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KOHN-ROSSI COHOMOLOGY AND ITS APPLICATION TO THE COMPLEX PLATEAU PROBLEM, II

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Dedicated to Professor J.J. Kohn on the occasion of his 75th birthday

Abstract

Let X be a compact connected strongly pseudoconvex CR Manifold of real dimension $2n-1$ in $\mathbb{C}^{n+1}$. Tanaka introduced a spectral sequence $E_r^{(p,q)}(X)$ with $E_1^{(p,q)}(X)$ being the Kohn-Rossi cohomology group and $E_2^{(k,0)}(X)$ being the holomorphic De Rham cohomology denoted by $H_h^k(X)$. We study the holomorphic De Rham cohomology in terms of the s-invariant of the isolated singularities of the variety V bounded by X. We give a characterization of the singularities with vanishing s-invariants. For $n \geq 3$, Yau used the Kohn-Rossi cohomology groups to solve the classical complex Plateau problem in 1981. For $n = 2$, the problem has remained unsolved for over a quarter of a century. In this paper, we make progress in this direction by putting some conditions on X so that V will have very mild singularities. Specifically, we prove that if $\dim X = 3$ and $H_h^2(X) = 0$, then X is a boundary of complex variety V with only isolated quasi-homogeneous singularities such that the dual graphs of the exceptional sets in the resolution are star shaped and all curves are rational.

1. Introduction

One of the natural fundamental questions of complex geometry is to study the boundaries of complex varieties. For example, the famous classical complex Plateau problem asks which odd dimensional real submanifolds of $\mathbb{C}^N$ are boundaries of complex submanifolds in $\mathbb{C}^N$. In 1975, Harvey and Lawson [**Ha-La**] showed that for any compact connected CR manifold X in $\mathbb{C}^N$, there is a unique complex variety V in $\mathbb{C}^N$ such that the boundary of V is X. Therefore a natural and important question is to study V in terms of X explicitly.

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If X is a strongly pseudoconvex CR manifold of dimension $2n-1$, $n \geq 2$, contained in the boundary of a bounded strongly pseudoconvex domain D in $\mathbb{C}^N$, then V has boundary regularity at every point of X and V has only isolated singularities in $V - X$ (cf. [**Lu-Ya**]). The next fundamental question is to determine when X is a boundary of a complex submanifold in $\mathbb{C}^N$, i.e., when V is smooth. In [**Do**], Donnelly first found a necessary condition depending on eta invariants of Atiyah and Singer. In 1963, J.J. Kohn [**Ko**] solved the famous $\overline{\partial}$-Neumann problem. Based on this work, Kohn and Rossi [**Ko-Ro**] introduced the fundamental CR-invariant, the Kohn-Rossi cohomology groups $H^{pq}_{KR}(X)$. They proved the finite dimensionality of their cohomology groups. A strongly pseudoconvex manifold M is a modification of a Stein space V with isolated singularities. In [**Ko-Ro**], Kohn and Rossi conjectured that in general, either there is no boundary cohomology of the boundary X in degree (p,q) for $q \neq 0$, $n-1$, or it must result from the interior singularities of V. Yau [**Ya1**] solved the Kohn-Rossi conjecture affirmatively.

Theorem 1. *Let M be a strongly pseudoconvex manifold of dimension $n(n \geq 3)$ which is a modification of a Stein space V at the isolated singularities $x_1, \dots, x_m$. Then $\dim H^{p,q}_{KR}(X) = \sum_{i=1}^{m} b^{p,q+1}_{x_i}$ for $1 \leq q \leq n-2$, where $b^{p,q+1}_{x_i} = \dim H^{q+1}_{\{x_i\}}(V, \Omega^p_V)$ is a local invariant of the singularity x_i. Suppose that $x_1, \dots, x_m$ are hypersurface singularities. Then for $1 \leq q \leq n-2$,*

$$\dim H^{p,q}_{KR}(X) = \begin{cases} 0 & \text{if } p+q \leq n-2 \text{ or } \ p+q \geq n+1 \\ \tau_1 + \cdots + \tau_m & \text{if } p+q = n-1 \text{ or } \ p+q = n \end{cases}$$

where τ_i is the number of moduli of V at x_i and can be computed explicitly.

As a result of the above theorem, Yau [**Ya1**] answers the classical complex Plateau problem for some cases.

Theorem 2. *Let X be a compact connected strongly pseudoconvex CR-manifold of real dimension $2n-1$, $n \geq 3$, in the boundary of a bounded strongly pseudoconvex domain D in $\mathbb{C}^{n+1}$. Then X is a boundary of the complex submanifold $V \subset D - X$ if and only if Kohn-Rossi cohomology groups $H^{p,q}_{KR}(X)$ are zero for $1 \leq q \leq n-2$.*

For $n = 2$ in Theorem 2, X is a 3-dimensional CR manifold. The classical complex Plateau problem remains unsolved for over a quarter of a century. The main difficulty is that the Kohn-Rossi cohomology groups are infinite dimensional in this case. Let V be the complex variety with X as its boundary. Then the singularities of V are surface singularities. In order to solve the classical complex Plateau problem for $n = 2$, one would like to ask under what kind of condition on X will V have only very mild singularities. This paper solves the problem and is a natural continuation of [**Ya1**]. Our basic observation is the following. Although Kohn-Rossi cohomology groups are infinite dimensional, we can derive from them the holomorphic De Rham cohomology. In fact, in [**Ta**], Tanaka introduced a spectral sequence $E_r^{p,q}(X)$ with $E_1^{p,q}(X)$ being the Kohn-Rossi cohomology group and $E_2^{k,0}(X)$ being the holomorphic De Rham cohomology denoted by $H_h^k(X)$. It is this holomorphic De Rham cohomology which plays a central role in the above problem.

