=1 to n do if q[i, j]=1 then writeln(j);
sum[1]:=a[1];
for i:=2 to n do sum[i]:=sum[i-1]+a[i];
f:=0;
for i:=1 to n-1 do
    for j:=i+1 to n do
        f:=f+abs(sum[j]-sum[i])+abs(sum[i]+sum[n]-sum[j]);
inc(f);
writeln('f=', f);
close(output);
readln;
end.

```

Results:

$n=3$ a_1 (vertical), a_2 (horizontal)

$n=4$ a_1 (horizontal), a_2 (vertical), $a_3=0$

	1	2	3	4	5	6	7	8	9
1	3	2	1	3	2	1	3	2	1
2	1	3	2	1	3	2	1	3	2
3	2	1	3	2	1	3	2	1	3
4	3	2	1	3	2	1	3	2	1
5	1	3	2	1	3	2	1	3	2
6	2	1	3	2	1	3	2	1	3
7	3	2	1	3	2	1	3	2	1
8	1	3	2	1	3	2	1	3	2
9	2	1	3	2	1	3	2	1	3

	-5	-4	-3	-2	-1	0
0	3	1	3	1	3	1
1	4	2	4	2	4	2
2	1	3	1	3	1	3
3	2	4	2	4	2	4
4	3	1	3	1	3	1
5	4	2	4	2	4	2