J. DIFFERENTIAL GEOMETRY
75 (2007) 387-401

NORMAL GEODESIC GRAPHS OF CONSTANT MEAN CURVATURE

Luis J. Alías & Marcos Dajczer

Abstract

We extend recent existence results for compact constant mean curvature normal geodesic graphs with boundary in space forms in a unified way to a large class of ambient spaces. That extension is achieved by solving a constant mean curvature existence problem in a general setting presented in two isometrically equivalent versions.

1. Introduction

Various existence results for compact with boundary constant mean curvature normal (geodesic) graphs Σ^n in flat Euclidean space $\mathbb{R}^{n+1}$ and standard hyperbolic space $\mathbb{H}^{n+1}$ have been recently obtained. The common context for these results is that of a bounded domain Ω contained in a nonnegatively curved totally umbilical hypersurface $\mathbb{P}^n$ of the ambient space form. In addition, the boundary Γ of Ω is mean convex, that is, its mean curvature H_Γ as a submanifold of $\mathbb{P}^n$ is positive with respect to the inner orientation. Then, for any number H satisfying $-H_\Gamma < H \leqslant 0$, it is shown that there exists a normal graph Σ^n over Ω with boundary Γ and constant mean curvature H.

The case of normal graphs over a domain in a round sphere $\mathbb{S}^n$ contained in $\mathbb{R}^{n+1}$ was solved in [**6**] when the closure of the domain is contained in an open hemisphere. For $n=2$, this also follows from the general result given in [**3**]. A proof for normal graphs in $\mathbb{H}^{n+1}$ over a domain in an horosphere under the weaker assumption $-H_\Gamma < H \leqslant 1$ was obtained in [**7**] after the earlier result in [**9**] for $H \in (0,1)$. Moreover, the case of a domain with closure contained in an open hemisphere of a geodesic sphere in $\mathbb{H}^{n+1}$ was solved in [**5**].

Constant mean curvature normal geodesic graphs have also been considered outside ambient space forms, namely, the result in [**2**] applies to vertical graphs in product manifolds $M^3 = \mathbb{P}^2 \times \mathbb{R}$.

L.J. Alías was partially supported by MEC/FEDER Grant MTM2004-04934-C04-02 and Fundación Séneca Grant 00625/PI/04, Spain. M. Dajczer was partially supported by CNPq - 200299/2004-2, Brazil, and MEC Grant SAB2003-0275, Spain.
Received 08/10/2005.

In all of the above cases, it is shown that there exists a smooth function u on Ω vanishing at Γ such that the corresponding normal geodesic graph (i.e., the set of points at distance $u(x)$ along the unit speed normal geodesic starting at $x \in \Omega$ in the direction of the mean curvature vector of $\mathbb{P}^n$) is a hypersurface of constant mean curvature H.

Our goal in this article is to extend the aforementioned results in a unified way to a large class of Riemannian ambient spaces. In the process, some of them are also generalized. Our extension is achieved by solving a constant mean curvature existence problem in a general setting presented in two isometrically equivalent versions related by a generalized Mercator projection, as seen in [**1**]. The first version is for hypersurfaces in warped product spaces, and the second one in ambient spaces conformal to Riemannian products. A precise description of both versions of our setting, as well as the condition for equivalence, is given in the first section. In the following two sections of the paper we state and prove our results.

Finally, we would like to thank Jaime Ripoll for several useful comments.

2. The setting

We can describe with further details the common framework of the results in the introduction as being that of a bounded domain Ω contained in a leaf of a foliation orthogonal to a closed conformal Killing field of the ambient space. Moreover, the leaves are parallel totally umbilical constant mean curvature hypersurfaces and the flow of normal geodesics acts homethetically on them.

If the ambient is just a space form, the geometric context has the following elementary description. Associated to any given complete totally umbilical hypersurface $\mathbb{P}^n$ of either $\mathbb{R}^{n+1}$, $\mathbb{S}^{n+1}$ or $\mathbb{H}^{n+1}$, there is a warped product representation of the space form (possibly up to one or two points) as a warped product $\mathbb{I} \times_\varrho \mathbb{P}^n$. Here $\mathbb{I} \subset \mathbb{R}$ is an open interval and the warping function $\varrho \in \mathcal{C}^\infty(\mathbb{I})$ can be seen as a height function when considered in the proper setting; see [**10**] for details. Then $\mathcal{T} = -\varrho(t)\partial/\partial t$ is a closed conformal Killing field orthogonal to the foliation $t \in \mathbb{I} \mapsto \{t\} \times \mathbb{P}^n$ determined by the representation. For instance, $\mathbb{R}^{n+1} = \mathbb{R}_+ \times_t \mathbb{S}^n$ in the Euclidean case. In the hyperbolic case, we have $\mathbb{H}^{n+1} = \mathbb{R} \times_{e^t} \mathbb{R}^n$ if the foliation is by horospheres and $\mathbb{H}^{n+1} = \mathbb{R}_+ \times_{\sinh t} \mathbb{S}^n$ if by geodesic hyperspheres.

In view of the results referred to in the introduction it is natural to work in the following general setting.

2.1. The $1^{\underline{\text{st}}}$ version. Consider the product manifold $\mathbb{I} \times \mathbb{P}^n$, where $\mathbb{I} \subset \mathbb{R}$ is an open interval and $(\mathbb{P}^n, \langle\ ,\ \rangle_{\mathbb{P}})$ an n-dimensional Riemannian manifold. Let

$$M^{n+1} = \mathbb{I} \times_\varrho \mathbb{P}^n$$

be the product manifold endowed with the Riemannian warped product metric

$$\langle\ ,\ \rangle_M = \pi_{\mathbb{I}}^*(dt^2) + \varrho^2(\pi_{\mathbb{I}})\pi_{\mathbb{P}}^*(\langle\ ,\ \rangle_{\mathbb{P}}),$$

where $\pi_{\mathbb{I}}$ and $\pi_{\mathbb{P}}$ denote the projections onto the corresponding factor and $\varrho\colon \mathbb{I} \to \mathbb{R}_+$ is a given smooth function.

