Generalization, Applications and Inequality Chain of the Idempotent Sum of Three Sides of the Right-angled Triangular

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Abstract：In a class of mathematic competition in our school, Mr. Yang explained for us a problem in 2006 Iranian Maths Olympics (Example 1) and the problem of the 176 in Medium Mathematics 2006.4.At the end of which he raised a question that whether we can make up an inequality chain, which interested us very much. So with the instruction of Mr. Yang, we obtained Theorems 1-6 by means of Bottema 2009 which is mdae by Professor Yang Lu ,a researcher of Chinese Academy of Social Sciences), computers and calculators. And in this line, we considered that whether they could be extended to situations of high power. After making great endeavors, we endeavor Theorem 7. Finally, we thought about whether it could be extended to high dimensional situation and obtained the creative Theorem 8.

The objective of this paper is to further study a type of inequalities of the sum of equal powers of right triangle's 3 edges to the and has successfully obtained a series of inequalities to the power of 3 to 6 and some of higher order, among which some inequalities of lower order have been proved true by means of the software of Bottema 2009 that developed by Doctor Yang Lu, who is a researcher of Chinese Academy of Science. And we have obtained the inequalities about n dimensional simplex. Meanwhile we studied the application of these inequalities in mathematical competitions and teaching. In the last section, we also proposed two conjectures of more generality, among which conjecture 2 is about n dimensional simplex, for those who are interested in it. The study of this type of problems is beneficial to mathematical competitions and researches of primary and advanced mathematics.

As for the application of these theorems, we can discover their value in geometries like right-angled triangle, rectangle,round,ellipse, hyperbola,cube, hypocycloid,right-angled triangular pyramid, globe, ellipsoid, hyperboloid of two sheet, elliptic cone and so on. We can also study about their value in teaching and competitions.

In the end, we proposed more general conjectures while the second discusses about situations of n dimensional simplex and both of which are for those readers who are interested in it. This type of problems is beneficial for competitions and study of primary and advanced mathematics.

Highlight：We systematically study the inequality of the sum of 3 right-angled triangular sides （Theorems1-6）and extend them to the right-angled tetrahedron and propose some difficult conjectures.

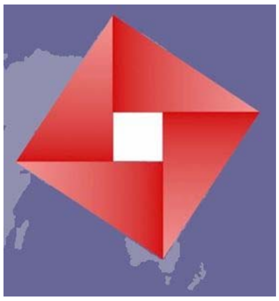
Innovation：We get some inequalities of the sum of 3 right-angled triangular sides of high order (4 and above(Theorem 7))，and extend them to the n dimensional space (theorem 8)，and proposed conjectures in n dimensional space (conjecture 1,2) with creative methods .

Key words: Right-angled Triangle, the Sum of Equal Power, Inequality Chain, N Dimensional Simplex.

1 **Introduction and Main results**

In a class of mathematics competitions in our school, Mr. Yang explained to us a problem in 2006 Iranian Maths Olympics (Example 1) and the senior 176 in *Medium Mathematics* 2006.4., at the end of which he raised a question whether they can make up an inequality chain, which interested us very much. So under the instruction of Mr. Yang, we obtained Theorems 1-6 by means of Bottema 2009, computers and calculators. Along this line, what came to our minds was whether they could be extended to situations of high order. After making great endeavors, we got Theorem 7. Finally, we thought about whether they could be extended to situations of high dimensions and then Theorem 8 is obtained .

Pythagorean is a great theorem in the history of mathematics with a lot of applications in daily life. There are more than 400 approaches to the proof of Pythagorean, also called the *Historic Theorem*！ The logo of the 2002 International Conference of Mathematics (**ICM2002**) in Beijing is designed on the basis of Chord Chart of Zhao Shuang, a Chinese mathematician, which consists of four Congruent right-angled triangle and a square that make up a larger square (shown in the picture). On October 4, 1957, on the first satellite sent up by the Soviet Union included a picture of Pythagorean.



ICM2002

Some problems of the solutions and proofs of inequalities about Pythagorean（three right-angled triangle edges： $a^2+b^2=c^2$ ）

In some national and international Olympic competitions, there are many problems of inequalities on 3 triangular sides（ $a^2+b^2=c^2$ ）, especially the problems of 3 sides to the power of 3 of 3 order.

Therefore we need to study this kind of problems, and got a series of inequality chain of right-angled triangle.

Theorem 1:In $Rt\Delta ABC$, if $a^2+b^2=c^2$,then

$$a^2+b^2+c^2 \ge (6-4\sqrt{2})(a+b+c)^2 \ge \frac{1}{7}(8\sqrt{2}-4)(ab+bc+ac).$$

Theorem 2:In $Rt\Delta ABC$, if $a^2+b^2=c^2$,then

$$a^3+b^3+c^3 \ge \frac{\sqrt{2}}{4}(a+b+c)(a^2+b^2+c^2)$$

$$\ge \frac{1}{7}(2\sqrt{2}+1)[a(b^2+c^2)+b(a^2+c^2)+c(a^2+b^2)]$$

$$\ge \frac{1}{2}(3\sqrt{2}-4)(a+b+c)^3 \ge (2+\sqrt{2})abc.$$

By applying the conclusion above， we can also get the following inequality chain：

Theorem 3:In $Rt\Delta ABC$, if $a^2+b^2=c^2$,then

$$a^4+b^4+c^4 \ge \begin{Bmatrix} \frac{1}{2}(9\sqrt{2}-12)(a+b+c)(a^3+b^3+c^3) \\ \frac{3}{8}(a^2+b^2+c^2)^2 \end{Bmatrix}$$

$$\ge \frac{1}{14}(24-15\sqrt{2})(a+b+c)[a(b^2+c^2)+b(a^2+c^2)+c(a^2+b^2)]$$

$$\ge \frac{1}{2}(51-36\sqrt{2})(a+b+c)^4 \ge 3(\sqrt{2}-1)(a+b+c)abc.$$

Theorem 4 In $Rt\Delta ABC$, if $a^2+b^2=c^2$, then

$$a^5+b^5+c^5 \ge \begin{Bmatrix} \frac{1}{4}(3-\sqrt{2})(a^2+b^2+c^2)(a^3+b^3+c^3) \\ \frac{1}{6}(3\sqrt{2}-2)(a+b+c)(a^4+b^4+c^4) \end{Bmatrix} \ge (\sqrt{2}-1)(ab+bc+ca)(a^3+b^3+c^3).$$

Theorem 5 In $Rt\Delta ABC$, if $a^2+b^2=c^2$, let

$A_1=a^6+b^6+c^6$, $A_2=\frac{1}{2}(15-10\sqrt{2})(a^3+b^3+c^3)^2$,

$A_3=\frac{1}{14}(20\sqrt{2}-25)(a^3+b^3+c^3)[a(b^2+c^2)+b(a^2+c^2)+c(a^2+b^2)]$,

$A_4=\frac{1}{4}(85\sqrt{2}-120)(a^3+b^3+c^3)(a+b+c)^3$,

$A_5=\frac{1}{2}(10-5\sqrt{2})(a^3+b^3+c^3)abc$,

$A_6=\frac{1}{98}(55-30\sqrt{2})[a(b^2+c^2)+b(a^2+c^2)+c(a^2+b^2)]^2$,

$A_7=\frac{1}{28}(220-155\sqrt{2})[a(b^2+c^2)+b(a^2+c^2)+c(a^2+b^2)](a+b+c)^3$,

$A_8=\frac{1}{14}(15\sqrt{2}-10)[a(b^2+c^2)+b(a^2+c^2)+c(a^2+b^2)]abc$,

$A_9=\frac{1}{4}(495-350\sqrt{2})(a+b+c)^6$,

$A_{10}=\frac{1}{2}(25\sqrt{2}-35)(a+b+c)^3abc$, $A_{11}=5(abc)^2$;

then

A_1	$\ge A_2$	$\ge A_3$	$\ge A_4$	$\ge A_5$
		$\ge A_6$	$\ge A_7$	$\ge A_8$
			$\ge A_9$	$\ge A_{10}$
				$\ge A_{11}$

Thus we get several inequality chains. Arrange them in the order in the table above. And then we find that the chain holds in both horizontal and vertical order. (Note: the relationship between the ending item of each line and the starting one of the next line is still uncertain. For example A_5 could be smaller than A_6).

From Theorems 1-5, we can get the following inequality chain:

Theorem 6 In $Rt\Delta ABC$, if $a^2+b^2=c^2$,then

$$a^6+b^6+c^6 \ge \left\{ \begin{array}{l} \frac{1}{14}(25\sqrt{2}-30)(a+b+c)(a^5+b^5+c^5) \\ \frac{5}{12}(a^2+b^2+c^2)(a^4+b^4+c^4) \\ \frac{1}{2}(15-10\sqrt{2})(a^3+b^3+c^3)^2 \end{array} \right\}$$

$$\ge \frac{1}{21}(10\sqrt{2}-5)(ab+bc+ca)(a^4+b^4+c^4)$$

$$\ge \frac{1}{14}(80-55\sqrt{2})(a+b+c)(ab+bc+ca)(a^3+b^3+c^3)$$

$$\ge \frac{1}{14}(15\sqrt{2}-20)(a+b+c)(ab+bc+ca)[a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)]$$

$$\ge \frac{1}{14}(230\sqrt{2}-325)(ab+bc+ca)(a+b+c)^4$$

$$\ge \frac{1}{2}(25\sqrt{2}-35)abc(a+b+c)^3$$

$$\ge \frac{1}{14}(25-15\sqrt{2})(a+b+c)(ab+bc+ca)abc .$$

To consider a bit further, we write the inequalities of 7 order or above as follows.

Theorem 7 In $Rt\Delta ABC$, if $a^2+b^2=c^2$, $n \in \{1\} \bigcup [2,+\infty)$,then

$$a^n+b^n+c^n \ge \frac{2+\sqrt{2^n}}{(2+\sqrt{2})^n}(a+b+c)^n .$$

Theorem 8 If $a_i \in R^+, i=1,2,\cdots,m+1, m\in N^+, m\ge 2$,and $\sum_{i=1}^{m} a_i^{\,k} = a_{m+1}^{\,k}$, $k \ge t > 0$,or $t \le k < 0$, then

$$\sum_{i=1}^{m+1} a_i^{k+t} \ge \frac{1+m^{\frac{t}{k}}}{2(m+m^{\frac{t}{k}})}(\sum_{i=1}^{m+1} a_i^{\,k})(\sum_{i=1}^{m+1} a_i^{\,t}) .$$

2 Lemmas and their proofs

Lemma 1[1] （Newton's formula）To a discretionary symmetric polynomial of n parameters $f(x_1, x_2, \cdots, x_n)$, they can all be represented as the polynomial $f(\sigma_1, \sigma_2, \cdots, \sigma_n)$ of basic symmetric polynomials $\sigma_1, \sigma_2, \cdots, \sigma_n$, and the representation is exclusive. Finally we introduce Newton's formula.

Assume that

$f(x) = (x - x_1)(x - x_2)\cdots(x - x_n) = x^n - \sigma_1 x^{n-1} + \sigma_2 x^{n-2} + \cdots + (-1)^n \sigma_n$,

$s_k = x_1{}^k + x_2{}^k + \cdots, x_n{}^k, k \in N^*$,

They are connected in：

(1) $x^{k+1} \sum_{i=1}^{n} \frac{(x - x_1)(x - x_2)\cdots(x - x_n)}{x - x_i}$

$= (s_0 x^k + s_1 x^{k-1} + \cdots + s_{k-1} x + s_k) f(x) + g(x)$, and $g(x)$ is a polynomial，degg<n.