Let M be a n-dimensional complex manifold. The q-th holomorphic De-Rham cohomology $H_h^q(M)$ of M is defined to be the d-closed holomorphic q-forms quotient by the d-exact holomorphic q-forms. Holomorphic De-Rham cohomology was studied by Hörmander [**Hö**]. It is well known that if M is a Stein manifold, then the holomorphic De-Rham cohomology coincides with the ordinary De-Rham cohomology [**Hö**]. Let X be a compact connected strongly pseudoconvex CR manifold. Suppose that X is the boundary of a strongly pseudoconvex manifold M which is a modification of a Stein space V with only isolated singularities. Let A be the maximal compact analytic set in M. One natural question is to find the relationship between $H_h^q(X), H_h^q(M \setminus A)$ and $H_h^q(M)$.

Let (V, x) be an isolated singularity of dimension n. Let $\pi : (M, A) \to (V, x)$ be a resolution of singularity with A as exceptional set. The number

$$s = \dim \Gamma(M \setminus A, \Omega^n) \big/ (d\Gamma(M \setminus A, \Omega^{n-1}) + \Gamma(M, \Omega^n))$$

is an invariant of the singularity (V, x). It turns out that the s-invariant plays an important role in the relationship between $H_h^n(M \setminus A)$ and $H_h^n(M)$.

Theorem A. *Let X be a compact connected $(2n-1)$-dimensional $(n \geqslant 2)$ strongly pseudoconvex CR manifold. Suppose that X is the boundary of a n-dimensional strongly pseudoconvex manifold M which is a modification of a Stein space V with only isolated singularities $\{x_1, \ldots, x_m\}$. Let A be the maximal compact analytic set in M which can be blown down to $\{x_1, \ldots, x_m\}$. Then*

(1) $H_h^q(X) \cong H_h^q(M \setminus A) \cong H_h^q(M)$ *for* $1 \leqslant q \leqslant n-1$
(2) $H_h^n(X) \cong H_h^n(M \setminus A)$, $\dim H_h^n(M \setminus A) = \dim H_h^n(M) + s$, *where* $s = s_1 + \cdots + s_m$, s_i *is the s-invariant of the singularity* (V, x_i).

Remark 1.1. The above theorem in particular asserts that up to degree $n-1$, the holomorphic De-Rham cohomology can extend across the maximal compact analytic set.

A normal surface singularity $(V, 0)$ is Gorenstein if there exists a nowhere vanishing holomorphic 2-form on $V \setminus \{0\}$. Recall that isolated hypersurface or complete intersection singularities are Gorenstein.

It is a natural question to ask for a characterization of Gorenstein surface singularities with vanishing s-invariant.

Theorem B. *Let $(V,0)$ be a Gorenstein surface singularity. Let $\pi : M \to V$ be a good resolution with $A = \pi^{-1}(0)$ as exceptional set. Assume that M is contractible to A. If $s = 0$, then $(V,0)$ is a quasi-homogeneous singularity, $H^1(A,\mathbb{C})=0$, $\dim H^1(M,\Omega^1)=\dim H^2(A,\mathbb{C}) + \dim H^1(M,\mathcal{O})$, and $H^1_h(M) = H^2_h(M) = 0$.*

Conversely, if $(V,0)$ is a two dimensional quasi-homogeneous Gorenstein singularity and $H^1(A,\mathbb{C}) = 0$, then the s-invariant vanishes.

Remark 1.2.

(a) $H^1(A,\mathbb{C}) = 0$ is equivalent to the fact that all the curves in A are rational and the first betti number of the dual graph of A is zero (i.e., there are no loops in the dual graph of A).

(b) Quasi-homogeneity of $(V,0)$ plus $H^1(A,\mathbb{C}) = 0$ imply the dual graph of A is star-shaped and all the curves are rational.

Definition 1.3. Let X be a CR manifold of real dimension $2n-1$; X is said to be a Calabi-Yau CR manifold if there exists a nowhere vanishing holomorphic section in $\Gamma(\Lambda^n\widehat{T}(X)^*)$ where $\widehat{T}(X)$ is the holomorphic tangent bundle of X (cf. Remark 2.4 and the paragraph above it).

Remark 1.4.

(a) Let X be a CR manifold of real dimension $2n-1$ in $\mathbb{C}^n$. Then X is a Calabi-Yau CR manifold.

(b) Let X be a strongly pseudoconvex CR manifold of real dimension $2n-1$ contained in the boundary of bounded strongly pseudoconvex domain in $\mathbb{C}^{n+1}$. Then X is a Calabi-Yau CR manifold.

The following theorem is a fundamental theorem toward the complete solution of the classical complex Plateau problem for 3-dimensional strongly pseudoconvex Calabi-Yau CR manifold in $\mathbb{C}^n$. The theorem is interesting in its own right.