Each leaf of the foliation

$$t \in \mathbb{I} \mapsto \mathbb{P}_t := \{t\} \times \mathbb{P}^n$$

of M^{n+1} is a totally umbilical hypersurface with constant mean curvature

$$\mathcal{H}(t) = \varrho'(t)/\varrho(t)$$

pointing in direction of $-T$, where $T = \partial/\partial t \in TM$. Notice that we denote equally functions on $\mathbb{I}$ and their lift to M^{n+1}.

Moreover, we have that $\mathcal{T} = \varrho T$ is a *closed conformal vector field* on M^{n+1} since

$$\bar\nabla_X \mathcal{T} = \varrho' X \quad \text{for any } X \in TM, \tag{1}$$

where $\bar\nabla$ stands for the Levi-Civita connection in M^{n+1}. In [**8**, §3] it was shown that any Riemannian manifold with a closed conformal vector field is locally isometric to a warped product manifold $\mathbb{I} \times_\varrho \mathbb{P}^n$ as above.

Throughout the paper Ω stands for a domain of compact closure contained in a normal geodesic ball of $\mathbb{P}^n$. By the *normal graph* $\Sigma^n = \Sigma^n(u)$ in $M^{n+1} = \mathbb{I} \times_\varrho \mathbb{P}^n$ over $\Omega \subset \mathbb{P}_c$ determined by a continuous function $u\colon \Omega \to \mathbb{I}$ vanishing at $\Gamma = \partial\Omega$, we mean the compact hypersurface with boundary Γ defined as

$$\Sigma^n(u) = \{(c + u(x), x) : x \in \Omega\}.$$

A straightforward computation shows that $\Sigma^n(u)$ has constant mean curvature H and boundary Γ if $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\bar\Omega)$ is a solution of the Dirichlet problem

$$\begin{cases} T(u) = \operatorname{Div}\left(\frac{Du}{\sqrt{\varrho^2(u)+|Du|^2}}\right) + n\varrho(u)\left(H - \frac{\varrho'(u)}{\sqrt{\varrho^2(u)+|Du|^2}}\right) = 0 \\ u|_\Gamma = 0, \end{cases} \tag{2}$$

where Div and D denote the divergence and gradient operators in $\mathbb{P}^n$.

2.2. The 2$^{\underline{\text{nd}}}$ version. We describe next an isometrically equivalent second version of the setting where the Dirichlet problem (2) takes the more convenient form of a divergence type second order quasilinear elliptic partial differential equation.

Consider the product manifold $\bar{M}^{n+1} = \mathbb{J} \times \mathbb{P}^n$, where $\mathbb{J} \subset \mathbb{R}$ is an open interval, endowed with the conformal metric

$$\langle\ ,\ \rangle = \lambda^2(s)\left(ds^2 + \langle\ ,\ \rangle_{\mathbb{P}^n}\right),$$

where the conformal factor $\lambda\colon\ \mathbb{J} \to \mathbb{R}_+$ is any given smooth function. Each leaf of the foliation $s \in \mathbb{J} \mapsto \mathbb{P}_s$ of $\bar{M}^{n+1} = (\bar{M}^{n+1}, \lambda)$ is a hypersurface with constant mean curvature $\bar{\mathcal{H}}(s) = \lambda'(s)/\lambda^2(s)$ pointing in direction of $-\bar{\mathcal{T}}$, where $\bar{\mathcal{T}} = \partial/\partial s$. Moreover, we have that $\bar{\mathcal{T}}$ is a closed conformal vector field since it satisfies

$$\bar{\nabla}_X \bar{\mathcal{T}} = \frac{\lambda'}{\lambda} X \quad \text{for any} \;\; X \in TM.$$

In this version of the setting the normal graph $\Sigma^n(v)$ has constant mean curvature H and boundary $\Gamma \subset \mathbb{P}_c$ if $v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$ is a solution of the Dirichlet problem

$$\text{(3)} \qquad \begin{cases} Q(v) = \operatorname{Div}\left(\frac{Dv}{\sqrt{1+|Dv|^2}}\right) + n\left(H\lambda(v) - \frac{\lambda'(v)}{\lambda(v)\sqrt{1+|Dv|^2}}\right) = 0 \\ v|_\Gamma = 0, \end{cases}$$

where Div and D denote the divergence and gradient operators in $\mathbb{P}^n$.

2.3. The equivalence. Next, we establish conditions for an isometric equivalence between the above two versions of the setting. In that regard, fix $c \in \mathbb{I}$ and let $\phi\colon\ \mathbb{I} \to \mathbb{J} := \phi(\mathbb{I})$ be the increasing diffeomorphism given by

$$\text{(4)} \qquad \phi(t) = c + \int_c^t \frac{1}{\varrho(r)} dr.$$

Then, the map $\mathcal{J}\colon\ \mathbb{I} \times \mathbb{P}^n \to \mathbb{J} \times \mathbb{P}^n$ defined as

$$\mathcal{J}(t,x) = (s = \phi(t), x)$$

is an *isometry* between $M^{n+1} = \mathbb{I} \times_\varrho \mathbb{P}^n$ and $\bar{M}^{n+1} = (\mathbb{J} \times \mathbb{P}^n, \lambda)$ if and only if

$$\text{(5)} \qquad \lambda(s) = \varrho(\phi^{-1}(s)).$$

Notice that the inverse $\mathcal{J}^{-1}\colon\ \bar{M}^{n+1} \to M^{n+1}$ is $\mathcal{J}^{-1}(s,x) = (\phi^{-1}(s), x)$, where

$$\text{(6)} \qquad \phi^{-1}(s) = c + \int_c^s \lambda(r) dr.$$

Examples 1.

