ERRATUM: A LOCAL PROOF OF PETRI’S CONJECTURE AT THE GENERAL CURVE

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Abstract

A generation of symbols asserted for \( n \geq 0 \) in the proof of Theorem 3.3 of the original paper in fact only holds for \( n > 0 \), thus undermining the proof of the theorem. A new version of Section 3.5 of the original paper is given, culminating in a corrected proof of Theorem 3.3. The author thanks Deepak Khosla for pointing out the gap in the previous version of the proof.

3.5 Extendable linear systems on curves.

If \( M \) denotes a sufficiently small analytic neighborhood of a general point in the moduli space of curves of genus \( g \), with universal curve \( C/M \), there is a stratification of the locus

\[
Z^r_d = \{ L : L \text{ globally generated}, \; h^0 (L) = r + 1 \} \subseteq \text{Pic}^d (C/M)
\]

such that all strata are smooth and the projection of each to \( M \) is submersive with diffeomorphic fibers. Next consider the induced stratification of the pre-image of \( Z^r_d \) under the Abel-Jacobi map

\[
\alpha : C^{(d)}/M \rightarrow \text{Pic}^d (C/M).
\]

By considering the contact locus between this pre-image stratification and the various diagonal loci in \( C^{(d)}/M \), one can construct a refinement of the stratification of

\[
\alpha^{-1} (Z^r_d) \subseteq C^{(d)}/M
\]

such that all strata are smooth and the projection of each to \( M \) is submersive with diffeomorphic fibers and having the additional property that, beginning with the initial element \( (d) \) of the partially ordered set \( \{(d_1, \ldots, d_s)\} \) of all partitions of \( d \), the stratification is compatible with each set

\[
\text{diag}_{(d_1, \ldots, d_s)} \left( C^{(d)}/M_g \right) \cap \alpha^{-1} (Z^r_d).
\]

Suppose now that \( C_0 \) is a compact Riemann surface of genus \( g \) of general moduli and that \( L_0 \) is a line bundle of degree \( d \) on \( C_0 \) such that the linear system \( \mathbb{P}_0 := \mathbb{P} (H^0 (L_0)) \) is basepoint-free. Let \( C_\beta/\Delta \) be a Schiffer variation supported at a finite set \( A_0 \subseteq C_0 \). Then, by genericity

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of $C_0$ and the remarks just above, there is a deformation $\mathbb{P}_\Delta \subseteq C^{(d)}_\beta$ over $\Delta$ of $\mathbb{P}_0 \subseteq C^{(d)}_0$ for which there exists a trivialization
\begin{equation}
T : \mathbb{P}_\Delta \to \mathbb{P}_0 \times \Delta
\end{equation}
compatible with each partition locus of $d$, that is, for each partition $(d_1, \ldots, d_s)$ of $d$,
\begin{equation}
T \left( \operatorname{diag}(d_1, \ldots, d_s) \left( C^{(d)}_\beta \times C^{(d)}_\beta \mathbb{P}_\Delta \right) \right) = \left( \operatorname{diag}(d_1, \ldots, d_s) \left( C^{(d)}_0 \times C^{(d)}_0 \mathbb{P}_0 \right) \right) \times \Delta.
\end{equation}
Notice that $T$ is a $C^\infty$-map, and is not in general analytic. However $T$ can be chosen so that, for each $p \in \mathbb{P}_0$, $T^{-1}(\{p\} \times \Delta)$ is a proper analytic subvariety of $\mathbb{P}_\Delta$.

Now the tautological section $\tilde{f}_0$ of $\tilde{L}_0(1) = O_{\mathbb{P}_0} \boxtimes L_0$ defined in (27) has divisor $D_0 \subseteq \mathbb{P}_0 \times C_0$.

Let $D \subseteq \mathbb{P}_\Delta \times C_\beta$ denote the divisor of the tautological section $\tilde{f}$ of $\tilde{L}(1) := O_{\mathbb{P}_\Delta}(1) \boxtimes L$.