(2）to $k > n$, $s_k - \sigma_1 s_{k-1} + \sigma_2 s_{k-2} + \cdots + (-1)^n \sigma_n s_{k-n} = 0$;

To $1 \le k \le n$, $s_k - \sigma_1 s_{k-1} + \cdots + (-1)^{k-1} \sigma_{k-1} s_1 + (-1)^k k \sigma_k = 0$.

Detailed proof is available in paper[1] which is omitted here. Conclusion (2）is called Newton’s sum of powers.

Deduction: $f(n) = a^n + b^n + c^n, n \in N^*, n \ge 2$, then

$$f(n+2) = (a+b+c) f(n+1) - (ab+bc+ca) f(n) + abcf(n-1) .$$

Lemma 2 To $n \in N^*, n \ge 3$, then

$$1 + \sin^n \theta + \cos^n \theta = 1 + [\frac{1}{2}(t + \sqrt{2 - t^2})]^n + [\frac{1}{2}(t - \sqrt{2 - t^2})]^n .$$

here， $t = \sin\theta + \cos\theta = \sqrt{2}\sin(\theta + \frac{\pi}{4}) \in (1, \sqrt{2}]$ （ $\theta \in (0, 90^\circ)$ ）.

Proof: Assume that $a_n = 1 + \sin^n \theta + \cos^n \theta$, $t = \sin\theta + \cos\theta$, then

$$\sin\theta\cos\theta = \frac{t^2 - 1}{2} ,$$

$$a_1 = 1 + \sin\theta + \cos\theta = 1 + t ,$$

$$a_2 = 1 + \sin^2\theta + \cos^2\theta = 2 ,$$

$$a_3 = 1 + \sin^3\theta + \cos^3\theta = 1 + \frac{t(3 - t^2)}{2} ;$$

From the deduction above, we know

$$a^{n+2} + b^{n+2} + c^{n+2}$$

$$= (a+b+c)(a^{n+1} + b^{n+1} + c^{n+1}) - (ab+bc+ca)(a^n + b^n + c^n) + abc(a^{n-1} + b^{n-1} + c^{n-1})$$

Get both sides of the equation divided by c^{n+2} and we get

$$a_{n+2}=(1+\sin\theta+\cos\theta)\cdot a_{n+1}-(\sin\theta+\cos\theta+\sin\theta\cos\theta)\cdot a_n+\sin\theta\cos\theta\cdot a_{n-1},$$

namely $a_{n+2}=(1+t)\cdot a_{n+1}-(t+\frac{t^2-1}{2})\cdot a_n+\frac{t^2-1}{2}\cdot a_{n-1}$,

And the characteristic equation of Array $\{a_n\}$ is

$x^3-(1+t)x^2+(t+\frac{t^2-1}{2})x-\frac{t^2-1}{2}=0$, the three roots of this formula are

$x_1=1, x_2=\frac{t+\sqrt{2-t^2}}{2}, x_3=\frac{t-\sqrt{2-t^2}}{2}$. Supposing $a_n=c_1x_1^n+c_2x_2^n+c_3x_3^n$, we substitute a_1,a_2,a_3 with it and we get $c_1=1, c_2=1, c_3=1$.

Lemma 3[2] (**the deduction of Theorem of Remainders**) the necessary and sufficient condition for α which is the root of the polynomial $f(x)$ is $(x-\alpha)\mid f(x)$.

Lemma 4[3] (**Chebyshev inequality**) If $a_1\le a_2\le\cdots\le a_n, b_1\le b_2\le\cdots\le b_n$, or $a_1\ge a_2\ge\cdots\ge a_n, b_1\ge b_2\ge\cdots\ge b_n$. then

$$\frac{1}{n}\sum_{i=1}^{n}a_ib_i\ge(\frac{1}{n}\sum_{i=1}^{n}a_i)\cdot(\frac{1}{n}\sum_{i=1}^{n}b_i);$$

If $a_1\le a_2\le\cdots\le a_n, b_1\ge b_2\ge\cdots\ge b_n$, then

$$\frac{1}{n}\sum_{i=1}^{n}a_ib_i\le(\frac{1}{n}\sum_{i=1}^{n}a_i)\cdot(\frac{1}{n}\sum_{i=1}^{n}b_i).$$

Lemma 5[3] **(an inequality for power mean)** Supposing $0<\alpha\le\beta$, $n\in N^+$, $a_1,a_2,\cdots,a_n\in R^+$, then

$$\left(\frac{a_1^\alpha+a_2^\alpha+\cdots+a_n^\alpha}{n}\right)^{\frac{1}{\alpha}}\le\left(\frac{a_1^\beta+a_2^\beta+\cdots+a_n^\beta}{n}\right)^{\frac{1}{\beta}},$$

Only when $a_1=a_2=\cdots=a_n$ is the equal sign tenable.

3 Proofs of Theorems

3.1 the proof of Theorem 1

Proof:Supposing $a=\cos\theta$, $b=\sin\theta$, $c=1$, and $t=\cos\theta+\sin\theta\in(1,\sqrt{2}]$, from Lemma 2, we get

$$a^2+b^2+c^2=\cos^2\theta+\sin^2\theta+1=1+\cos^2\theta+\sin^2\theta=2,$$

$$(a+b+c)^2=(\cos\theta+\sin\theta+1)^2=(t+1)^2,$$

$$ab+bc+ac=\cos\theta\sin\theta+\cos\theta+\sin\theta=\frac{1}{2}(t^2-1)+t.$$

The inequality $a^2+b^2+c^2\ge(6-4\sqrt{2})(a+b+c)^2$ equals

$$2\ge(6-4\sqrt{2})(t+1)^2$$

$$\Leftrightarrow 3+2\sqrt{2}\ge(t+1)^2$$

$$\Leftrightarrow (\sqrt{2}+1)^2\ge(t+1)^2$$

$\Leftrightarrow \sqrt{2}\ge t$. Apparently.

The inequality $(6-4\sqrt{2})(a+b+c)^2\ge\frac{1}{7}(8\sqrt{2}-4)(ab+bc+ac)$ equals

$$(6-4\sqrt{2})(t+1)^2\ge\frac{1}{7}(8\sqrt{2}-4)[\frac{1}{2}(t^2-1)+t]$$

$$\Leftrightarrow 7(3-2\sqrt{2})(t+1)^2\ge(2\sqrt{2}-1)[(t^2-1)+2t]$$

$$\Leftrightarrow (11-8\sqrt{2})t^2+(22-16\sqrt{2})t+10-6\sqrt{2}\ge 0$$

$$\Leftrightarrow (11-8\sqrt{2})(t+2+\sqrt{2})(\sqrt{2}-t)\ge 0.$$

So the original inequality holds. Only when $a=b$ is the equal sign tenable.

3.2 the proof of Theorem 2

Let $A=a^3+b^3+c^3$,

$$B=\frac{1}{7}(2\sqrt{2}+1)[a(b^2+c^2)+b(a^2+c^2)+c(a^2+b^2)],$$

$$C=\frac{1}{2}(3\sqrt{2}-4)(a+b+c)^3,\quad D=(2+\sqrt{2})abc.$$

May wish to set up $a=\cos\theta$, $b=\sin\theta$, $c=1$, then

$$A=a^3+b^3+c^3=(\cos^3\theta+\sin^3\theta+1),$$

$$a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)$$

$$=\sin\theta(\cos^2\theta+1)+\cos\theta(\sin^2\theta+1)+1\cdot(\cos^2\theta+\sin^2\theta)$$

$$=1+\sin\theta+\cos\theta+\sin\theta\cos\theta(\sin\theta+\cos\theta),$$

$(a+b+c)^3=(\cos\theta+\sin\theta+1)^3$,

$abc=\cos\theta\sin\theta$,

To prove: $A\geq B$.

The original inequality is equivalent to

$\cos^3\theta+\sin^3\theta+1\geq\frac{1}{7}(2\sqrt{2}+1)[1+\sin\theta+\cos\theta+\sin\theta\cos\theta(\sin\theta+\cos\theta)]$.

Also by Lemma 2 may, $1+\sin^3\theta+\cos^3\theta=1+[\frac{1}{2}(t+\sqrt{2-t^2})]^3+[\frac{1}{2}(t-\sqrt{2-t^2})]^3$

$$=1+(\frac{t+\sqrt{2-t^2}}{2}+\frac{t-\sqrt{2-t^2}}{2})[(\frac{t+\sqrt{2-t^2}}{2})^2-\frac{t-\sqrt{2-t^2}}{2}\cdot\frac{t+\sqrt{2-t^2}}{2}+(\frac{t-\sqrt{2-t^2}}{2})^2]$$

$$=1+t[(\frac{t+\sqrt{2-t^2}}{2})^2-\frac{2t^2-2}{2}+(\frac{t-\sqrt{2-t^2}}{2})^2]=\frac{1}{2}t(3-t^2)+1$$

Therefore, the original inequality is equivalent to

$$\frac{1}{2}t(3-t^2)+1\geq\frac{1}{7}(2\sqrt{2}+1)[1+t+\frac{1}{2}t(t^2-1)]$$

$$\Leftrightarrow -(4+\sqrt{2})t^3+(10-\sqrt{2})t+6-\sqrt{2}\geq 0.$$

（we can expand the polynomial by means of the expand function in Maple 12 and arrange the items in the descending order of **t**'s power by means of factor function）

From the fact that $A\geq B$ is a symmetric inequality about a,b, we know that the condition for A=B is $a=b$, namely $Rt\Delta ABC$ is isosceles right triangle. Thus, $\theta=45^\circ$, $t=\cos\theta+\sin\theta$ $=\cos 45^\circ+\sin 45^\circ=\sqrt{2}$ .So only when $t=\sqrt{2}$ is the equal sign in（*） is true. So from Lemma 3, $t-\sqrt{2}$ is a factor of the polynomial to power 3 on the left of (*). Using synthetic division, we can put it this way（by means of maple12's factor function）

$$(\sqrt{2}-t)(t+1)(t-1+\sqrt{2})\geq 0.$$

又 $t=\sin\theta+\cos\theta=\sqrt{2}\sin(\theta+\frac{\pi}{4})\in(1,\sqrt{2}]$.

∴The inequality is holds.

If and only if $a=b$,the inequality becomes equation.

To prove $B\geq C$.

The original inequality is equivalent to the

$$\frac{1}{7}(2\sqrt{2}+1)[\cos\theta(\sin^2\theta+1)+\sin\theta(\cos^2\theta+1)+1]$$

$$\geq\frac{1}{2}(3\sqrt{2}-4)(\cos\theta+\sin\theta+1)^3.$$

Also by Lemma 2 way： $\cos\theta(\sin^2\theta+1)+\sin\theta(\cos^2\theta+1)+1=\frac{1}{2}t(t^2-1)+t+1,$

$(\cos\theta+\sin\theta+1)^3=(t+1)^3$.

Then the original inequality is equivalent to

$\frac{1}{7}(2\sqrt{2}+1)[(\frac{1}{2}t(t^2-1)+t+1]\ge\frac{1}{2}(3\sqrt{2}-4)(t+1)^3$

$\Leftrightarrow\frac{1}{7}(2\sqrt{2}+1)(t^2-t+2)(t+1)\ge(3\sqrt{2}-4)(t+1)^3$

$\Leftrightarrow t^2-t+2\ge(16-11\sqrt{2})(t+1)^2$

$\Leftrightarrow(-15+11\sqrt{2})t^2+(-33+22\sqrt{2})t-14+11\sqrt{2}\ge 0$.

Let $f(t)=(-15+11\sqrt{2})t^2+(-33+22\sqrt{2})t-14+11\sqrt{2}\ge 0,$

Derivative obtained was： $f'(t)=(-30+22\sqrt{2})t-33+22\sqrt{2}$,

$f'(t)\le f'(\sqrt{2})=11-8\sqrt{2}<0$, then to the minimum value is $f(\sqrt{2})$, and $f(\sqrt{2})=0$. The proof is over.