(a) If $M^{n+1} = \mathbb{R}_+ \times_t \mathbb{P}^n$ and $c > 0$, then

$$\phi(t) = c + \log(t - c).$$

Thus $\bar{M}^{n+1} = \mathbb{R} \times \mathbb{P}^n$ with conformal factor $\lambda(s) = c + e^{s-c}$.

(b) If $M^{n+1} = \mathbb{R} \times_{e^t} \mathbb{P}^n$ and $c = 0$, then

$$\phi(t) = 1 - e^{-t}.$$

Thus $\bar{M}^{n+1} = \mathbb{J} \times \mathbb{P}^n$ where $\mathbb{J} = (-\infty, 1)$ and $\lambda(s) = 1/(1-s)$. The change of parameter $r = 1 - s$ reverses orientation, and for $\mathbb{P}^n = \mathbb{R}^n$ yields the standard parametrization of $\mathbb{H}^{n+1}$ in the half-space model.

(c) If $M^{n+1} = \mathbb{R}_+ \times_{\sinh t} \mathbb{P}^n$ and $c > 0$, then

$$\phi(t) = c + \log(b \tanh(t/2))$$

where $b^{-1} = \tanh(c/2)$. Thus $\bar{M}^{n+1} = \mathbb{J} \times \mathbb{P}^n$ where $\mathbb{J} = (-\infty, c - b^{-1})$ and $\lambda(s) = \sinh(2\,\mathrm{arctanh}(b^{-1}e^{s-c}))$.

3. The general result

In this section, we first state our general result in both versions of the setting. Then we proceed to give the proof.

In the sequel, we always work with an interval $\mathbb{I} = (a, +\infty)$ where $a \geq -\infty$ and, without loss of generality, we parametrize $\mathbb{I}$ such that $a < 0$. Thus $0 \in \mathbb{I}$, and we always take

$$\Omega \subset \mathbb{P}_0 = \{0\} \times \mathbb{P}^n \subset M^{n+1}.$$

To establish the equivalence between both versions of the setting, we choose $c = 0$ in (4) so that $0 = \phi(0) \in \mathbb{J} = \phi(\mathbb{I})$. In particular, we also have that $\Omega \subset \mathbb{P}_0 \subset \bar{M}^{n+1}$. Observe that in both cases $\mathbb{P}_0$ carries the induced metric

$$\langle\ ,\ \rangle_{\mathbb{P}_0} = \mu^2 \langle\ ,\ \rangle_{\mathbb{P}},$$

where $\mu := \varrho(0) = \lambda(0)$.

From now on, we denote by H_Γ the mean curvature function of the boundary Γ with respect to the inward pointing unit conormal vector field as a hypersurface of $\mathbb{P}_0$ endowed with the above metric.

One of the assumptions we will require is $T = \partial/\partial t = (1/\lambda(s))\partial/\partial s$ to be the direction of least Ricci curvature of M^{n+1}, i.e.,

$$(*) \qquad \mathrm{Ric}_M(X) \geqslant \mathrm{Ric}_M(T) \quad \text{for all } \ X \in TM.$$

This condition was first considered in [**8**] and then in [**1**]. The relation between the Ricci tensors of M^{n+1} and $\mathbb{P}^n$ is given by

$$(7) \qquad \mathrm{Ric}_M(X,Y) = \mathrm{Ric}_P(\pi_{\mathbb{P}*}X, \pi_{\mathbb{P}*}Y) - (n\mathcal{H}^2 + \mathcal{H}')\langle X, Y\rangle - (n-1)\mathcal{H}'\langle X, T\rangle\langle Y, T\rangle.$$

Therefore, condition $(*)$ in terms of the Ricci curvature of $\mathbb{P}^n$ and either $\varrho(t)$ or $\lambda(s)$ means that

$$(**) \qquad \mathrm{Ric}_{\mathbb{P}} \geqslant \sup\,(\varrho'^2 - \varrho\varrho'') = \sup\,(-\varrho^2\mathcal{H}')$$

or, equivalently, that

$$\mathrm{Ric}_{\mathbb{P}} \geqslant \sup\,\left(\frac{2\lambda'^2 - \lambda\lambda''}{\lambda^2}\right) = \sup\,(-\lambda\bar{\mathcal{H}}').$$

Of course, condition $(*)$ (or the equivalent condition $(**)$) is trivially satisfied if M^{n+1} is a space form. Notice that $\mathrm{Ric}_{\mathbb{P}} \geqslant 0$ implies condition $(*)$ if $\mathcal{H}' \geqslant 0$ and, for later use, that $(*)$ reduces to $\mathrm{Ric}_{\mathbb{P}} \geqslant 0$ if $\mathcal{H}$ is constant.

Next we state our general result in the first version of the setting.

Theorem 2. *Assume that $M^{n+1} = \mathbb{I} \times_\varrho \mathbb{P}^n$ with $\mathrm{Ric}_{\mathbb{P}} \geqslant 0$ satisfies:*

(i) $\lim_{t\to a^+} \varrho(t) = 0$,

(ii) $\varrho'(t) > 0$ *and* $\varrho''(t) \geq 0$ *for all* $t \in \mathbb{I}$,

(iii) *condition* $(*)$ *holds on the subset* $(a, 0] \times \mathbb{P}^n \subset M^{n+1}$.

Suppose that Γ in $\mathcal{C}^{2,\alpha}$ is mean convex and let H satisfy $-H_\Gamma < H \leqslant 0$. Then there exists a function $u \in \mathcal{C}^{2,\alpha}(\bar{\Omega})$ whose normal graph is a hypersurface of constant mean curvature H in M^{n+1} with boundary Γ.