Then, by (31), the “product” trivialization
\begin{equation}
(T, F_\beta) : \mathbb{P}_\Delta \times C_\beta \to \mathbb{P}_0 \times C_0 \times \Delta
\end{equation}
is compatible with the trivialization $T$ in (30), that is, for each $p \in \mathbb{P}_0$,
\begin{equation}
(T, F_\beta)^{-1}(\{p\} \times C_0 \times \Delta) = T^{-1}(\{p\} \times \Delta) \times \mathbb{P}_\Delta(\mathbb{P}_\Delta \times C_\beta).
\end{equation}
That is, we have the commutative diagram
\begin{equation}
\begin{array}{ccc}
\mathbb{P}_\Delta \times C_\beta & \xrightarrow{(T, F_\beta)} & \mathbb{P}_0 \times C_0 \times \Delta \\
\downarrow & & \downarrow \\
\mathbb{P}_\Delta & \xrightarrow{T} & \mathbb{P}_0 \times \Delta
\end{array}
\end{equation}

Furthermore, by (31), we can adjust $(T, F_\beta)$ “in the $C_0$-direction” to obtain a trivialization
\begin{equation}
\begin{array}{ccc}
\mathbb{P}_\Delta \times C_\beta & \xrightarrow{F} & \mathbb{P}_0 \times C_0 \times \Delta \\
\downarrow & & \downarrow \\
\mathbb{P}_\Delta & \xrightarrow{T} & \mathbb{P}_0 \times \Delta
\end{array}
\end{equation}
which maintains the property
\begin{equation}
F^{-1}(\{p\} \times C_0 \times \Delta) = T^{-1}(\{p\} \times \Delta) \times \mathbb{P}_\Delta(\mathbb{P}_\Delta \times C_\beta).
\end{equation}
and achieves in addition that
\begin{equation}
F^{-1}(D_0 \times \Delta) = D.
\end{equation}
Finally, we can choose the adjustments to be holomorphic in the $C_0$-direction in a small neighborhood of $\mathbb{P}_\Delta \times A_0 \times \Delta$.

Thus referring to Lemma 2.7 there is a $C^\infty$-vector field

$$\gamma = \sum_{n>0} \gamma_n t^n$$
on $\mathbb{P}_0 \times C_0 \times \Delta$ of type $(1, 0)$ such that

1) each $\gamma_n$ annihilates functions pulled back from $\mathbb{P}_0$, that is, it is an $\mathcal{O}_{\mathbb{P}_0}$-linear operator,

2) for each $n$ and each $p \in \mathbb{P}_0$,

$$\gamma_n|_{\{p\} \times C_0}$$
is meromorphic on a neighborhood of $\{p\} \times A_0$,

3) given a function

$$g = \sum_{k=0}^{\infty} g_k t^k : \mathbb{P}_0 \times C_0 \times \Delta \to \mathbb{C}$$with each $g_k$ a $C^\infty$-function on (an open set in) $\mathbb{P}_0 \times C_0$ and any point $p \in \mathbb{P}_0$,

$$g \circ F|_{T^{-1}(\{p\}) \times \Delta} = 0.$$
is holomorphic if and only if

$$[\partial_0, e^{L-\gamma}] (g)|_{\{p\} \times C_0 \times \Delta} = 0.$$

Again, following Lemma 3.2, there is a trivalization

$$\begin{pmatrix}
\tilde{L} (1)^\vee & \tilde{F} & \tilde{L}_0 (1)^\vee \\
\downarrow & \downarrow & \downarrow \\
\mathbb{P}_\Delta \times \Delta C_\beta & \tilde{F} & \mathbb{P}_0 \times C_0 \times \Delta
\end{pmatrix}$$
of $\tilde{L} (1)$ and a lifting $\tilde{\gamma}$ of $\gamma$ such that, for the tautological sections $\tilde{f}_0$ and $\tilde{f}$ defined earlier in this section,

$$\tilde{f} = \tilde{F} \circ \tilde{f}_0.$$Thus, for each $p \in \mathbb{P}_0$,

$$(34) \quad [\partial_0, e^{L-\gamma}] (\tilde{f}_0)|_{\{p\} \times C_0 \times \Delta} = 0.$$