The results hold up, if and only $a=b$ when the establishment of an equal sign.

The last to prove $C\ge D$.

Then the original inequality is equivalent to

$$\frac{1}{2}(3\sqrt{2}-4)(\cos\theta+\sin\theta+1)^3\ge(2+\sqrt{2})\sin\theta\cos\theta,$$

From Lemma 2, we get that, $(\cos\theta+\sin\theta+1)^3=(t+1)^3$, $\sin\theta\cos\theta=\frac{1}{2}(t^2-1)$.

With t substitution, the original inequality is equivalent to

$\frac{1}{2}(3\sqrt{2}-4)(t+1)^3\ge(2+\sqrt{2})\cdot\frac{1}{2}(t^2-1)$,

$\Leftrightarrow(3\sqrt{2}-4)(t+1)^2-(2+\sqrt{2})(t-1)\ge 0$

$\Leftrightarrow(3\sqrt{2}-4)t^2+(5\sqrt{2}-10)t+4\sqrt{2}-2\ge 0$.

Let $f(t)=(3\sqrt{2}-4)t^2+(5\sqrt{2}-10)t+4\sqrt{2}-2$,

Axis of symmetry is $x=\frac{5\sqrt{2}+5}{2}>\sqrt{2}$,

∴Let $f(t)=(3\sqrt{2}-4)t^2+(5\sqrt{2}-10)t+4\sqrt{2}-2$,and when in $(1,\sqrt{2}]$ is descending,

∴ $f(t)_{\min}=f(\sqrt{2})=0$ .May permit.

The result holds up，if and only if $a=b$ when the establishment of an equal sign.

3.3 the proof of Theorem 3

Let $E=\mathrm{a}^4+b^4+c^4$, $F=\frac{1}{2}(9\sqrt{2}-12)(a+b+c)(a^3+b^3+c^3)$,

$G=\frac{1}{14}(24-15\sqrt{2})(a+b+c)[a(b^2+c^2)+b(a^2+c^2)+c(a^2+b^2)]$,

$H=\frac{1}{2}(51-36\sqrt{2})(a+b+c)^4$, $I=3(\sqrt{2}-1)(a+b+c)abc$.

To prove $E\ge F$.

May wish to set up

$a=c\cos\theta$, $b=c\sin\theta$,

Then the original inequality is equivalent to

$$\cos^4\theta+\sin^4\theta+1\ge\frac{1}{2}(9\sqrt{2}-12)(1+\sin\theta+\cos\theta)(\cos^3\theta+\sin^3\theta+1).$$

Know from Lemma 2，$\cos^4\theta+\sin^4\theta+1=2-2[\frac{1}{2}(t^2-1)]^2$,

$$(1+\sin\theta+\cos\theta)(\cos^3\theta+\sin^3\theta+1)=\{t[1-\frac{1}{2}(t^2-1)]+1\}(t+1).$$

The original inequality is equivalent to

$$2-2[\frac{1}{2}(t^2-1)]^2\ge\frac{1}{2}(9\sqrt{2}-12)\{t[1-\frac{1}{2}(t^2-1)]+1\}(t+1)$$

$$\Leftrightarrow 4-(t^2-1)^2\ge\frac{1}{2}(9\sqrt{2}-12)(-t^3+3t-2)(t+1),$$

$$\Leftrightarrow (3-t^2)(t^2+1)\ge\frac{1}{2}(9\sqrt{2}-12)(t+1)^3(2-t)$$

$$\Leftrightarrow \frac{(4+3\sqrt{2})(3-t^2)(t^2+1)}{3(t+1)^3(2-t)}\ge 1$$

$$\Leftrightarrow (4+3\sqrt{2})(3-t^2)(t^2+1)-3(t+1)^3(2-t)\ge 0$$

$$\Leftrightarrow (-3\sqrt{2}-1)t^4+3t^3+(6\sqrt{2}-1)t^2-15t+6+9\sqrt{2}\ge 0 .(*)$$

（We can expand the polynomial by means of the expand function in Maple 12 and arrange the items in the decreasing order of t’s power by means of factor function）

Known from Lemma 3，On the type decomposable（Can make use of the software factor function maple12）to

$$\frac{1}{51}(1+3\sqrt{2})(\sqrt{2}-t)[17t^3+(3+8\sqrt{2})t^2+(12\sqrt{2}-21)t+9+24\sqrt{2}]\ge 0.$$

Owing to $t=\sin\theta+\cos\theta\in\left(1,\sqrt{2}\right]$，So we only need to prove

$$17t^3+(3+8\sqrt{2})t^2+(12\sqrt{2}-21)t+9+24\sqrt{2}\ge 0,$$

Let $F(t)=17t^3+(3+8\sqrt{2})t^2+(12\sqrt{2}-21)t+9+24\sqrt{2}$,

Then $F'(t)=51t^2+(6+16\sqrt{2})t+12\sqrt{2}-21$,

Also $F'(t)\ge F'(1)=36+28\sqrt{2}>0$， $F(1)=8+44\sqrt{2}$,

So $F(t)$ In the definition domain is greater than 0. The proof is over.

The establishment of the original inequality, if and only if a =b, when the establishment of an equal sign.

After proving $E\ge F$, Right triangle by the three parties before the conclusion of substitution $F\ge G\ge H\ge I$ can be shown, if and only if $a=b$, when the establishment of an equal sign.（1）is proved.

3.4 the proof of Theorem 4

Let $A=a^5+b^5+c^5, B=\frac{1}{4}(3-\sqrt{2})(a^2+b^2+c^2)(a^3+b^3+c^3)$,

$C=\frac{1}{6}(3\sqrt{2}-2)(a+b+c)(a^4+b^4+c^4)$， $D=(\sqrt{2}-1)(ab+bc+ca)(a^3+b^3+c^3)$.

To prove $A\ge B$.

Without loss of generality we assume that

$a=c\cos\theta$， $b=c\sin\theta$， Then the original inequality is equivalent to

$$\cos^5\theta+\sin^5\theta+1\ge\frac{1}{4}(3-\sqrt{2})(\cos^2\theta+\sin^2\theta+1)(\cos^3\theta+\sin^3\theta+1)$$

$$\Leftrightarrow (\cos^2\theta+\sin^2\theta)(\cos^3\theta+\sin^3\theta)-\cos^2\theta\sin^2\theta(\cos\theta+\sin\theta)+1$$

$$\ge\frac{1}{4}(3-\sqrt{2})(\cos^2\theta+\sin^2\theta+1)(\cos^3\theta+\sin^3\theta+1)$$

Know from Lemma 2,

$$(\cos^2\theta+\sin^2\theta)(\cos^3\theta+\sin^3\theta)-\cos^2\theta\sin^2\theta(\cos\theta+\sin\theta)+1$$

$$=t[1-\frac{1}{2}(t^2-1)]-t[\frac{1}{2}(t^2-1)]^2+1,$$

$$(\cos^2\theta+\sin^2\theta+1)(\cos^3\theta+\sin^3\theta+1)=2\{t[1-\frac{1}{2}(t^2-1)]+1\}.$$

Then the original inequality is equivalent to

$$t[1-\frac{1}{2}(t^2-1)]-t[\frac{1}{2}(t^2-1)]^2+1\ge\frac{1}{2}(3-\sqrt{2})\{t[1-\frac{1}{2}(t^2-1)]+1\}$$

$$\Leftrightarrow -t^5+(3-\sqrt{2})t^4+(3\sqrt{2}-4)t^3+2\sqrt{2}-2\ge 0 .$$

（we can expand the polynomial by means of the expand function in Maple 12 and arrange the items in the decreasing order of **t**’s power by means of factor function）

Know from Lemma 3，On the type decomposable（Can make use of the software factor function maple12）to

$$\frac{1}{4}(t-\sqrt{2})(t+1)^2[t^2-(2-\sqrt{2})t+2-\sqrt{2}]\le 0.$$

Owning to $(t+1)^2>0$, then to prove $t^2-(2-\sqrt{2})t+2-\sqrt{2}>0$ when in $t\in\left(1,\sqrt{2}\right]$ set，Let $F(t)=t^2-(2-\sqrt{2})t+2-\sqrt{2}$,

Then its axis of symmetry of $t=\frac{1}{2}(2-\sqrt{2})<1$，And $F(t)>F(1)=1$,

The definition domain $F(t)$ is greater than 0, so the original inequality

The inequality $a^5+b^5+c^5\ge\frac{1}{4}(3-\sqrt{2})(a^2+b^2+c^2)(a^3+b^3+c^3)$ set. If, and only $a=b$ when the establishment of an equal sign.

To prove $B\ge D$. This is derived from Theorem 1.

To prove $A\ge C$.

May wish to set up $a=c\cos\theta$， $b=c\sin\theta$,

Then the original inequality is equivalent to

$$\cos^5\theta+\sin^5\theta+1\ge\frac{1}{6}(3\sqrt{2}-2)(\cos\theta+\sin\theta+1)(\cos^4\theta+\sin^4\theta+1),$$

$$\Leftrightarrow(\cos^2\theta+\sin^2\theta)(\cos^3\theta+\sin^3\theta)-\cos^2\theta\sin^2\theta(\cos\theta+\sin\theta)+1$$

$$\ge\frac{1}{6}(3\sqrt{2}-2)(\cos\theta+\sin\theta+1)(\cos^4\theta+\sin^4\theta+1)$$

$$\Leftrightarrow(\cos^3\theta+\sin^3\theta)-\cos^2\theta\sin^2\theta(\cos\theta+\sin\theta)+1$$

$$\ge\frac{1}{6}(3\sqrt{2}-2)(\cos\theta+\sin\theta+1)(\cos^4\theta+\sin^4\theta+1)$$

Know from Lemma 2,

$$(\cos^3\theta+\sin^3\theta)-\cos^2\theta\sin^2\theta(\cos\theta+\sin\theta)+1$$

$$=\frac{1}{2}t(3-t^2)-t[\frac{1}{2}(t^2-1)]^2+1,$$

$$(\cos\theta+\sin\theta+1)(\cos^4\theta+\sin^4\theta+1)$$

$$=(t+1)\{1-2[\frac{1}{2}(t^2-1)]^2+1\}.$$

The original inequality is equivalent to the：

$$t[1-\frac{1}{2}(t^2-1)]-t[\frac{1}{2}(t^2-1)]^2+1\ge\frac{1}{6}(3\sqrt{2}-2)(t+1)\{2-2[\frac{1}{2}(t^2-1)]^2\},$$

$$\Leftrightarrow(3\sqrt{2}-5)t^5+(3\sqrt{2}-2)t^4+(4-6\sqrt{2})t^3+(4-6\sqrt{2})t^2$$

$$+(21-9\sqrt{2})t+18-9\sqrt{2}\ge 0.$$

（we can expand the polynomial by means of the expanding function in Maple 12 and arrange the items in the decreasing order of it's power by means of factor function）.

By Lemma 3 we can see that the style can be broken down (can be a factor with maple12 software function) for the

$$\frac{1}{84}(-5+3\sqrt{2})(t-\sqrt{2})(t+1)[7t^3-(15+2\sqrt{2})t^2+(27+12\sqrt{2})t+9+18\sqrt{2}]\geq 0.$$

又 $\frac{1}{84}(-5+3\sqrt{2})<0$,

To prove $7t^3-(15+2\sqrt{2})t^2+(27+12\sqrt{2})t+9+18\sqrt{2}>0$ when set in $t\in\left(1,\sqrt{2}\right]$.