It is clear that Theorem 2 applies if $\mathbb{P}^n$ is a round sphere in $\mathbb{R}^{n+1}$. It also applies in $\mathbb{H}^{n+1}$ if $\mathbb{P}^n$ is either a horosphere or a geodesic sphere. However, two results referred in the introduction are not covered by this result. In fact, our assumptions on ϱ are not satisfied if the ambient space is just a Riemannian product. Moreover, Theorem 2 does not allow H to take positive values as in the result for horospheres in $\mathbb{H}^{n+1}$. Nevertheless, in the last section we give two general results that show that the conclusions of Theorem 2 still hold in both cases.

We proceed to state our result in the second version of the setting. If we denote $\mathbb{J} = (d, e)$, then from (6) we have that

$$+\infty = \phi^{-1}(e) = \int_0^e \lambda(s)ds.$$

Moreover, we obtain from (5) that

$$(8) \qquad \varrho'(\phi^{-1}(s)) = \lambda'(s)/\lambda(s)$$

and, in particular, we have $\mathcal{H} = \lambda'/\lambda^2$.

In the second version of the setting it is easily seen that Theorem 2 reads as follows.

Theorem 2′. *Assume that $\bar{M}^{n+1} = (\mathbb{J} \times \mathbb{P}^n, \lambda)$ with $\mathrm{Ric}_{\mathbb{P}} \geqslant 0$ satisfies:*

(i) $\lim_{s\to d^+} \lambda(s) = 0$ *and* $\int_0^e \lambda(s)ds = +\infty$,

(ii) *$\lambda'(s) > 0$ and $(\lambda'(s)/\lambda(s))' \geq 0$ for all $s \in \mathbb{J}$,*

(iii) *Condition $(*)$ holds on the subset $(d, 0] \times \mathbb{P}^n \subset \bar{M}^{n+1}$.*

Suppose that Γ in $\mathcal{C}^{2,\alpha}$ is mean convex and let H satisfy $-H_\Gamma < H \leqslant 0$. Then there exists a function $u \in \mathcal{C}^{2,\alpha}(\bar{\Omega})$ whose normal graph is a hypersurface of constant mean curvature H in $\bar{M}^{n+1}$ with boundary Γ.

Proof of Theorem 2. It follows easily from (3), (5) and (8) that u is a solution of the Dirichlet problem (2) if and only if $v = \phi(u)$ is a solution of

$$\begin{cases} Q(v) = \operatorname{Div}\left(\frac{Dv}{\sqrt{1+|Dv|^2}}\right) + n\left(H\varrho(\phi^{-1}(v)) - \frac{\varrho'(\phi^{-1}(v))}{\sqrt{1+|Dv|^2}}\right) = 0 \\ v|_\Gamma = 0. \end{cases} \tag{9}$$

Consider on $\mathbb{I} \times \mathbb{P}^n$ the one-parameter family of warped product metrics

$$\varrho_\tau = \tau\varrho \quad \text{for} \quad \tau \in (0, 1] \tag{10}$$

and notice that $\phi_\tau^{-1}(s) = \phi^{-1}(\tau s)$. From (9) we have the family of Dirichlet problems

$$\begin{cases} Q_\tau(v_\tau) = 0 \\ v_\tau|_\Gamma = 0 \end{cases} \tag{11}$$

where

$$Q_\tau(v_\tau) = \operatorname{Div}\left(\frac{Dv_\tau}{\sqrt{1+|Dv_\tau|^2}}\right) + \tau n\left(H\varrho(\phi^{-1}(\tau v_\tau)) - \frac{\varrho'(\phi^{-1}(\tau v_\tau))}{\sqrt{1+|Dv_\tau|^2}}\right),$$

and Div and D denote the divergence and gradient operators in $\mathbb{P}^n$.

To prove that (3) can be solved we apply the continuity method for $\tau \in [0, 1]$. Namely, we have to show that

$$J = \{\tau \in [0, 1] : \text{ The problem (11) can be solved for } \tau\}$$

is a nonvoid, open and closed subset of $[0, 1]$. Thus $J = [0, 1]$, and we are done.

To conclude that J is closed we have to obtain a priori $\mathcal{C}^{2,\alpha}$ estimates of any solution of the family of Dirichlet problems (11). In fact, standard theory for divergence type quasilinear elliptic equations and Schauder theory guarantee that it is sufficient to obtain a priori $\mathcal{C}^1$ estimation; see [**4**]. For $\tau = 0$, the problem has $v = 0$ as its unique solution. In particular, this shows that J is nonempty. Hence, to obtain closeness it suffices to deal with $\tau \in (0, 1]$ and prove the existence of a constant $K = K(\Omega)$ independent of τ such that any solution $v_\tau \in \mathcal{C}^{2,\alpha}(\bar{\Omega})$ of (11) satisfies

$$\|v_\tau\|_{\mathcal{C}^1(\bar{\Omega})} = \sup_\Omega |v_\tau| + \sup_\Omega |Dv_\tau| < K. \tag{12}$$

Our proof of Theorem 2 relays on two basic results given in [**1**]. The first one is [**1**, Proposition 18]; a half-space type result proved without any assumption on $\mathrm{Ric}_{\mathbb{P}}$ or the sign of H as a consequence of a tangency principle. For the proof of (12) it is convenient to argue for $u_\tau = \phi_\tau^{-1}(v_\tau)$ in the first version of the setting. It follows from our hypotheses and $\mathcal{H}_\tau = \mathcal{H}$ that $H \leqslant 0 \leqslant \inf_{[0,+\infty)} \mathcal{H}_\tau$ for every $\tau \in (0,1]$. Thus part (i) of [**1**, Proposition 18] applies, and yields the a priori estimate

$$u_\tau \leqslant 0 \quad \text{on } \Omega. \tag{13}$$

Before proceeding with the proof, we briefly describe the second basic result from [**1**] to be used next. Let Θ be the angle function on $\Sigma^n(u)$ defined as

$$\Theta(p) = \langle N(p), T\rangle$$

where the orientation N of $\Sigma^n(u)$ is chosen so that $\Theta < 0$. Take any primitive $\sigma \in \mathcal{C}^\infty(\mathbb{I})$ of ϱ and consider the function ψ on $\Sigma^n(u)$ defined by

$$\psi = \sigma(u)H + \varrho(u)\Theta,$$

where u is seen as the height function $u(p) = \pi_{\mathbb{I}}|_{\Sigma^n}(p)$. Then [**1**, Theorem 13] assumes condition $(*)$ and states that the function ψ on $\Sigma^n(u)$ is subharmonic. Although the result in [**1**] assumes condition $(*)$ on all of $\mathbb{I}$ it can be easily seen that the proof only requires $(*)$ to hold on the interval $\pi_{\mathbb{I}}(\Sigma^n(u))$.