Theorem C. *Let X be a strongly pseudoconvex compact Calabi-Yau CR manifold of dimension 3. Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in $\mathbb{C}^n$. If the holomorphic De Rham cohomology $H^2_h(X) = 0$, then X is a boundary of a complex variety V in D with boundary regularity and V has only isolated singularities in the interior and the normalizations of these singularities are Gorenstein surface singularities with vanishing s-invariant.*

Corollary D. *Let X be a strongly pseudoconvex compact CR manifold of dimension 3. Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in $\mathbb{C}^3$. If the holomorphic DeRham cohomology $H^2_h(X) = 0$, then X is a boundary of a complex* *variety V in D with boundary regularity and V has only isolated quasi-homogeneous singularities such that the dual graphs of the exceptional sets in the resolution are star shaped and all the curves are rational.*

In Section 2, we shall recall the definition of holomorphic De-Rham cohomology for a CR manifold. In Section 3, we shall recall the Siu complex and prove that the s-invariant is indeed an invariant of the singularity. Theorem A is also proved in this section. In Section 4, we give a characterization for a two dimensional isolated Gorenstein singularity to be a singularity with vanishing s-invariant. In particular, we prove Theorem B and Theorem C.

2. Holomorphic De-Rham Cohomology of CR Manifolds

Kohn-Rossi cohomology was first introduced by Kohn-Rossi. Following Tanaka [**Ta**], we reformulate the definition in a way independent of the interior manifold.

Definition 2.1. Let X be a connected orientable manifold of real dimension $2n-1$. A CR structure on X is an $(n-1)$-dimensional subbundle S of $\mathbb{C}T(X)$ (complexified tangent bundle) such that

(1) $S \cap \overline{S} = \{0\}$.
(2) If L, L' are local sections of S, then so is $[L, L']$.

Such a manifold with a CR structure is called a CR manifold. There is a unique subbundle $\mathcal{H}$ of $T(X)$ such that $\mathbb{C}\mathcal{H} = S \oplus \overline{S}$. Furthermore, there is a unique homomorphism $J : \mathcal{H} \to \mathcal{H}$ such that $J^2 = -1$ and $S = \{v - iJv : v \in \mathcal{H}\}$. The pair $(\mathcal{H}, J)$ is called the real expression of the CR structure.

Let X be a CR manifold with structure S. For a complex valued C^∞ function u defined on X, the section $\overline{\partial}_b u \in \Gamma(\overline{S}^*)$ is defined by

$$(\overline{\partial}_b u)(\overline{L}) = \overline{L}(u) , \qquad L \in S.$$

The differential operator $\overline{\partial}_b$ is called the (tangential) Cauchy-Riemann operator, and a solution u of the equation $\overline{\partial}_b u = 0$ is called a holomorphic function.

Definition 2.2. A complex vector bundle E over X is said to be holomorphic if there is a differential operator

$$\overline{\partial}_E : \Gamma(E) \longrightarrow \Gamma(E \otimes \overline{S}^*)$$

satisfying the following conditions:

(a) $\overline{\partial}_E(fu)(\overline{L}_1) = (\overline{\partial}_b f)(\overline{L}_1)u + f(\overline{\partial}_E u)(\overline{L}_1)$
$\qquad = (\overline{L}_1 f)u + f(\overline{\partial}_E u)(\overline{L}_1)$
(b) $(\overline{\partial}_E u)[\overline{L}_1, \overline{L}_2] = \overline{\partial}_E(\overline{\partial}_E u(\overline{L}_2))(\overline{L}_1) - \overline{\partial}_E(\overline{\partial}_E u(\overline{L}_1))(\overline{L}_2)$,

where $u \in \Gamma(E)$, $f \in C^\infty(X)$ and $L_1, L_2 \in \Gamma(S)$.

The operator $\overline{\partial}_E$ is called the Cauchy-Riemann operator and a solution u of the equation $\overline{\partial}_E u = 0$ is called a holomorphic cross section.

Remark 2.3.

(a) The trivial bundle $X \times \mathbb{C}$ is holomorphic with respect to the operator $\overline{\partial}_b$ defined above.
(b) In the case where X is a complex manifold, the above definition of a holomorphic vector bundle is equivalent to the usual one in terms of holomorphic transition functions, as can be verified by standard application of Newlander-Nirenberg's theorem.

A basic holomorphic vector bundle over a CR manifold X is the vector bundle $\widehat{T}(X) = \mathbb{C}T(X)/\overline{S}$. The corresponding operator $\overline{\partial} = \overline{\partial}_{\widehat{T}(X)}$ is defined as follows. Let p be the projection from $\mathbb{C}T(X)$ to $\widehat{T}(X)$. Take any $u \in \Gamma(\widehat{T}(X))$ and express it as $u = p(Z)$, $Z \in \Gamma(\mathbb{C}T(X))$. For any $L \in \Gamma(S)$, define a cross section $(\overline{\partial}u)(\overline{L})$ of $\widehat{T}(X)$ by

$$(\overline{\partial}u)(\overline{L}) = p([\overline{L}, Z]).$$

One can show that $(\overline{\partial}u)(\overline{L})$ does not depend on the choice of Z and that $\overline{\partial}u$ gives a cross section of $\widehat{T}(X) \otimes \overline{S}^*$. Furthermore one can show that the operator $u \longmapsto \overline{\partial}u$ satisfies (a) and (b) of Definition 2.2, using the Jacobi identity in the Lie algebra $\Gamma(\mathbb{C}T(X))$. The resulting holomorphic vector bundle $\widehat{T}(X)$ is called the holomorphic tangent bundle of X.