Let $F(t)=7t^3-(15+2\sqrt{2})t^2+(27+12\sqrt{2})t+9+18\sqrt{2}$,

Then $F'(t)=21t^2-(30+4\sqrt{2})t+27+12\sqrt{2}$,

Owning to $F'(t)$ Symmetry axis is $t=\frac{1}{21}(15+2\sqrt{2})<1$, and $F'(1)=18+8\sqrt{2}>0$,

So $F(t)$ is in the last increment of $t\in\left(1,\sqrt{2}\right]$.

And $F(t)>F(1)=28+28\sqrt{2}$,

So $7t^3-(15+2\sqrt{2})t^2+(27+12\sqrt{2})t+9+18\sqrt{2}>0$ set in $t\in\left(1,\sqrt{2}\right]$.

$A\geq C$ set when $t\in\left(1,\sqrt{2}\right]$, If and only if $a=b$ when the establishment of an equal sign.

The last to prove $C\geq D$.

Without loss of generality we assume that $a=c\cos\theta$, $b=c\sin\theta$,

The original inequality is equivalent to

$$\frac{1}{6}(3\sqrt{2}-2)(\cos\theta+\sin\theta+1)(\cos^4\theta+\sin^4\theta+1)$$

$$\geq(\sqrt{2}-1)(\cos\theta\sin\theta+\cos\theta+\sin\theta)(\cos^3\theta+\sin^3\theta+1).$$

Know form Lemma 2 ,

$$(\cos\theta+\sin\theta+1)(\cos^4\theta+\sin^4\theta+1)$$

$$=(t+1)\{1-2[\frac{1}{2}(t^2-1)]^2+1\},$$

$$(\cos\theta\sin\theta+\cos\theta+\sin\theta)(\cos^3\theta+\sin^3\theta+1)$$

$$=[t+\frac{1}{2}(t^2-1)][\frac{1}{2}t(3-t^2)+1].$$

The inequality is equivalent to

$$\frac{1}{6}(3\sqrt{2}-2)(t+1)\{1-2[\frac{1}{2}(t^2-1)]^2+1\}\geq(\sqrt{2}-1)[t+\frac{1}{2}(t^2-1)][\frac{1}{2}t(3-t^2)+1].$$

$$\Leftrightarrow -2t^5+(6\sqrt{2}-8)t^4+(16-12\sqrt{2})t^3+(40-36\sqrt{2})t^2$$

$$+(12\sqrt{2}-6)t+30\sqrt{2}-24\geq 0$$

（we can expand the polynomial by means of the expand function in Maple 12 and arrange the items in the decreasing order of **t**'s power by means of factor function）.

By Lemma 3 we can see that the style can be broken down (can be a factor with maple12 software function) for the

$$\frac{1}{12}(\sqrt{2}-t)(t+1)[t^3+(3-2\sqrt{2})t^2+(12\sqrt{2}-15)t+15-6\sqrt{2}]\geq 0.$$

To prove $t^3+(3-2\sqrt{2})t^2+(12\sqrt{2}-15)t+15-6\sqrt{2}>0$ setting when $t\in\left(1,\sqrt{2}\right]$.

Let $F(t)=t^3+(3-2\sqrt{2})t^2+(12\sqrt{2}-15)t+15-6\sqrt{2}$,

Then $F'(t)=3t^2+(6-4\sqrt{2})t+(12\sqrt{2}-15)$, the axis of symmetry is

$t=\frac{1}{3}(3-2\sqrt{2})$,

And $F'(t)>F'(1)=8\sqrt{2}-6$, so $F(t)>F(1)=4\sqrt{2}+4>0$, May permit.

So the inequality

$\frac{1}{6}(3\sqrt{2}-2)(a+b+c)(a^4+b^4+c^4)\geq(\sqrt{2}-1)(ab+bc+ca)(a^3+b^3+c^3)$ set. If and only if $a=b$ when the establishment of an equal sign.

3.5 the proof of Theorem 5

To prove $A_1\geq A_2$.

Without loss of generality we assume that $a=c\cos\theta$, $b=c\sin\theta$.

The inequality is equivalent to

$$\cos^6\theta+\sin^6\theta+1\geq\frac{1}{2}(15-10\sqrt{2})(\cos^3\theta+\sin^3\theta+1)^2,$$

Know from Lemma 2, $\cos^6\theta+\sin^6\theta+1=2-\frac{3}{4}(t^2-1)^2$,

$$(\cos^3\theta+\sin^3\theta+1)^2=\{t[\frac{1}{2}(3-t^2)]+1\}^2.$$

The inequality is equivalent to

$$2-\frac{3}{4}(t^2-1)^2\geq\frac{1}{2}(15-10\sqrt{2})[t\cdot\frac{1}{2}(3-t^2)+1]^2$$

$$\Leftrightarrow(10\sqrt{2}-15)t^6+(84-60\sqrt{2})t^4+(60-4\sqrt{2})t^3+(90\sqrt{2}-123)t^2$$

$$+(120\sqrt{2}-180)t+40\sqrt{2}-50\geq 0.$$

（we can expand the polynomial by means of the expand function in Maple 12 and arrange the items in the decreasing order of **t**'s power by means of factor function）

By Lemma 3 we can see that the style can be broken down (can be a factor with maple12 software function) for the

$$\frac{1}{8}(-3+2\sqrt{2})(t-\sqrt{2})[5t^5+5\sqrt{2}t^4+(12\sqrt{2}-2)t^3+(4-2\sqrt{2})t^2+(5-20\sqrt{2})t+20+5\sqrt{2}]\geq 0.$$

Obviously, the establishment of an equal sign sets when $t=\sqrt{2}$.

To prove $5t^5+5\sqrt{2}t^4+(12\sqrt{2}-2)t^3+(4-2\sqrt{2})t^2+(5-20\sqrt{2})t+20+5\sqrt{2}>0$ setting when $t\in\left(1,\sqrt{2}\right]$.

Let $F(t)=5t^5+5\sqrt{2}t^4+(12\sqrt{2}-2)t^3+(4-2\sqrt{2})t^2+(5-20\sqrt{2})t+20+5\sqrt{2}$,

Then $F'(t)=25t^4+20\sqrt{2}t^3+(36\sqrt{2}-6)t^2+(8-4\sqrt{2})t+5-20\sqrt{2}$,

Owning to $36\sqrt{2}-6>0$ 、 $8-4\sqrt{2}>0$ ， $F'(1)=32+32\sqrt{2}>0$

So $F'(t)>F'(1)>0$ ，it's an increasing function.

And $F(1)=32>0$,

So $5t^5+5\sqrt{2}t^4+12\sqrt{2}t^3+(4-2\sqrt{2})t^2+(5-20\sqrt{2})t+20+5\sqrt{2}>0$ set when $t\in\left(1,\sqrt{2}\right]$.

Foregoing，the original inequality $a^6+b^6+c^6\geq\frac{1}{2}(15-10\sqrt{2})(a^3+b^3+c^3)^2$ set.

If and only if a = b when the establishment of an equal sign.

After proving $A_1\geq A_2$, From Theorem 1-4, substituting into the table ranging from easy to prove the relationship between the establishment of an equal sign if and only time, (2) may permit several inequality chain.

3.6 the proof of Theorem 6

To prove $a^6+b^6+c^6\geq\frac{1}{14}(25\sqrt{2}-30)(a+b+c)(a^5+b^5+c^5)$.

Without loss of generality we assume that $a=c\cos\theta$ ， $b=c\sin\theta$.
The original inequality is equivalent to

$$\cos^6\theta+\sin^6\theta+1\geq\frac{1}{14}(25\sqrt{2}-30)(\cos\theta+\sin\theta+1)(\cos^5\theta+\sin^5\theta+1).$$

Know from Lemma 2，

$$\cos^6\theta+\sin^6\theta+1=2-\frac{3}{4}(t^2-1)^2,$$

$$(\cos\theta+\sin\theta+1)(\cos^5\theta+\sin^5\theta+1)$$

$$=(t+1)\{t[1-\frac{1}{2}(t^2-1)]-t[\frac{1}{2}(t^2-1)]^2+1\}.$$

So the original inequality is equivalent to t

$$2-\frac{3(t^2-1)^2}{4}\geq\frac{25\sqrt{2}-30}{14}(t+1)[t(1-\frac{t^2-1}{2})-t(\frac{t^2-1}{2})^2+1]$$

$$\Leftrightarrow 2-\frac{3}{4}(t^2-1)^2\geq\frac{1}{14}(25\sqrt{2}-30)(t+1)[t\cdot\frac{1}{4}(5-t^4)+1]$$

$$\Leftrightarrow(25\sqrt{2}-30)t^6+(25\sqrt{2}-30)t^5-42t^4+(234-125\sqrt{2})t^3$$

$$+(270-225\sqrt{2})t+190-100\sqrt{2}\geq 0.$$

（we can expand the polynomial by means of the expand function in Maple 12 and arrange the items in the decreasing order of **t**'s power by means of factor function）

By Lemma 3 we can see that the style can be broken down (can be a factor with maple12 software function) for the

$$\frac{1}{56}(5\sqrt{2}-6)(t-\sqrt{2})[5t^5+(5+5\sqrt{2})t^4-(10\sqrt{2}+8)t^3-(20+8\sqrt{2})t^2+(10\sqrt{2}-5)t-25-5\sqrt{2}]\geq 0,$$

Which $5\sqrt{2}-6>0$.

To prove $5t^5+(5+5\sqrt{2})t^4-(10\sqrt{2}+8)t^3-(20+8\sqrt{2})t^2+(10\sqrt{2}-5)t-25-5\sqrt{2}<0$

Let $F(t)=5t^5+(5+5\sqrt{2})t^4-(10\sqrt{2}+8)t^3-(20+8\sqrt{2})t^2+(10\sqrt{2}-5)t-25-5\sqrt{2}$,

Then $F^{(1)}(t)=25t^4+(20+20\sqrt{2})t^3-(30\sqrt{2}+24)t^2-(40+16\sqrt{2})t+(10\sqrt{2}-5)$

$$F^{(2)}(t)=100t^3+(60+60\sqrt{2})t^2-(60\sqrt{2}+48)t-(40+16\sqrt{2})$$

$$F^{(3)}(t)=300t^2+(120+120\sqrt{2})t-(48\sqrt{2}+60).$$

Obviously ， the axis of symmetry is $t=\frac{1}{300}(120+120\sqrt{2})=\frac{1}{5}(2+2\sqrt{2})<1$，

And $F^{(3)}(t)>F^{(3)}(1)=372+60\sqrt{2}$， so $F^{(2)}(t)$ is incremental，

And $t\in(1,\sqrt{2}]$， $100+60+60\sqrt{2}-60\sqrt{2}-48\sqrt{2}-40-16\sqrt{2}=120-64\sqrt{2}>0$

So $F^{(2)}(t)>F^{(2)}(1)=72\text{-}16\sqrt{2}>0$， and $F^{(1)}(t)$ is incremental.

Owning to $F^{(1)}(1)=-24+16\sqrt{2}<0$， $F^{(1)}(\sqrt{2})=95-50\sqrt{2}>0$，

So $F(t)$ has the existence of minimum.

And $F(1)=-48-8\sqrt{2}<F(\sqrt{2})=-65-2\sqrt{2}$， May permit.

$5t^5+(5+5\sqrt{2})t^4-(10\sqrt{2}+8)t^3-(20+8\sqrt{2})t^2+(10\sqrt{2}-5)t-25-5\sqrt{2}<0$ 成立.

Then the original inequality

$a^6+b^6+c^6\geq\frac{1}{14}(25\sqrt{2}-30)(a+b+c)(a^5+b^5+c^5)$ set. If and only if $a=b$ When the establishment of an equal sign.