Coming back to our proof, on $\Sigma^n_\tau = \Sigma^n(u_\tau)$ we consider the function $\psi_\tau = \sigma_\tau(u_\tau)H + \varrho_\tau(u_\tau)\Theta_\tau$ where we choose

$$\sigma_\tau(t) = \int_0^t \varrho_\tau(r)dr = \tau\sigma(t).$$

Then [**1**, Theorem 13] applies since the estimate (13) holds, and we have $\mathrm{Ric}_{\mathbb{P}} \geqslant \sup_{(a,0]}\left(-\varrho_\tau^2\mathcal{H}'_\tau\right)$ from (10) and our hypotheses on $\mathbb{P}^n$. Thus ψ_τ is subharmonic, and we obtain from the maximum principle that

$$\psi_\tau \leqslant \max_{\partial\Sigma^n_\tau} \psi_\tau = \tau\mu\Theta_\tau(q_\tau)$$

at a certain $q_\tau \in \Gamma$. Equivalently, we have

$$\sigma(u_\tau)H + \varrho(u_\tau)\Theta_\tau \leqslant \mu\Theta_\tau(q_\tau). \tag{14}$$

The functions σ and ϱ are, in particular, nondecreasing. It follows from (13) that $\sigma(u_\tau) \leqslant \sigma(0) = 0$ and $\varrho(u_\tau) \leqslant \mu$, and we conclude from (14) and $H \leqslant 0$ that

$$\Theta_\tau \leqslant \Theta_\tau(q_\tau). \tag{15}$$

The next step of the proof is to give a gradient estimate along the boundary of the domain. The following step uses this result to provide a height estimate for the graph. We want to show that there exists an explicitly computable constant $C(\tau) > 0$ such that

$$\Theta_\tau(q_\tau) = \max_\Gamma \Theta_\tau \leqslant -C(\tau) < 0. \tag{16}$$

To see this, first observe that the gradient of $\pi_{\mathbb{I}}$ in M^{n+1} is $\bar{\nabla}\pi_{\mathbb{I}} = T$. Therefore,

$$\nabla u_\tau = T - \Theta_\tau N \tag{17}$$

where ∇ denotes the gradient on Σ^n_τ. We obtain using (1) that

$$\nabla \varrho_\tau(u_\tau)\Theta_\tau = \nabla\langle N, \mathcal{T}_\tau\rangle = -A(\mathcal{T}_\tau - \langle \mathcal{T}_\tau, N\rangle N) = -\varrho_\tau(u_\tau) A\nabla u_\tau, \tag{18}$$

where $A = A_N$ stands for the second fundamental form of Σ^n_τ. It follows that

$$\nabla\psi_\tau = \varrho_\tau(u_\tau) H \nabla u_\tau - \varrho_\tau(u_\tau) A\nabla u_\tau.$$

The maximum principle yields

$$\langle \nabla\psi_\tau(q_\tau), \nu_{q_\tau}\rangle = -\tau\mu\langle A\nabla u_\tau(q_\tau) - H\nabla u_\tau(q_\tau), \nu_{q_\tau}\rangle \leqslant 0, \tag{19}$$

where ν denotes the inward pointing unit conormal vector field along Γ. From (17) and $u_\tau|_\Gamma = 0$, we obtain $\nabla u_\tau = \langle \nabla u_\tau, \nu\rangle\nu = \langle T, \nu\rangle\nu$ along Γ. Then (19) gives

$$\langle T, \nu_{q_\tau}\rangle\langle A\nu_{q_\tau} - H\nu_{q_\tau}, \nu_{q_\tau}\rangle \geqslant 0. \tag{20}$$

Moreover, $\langle \nabla u_\tau, \nu\rangle = \langle T, \nu\rangle \leqslant 0$ along Γ since $u_\tau \leqslant u_\tau|_\Gamma$ on Σ^n_τ. In fact, we may assume that $\langle T, \nu_{q_\tau}\rangle < 0$ since, otherwise, we obtain using (17) that $\Theta_\tau(q_\tau) = -1$, and we are done. Thus (20) yields

$$\langle A\nu_{q_\tau}, \nu_{q_\tau}\rangle \leqslant H.$$

Choosing an orthonormal basis $\{e_1, \ldots, e_{n-1}\}$ of $T_{q_\tau}\Gamma$, from

$$nH = \sum_i \langle Ae_i, e_i\rangle + \langle A\nu_{q_\tau}, \nu_{q_\tau}\rangle,$$

we obtain

$$(n-1)H \leqslant \sum_i \langle Ae_i, e_i\rangle. \tag{21}$$

Observe that $\partial\Sigma^n_\tau = \Gamma \subset \mathbb{P}_0(\tau)$, where $\mathbb{P}_0(\tau)$ denotes $\mathbb{P}_0$ seen as a hypersurface of $\mathbb{I} \times_{\varrho_\tau} \mathbb{P}^n$, and that $\mathbb{P}_0(\tau)$ is homothetic to $\mathbb{P}_0$ with homothety factor τ^2. In particular,

$$H^\tau_\Gamma = \frac{1}{\tau} H_\Gamma. \tag{22}$$

Let η denote the inward pointing unit conormal η along Γ in $\mathbb{P}_0(\tau)$. Then