Remark 2.4. If X is a real hypersurface in a complex manifold M, we may identify $\widehat{T}(M)$ with the holomorphic vector bundle of all (1,0) tangent vectors to M and $\widehat{T}(X)$ with the restriction of $\widehat{T}(M)$ to X. In fact, since the structure S of X is the bundle of all (1,0) tangent vectors to X, the inclusion map $\mathbb{C}T(X) \longrightarrow \mathbb{C}T(M)$ induces a natural map $\widehat{T}(X) \xrightarrow{\phi} \widehat{T}(M)|_X$ which is a bundle isomorphism satisfying $\overline{\partial}(\phi(u))(\overline{L}) = \phi(\overline{\partial}u(\overline{L}))$, $u \in \Gamma(\widehat{T}(X))$, $L \in S$.

For a holomorphic vector bundle E over X, set

$$C^q(X,E) = E \otimes \wedge^q \overline{S}^* \ , \ \mathcal{C}^q(X,E) = \Gamma(C^q(X,E))$$

and define a differential operator

$$\overline{\partial}_E^q : \mathcal{C}^q(X,E) \longrightarrow \mathcal{C}^{q+1}(X,E)$$

by

$$\begin{aligned}(\overline{\partial}_E^q\phi)&(\overline{L}_1, \ldots, \overline{L}_{q+1}) \\ &= \sum_i (-1)^{i+1} \overline{\partial}_E(\phi(\overline{L}_1, \ldots, \widehat{\overline{L}}_i, \ldots, \overline{L}_{q+1}))(\overline{L}_i) \\ &\quad + \sum_{i<j} (-1)^{i+j} \phi([\overline{L}_i, \overline{L}_j]\ , \overline{L}_1, \ldots, \widehat{\overline{L}}_i, \ldots, \widehat{\overline{L}}_j, \ldots, \overline{L}_{q+1})\end{aligned}$$

for all $\phi \in \mathcal{C}^q(X,E)$ and $L_1,\ldots,L_{q+1} \in \Gamma(S)$. One shows by standard arguments that $\overline{\partial}_E^q\phi$ gives an element of $\mathcal{C}^{q+1}(X,E)$ and that $\overline{\partial}_E^{q+1}\overline{\partial}_E^q = 0$. The cohomology groups of the resulting complex $\{\mathcal{C}^q(X,E),\overline{\partial}_E^q\}$ is denoted by $H^q(X,E)$.

Let $\{\mathcal{A}^k(X),d\}$ be the De-Rham complex of X with complex coefficients, and let $H^k(X)$ be the De-Rham cohomology groups. There is a natural filtration of the De-Rham complex, as follows. For any integer p and k, put $A^k(X) = \wedge^k(\mathbb{C}T(X)^*)$ and denote by $F^p(A^k(X))$ the subbundle of $A^k(X)$ consisting of all $\phi \in A^k(X)$ which satisfy the equality

$$\phi(Y_1,\ldots,Y_{p-1},\overline{Z}_1,\ldots,\overline{Z}_{k-p+1}) = 0$$

for all $Y_1,\ldots,Y_{p-1} \in \mathbb{C}T(X)_x$ and $Z_1,\ldots,Z_{k-p+1} \in S_x$, x being the origin of ϕ. Then

$$\begin{aligned} A^k(X) = F^0(A^k(X)) \supset F^1(A^k(X)) \supset \cdots \\ \supset F^k(A^k(X)) \supset F^{k+1}(A^k(X)) = 0. \end{aligned}$$

Setting $F^p(\mathcal{A}^k(X)) = \Gamma(F^p(A^k(X)))$, we have

$$\begin{aligned} \mathcal{A}^k(X) = F^0(\mathcal{A}^k(X)) \supset F^1(\mathcal{A}^k(X)) \supset \cdots \\ \supset F^k(\mathcal{A}^k(X)) \supset F^{k+1}(\mathcal{A}^k(X)) = 0. \end{aligned}$$

Since clearly $dF^p(\mathcal{A}^k(X)) \subseteq F^p(\mathcal{A}^{k+1}(X))$, the collection $\{F^p(\mathcal{A}^k(X))\}$ gives a filtration of the De-Rham complex.

We denote by $H_{KR}^{p,q}(X)$ the groups $E_1^{p,q}(X)$ of the spectral sequence $\{E_r^{p,q}(X)\}$ associated with the filtration $\{F^p(\mathcal{A}^k(X))\}$. We call $H_{KR}^{p,q}(X)$ the Kohn-Rossi cohomology group of type (p,q). More explicitly, let

$$\begin{aligned} A^{p,q}(X) &= F^p(A^{p+q}(X)), & \mathcal{A}^{p,q}(X) &= \Gamma(A^{p,q}(X)) \\ C^{p,q}(X) &= A^{p,q}(X)/A^{p+1,q-1}(X), & \mathcal{C}^{p,q}(X) &= \Gamma(C^{p,q}(X)). \end{aligned}$$

Since $d : \mathcal{A}^{p,q}(X) \longrightarrow \mathcal{A}^{p,q+1}(X)$ maps $\mathcal{A}^{p+1,q-1}(X)$ into $\mathcal{A}^{p+1,q}(X)$, it induces an operator $d'' : \mathcal{C}^{p,q}(X) \longrightarrow \mathcal{C}^{p,q+1}(X)$. $H_{KR}^{p,q}(X)$ are then the cohomology groups of the complex $\{\mathcal{C}^{p,q}(X),d''\}$.