Let

$$\mathcal{D}_n^{\mathbb{P}_0} \left( \tilde{L}_0 (1) \right) \subseteq \mathcal{D}_n \left( \tilde{L}_0 (1) \right)$$
denotes the subsheaf of $\mathcal{O}_{\mathbb{P}_0}$-linear operators. Then

$$[\partial_0, e^{L-\gamma}]$$is a $\partial_0$-closed element of

$$\sum_{n>0} H^1 \left( \mathcal{D}_n^{\mathbb{P}_0} \left( \tilde{L}_0 (1) \right) \right) t^n.$$
Now, referring to (29), we need to analyze
\[ \rho^* [\overline{\partial}_0, e^{L-\gamma}] \in \sum_{n>0} H^1 (\mathcal{D}'_n) t^n \]
\[ = \sum_{n>0} H^1 (\mathcal{D}_n (L_0)) \otimes \text{End} (H^0 (L_0)) t^n. \]
In fact, by construction, this element lies in the image of
\[ \sum_{n>0} H^1 (\mathcal{D}_n (L_0)) t^n. \]
Now
\[ H^1 (\tilde{L}_0 (1)) = \text{Hom} (H^0 (L_0), H^1 (L_0)). \]
But by (34), the image of
\[ \{ [\overline{\partial}_0, e^{L-\gamma}] (f_0) \}_{p \times C_0 \times \Delta} \in \sum_{n>0} H^1 (L_0) \cdot t^n. \]
is zero for each \( p \in \mathbb{P}_0 \). Thus
\[ (35) \quad \rho^* [\overline{\partial}_0, e^{L-\gamma}] (\rho^* f_0) = 0 \in \sum_{n>0} \text{Hom} (H^0 (L_0), H^1 (L_0)) t^n. \]

**Theorem 3.3.** Suppose \( X_0 \) is a curve of genus \( g \) of general moduli. Suppose further that, by varying the choice of \( \beta \) in the Schiffer-type deformation in Section 3.3, the coefficients to \( t^{n+1} \) in all expressions
\[ [\overline{\partial}, e^{-L \beta}] \]
generate \( H^1 (S^{n+1} (T_{X_0})) \) for each \( n \geq 0 \). (For example we allow the divisor \( A_0 \subseteq X_0 \) to move.) Then the maps
\[ \mu^{n+1} : H^1 (S^{n+1} T_{X_0}) \to \text{Hom} (H^0 (L_0), H^1 (L_0)) \]
are zero for all \( n \geq 0 \).

**Proof.** Let
\[ \rho_* [\overline{\partial}_0, e^{L-\gamma}]_{n+1}, \]
denote the coefficient of \( t^{n+1} \) in \( \rho_* [\overline{\partial}_0, e^{L-\gamma}] \). Referring to (29) and the fact the the operators take values in the sheaf \( \mathcal{D}'_n (\tilde{L}_0 (1)) \), we have that
\[ (36) \quad \text{symbol} \left( (\rho_* [\overline{\partial}_0, e^{L-\gamma}])_{n+1} \right) \]
\[ = (\overline{\partial} \beta^{n+1} \otimes 1) \oplus 0 \in S^{n+1} (T_{X_0}) \oplus (S^n (T_{X_0}) \otimes \text{End}^0 (H^0 (L_0))). \]
where
\[ \beta = \sum_{j>0} \beta_j t^j. \]

By (36) and the hypothesis that the elements \( \partial \beta^{n+1}_1 \) generate \( H^1(S^{n+1}_n T_{X_0}) \), we have that, by varying \( \beta \), the elements
\[ \text{symbol } \left( \rho_* \left[ \partial_0, e^{L-\gamma} \right]_{n+1} \right) \]
generate
\[ S^{n+1}(T_{X_0}) \]
for each \( n \geq 0 \).

Thus, by (29) and (35), the map \( \tilde{\nu}^{n+1} \) given by
\[
H^1(\mathcal{D}_{n+1}(L_0)) \to \frac{\text{Hom}(H^0(L_0), H^1(L_0))}{\text{image } (\tilde{\nu}^n)} \quad D \mapsto D\left(\tilde{f}_0\right)
\]
is zero for all \( n \geq 0 \). q.e.d.

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