$$\langle N, \eta\rangle = \langle T, \nu\rangle = -\sqrt{1 - \Theta^2_\tau},$$

and hence

$$N = -\sqrt{1 - \Theta^2_\tau}\,\eta + \Theta_\tau T. \tag{23}$$

Taking into account that $\mathbb{P}_0(\tau)$ is totally umbilical in $\mathbb{I}\times_{\varrho_\tau}\mathbb{P}^n$ with mean curvature vector field $-\mathcal{H}_\tau(0)T = -\mathcal{H}(0)T$ and using (23), we have

$$\langle Ae_i, e_i\rangle = \langle \bar\nabla_{e_i} e_i, N\rangle = -\langle B_\eta e_i, e_i\rangle\sqrt{1-\Theta_\tau^2(q_\tau)} - \mathcal{H}(0)\Theta_\tau(q_\tau) \tag{24}$$

where B_η stands for the second fundamental form of Γ in $\mathbb{P}_0(\tau)$. We conclude from (21), (22) and (24) that

$$H + \mathcal{H}(0)\Theta_\tau(q_\tau) + \frac{\kappa}{\tau}\sqrt{1-\Theta_\tau^2(q_\tau)} \leqslant 0, \tag{25}$$

where $\kappa := \min_\Gamma H_\Gamma$.

In view of (25) consider the equation

$$P(x) := H + \mathcal{H}(0)x + \frac{\kappa}{\tau}\sqrt{1-x^2} = 0. \tag{26}$$

Our hypotheses yield

$$P(-1) = H - \mathcal{H}(0) \leqslant 0 \quad \text{and} \quad P(0) = H + \frac{\kappa}{\tau} \geqslant H + \kappa > 0.$$

It is easy to see that (26) has a unique root $-C(\tau) \in [-1,0)$ where

$$C(\tau) = \frac{H\mathcal{H}(0)\tau^2 + \kappa\sqrt{\kappa^2 + \tau^2(\mathcal{H}^2(0) - H^2)}}{\kappa^2 + \tau^2\mathcal{H}^2(0)}.$$

But (25) simply means that $P(\Theta_\tau(q_\tau)) \leqslant 0$, and we conclude that (16) holds.

We obtain from (15) and (16) that $\sup_{\Sigma_\tau^n}\Theta_\tau \leqslant -C(\tau) < 0$. A straightforward computation gives

$$\frac{dC}{d\tau} = \frac{-\tau\kappa\left(\kappa H - \mathcal{H}(0)\sqrt{\kappa^2 + \tau^2(\mathcal{H}^2(0) - H^2)}\right)^2}{\left(\kappa^2 + \tau^2\mathcal{H}^2(0)\right)^2\sqrt{\kappa^2 + \tau^2(\mathcal{H}^2(0) - H^2)}}, \tag{27}$$

and hence $dC/d\tau \leq 0$. Therefore,

$$\sup_{\Sigma_\tau^n}\Theta_\tau \leqslant -C \quad \text{for any} \quad \tau \in (0,1], \tag{28}$$

where $C := C(1) \in (0,1]$ is given by

$$C = \frac{H\mathcal{H}(0) + \kappa\sqrt{\kappa^2 + \mathcal{H}^2(0) - H^2}}{\kappa^2 + \mathcal{H}^2(0)}.$$

In the second version of the setting it is easy to see that

$$\Theta_\tau = \frac{-1}{\sqrt{1 + |Dv_\tau|^2}}, \tag{29}$$

and we obtain from (28) the gradient estimate

$$\sup_\Omega |Dv_\tau| \leqslant K_1 := \frac{\sqrt{1-C^2}}{C}. \tag{30}$$

It remains to estimate $\sup_\Omega |v_\tau|$. First observe that (13) is equivalent to

$$v_\tau = \phi_\tau(u_\tau) \leqslant \phi_\tau(0) = 0 \text{ on } \Omega. \tag{31}$$

Thus, it suffices to estimate $\inf_\Omega v_\tau$ or, equivalently, $\min_{\Sigma^n_\tau} u_\tau$. Let p_τ be a point in the interior of Σ^n_τ where

$$u_\tau(p_\tau) = \underline{u}_\tau = \min_{\Sigma^n_\tau} u_\tau.$$

Hence $\nabla u_\tau(p_\tau) = 0$ and $\Theta_\tau(p_\tau) = -1$ by (17). Then (14) jointly with (16) give

$$\sigma(\underline{u}_\tau)H - \varrho(\underline{u}_\tau) \leqslant \mu\Theta_\tau(q_\tau) \leqslant -\mu C(\tau). \tag{32}$$

Using that $\sigma(\underline{u}_\tau)H \geqslant 0$, we obtain

$$\varrho(\underline{u}_\tau) \geqslant \sigma(\underline{u}_\tau)H + \mu C(\tau) \geqslant \mu C(\tau) > 0.$$

By our hypotheses $\inf_{\mathbb{I}} \varrho = 0$ and ϱ is strictly increasing. We obtain that

$$\min_{\Sigma^n_\tau} u_\tau = \underline{u}_\tau \geqslant \varrho^{-1}(\mu C(\tau)) > a.$$

Recall that $v_\tau = \phi_\tau(u_\tau)$ where $\phi_\tau = (1/\tau)\phi$ is strictly increasing. Thus,

$$\inf_\Omega v_\tau = \phi_\tau(\underline{u}_\tau) \geqslant \phi_\tau(\varrho^{-1}(\mu C(\tau))) = \alpha(\tau), \tag{33}$$

where $\alpha(\tau) = (1/\tau)\beta(\tau)$ and $\beta(\tau) = \phi(\varrho^{-1}(\mu C(\tau)))$.