Alternatively $H_{KR}^{p,q}(X)$ may be described in terms of the vector bundle $E^p = \wedge^p(\widehat{T}(X)^*)$. If for $\phi \in \Gamma(E^p)$, $u_1,\ldots,u_p \in \Gamma(\widehat{T}(X))$, $Y \in S$, we define $(\overline{\partial}_{E^p}\phi)(\overline{Y}) = \overline{Y}\phi$ by

$$\begin{aligned} &(\overline{Y}\phi)(u_1,\ldots,u_p) \\ &\qquad = \overline{Y}(\phi(u_1,\ldots,u_p)) + \sum_i (-1)^i \phi(\overline{Y}u_i,u_1,\ldots,\widehat{u}_i,\ldots,u_p) \end{aligned}$$

where $\overline{Y}u_i = (\overline{\partial}_{\widehat{T}(X)}u_i)(\overline{Y})$, then we easily verify that E^p with $\overline{\partial}_{E^p}$ is a holomorphic vector bundle. Tanaka [**Ta**] proves that $C^{p,q}(X)$ may be identified with $C^q(X, E^p)$ in a natural manner such that

$$d''\phi = (-1)^p \overline{\partial}_{E^p}\phi, \quad \phi \in \mathcal{C}^{p,q}(X).$$

Thus, $H^{p,q}_{KR}(X)$ may be identified with $H^q(X, E^p)$.

We denote by $H^k_h(X)$ the groups $E^{k,0}_2(X)$ of the spectral sequence $\{E^{p,q}_r(X)\}$ associated with the filtration $\{F^p(\mathcal{A}^k(X))\}$. We call $H^k_h(X)$ the holomorphic De-Rham cohomology groups. The groups $H^k_h(X)$ are the cohomology groups of the complex $\{\mathcal{S}^k(X), d\}$, where we put $\mathcal{S}^k(X) = E^{k,0}_1(X)$ and $d = d_1 : E^{k,0}_1 \longrightarrow E^{k+1,0}_1$. Recall that $\mathcal{S}^k(X)$ is the kernel of the following mapping:

$$\begin{aligned} d_0 : E^{k,0}_0 = F^k\mathcal{A}^k = \mathcal{A}^{k,0}(X) \to E^{k,1}_0 &= F^k\mathcal{A}^{k+1}/F^{k+1}\mathcal{A}^{k+1} \\ &= \mathcal{A}^{k,1}(X)/\mathcal{A}^{k+1,0}. \end{aligned}$$

Note that $\mathcal{S}^k(X)$ may be characterized as the space of holomorphic k-forms, namely holomorphic cross sections of E^k. Thus the complex $\{\mathcal{S}^k(X), d\}$ (respectively, the groups $H^k_h(X)$) will be called the holomorphic De-Rham complex (respectively, the holomorphic De-Rham cohomology groups).

Definition 2.5. Let $L_1, \ldots, L_{n-1}$ be a local frame of the CR structure S on X so that $\overline{L}_1, \ldots, \overline{L}_{n-1}$ is a local frame of $\overline{S}$. Since $S \oplus \overline{S}$ has complex codimension one in $\mathbb{C}T(X)$, we may choose a local section N of $\mathbb{C}T(X)$ such that $L_1, \ldots, L_{n-1}, \overline{L}_1, \ldots, \overline{L}_{n-1}, N$ span $\mathbb{C}T(X)$. We may assume that N is purely imaginary. Then the matrix (c_{ij}) defined by

$$[L_i, \overline{L}_j] = \sum_k a^k_{ij} L_k + \sum_k b^k_{ij} \overline{L}_k + c_{ij} N$$

is Hermitian, and is called the Levi form of X.

The Levi form is noninvariant; however, its essential features are invariant.

Proposition 2.6. *The number of non-zero eigenvalues and the absolute value of the signature of (c_{ij}) at each point are independent of the choice of $L_1, \ldots, L_{n-1}, N$.*

Definition 2.7. X is said to be strongly pseudoconvex if the Levi form is positive definite at each point of X.

3. Siu complex, s-invariant and holomorphic De-Rham cohomology

Let V be a n-dimensional complex analytic subvariety in $\mathbb{C}^N$ with only isolated singularities. In [**Ya2**], Yau considered two kinds of sheaves of germs of holomorphic p-forms

(i) $\overline{\Omega}^p_V := R^0\pi_*\Omega^p_M$, where $\pi\colon M \longrightarrow V$ is a resolution of singularities of V,

(ii) $\overline{\overline{\Omega}}^p_V := \theta_*\Omega^p_{V\setminus \text{Sing } V}$, where $\theta : V\setminus\text{Sing } V \longrightarrow V$ is the inclusion map and Sing V is the singular set of V.

Clearly $\overline{\Omega}^p_V$ is a coherent sheaf because π is a proper map. $\overline{\overline{\Omega}}^p_V$ is also a coherent sheaf by a Theorem of Siu [**Si**]. In case V is a normal variety, the dualizing sheaf ω_V of Grothendieck is actually the sheaf $\overline{\overline{\Omega}}^n_V$.