To prove $\frac{25\sqrt{2}-30}{14}(a+b+c)(a^5+b^5+c^5)$

$\geq\frac{10\sqrt{2}-5}{21}(ab+bc+ca)(a^4+b^4+c^4)$.

The original inequality equals

$$(a+b+c)(a^5+b^5+c^5)\geq\frac{2+\sqrt{2}}{3}(ab+bc+ca)(a^4+b^4+c^4)$$

$$\Leftrightarrow (1+\cos\theta+\sin\theta)(1+\cos^5\theta+\sin^5\theta)$$

$$\geq\frac{2+\sqrt{2}}{3}(1+\cos\theta+\sin\theta+\sin\theta\cos\theta)(1+\cos^4\theta+\sin^4\theta)$$

Noted that $\sin^4\theta+\cos^4\theta=(\sin^2\theta+\cos^2\theta)^2-2(\sin\theta\cos\theta)^2$

$=1-2(\frac{t^2-1}{2})^2=\frac{1}{2}[2-(t^2-1)^2]$,

$$\sin^5\theta+\cos^5\theta=(\sin^2\theta+\cos^2\theta)(\sin^3\theta+\cos^3\theta)-\sin^2\theta\cos^2\theta(\sin\theta+\cos\theta)$$

$$=\sin^3\theta+\cos^3\theta-\sin^2\theta\cos^2\theta(\sin\theta+\cos\theta)$$

$$=\frac{1}{2}t(3-t^2)-(\frac{t^2-1}{2})^2t=\frac{1}{4}t[2(3-t^2)-(t^2-1)^2].$$

Therefore , the formula above equals

$$(1+t)\cdot\{1+\frac{1}{4}t[2(3-t^2)-(t^2-1)^2]\}$$

$$\geq\frac{2+\sqrt{2}}{3}[t+\frac{1}{2}(t^2-1)]\cdot\{1+\frac{1}{2}[2-(t^2-1)^2]\}$$

$$\Leftrightarrow (\sqrt{2}-1)t^6+(2\sqrt{2}+1)t^5-(6+3\sqrt{2})t^4-(8+4\sqrt{2})t^3$$

$$+(13-\sqrt{2})t^2+(15-6\sqrt{2})t+18+3\sqrt{2}\ge 0$$

$$\Leftrightarrow (\sqrt{2}-1)[-t^5-(5+4\sqrt{2})t^4+(4+4\sqrt{2})t^3+(24+16\sqrt{2})t^2$$

$$+(21+12\sqrt{2})t+21+12\sqrt{2}](\sqrt{2}-t)\ge 0$$

So we should only prove

$$-t^5-(5+4\sqrt{2})t^4+(4+4\sqrt{2})t^3+(24+16\sqrt{2})t^2$$

$$+(21+12\sqrt{2})t+21+12\sqrt{2}\ge 0 .$$

$$\Leftrightarrow t^3(2-t^2)+(4+\sqrt{2})t^3(\sqrt{2}-t)+2t^2(2-t^2)+(18+16\sqrt{2})t^2$$

$$+(21+12\sqrt{2})t+21+12\sqrt{2}\ge 0 .$$

From $t\in(1,\sqrt{2}]$, we know the inequality above is tenable.

Now prove $(a^3+b^3+c^3)^2\ge\frac{10+8\sqrt{2}}{21}(ab+bc+ca)(a^4+b^4+c^4)$.

$$\Leftrightarrow (1+\cos^3\theta+\sin^3\theta)^2\ge\frac{10+8\sqrt{2}}{21}(1+\cos\theta+\sin\theta+\sin\theta\cos\theta)(1+\cos^4\theta+\sin^4\theta)$$

Noted that $\sin^3\theta+\cos^3\theta=(\sin\theta+\cos\theta)(\sin^2\theta+\cos^2\theta-\sin\theta\cos\theta)$

$$=t(1-\frac{t^2-1}{2})=\frac{1}{2}t(3-t^2) ,$$

$$\sin^4\theta+\cos^4\theta=(\sin^2\theta+\cos^2\theta)^2-2(\sin\theta\cos\theta)^2$$

$$=1-2(\frac{t^2-1}{2})^2=\frac{1}{2}[2-(t^2-1)^2] ,$$

Therefore, the formula above equals

$$[1+\frac{1}{2}t(3-t^2)]^2\ge\frac{10+8\sqrt{2}}{21}[t+\frac{1}{2}(t^2-1)]\cdot\{1+\frac{1}{2}[2-(t^2-1)^2]\}$$

$$\Leftrightarrow (31+8\sqrt{2})t^6+(20+16\sqrt{2})t^5-(156+24\sqrt{2})t^4-(124+32\sqrt{2})t^3$$

$$+(178-8\sqrt{2})t^2+(192-48\sqrt{2})t+114+24\sqrt{2}\ge 0$$

$$\Leftrightarrow \frac{1}{119}(31+8\sqrt{2})[-119t^5-(52+167\sqrt{2})t^4+(302-12\sqrt{2})t^3$$

$$+(228+302\sqrt{2})t^2+(468\sqrt{2}-207)t+225\sqrt{2}-24](\sqrt{2}-t)\ge 0$$

So we should only prove

$$-119t^5-(52+167\sqrt{2})t^4+(302-124\sqrt{2})t^3+(228+302\sqrt{2})t^2$$

$$+(468\sqrt{2}-207)t+225\sqrt{2}-24\geq 0.$$

$$\Leftrightarrow [119t^4+(52+286\sqrt{2})t^3+(270+176\sqrt{2})t^2+(124-32\sqrt{2})t](\sqrt{2}-t)$$

$$(143-344\sqrt{2})(\sqrt{2}-t)+664+82\sqrt{2}\geq 0.$$

From Lemma 3, we know the inequality above equals

$$[119t^4+(52+286\sqrt{2})t^3+(270+176\sqrt{2})t^2+(124-32\sqrt{2})t](\sqrt{2}-t)$$

$$(344\sqrt{2}-143)t+225\sqrt{2}-24\geq 0.$$

From $t\in(1,\sqrt{2}]$, we know the inequality above is tenable.

The inequality $a^6+b^6+c^6\geq\frac{5}{12}(a^2+b^2+c^2)(a^4+b^4+c^4)$ is a deduction of Theorem 8, and the inequality $\frac{5}{12}(a^2+b^2+c^2)(a^4+b^4+c^4)\geq\frac{1}{21}(10\sqrt{2}-5)(bc+ca+ab)(a^4+b^4+c^4)$ is another form of Theorem 1.

（**2**）Clearly, by Theorem 1 to 4 before the substitution, easy to get the remaining inequality chain of certificates.

3.7 the proof of Theorem 7

Proof 1 :As the pending permit inequality on a and b symmetrical of a and b.

Without loss of generality we assume that $a\geq b$,then $a=c\cos\theta$，$b=c\sin\theta$，$\theta\in(0^\circ,45^\circ]$.

Then the original inequality is equivalent to

$$\frac{\sin^n\theta+\cos^n\theta+1}{(\sin\theta+\cos\theta+1)^n}\geq\frac{2+\sqrt{2^n}}{(2+\sqrt{2})^n}.$$

When $n=1$, the inequality is clearly established.

To prove the inequality is clearly established when $n\geq 2$.

Let $f(\theta)=\frac{\sin^n\theta+\cos^n\theta+1}{(\sin\theta+\cos\theta+1)^n}$， $\theta\in(0^\circ,90^\circ)$,then

$$f'(\theta)=\frac{n\sin\theta\cos\theta(\sin^{n-2}\theta-\cos^{n-2}\theta)+n(\sin\theta-\cos\theta)(\sin^n\theta+\cos^n\theta+1)(\sin\theta+\cos\theta+1)}{(\sin\theta+\cos\theta+1)^n}$$

When $\theta\in(0^\circ,45^\circ]$， $0<\sin\theta\leq\cos\theta<1, \sin\theta-\cos\theta\leq 0$, $\sin^{n-2}\theta-\cos^{n-2}\theta\leq 0$,

then $f'(\theta)\leq 0$.

$f(\theta) \geq f(45^\circ) = \dfrac{2+\sqrt{2^n}}{(2+\sqrt{2})^n}$ .May permit.

Proof 2: When $n=1$, the inequality is clearly established.
To prove the inequality is clearly established when $n \geq 2$.
The original inequality equals

$$a^n + b^n + (a^2+b^2)^{\frac{n}{2}} \geq \frac{2+2^{\frac{n}{2}}}{(2+2^{\frac{1}{2}})^n}(a+b+\sqrt{a^2+b^2})^n ,$$

$\because$ From Lemma 5, we know $a^n + b^n \geq 2^{1-\frac{n}{2}}(a^2+b^2)^{\frac{n}{2}} \quad (n \geq 2)$,

$\therefore$ we should only prove $(1+2^{1-\frac{n}{2}})(a^2+b^2)^{\frac{n}{2}} \geq \dfrac{2+2^{\frac{n}{2}}}{(2+2^{\frac{1}{2}})^n}(a+b+\sqrt{a^2+b^2})^n$,

Namely $(\dfrac{a^2+b^2}{2})^{\frac{n}{2}} \geq (\dfrac{a+b+\sqrt{a^2+b^2}}{2+\sqrt{2}})^n$, which equals

$$\frac{a^2+b^2}{2} \geq (\frac{a+b+\sqrt{a^2+b^2}}{2+\sqrt{2}})^2$$, which equals

$$\frac{a^2+b^2}{2} \geq (\frac{a+b}{2})^2,$$

From Lemma 5, we know it's tenable.

3.8 the proof of Theorem 8

Assume that $a_1 \leq a_2 \leq \cdots \leq a_m < a_{m+1}$,

When $k \geq t > 0$, we know $a_1^k \leq a_2^k \leq \cdots \leq a_m^k < a_{m+1}^k$, $a_1^t \leq a_2^t \leq \cdots \leq a_m^t < a_{m+1}^t$;

When $t \leq k < 0$, we know $a_1^k \geq a_2^k \geq \cdots \geq a_m^k > a_{m+1}^k$, $a_1^t \geq a_2^t \geq \cdots \geq a_m^t > a_{m+1}^t$.