We claim that $\alpha(\tau)$ is continuous on $[0,1]$. Actually, we obtain from $C(0) = 1$ that $\beta(0) = 0$. Moreover, using that $dC/d\tau(0) = 0$ from (27), we also have that $\lim_{\tau\to 0}\alpha(\tau) = \lim_{\tau\to 0}\beta'(\tau) = 0$, and the claim follows. From (33) and the claim we obtain that $\inf_\Omega v_\tau \geqslant -K_2$, where $K_2 := -\min_{[0,1]}\alpha$ is independent of τ and satisfies $K_2 > 0$. This and (31) give

$$\sup_\Omega |v_\tau| \leqslant K_2, \tag{34}$$

and (12) follows from (30) and (34).

To show that J is open we use the implicit function theorem for elliptic partial differential equations. For $\tau = 0$, the operator Q_0 in (11) is trivially invertible. Therefore it suffices to deal with $\tau \in (0,1]$. Recall that the linearized mean curvature operator about a normal geodesic graph in a Riemannian manifold M is given by

$$\mathcal{L} = \Delta + \|A\|^2 + \mathrm{Ric}_M(N,N),$$

where Δ is the Laplace-Beltrami operator on the graph and $\|A\|$ denotes the norm of its second fundamental form. To prove that the operator Q_τ is invertible, it suffices to show that $\mathcal{L}f \geq 0$ for some function f on Σ^n_τ satisfying $f < 0$. Similarly as in [**7**] for hyperbolic space, we work with the negative valued function $f = \varrho(u_\tau)\Theta_\tau$. A straightforward computation using (18) and the Codazzi equation for the graph (for details see the proof of [**1**, Theorem 13]) give

$$\Delta f = \varrho(u_\tau)\mathrm{Ric}_M(N, \nabla u_\tau) - n\varrho'(u_\tau)H - \|A\|^2 f.$$

It follows using (17) that

$$\mathcal{L}f = -n\varrho'(u_\tau)H + \varrho(u_\tau)\mathrm{Ric}_M(N, T).$$

It is easy to verify from (7) that $\mathrm{Ric}_M(N,T) = -n\varrho''(u_\tau)\Theta_\tau/\varrho(u_\tau)$. Hence,

$$\mathcal{L}f = -n(\varrho'(u_\tau)H + \varrho''(u_\tau)\Theta_\tau). \tag{35}$$

Thus $\mathcal{L}f \geq 0$ since $\varrho', \varrho'' \geq 0$, and this concludes the proof of Theorem 2. q.e.d.

4. Two further cases

There are two cases in which all leaves of the umbilical foliation of a warped product $M^{n+1} = \mathbb{I}\times_\varrho\mathbb{P}^n$ have the same constant mean curvature $\mathcal{H}$. The first case is when the ambient space is just a Riemannian product $\mathbb{R} \times \mathbb{P}^n$, and hence leaves are totally geodesic. In this case Theorem 2 does not apply but the result still holds.

Theorem 3. *Let $M^{n+1} = \mathbb{R} \times \mathbb{P}^n$ be a Riemannian product with $\mathrm{Ric}_\mathbb{P} \geqslant 0$. Assume that Γ in $\mathcal{C}^{2,\alpha}$ is mean convex and let H satisfy $-H_\Gamma < H \leqslant 0$. Then there exists a function $u \in \mathcal{C}^{2,\alpha}(\bar{\Omega})$ whose vertical graph is a hypersurface of constant mean curvature H in M^{n+1} with boundary Γ.*

Proof. By [**1**, Proposition 18] the solution for $H = 0$ is the trivial graph. Thus, we may assume $-H_\Gamma < H < 0$. The proof of Theorem 2 works step by step in this product case up to equation (32) included, that reads as

$$\underline{u}_\tau H - 1 \leq -C(\tau),$$

where $\underline{u}_\tau = \min_{\Sigma_\tau^n} u_\tau$ and

$$0 < C(\tau) = \frac{1}{\kappa}\sqrt{\kappa^2 - \tau^2 H^2} < 1.$$

Therefore, $\underline{u}_\tau \geqslant (1 - C(\tau))/H$. Using $v_\tau = \phi_\tau(u_\tau)$ and $\phi_\tau(t) = t/\tau$, this gives

$$\inf_\Omega v_\tau = (1/\tau)\underline{u}_\tau \geqslant \alpha(\tau),$$

where $\alpha(\tau) = \beta(\tau)/\tau$ and

$$\beta(\tau) = \frac{1}{H}\left(1 - \frac{1}{\kappa}\sqrt{\kappa^2 - \tau^2 H^2}\right).$$

Again the function $\alpha(\tau)$ is continuous on $[0,1]$, and we obtain the estimate $\inf_\Omega v_\tau \geqslant -K_2$, where $K_2 := -\min_{[0,1]} \alpha$ is independent of τ and satisfies $K_2 > 0$. The proof then finishes as in Theorem 2, observing from (35) that in this case $f = \Theta_\tau < 0$ satisfies $\mathcal{L}f = 0$. q.e.d.

The second case of whether $\mathcal{H} \neq 0$ is constant (after normalization for simplicity) corresponds to $\varrho(t) = e^t$ ($\lambda(s) = 1/s$) and, hence, the ambient space belongs to a class referred to as *pseudo-hyperbolic* manifolds in [**11**]; see [**1**] or [**8**] for a brief discussion of the class in the context of this paper.

Theorem 4. *Assume that $M^{n+1} = \mathbb{R} \times_{e^t} \mathbb{P}^n$ has $\mathrm{Ric}_{\mathbb{P}} \geqslant 0$. Suppose that Γ in $\mathcal{C}^{2,\alpha}$ is mean convex and let H satisfy $-H_\Gamma < H \leqslant 1$. Then there exists a function $u \in \mathcal{C}^{2,\alpha}(\bar{\Omega})$ whose normal graph is a hypersurface of constant mean curvature H in M^{n+1} with boundary Γ.*

Proof. For $H \leqslant 0$ the result follows from Theorem 2, and for $H = 1$ the solution is the trivial graph by [**1**, Proposition 18]. In the sequel, we see that the proof of Theorem 2 can be adapted to the case $0 < H < 1$.