Definition 3.1. The Siu complex is a complex of coherent sheaves $J^\bullet$ supported on the singular points of V which is defined by the following exact sequence

$$0 \to \overline{\Omega}^\bullet_V \to \overline{\overline{\Omega}}^\bullet_V \to J^\bullet \to 0. \tag{3.1}$$

Definition 3.2. Let V be a n-dimensional Stein space with x as its only singular point. Let $\pi : (M, A) \to (V, x)$ be a resolution of the singularity with A as exceptional set. The geometric genus p_g and the irregularity q of the singularity are defined as follows (cf. [**Ya2**], [**St-St**]):

$$p_g := \dim\ \Gamma(M \setminus A, \Omega^n)/\Gamma(M, \Omega^n) \tag{3.2}$$

$$q := \dim\ \Gamma(M \setminus A, \Omega^{n-1})/\Gamma(M, \Omega^{n-1}). \tag{3.3}$$

The s-invariant of the singularity is defined as follows

$$s := \dim\ \Gamma(M \setminus A, \Omega^n)/[\Gamma(M, \Omega^n) + d\Gamma(M \setminus A, \Omega^{n-1})]. \tag{3.4}$$

The following Lemma follows from a deep theorem of Straten and Steenbrink [**St-St**].

Lemma 3.3. *Let V be a n-dimensional Stein space with x as its only singular point. Let $\pi : (M, A) \to (V, x)$ be a resolution of the singularity with A as exceptional set. Let $J^\bullet$ be the Siu complex of coherent sheaves supported on x. Then*

(i) $\dim J^n = p_g$

(ii) $\dim J^{n-1} = q$

(iii) $\dim J^i = 0$ *for* $1 \leqslant i \leqslant n-2$.

Proof. By Cartan Theorem A, the long cohomology exact sequence of (3.1) at i level gives

$$0 \to \Gamma(V, \overline{\Omega}^i_V) \to \Gamma(V, \overline{\overline{\Omega}}^i_V) \to \Gamma(V, J^i) \to 0. \tag{3.5}$$

Therefore

$$\begin{aligned} \dim J^i &= \dim \Gamma(V, \overline{\overline{\Omega}}^i_V)/\Gamma(V, \overline{\Omega}^i_V) \\ &= \dim \Gamma(M \setminus A, \Omega^i)/\Gamma(M, \Omega^i) \end{aligned} \tag{3.6}$$

(i) and (ii) of the Lemma follow immediately from (3.6) while (iii) of the Lemma is a consequence of Theorem 1.3 in [**St-St**]. q.e.d.

Proposition 3.4. *Let V be a n-dimensional Stein space with x as its only singular point. Let $\pi : (M, A) \to (V, x)$ be a resolution of the singularity with A as exceptional set. Let $J^\bullet$ be the Siu complex of coherent sheaves supported on x. Then the s-invariant is given by*

$$s = \dim H^n(J^\bullet) = p_g - q \tag{3.7}$$

and

$$\dim H^{n-1}(J^\bullet) = 0 \tag{3.8}$$

where p_g and q are respectively the geometric genus and the irregularity of the singularity.

Proof. The proof of Lemma 3.3 gives the following exact sequence of complexes:

$$0 \to \Gamma(V, \overline{\Omega}_V^\bullet) \to \Gamma(V, \overline{\overline{\Omega}}_V^\bullet) \to \Gamma(V, J^\bullet) \to 0. \tag{3.9}$$

In view of Lemma 3.3, the cohomology exact sequence of (3.9) gives

$$\begin{aligned} 0 &\to H^{n-1}(\Gamma(V, \overline{\Omega}_V^\bullet)) \to H^{n-1}(\Gamma(V, \overline{\overline{\Omega}}_V^\bullet)) \to H^{n-1}(\Gamma(V, J^\bullet)) \\ &\to H^n(\Gamma(V, \overline{\Omega}_V^\bullet)) \to H^n(\Gamma(V, \overline{\overline{\Omega}}_V^\bullet)) \to H^n(\Gamma(V, J^\bullet)) \to 0. \end{aligned} \tag{3.10}$$

Since $J^\bullet$ is supported on x, we have

$$H^{n-1}(J^\bullet) = H^{n-1}(\Gamma(V, J^\bullet)),\ H^n(J^\bullet) = H^n(\Gamma(V, J^\bullet)). \tag{3.11}$$

In view of (3.6) and Lemma 3.3, $J^\bullet$ is a complex with only possibly two nonzero terms on $n-1$ and n levels

$$\begin{array}{ccccccc} 0 & \to & \Gamma(M \setminus A, \Omega^{n-1})/\Gamma(M, \Omega^{n-1})) & \xrightarrow{d} & \Gamma(M \setminus A, \Omega^n)/\Gamma(M, \Omega^n) & \to & 0 \\ & & \| & & \| & & \\ & & J^{n-1} & & J^n & & \end{array}$$

By Corollary 1.4 of [**St-St**], we know that the composition of the above map d with the natural map

$$\Gamma(M \setminus A, \Omega^n)/\Gamma(M, \Omega^n) \longrightarrow \Gamma(M \setminus A, \Omega^n)/\Gamma(M, \Omega^n(\log A))$$

is injective. Therefore d is injective. It follows that $H^{n-1}(J^\bullet) = 0$ and

$$\begin{aligned} \dim H^n(J^\bullet) &= \dim \Gamma(M \setminus A, \Omega^n)/[\Gamma(M, \Omega^n) + d\Gamma(M \setminus A, \Omega^{n-1})] \\ &= s. \end{aligned}$$

Finally, since the Euler characteristic of a complex is equal to the Euler characteristic of its cohomology, we have $s = p_g - q$. q.e.d.