Know from Lemma 4,

$$\sum_{i=1}^{m+1} a_i^{k+t} = \sum_{i=1}^{m} a_i^{k+t} + a_{m+1}^{k+t} \geq \frac{1}{m}(\sum_{i=1}^{m} a_i^k)(\sum_{i=1}^{m} a_i^t) + a_{m+1}^k \cdot a_{m+1}^t$$

$$= \frac{1}{m}(\sum_{i=1}^{m} a_i^k)(\sum_{i=1}^{m} a_i^t) + \sum_{i=1}^{m} a_i^k \cdot a_{m+1}^t$$

$$= (\sum_{i=1}^{m} a_i^k)(\frac{1}{m}\sum_{i=1}^{m} a_i^t + a_{m+1}^t)$$

$$= 2\sum_{i=1}^{m} a_i^k (\frac{1}{2m}\sum_{i=1}^{m} a_i^t + \frac{1}{2}a_{m+1}^t)$$

$$= (\sum_{i=1}^{m} a_i^k + a_{m+1}^k)(\frac{1}{2m}\sum_{i=1}^{m} a_i^t + \frac{1}{2}a_{m+1}^t)$$

$$=\sum_{i=1}^{m+1} a_i{}^k \left(\frac{1}{2m}\sum_{i=1}^{m} a_i{}^t + \frac{1}{2}a_{m+1}{}^t\right).$$

To this end, to prove theorem, just to prove

$$\frac{1}{2m}\sum_{i=1}^{m} a_i{}^t + \frac{1}{2}a_{m+1}{}^t \geq \frac{1+m^{\frac{t}{k}}}{2(m+m^{\frac{t}{k}})}\sum_{i=1}^{m+1} a_i{}^t$$,is to prove

$$\frac{1}{m}\sum_{i=1}^{m} a_i{}^t + a_{m+1}{}^t \geq \frac{1+m^{\frac{t}{k}}}{m+m^{\frac{t}{k}}}\left(\sum_{i=1}^{m} a_i{}^t + a_{m+1}{}^t\right)$$

$$\Leftrightarrow \left(1-\frac{1+m^{\frac{t}{k}}}{m+m^{\frac{t}{k}}}\right)a_{m+1}{}^t \geq \left(\frac{1+m^{\frac{t}{k}}}{m+m^{\frac{t}{k}}}-\frac{1}{m}\right)\sum_{i=1}^{m} a_i{}^t$$

$$\Leftrightarrow \frac{m-1}{m+m^{\frac{t}{k}}}a_{m+1}{}^t \geq \frac{(m-1)m^{\frac{t}{k}}}{m(m+m^{\frac{t}{k}})}\sum_{i=1}^{m} a_i{}^t$$

$$\Leftrightarrow a_{m+1}{}^t \geq \frac{m^{\frac{t}{k}}}{m}\sum_{i=1}^{m} a_i{}^t$$

$$\Leftrightarrow (a_{m+1}{}^k)^{\frac{t}{k}} \geq \frac{m^{\frac{t}{k}}}{m}\sum_{i=1}^{m} a_i{}^t$$

$$\Leftrightarrow \left(\sum_{i=1}^{m} a_i{}^k\right)^{\frac{t}{k}} \geq \frac{m^{\frac{t}{k}}}{m}\sum_{i=1}^{m} a_i{}^t$$

$$\Leftrightarrow \left(\frac{1}{m}\sum_{i=1}^{m} a_i{}^k\right)^{\frac{1}{k}} \geq \left(\frac{1}{m}\sum_{i=1}^{m} a_i{}^t\right)^{\frac{1}{t}}.$$

Finally a type shall lemma 5, theorem 8 may permit.

4 Application

With the conclusions above, we can solve many relevant problems. For example, Theorem 2 can make $C_4^2 = 6$ inequalities and many problems can be quickly solved.

With the inequality chain of **Theorem 2**, we can get $C_4^2 = 6$ inequalities, which are applied in many problems.

Example 1[4]：to get the maximum which enables every set of the 3 triangular sides to make $a^3+b^3+c^3 \geq K(a+b+c)^3$ tenable.（***2006 Iranian Maths Olympic***）

Analysis：If we use regular methods, we need to substitute the parameters with triangular ones. With Theorem 1, the problem will be solved.

Key: $K_{\max} = \dfrac{3\sqrt{2}-4}{2}$.

Proof: solution 1：first consider c as the hypotenuse.

From the inequality of **Theorem 2** $a^3+b^3+c^3 \geq \frac{3\sqrt{2}-4}{2}(a+b+c)^3$， it is clear that this inequality is tenable. Only when $a=b$ is the equal-sign tenable. $K_{\max}=\frac{3\sqrt{2}-4}{2}$.

Solution 2：using regular methods, as we prove Theorem 1, we need to substitute the parameters with triangular ones twice.

Example 2[5]：Known that a,b,c are positive numbers that make $a^2+b^2=c^2$ tenable.

Try to prove：

$$a^3+b^3+c^3 \geq \frac{2\sqrt{2}+1}{7}[a(b^2+c^2)+b(a^2+c^2)+c(a^2+b^2)] .$$

（Maths Olympics problem, senior 176，《Medium Mathematics》2006.4.）

Proof：**solution 1**: From the inequality of the lemma, we can know

$a^3+b^3+c^3 \geq \frac{2\sqrt{2}+1}{7}[a(b^2+c^2)+b(a^2+c^2)+c(a^2+b^2)]$ is tenable. Only when $a=b$ is the equal-sign tenable.

Solution 2：using regular methods, as we prove Theorem 1, we need to substitute the parameters with triangular ones twice.

Example 3[6]：supposing a,b,c are 3 triangular sides and c is the hypotenuse.Try to get the maximum of k that makes $\frac{a^3+b^3+c^3}{abc} \geq k$ tenable. ***（test questions of the 4th Northern China Math Olympic 2008）***

Key: $K_{\max}=2+\sqrt{2}$.

Proof ：**solution 1**：from the proved inequality, we know that

$a^3+b^3+c^3 \geq (2+\sqrt{2})abc$， Only when $a=b$ is the equal-sign tenable.

Solution 2：using substitution of the parameters with triangular ones

Example 4：supposing $a \leq b \leq c$ and they are 3 triangular sides of $Rt\Delta ABC$ .Try to get the maximum M that makes

$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{M}{a+b+c}$ tenable eternally. ***（test questions of national maths training team of China 1991）***

key： $M_{\max}=2+\sqrt{3}$.

Proof ：the original inequality equals $(a+b+c)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}) \geq 2+\sqrt{3}$，

$$\Leftrightarrow \frac{(bc+ac+ab)(a+b+c)}{abc} \ge 2+\sqrt{3},$$

$$\Leftrightarrow \frac{a(b^2+c^2)+b(a^2+c^2)+c(a^2+b^2)}{abc} \ge 2+\sqrt{3},$$

From the inequality of Theorem 1

$$\frac{2\sqrt{2}+1}{7}[a(b^2+c^2)+b(a^2+c^2)+c(a^2+b^2)] \ge (2+\sqrt{2})abc,$$ It is known that

$$\frac{a(b^2+c^2)+b(a^2+c^2)+c(a^2+b^2)}{abc} \ge \frac{2+\sqrt{2}}{\frac{2\sqrt{2}+1}{7}} = 2+\sqrt{3}.$$

So $M_{\max} = 2+\sqrt{3}$. Only when $a=b$ is the equal-sign tenable.

Example 5: shown in Picture 1, $P(x,y)$ is a dot of astroid whose formula is $x^{2/3}+y^{2/3}=a^{2/3}(a>0)$. Try to find the minimum of x^2+y^2.

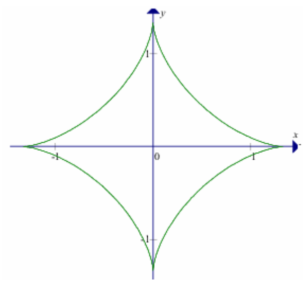


Fig.1

Solution: from Theorem 6, we know

$$a^6+b^6+c^6 \ge \frac{5}{12}(a^2+b^2+c^2)(a^4+b^4+c^4),$$

From Theorem 4, we know $a^4+b^4+c^4 \ge \frac{3}{8}(a^2+b^2+c^2)^2$,

Therefore, $a^6+b^6+c^6 \ge \frac{5}{32}(a^2+b^2+c^2)^3$, Let $a=x^{1/3}, b=y^{1/3}, c=a^{1/3}$, then

$x^2+y^2+a^2 \ge \frac{5}{32}(x^{2/3}+y^{2/3}+a^{2/3})^3 = \frac{5}{32}(2a^{2/3})^3 = \frac{5}{4}a^2$, then $x^2+y^2 \ge \frac{1}{4}a^2$..

So the minimum of x^2+y^2 is $\frac{1}{4}a^2$.

Appedix: Star line is one of hypocycloid within the cycloid (circle spiral), n is 3. a hypocycloid (Circle spiral) is All in the form of Curve : $\begin{cases} x=\cos t+\frac{1}{n}\cos nt, \\ y=\sin t-\frac{1}{n}\sin nt. \end{cases}$ Where n is positive real number.

Example 6: Shown in picture 2 is the emblem of the 7th International Conference of Mathematical Education (ICME for short). Its body is developed by a series of right-angled triangles in picture 3. And $OA_1=A_1A_2=A_2A_3=\ldots A_7A_8=1$, if we keep making triangles that way and call off its limit of $OA_1=A_1A_2=\cdots=A_nA_{n+1}=1$, we get what is in picture 4, and let

$OA_1 = a_1, A_1A_2 = a_2, \cdots, A_nA_{n+1} = a_n, A_{n+1}O = a_{n+1}$, try to prove:

$$a_1^4 + a_2^4 + \cdots + a_n^4 + a_{n+1}^4 \geq \frac{n+1}{4n}(a_1^2 + a_2^2 + \cdots + a_n^2 + a_{n+1}^2)^2.$$

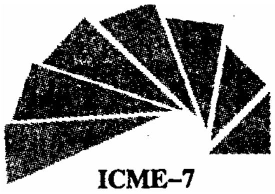


Fig.2

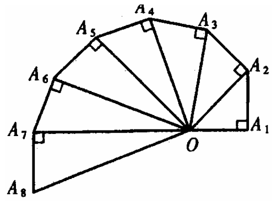


Fig.3

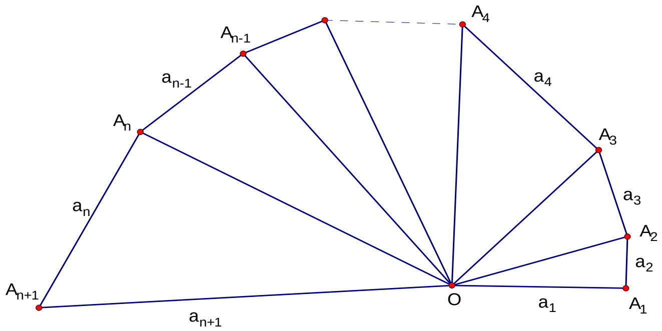


Fig.5

Proof: In picture 6, from Pythagorean, we know

$$OA_i^2 + A_iA_{i+1}^2 = OA_{i+1}^2 \quad (i = 1, 2, \cdots, n),$$

Add the equations with i from 1 to n and we get

$OA_1^2 + A_1A_2^2 + A_2A_3^2 \cdots + A_nA_{n+1}^2 = OA_{n+1}^2$, then

$$a_1^2 + a_2^2 + \cdots + a_n^2 = a_{n+1}^2.$$

In theorem 8, let $k = t = 2$ then

$$a_1^4 + a_2^4 + \cdots + a_n^4 + a_{n+1}^4 \geq \frac{n+1}{4n}(a_1^2 + a_2^2 + \cdots + a_n^2 + a_{n+1}^2)^2.$$

Example 7: In the cuboid shown in Picture 6, the length is a ,the width b, the height c and the body diagonal d. Try to prove:

$$a^4 + b^4 + c^4 + d^4 \geq \frac{1}{3}(a^2 + b^2 + c^2 + d^2)^2.$$

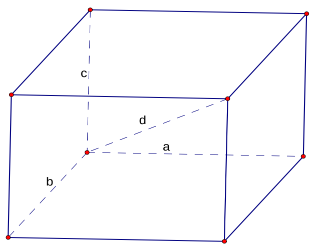


Fig.6

Proof : From the fact that the square of the body diagonal equals the sum of the squares of the length, width and height, we get $a^2 + b^2 + c^2 = d^2$.

In Theorem 8, let $m = 3$,

$a_1 = a, a_2 = b, a_3 = c, a_4 = d$. So

$$a^4 + b^4 + c^4 + d^4 \geq \frac{1}{3}(a^2 + b^2 + c^2 + d^2)^2 .$$

Example 8: In the cuboid shown in Picture 7，the angles that the body diagonal is at to the 3 surfaces which are vertical to each are α、β、γ ,Try to prove $\cos^4\alpha + \cos^4\beta + \cos^4\gamma \geq \frac{4}{3}$.