In the present situation $\sigma(t) = e^t - 1$, and $\psi_\tau = \tau(e^{u_\tau}(H + \Theta_\tau) - H)$. Then (14) reduces to

$$e^{u_\tau}(H + \Theta_\tau) \leqslant H + \Theta_\tau(q_\tau). \tag{36}$$

Moreover, (25) takes the form

$$H + \Theta_\tau(q_\tau) \leqslant -\frac{\kappa}{\tau}\sqrt{1 - \Theta_\tau^2(q_\tau)} \leqslant 0. \tag{37}$$

It follows from (36) and (37) that $\Theta_\tau \leqslant -H$, and we conclude using (29) that

$$\sup_\Omega |Dv_\tau| \leqslant K_1 := \frac{\sqrt{1 - H^2}}{H}.$$

Equation (26) now reads

$$P(x) := H + x + \frac{\kappa}{\tau}\sqrt{1 - x^2} = 0,$$

and satisfies $P(-1) = H - 1 < 0$ and $P(0) = H + \frac{\kappa}{\tau} \geqslant H + \kappa > 0$. The only root $-C(\tau) \in (-1, 0)$ is given by

$$C(\tau) = \frac{H\tau^2 + \kappa\sqrt{\kappa^2 + \tau^2(1 - H^2)}}{\kappa^2 + \tau^2}.$$

Since (37) reads as $P(\Theta_\tau(q_\tau)) \leqslant 0$, we conclude that

$$\Theta_\tau(q_\tau) \leqslant -C(\tau) < 0. \tag{38}$$

By [**1**, Proposition 18] we have $u_\tau \leqslant 0$, and $u_\tau \neq 0$ since $H < 1$. As in the proof of Theorem 2, taking the point p_τ in the interior of Σ_τ^n where u_τ attains its minimum, we obtain from (36) and (38) that

$$e^{\underline{u}_\tau} \geqslant \frac{-H\kappa^2 + \kappa\sqrt{\kappa^2 + \tau^2(1 - H^2)}}{(1 - H)(\kappa^2 + \tau^2)}. \tag{39}$$

From $v_\tau = (1/\tau)(1 - e^{-u_\tau})$ we have

$$\inf_\Omega v_\tau = \frac{1}{\tau}(1 - e^{-\underline{u}_\tau}).$$

Using (39) it follows that

$$\inf_\Omega v_\tau \geqslant \alpha(\tau) = \beta(\tau)/\tau,$$

where

$$\beta(\tau) = \frac{\kappa\sqrt{\kappa^2 + \tau^2(1 - H^2)} - \kappa^2 + (H-1)\tau^2}{-H\kappa^2 + \kappa\sqrt{\kappa^2 + \tau^2(1 - H^2)}}.$$

Clearly $\beta(0) = \beta'(0) = 0$, and the proof that J is closed finishes as in Theorem 2.

It remains to see that J is open. In that sense, observe that by (35) $f = e^{u_\tau}\Theta_\tau < 0$ satisfies

$$\mathcal{L}f = -ne^{u_\tau}(H + \Theta_\tau).$$

It follows from (36) and (37) that $\mathcal{L}f \geq 0$, and this concludes the proof. q.e.d.

References

[1] L.J. Alías & M. Dajczer, *Constant mean curvature hypersurfaces in warped product spaces*, to appear in Proc. Edinb. Math. Soc.; available at www.preprint.impa.br (Serie A, 388/2005).

[2] M. Dajczer & J. Ripoll, *An extension of a theorem of Serrin to graphs in warped products*, J. Geom. Anal. **15** (2005) 193–205, MR 2152479.

[3] P. Fusieger & J. Ripoll, *Radial graphs of constant mean curvature and doubly connected minimal surfaces with prescribed boundary*, Ann. Global Anal. Geom. **23** (2003) 373–400, MR 1973266, Zbl 1042.53005.

[4] D. Gilbarg & N. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, New York, 2001, MR 1814364, Zbl 1042.35002.

[5] J.H.S. de Lira, *Radial graphs with constant mean curvature in the hyperbolic space*, Geom. Dedicata **93** (2002) 11–23, MR 1934682, Zbl 1037.53003.

[6] R. López, *A note on radial graphs with constant mean curvature*, Manuscripta Math. **110** (2003) 45–54, MR 1951799, Zbl 1030.53010.

[7] R. López & S. Montiel, *Existence of constant mean curvature graphs in hyperbolic space*, Calc. Var. Partial Differential Equations **8** (1999) 177–190, MR 1680662, Zbl 0945.53008.

[8] S. Montiel, *Unicity of constant mean curvature hypersurfaces in some Riemannian manifolds*, Indiana Univ. Math. J. **48** (1999) 711–748, MR 1722814, Zbl 0973.53048.

[9] B. Nelli & J. Spruck, *On the existence and uniqueness of constant mean curvature hypersurfaces in hyperbolic space*, Geometric analysis and the calculus of variations, 253–266, Internat. Press, Cambridge, MA, 1996, MR 1449411, Zbl 0936.35069.

[10] S. Nölker, *Isometric immersions of warped products*, Diff. Geom. and Appl. **6** (1996) 1–30, MR 1384876, Zbl 0881.53052.

[11] Y. Tashiro, *Complete Riemannian manifolds and some vector fields*, Trans. Amer. Math. Soc. **117** (1965) 251–275, MR 0174022, Zbl 0136.17701.

Departamento de Matematicas
Universidad de Murcia, Campus de Espinardo
E-30100 Espinardo, Murcia
Spain

E-mail address: ljalias@um.es

Instituto de Matematica Pura e Aplicada (IMPA)
Estrada Dona Castorina 110
22460-320 Rio de Janeiro
Brazil

E-mail address: marcos@impa.br