We are now ready to prove Theorem A of Section 1.

Proof of Theorem A. Recall that $H_h^q(X)$ is the q-th homology group of the complex $\{\mathcal{S}^\bullet(X), d\}$, where $\mathcal{S}^p(X)$ may be characterized as the space of holomorphic p-forms (i.e., holomorphic cross sections of $E^p = \wedge^p(\widehat{T}(X)^*)$, cf. §2 above). Let $\pi : M \longrightarrow V$ be a resolution of singularities of V with $\pi^{-1}(\{x_1, \ldots, x_m\}) = A$ as exceptional set. Take a one-convex exhaustion function ϕ on M such that $\phi \geqslant 0$ on M and $\phi(y) = 0$ if and only if $y \in A$. Set $M_r = \{y \in M : \phi(y) \geqslant r\}$. Since $X = \partial M$ is strictly pseudoconvex, any holomorphic p-form $\theta \in \mathcal{S}^p(X)$ can be extended to a one sided neighborhood of X in M. Hence, θ can be thought of as holomorphic p-form on M_r, i.e., an element in $\Gamma(M_r, \Omega^p_{M_r})$. By Andreotti and Grauert (Théorème 15 of [**An-Gr**]), $\Gamma(M_r, \Omega^p_{M_r})$ is isomorphic to $\Gamma(M \setminus A, \Omega^p)$, which is equal to $\Gamma(V, \overline{\overline{\Omega}}{}^p_V)$. We have shown that $\mathcal{S}^p(X)$ can be identified with $\Gamma(V, \overline{\overline{\Omega}}{}^p_V)$ which is equal to $\Gamma(M \setminus A, \Omega^p)$. Therefore

$$\begin{aligned} H^p_h(X) &= H^p(\mathcal{S}^\bullet(X)) = H^p(\Gamma(M \setminus A, \Omega^\bullet)) \\ &= H^p_h(M \setminus A) \qquad \text{for } 1 \leqslant p \leqslant n. \end{aligned}$$

For $1 \leqslant p \leqslant n-2$, $H^p_h(M \setminus A) \cong H^p_h(M)$ by Lemma 3.3 (iii) and (3.6). (3.8) and (3.10) imply

$$H^{n-1}_h(M) \cong H^{n-1}_h(M \setminus A) \qquad \text{and}$$

$$0 \to H^n(\Gamma(V, \overline{\Omega}{}^\bullet_V)) \to H^n(\Gamma(V, \overline{\overline{\Omega}}{}^\bullet_V)) \to H^n(\Gamma(V, J^\bullet)) \to 0. \tag{3.12}$$

Observe that $H^n(\Gamma(V, \overline{\Omega}{}^\bullet_V)) = H^n(\Gamma(M, \Omega^\bullet)) = H^n_h(M)$. $H^n(\Gamma(V, \overline{\overline{\Omega}}{}^\bullet_V)) = H^n(\Gamma(M \setminus A, \Omega^\bullet)) = H^n_h(M \setminus A)$. By (3.11) and (3.7), we have $s = \dim H^n(\Gamma(V, J^\bullet))$. It follows from (3.12) that

$$\dim H^n_h(M \setminus A) = \dim H^n_h(M) + s.$$

This completes the proof of our Theorem A in §1. q.e.d.

4. Complex Plateau problem for 3-dimensional CR manifolds

In this section, we shall prove Theorem B. To begin with, let us recall some basic notions and a deep result of Steenbrink on surface singularities which can be found in [**Wa**]. Let $(V, 0)$ be a normal surface singularity. Let $\pi : M \to V$ be a good resolution of singularity. Let $\pi^{-1}(0) = A = \cup A_i$, $1 \le i \le n$, be the irreducible decomposition of the exceptional set A into irreducible components. Let g_i = genus of A_i, $g = \Sigma g_i$, and denote by $\widetilde{A}$ the disjoint union of the A_i. Let Γ be the dual graph of A. Define b = first betti number of Γ, i.e., b = number of loops in Γ. Then $\dim H^1(A, \mathbb{C}) = 2g + b$.

The sheaf of germs of logarithmic 1-forms $\Omega^1_M(\log A)$ is defined by the kernel of the restriction map

$$0 \to \Omega^1_M(\log A)(-A) \to \Omega^1_M \to \Omega^1_{\widetilde{A}} \to 0. \tag{4.1}$$

It follows that $\wedge^2 \Omega^1_M(\log A) = \Omega^2_M(A)$, and there is an exact sequence

$$0 \to \Omega^1_M \to \Omega^1_M(\log A) \to \mathcal{O}_{\widetilde{A}} \to 0 \tag{4.2}$$

where the map on the right is the residue map. The following Lemma can be found in [**Wa**].

Lemma 4.1.

(a) *The composition* $H^0(\mathcal{O}_{\widetilde{A}}) \to H^1(\Omega^1_M) \to H^1(\Omega^1_{\widetilde{A}})$ *is an isomorphism.*

(b) $H^0(\Omega^1_M) \xrightarrow{\sim} H^0\big(\Omega^1_M(\log A)\big)$.