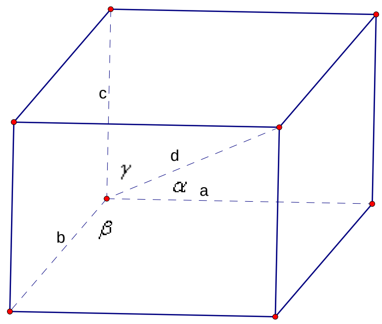


Fig.7

Proof: Make the length a, the width b, the height c, the body diagonal d, then $a^2 + b^2 + c^2 = d^2$,

$\cos\alpha = \frac{\sqrt{b^2 + c^2}}{d}$, $\cos\beta = \frac{\sqrt{c^2 + a^2}}{d}$,

$\cos\gamma = \frac{\sqrt{a^2 + b^2}}{d}$， then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{b^2 + c^2}{d^2} + \frac{c^2 + a^2}{d^2} + \frac{a^2 + b^2}{d^2} = \frac{2(a^2 + b^2 + c^2)}{d^2} = 2 ,$$

In theorem 8，let $m = 3$， $a_1 = \cos\alpha, a_2 = \cos\beta, a_3 = \cos\gamma, a_4 = \sqrt{2}$,then

$$\cos^4\alpha + \cos^4\beta + \cos^4\gamma + (\sqrt{2})^4 \geq 2[\cos^2\alpha + \cos^2\beta + \cos^2\gamma + (\sqrt{2})^2]^2 = 16$$,then

$$\cos^4\alpha + \cos^4\beta + \cos^4\gamma \geq \frac{4}{3}.$$

Only when $\cos\alpha = \cos\beta = \cos\gamma = \frac{\sqrt{6}}{3}$ is the equal-sign tenable.

Example9: If S—ABC is a triangular pyramid whose sides are vertical to each other，O is a dot in surface ABC，if ∠OSA= α ,∠OSB= β,∠OSC= γ， try to get the minimum of $\cos^4\alpha + \cos^4\beta + \cos^4\gamma$.

(adapted from Hunan mathematics competitions 2003)

Solution: Make a cube using $\overrightarrow{SA}$、$\overrightarrow{SB}$、$\overrightarrow{SC}$ as the basic vectors，and SO as the body diagonal，then the direction angles that SO is at to SA、SB、SC are α、β、γ, let the length a, the width b, the height c, the body diagonal d, then $a^2 + b^2 + c^2 = d^2$ ， and $\cos\alpha = \frac{a}{d}, \cos\beta = \frac{b}{d}, \cos\gamma = \frac{c}{d}$， then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{a^2}{d^2} + \frac{b^2}{d^2} + \frac{c^2}{d^2} = \frac{a^2 + b^2 + c^2}{d^2} = 1$$， then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 .$$

In theorem 8, let $m=3$, $a_1=\cos\alpha, a_2=\cos\beta, a_3=\cos\gamma, a_4=1$ then

$$\cos^4\alpha+\cos^4\beta+\cos^4\gamma+1^4 \ge \frac{1}{3}[\cos^2\alpha+\cos^2\beta+\cos^2\gamma+1^2]^2=\frac{4}{3},$$ then

$$\cos^4\alpha+\cos^4\beta+\cos^4\gamma \ge \frac{1}{3}.$$

Only when $\cos\alpha=\cos\beta=\cos\gamma=\frac{\sqrt{3}}{3}$ is the equal-sign tenable.

So, the minimum of $\cos^4\alpha+\cos^4\beta+\cos^4\gamma$ is $\frac{1}{3}$.

Example 10 In the geometry in planes, there is Pythagorean: 'If AC is vertical to BC in ΔABC, and a,b,c are length of 3 sides opposite A、B and C, then $a^2+b^2=c^2$'. Extended to space, in comparison with it, a correct conclusion that the areas of side surface S_1,S_2,S_3 and the bottom surface S_4 can be drawn in the right-angled tetrahedron,

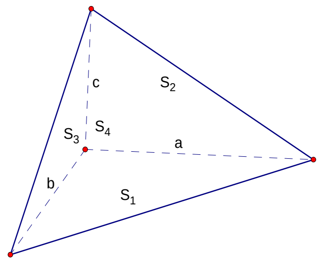


Fig.8

$$S_1^2+S_2^2+S_3^2=S_4^2.$$

(1) Try to prove: $S_1^2+S_2^2+S_3^2=S_4^2$;

(2) Try to prove: $S_1^4+S_2^4+S_3^4+S_4^4 \ge \frac{1}{3}(S_1^2+S_2^2+S_3^2+S_4^2)^2$.

(adapted from 2003 Xinkechengjuan fill-in adaptation)

Proof: (1) Shown in picture 8, from Helen's Formula,

$$S_4^2=p'(p'-a')(p'-b')(p'-c')$$

$$=\frac{1}{2}(a'+b'+c')\cdot\frac{1}{2}(-a'+b'+c')\cdot\frac{1}{2}(a'-b'+c')\cdot\frac{1}{2}(a'+b'-c')$$

$$=\frac{1}{16}[(a'+b'+c')\cdot(-a'+b'+c')]\cdot[(a'-(b'-c')]\cdot[a'+(b'-c')]$$

$$=\frac{1}{16}[(a'^2-(b'+c')^2]\cdot[(a'^2-(b'-c')^2]$$

$$=\frac{1}{16}(a'^2b'^2+b'^2c'^2+c'^2a'^2-a'^4-b'^4-c'^4)^2$$

$$=\frac{1}{16}[(a^2+b^2)(b^2+c^2)+(b^2+c^2)(c^2+a^2)+(c^2+a^2)(a^2+b^2)$$

$$-(a^2+b^2)^2-(b^2+c^2)^2-(c^2+a^2)^2]$$

$$=\frac{1}{4}(a^2b^2+b^2c^2+c^2a^2)=S_1^2+S_2^2+S_3^2$$

$(\frac{S_1}{S_4})^2+(\frac{S_2}{S_4})^2+(\frac{S_3}{S_4})^2=1$, simplify it this way: $S_1^{\ 2}+S_2^{\ 2}+S_3^{\ 2}=S_4^{\ 2}$.

（2）In theorem 8, let $m=3$, $a_1=S_1, a_2=S_2, a_3=S_3, a_4=S_4$ then

$S_1^{\ 4}+S_2^{\ 4}+S_3^{\ 4}+S_4^{\ 4}\geq\frac{1}{3}(S_1^{\ 2}+S_2^{\ 2}+S_3^{\ 2}+S_4^{\ 2})^2$.

Example 11 In the first of Octant shown in picture 4,on Unit ball there is a dot $M(x,y,z)$,try to get the minimum of $\frac{1+x^3+y^3+z^3}{1+x+y+z}$.

Solution: From the definition of a globe, we know $|OM|=1$, then

$\sqrt{x^2+y^2+z^2}=1$, then $x^2+y^2+z^2=1$.

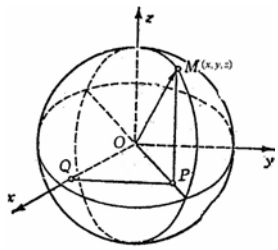


Fig.9

In theorem 8, let $m=3, k=2, t=1$,

$a_1=x, a_2=y, a_3=z, a_4=1$, then

$$1+x^3+y^3+z^3\geq\frac{1+3^{\frac{1}{2}}}{2(3+3^{\frac{1}{2}})}(1+x^2+y^2+z^2)(1+x+y+z),$$

then

$$\frac{1+x^3+y^3+z^3}{1+x+y+z}\geq\frac{1}{2\sqrt{3}}(1+x^2+y^2+z^2)=\frac{\sqrt{3}}{3}.$$

Only when $x=y=z=\frac{\sqrt{3}}{3}$ is the equal-sign tenable.

So the minimum of $\frac{1+x^3+y^3+z^3}{1+x+y+z}$ is $\frac{\sqrt{3}}{3}$.

例 **12** shown in Picture 10 is a ellipsoid whose standard formula is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, to any dot on ellipsoid $M(x_0,y_0,z_0)$, try to get the minimum of $\frac{x_0^{\ 4}}{a^4}+\frac{y_0^{\ 4}}{b^4}+\frac{z_0^{\ 4}}{c^4}$.

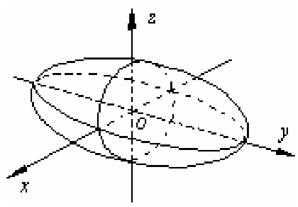


Fig.10

Solution: From the problem, we know $\frac{x_0^{\ 2}}{a^2}+\frac{y_0^{\ 2}}{b^2}+\frac{z_0^{\ 2}}{c^2}=1$.

In Theorem 8, let $m=4,k=2,t=2$， $a_1=\frac{|x_0|}{a},a_2=\frac{|y_0|}{b},a_3=\frac{|z_0|}{c},a_4=1$， then

$\frac{{x_0}^4}{a^4}+\frac{{y_0}^4}{b^4}+\frac{{z_0}^4}{c^4}+1\ge\frac{1}{3}(\frac{{x_0}^2}{a^2}+\frac{{y_0}^2}{b^2}+\frac{{z_0}^2}{c^2}+1)^2$， namely

$\frac{{x_0}^4}{a^4}+\frac{{y_0}^4}{b^4}+\frac{{z_0}^4}{c^4}+1\ge\frac{4}{3}$， which equals $\frac{{x_0}^4}{a^4}+\frac{{y_0}^4}{b^4}+\frac{{z_0}^4}{c^4}\ge\frac{1}{3}$.

So the minimum of $\frac{{x_0}^4}{a^4}+\frac{{y_0}^4}{b^4}+\frac{{z_0}^4}{c^4}$ is $\frac{1}{3}$.

例 **13** shown in Picture 11 is a hyperboloid of one sheet whose standard formula is $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ ， to any dot on hyperboloid of one sheet $N(x_0,y_0,z_0)$， try to prove：

$$\frac{{x_0}^6}{a^6}+\frac{{y_0}^6}{b^6}+\frac{(c^2+{z_0}^2)^3}{c^6}\ge\frac{5}{32}(\frac{{x_0}^2}{a^2}+\frac{{y_0}^2}{b^2}+\frac{{z_0}^2}{c^2}+1)^3.$$

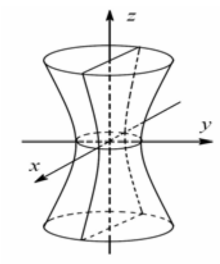


Fig.11

solution：from the problem，we know $\frac{{x_0}^2}{a^2}+\frac{{y_0}^2}{b^2}=\frac{c^2+{z_0}^2}{c^2}$.

In Theorems 6 and 3，

let $a=\frac{|x_0|}{a},b=\frac{|y_0|}{b},c=\frac{\sqrt{c^2+{z_0}^2}}{c}$， then

$$\frac{{x_0}^6}{a^6}+\frac{{y_0}^6}{b^6}+\frac{(c^2+{z_0}^2)^3}{c^6}=\frac{{x_0}^6}{a^6}+\frac{{y_0}^6}{b^6}+\frac{(\sqrt{c^2+{z_0}^2})^6}{c^6}$$

$$\ge\frac{5}{12}[\frac{{x_0}^4}{a^4}+\frac{{y_0}^4}{b^4}+\frac{(\sqrt{c^2+{z_0}^2})^4}{c^4}][\frac{{x_0}^2}{a^2}+\frac{{y_0}^2}{b^2}+\frac{(\sqrt{c^2+{z_0}^2})^2}{c^2}]$$

$$\ge\frac{5}{12}\cdot\frac{3}{8}[\frac{{x_0}^2}{a^2}+\frac{{y_0}^2}{b^2}+\frac{(\sqrt{c^2+{z_0}^2})^2}{c^2}]^2[\frac{{x_0}^2}{a^2}+\frac{{y_0}^2}{b^2}+\frac{(\sqrt{c^2+{z_0}^2})^2}{c^2}]$$

$$=\frac{5}{32}[\frac{{x_0}^2}{a^2}+\frac{{y_0}^2}{b^2}+\frac{c^2+{z_0}^2}{c^2}]^3$$

$$=\frac{5}{32}(\frac{{x_0}^2}{a^2}+\frac{{y_0}^2}{b^2}+\frac{c^2+{z_0}^2}{c^2})^3.$$

Besides, we may find out the applications of the theorems in ellipse,hyperbola,hyperboloid pairs**:**

$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$ ($a,b,c>0$), elliptic cone: $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0 \Leftrightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$ and so on. Due to the limit of size, we can't illustrate more..

5 Two Conjecture

To prove Theorems 1-6， we use maple 12 and Bottema2009,and its operation is as follows：

```
> read "bottema2009";
> a^(m+n)+b^(m+n)+c^(m+n)-(2+sqrt(2^(m+n)))/(2+sqrt(2^m))/(2+sqrt(2^n))*(a^m+b^m+c^m)*(a^n+b^n+c^n)>=0;
```

$$0 \le a^{(m+n)}+b^{(m+n)}+c^{(m+n)}-\frac{(2+\sqrt{2^{(m+n)}})(a^m+b^m+c^m)(a^n+b^n+c^n)}{(2+\sqrt{2^m})(2+\sqrt{2^n})}$$

```
> subs(c=1,a=cos(A),b=sin(A),%);
```

$$0 \le \cos(A)^{(m+n)}+\sin(A)^{(m+n)}+1 - \frac{(2+\sqrt{2^{(m+n)}})(\cos(A)^m+\sin(A)^m+1)(\cos(A)^n+\sin(A)^n+1)}{(2+\sqrt{2^m})(2+\sqrt{2^n})}$$

```
> subs(m=3,n=2,%);
```

$$0 \le \cos(A)^5+\sin(A)^5+1-\frac{(2+\sqrt{32})(\cos(A)^3+\sin(A)^3+1)(\cos(A)^2+\sin(A)^2+1)}{(2+\sqrt{8})(2+\sqrt{4})}$$

```
> prove(%,[aa]);
```

$$\left[0 \le \cos(A)^5+\sin(A)^5+1 - \frac{(2+\sqrt{32})(\cos(A)^3+\sin(A)^3+1)(\cos(A)^2+\sin(A)^2+1)}{(2+\sqrt{8})(2+\sqrt{4})}, 0 \le 2s-p-10\right]$$

$$\left[0 \le \cos(A)^5+\sin(A)^5+1 - \frac{(2+\sqrt{32})(\cos(A)^3+\sin(A)^3+1)(\cos(A)^2+\sin(A)^2+1)}{(2+\sqrt{8})(2+\sqrt{4})}, 0 \le 2s-p-10\right]$$

Found border curves..

$$(x-1)(y-1)(xy-y-x-1)(xy-1)(x^8+4x^6+22x^4-64x^3+100x^2-64x+17)(x^2-2x-1)xy$$

Start to project curves.. , 73.250

$[y, x]$

do 1-th partition...

Start to find the sample points. , 73.296

in 1-dimensional space....

finished in 1-dimensional space.

in 2-dimensional space....

finished in 2-dimensional space.

number(s) of sample points: , 2, 73.421

$$[y, x]$$

$$\left[\left[2, \frac{5}{3}\right], \left[\frac{3}{2}, 3\right]\right]$$

$$-\frac{2212000}{1419857} + \frac{35200}{4913\,(2+\sqrt{8})\,(2+\sqrt{4})} + \frac{17600\sqrt{32}}{4913\,(2+\sqrt{8})\,(2+\sqrt{4})}$$

OK

$$-\frac{4392}{3125} + \frac{864}{125\,(2+\sqrt{8})\,(2+\sqrt{4})} + \frac{432\sqrt{32}}{125\,(2+\sqrt{8})\,(2+\sqrt{4})}$$

OK

The inequality holds !

$$0 \le \left(1 - \frac{2y}{(xy-1)\left(\frac{x+y}{xy-1}+x\right)}\right)^5 + \frac{32\left(y+\frac{x+y}{xy-1}\right)^5}{\left(\frac{y(x+y)}{xy-1}+\frac{(x+y)x}{xy-1}+xy-1\right)^5} + 1$$

$$-\frac{2\left(1 - \frac{2y}{(xy-1)\left(\frac{x+y}{xy-1}+x\right)}\right)^5}{(2+\sqrt{8})\,(2+\sqrt{4})}$$

$$-\frac{8\left(1 - \frac{2y}{(xy-1)\left(\frac{x+y}{xy-1}+x\right)}\right)^3\left(y+\frac{x+y}{xy-1}\right)^2}{(2+\sqrt{8})\,(2+\sqrt{4})\left(\frac{y(x+y)}{xy-1}+\frac{(x+y)x}{xy-1}+xy-1\right)^2}$$

$$-\frac{2\left(1 - \frac{2y}{(xy-1)\left(\frac{x+y}{xy-1}+x\right)}\right)^3}{(2+\sqrt{8})\,(2+\sqrt{4})}$$

$$-\frac{16\left(y+\frac{x+y}{xy-1}\right)^3\left(1-\frac{2y}{(xy-1)\left(\frac{x+y}{xy-1}+x\right)}\right)^2}{(2+\sqrt{8})(2+\sqrt{4})\left(\frac{y(x+y)}{xy-1}+\frac{(x+y)x}{xy-1}+xy-1\right)^3}$$

$$-\frac{64\left(y+\frac{x+y}{xy-1}\right)^5}{(2+\sqrt{8})(2+\sqrt{4})\left(\frac{y(x+y)}{xy-1}+\frac{(x+y)x}{xy-1}+xy-1\right)^5}$$

$$-\frac{16\left(y+\frac{x+y}{xy-1}\right)^3}{(2+\sqrt{8})(2+\sqrt{4})\left(\frac{y(x+y)}{xy-1}+\frac{(x+y)x}{xy-1}+xy-1\right)^3}$$

$$-\frac{2\left(1-\frac{2y}{(xy-1)\left(\frac{x+y}{xy-1}+x\right)}\right)^2}{(2+\sqrt{8})(2+\sqrt{4})}$$

$$-\frac{8\left(y+\frac{x+y}{xy-1}\right)^2}{(2+\sqrt{8})(2+\sqrt{4})\left(\frac{y(x+y)}{xy-1}+\frac{(x+y)x}{xy-1}+xy-1\right)^2}-\frac{2}{(2+\sqrt{8})(2+\sqrt{4})}$$

$$-\frac{\sqrt{32}\left(1-\frac{2y}{(xy-1)\left(\frac{x+y}{xy-1}+x\right)}\right)^5}{(2+\sqrt{8})(2+\sqrt{4})}$$

$$-\frac{4\sqrt{32}\left(1-\frac{2y}{(xy-1)\left(\frac{x+y}{xy-1}+x\right)}\right)^3\left(y+\frac{x+y}{xy-1}\right)^2}{(2+\sqrt{8})(2+\sqrt{4})\left(\frac{y(x+y)}{xy-1}+\frac{(x+y)x}{xy-1}+xy-1\right)^2}$$

$$-\frac{\sqrt{32}\left(1-\frac{2y}{(xy-1)\left(\frac{x+y}{xy-1}+x\right)}\right)^3}{(2+\sqrt{8})(2+\sqrt{4})}$$

$$-\frac{8\sqrt{32}\left(y+\frac{x+y}{xy-1}\right)^3\left(1-\frac{2y}{(xy-1)\left(\frac{x+y}{xy-1}+x\right)}\right)^2}{(2+\sqrt{8})(2+\sqrt{4})\left(\frac{y(x+y)}{xy-1}+\frac{(x+y)x}{xy-1}+xy-1\right)^3}$$

$$-\frac{32\sqrt{32}\left(y+\frac{x+y}{xy-1}\right)^5}{(2+\sqrt{8})(2+\sqrt{4})\left(\frac{y(x+y)}{xy-1}+\frac{(x+y)x}{xy-1}+xy-1\right)^5}$$

$$-\frac{8\sqrt{32}\left(y+\frac{x+y}{xy-1}\right)^3}{(2+\sqrt{8})(2+\sqrt{4})\left(\frac{y(x+y)}{xy-1}+\frac{(x+y)x}{xy-1}+xy-1\right)^3}$$

$$-\frac{\sqrt{32}\left(1-\frac{2y}{(xy-1)\left(\frac{x+y}{xy-1}+x\right)}\right)^2}{(2+\sqrt{8})(2+\sqrt{4})}$$

$$-\frac{4\sqrt{32}\left(y+\frac{x+y}{xy-1}\right)^2}{(2+\sqrt{8})(2+\sqrt{4})\left(\frac{y(x+y)}{xy-1}+\frac{(x+y)x}{xy-1}+xy-1\right)^2}-\frac{\sqrt{32}}{(2+\sqrt{8})(2+\sqrt{4})}$$

when

$$0\le\frac{y(x+y)x}{xy-1}-xy-\frac{(x+y)x}{xy-1}+x-\frac{y(x+y)}{xy-1}+y+\frac{x+y}{xy-1}-1,\ 1<xy$$

And we have every reason to provide the following 2 conjectures for readers.

Conjecture 1: In $Rt\Delta ABC$, $a^2+b^2=c^2$, $m\ge 2\ge n>0$, then

$$a^{m+n}+b^{m+n}+c^{m+n}\ge\frac{2+\sqrt{2^{m+n}}}{(2+\sqrt{2^m})(2+\sqrt{2^n})}(a^m+b^m+c^m)(a^n+b^n+c^n).$$

More generally, what if it's a problem about dimensions of $n(n\ge 3)$?

Conjecture 2: In simple forms of $m(m\ge 3)$ dimensions,

$a_i\in R^+, i=1,2,\cdots,m+1$, and $\sum_{i=1}^{m}a_i^{\,k}=a_{m+1}^{\,k}$, $n\ge k\ge t>0$, or $n\le k\le t<0$, then

$$\sum_{i=1}^{m+1}a_i^{\,n+t}\ge\frac{m+m^{\frac{n+t}{k}}}{(m+m^{\frac{n}{k}})(m+m^{\frac{t}{k}})}\left(\sum_{i=1}^{m+1}a_i^{\,n}\right)\left(\sum_{i=1}^{m+1}a_i^{\,t}\right).$$

6 Conclusion and Prospect

Pythagorean is an ancient but charming research subject which people spend much time and energy to study. The problems about the sum of equal power of natural numbers are also heat topics and being studied. So we firmly believe that problems about the sum of equal powers about 3 triangular sides are also new subjects with bright future and broad prospects. So we establish this subject. It's delightful that we can get a series of propositions about the inequality of the sum of 3 right-angled triangular sides through great efforts.

As for the application of these theorems, we can discover their value in geometries like right-angled triangle, rectangle, round, ellipse, hyperbola, cube,hypocycloid,right-angled triangular pyramid, globe, ellipsoid, hyperboloid of two sheet, elliptic cone and so on.

Although our conjecture 1 and 2 remains to be solved, we believe that we can figure them out through protracted and unremitting efforts and we hope to extend it to simplex of $m(m \geq 3)$ dimensions.

At present, we have three thoughts of solution toward this problem: The first is to use the mathematical induction of primary mathematics; The second is to use step-by-step adjustment in math competition; The third is to use higher mathematics in the number of Lagrange's least squares method to deal with. However, these 3 methods require much operation.

7 Thanks

Shing-Tung Yau we first participated in Grand Middle School Mathematics Prize Papers Race, given our limited ability, lack of papers there are many Where the judges were invited to the exhibitions.

Like to take this opportunity, we sincerely thank you for helped us The experts and professors, teachers and students, as well as to our support Holding parents.

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