Recall that n is the number of components of A. We define

$$\gamma = rk\big(H^1_A(\Omega^1_M) \to H^1(\Omega^1_M)\big) - n. \tag{4.3}$$

Since $H^0_A(\mathcal{O}_{\widetilde{A}}) \xrightarrow{\sim} H^0(\mathcal{O}_{\widetilde{A}})$, the map $H^0(\mathcal{O}_{\widetilde{A}}) \to H^1(\Omega^1_M)$ factors via $H^1_A(\Omega^1_M)$. Therefore by Lemma 4.1(a), γ is a nonnegative integer. Besides γ, Steenbrink introduces two other invariants

$$\alpha := \dim H^0(\Omega^2_M)/dH^0\big(\Omega^1_M(\log A)(-A)\big) \tag{4.4}$$

$$\beta := \dim H^0(\Omega^1_{\widetilde{A}})/\mathrm{Im}\ H^0(\Omega^1_M). \tag{4.5}$$

Theorem 4.2 (Steenbrink)**.** *Let* $\pi : (M, A) \to (V, 0)$ *be a good resolution of a normal surface singularity, with geometric genus* p_g *and irregularity* q*. Let* $g = \Sigma g_i$*,* $g_i =$ *genus of* A_i*,* $b =$ *first betti number of the dual graph of* A*, and* $\alpha, \beta, \gamma \geq 0$ *as in* (4.3), (4.4) *and* (4.5)*. Then*

$$p_g - q = g + b + \alpha + \beta + \gamma. \tag{4.6}$$

We are now ready to give a characterization of Gorenstein surface singularities with vanishing s-invariant.

Proof of Theorem B. As M is smooth, $H^p(M, \mathbb{C})$ is the p-th hypercohomology of the De Rham complex $\Omega^\bullet_M$. The spectral sequence $E_1^{pq} = H^q(M, \Omega^p) \Rightarrow H^{p+q}(M, \mathbb{C})$ induces an exact sequence of small order terms

$$0 \to H^1_h(M) \to H^1(M, \mathbb{C}) \to E_2^{0,1} \to H^2_h(M). \tag{4.7}$$

Since $s = 0$, by Proposition 3.4, $p_g - q = 0$. In view of Theorem 4.2, $g = b = 0 = \alpha = \beta = \gamma$. Hence $\dim H^1(M, \mathbb{C}) = \dim H^1(A, \mathbb{C}) = 2g + b = 0$. By Theorem 3.2 of [**Wa**], $(V, 0)$ is quasi-homogeneous. On the other hand, it is easy to see that $H^0\big(\Omega^1_M(\log A)(-A)\big) \subseteq H^0(\Omega^1_M)$. Therefore

$$\begin{aligned} \dim H^2_h(M) &= \dim H^0(\Omega^2_M)/dH^0(\Omega^1_M) \\ &\leq \dim H^0(\Omega^2_M)/dH^0\big(\Omega^1_M(\log A)(-A)\big) = \alpha. \end{aligned}$$

It follows that $\dim H^2_h(M) = 0$. In view of (4.7), we have $\dim H^1_h(M) = 0$ and $E_2^{0,1} = 0$. Notice that $E_2^{0,1}$ in (4.7) is the kernel of the map $H^1(M, \mathcal{O}) \to H^1(M, \Omega^1)$. $E_2^{0,1} = 0$ from (4.7) implies that $H^1(M, \mathcal{O}) \to H^1(M, \Omega^1)$ is injective. The cokernel of this map is exactly $E_2^{1,1}$. Therefore $\dim E_2^{1,1} = \dim H^1(M, \Omega^1) - \dim H^1(M, \mathcal{O})$. Observe that $E_\infty^{0,2} = E_1^{0,2} = H^2(M, \mathcal{O}) = 0$, $E_2^{1,1} = E_\infty^{1,1}$, $E_\infty^{2,0} = E_2^{2,0} = H^2_h(M) = 0$, and $\dim H^2(M, \mathbb{C}) = \dim E_\infty^{0,2} + \dim E_\infty^{1,1} + \dim E_\infty^{2,0}$. Hence $\dim H^1(M, \Omega^1) = \dim H^2(M, \mathbb{C}) + \dim H^1(M, \mathcal{O})$.

Conversely if $(V,0)$ is a quasi-homogeneous singularity, then $\alpha=\beta=\gamma=0$ by Theorem 3.2 of [**Wa**]. $H^1(A,\mathbb{C})=0$ implies $b=0=g$. In view of Theorem 4.2 $p_g-q=0$. By Proposition 3.4, s-invariant vanishes.

q.e.d.

Proof of Theorem C. It is well known that X is a boundary of V in D with boundary regularity (see for example [**Lu-Ya**]). Let $\pi: M\to V$ be a good resolution of singularities with exceptional set A.

Since X is a Calabi-Yau CR manifold, there exists a nowhere vanishing holomorphic section ω of $\Lambda^n(\widehat{T}(X)^*)$. The same argument as in the proof of Theorem A shows that ω can be extended as a holomorphic section in $\Gamma(M\backslash A,\Omega^n)$ and as a meromorphic n-form on M. The zero divisor of ω is a compact analytic set since it is disjoint from X. It follows that the zero divisor of ω is contained in A as A is the maximal compact analytic set. We have shown that ω as a holomorphic section in $\Gamma(M\backslash A,\Omega^n)$ is nowhere vanishing. Thus the normalization of interior singularities of V are Gorenstein. By Theorem A and Proposition 3.4, we have $s=p_g-q=0$. q.e.d.

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