A PROPERTY OF THE SKEIN POLYNOMIAL WITH AN APPLICATION TO CONTACT GEOMETRY

A. Stoimenow

Abstract

We prove a finiteness property of the values of the skein polynomial of homogeneous knots that allows us to establish large classes of such knots to have arbitrarily unsharp Bennequin inequality (for the Thurston-Bennequin invariant of any of their Legendrian embeddings in the standard contact structure of $\mathbb{R}^3$). We also give a short proof that there are only finitely many such knots that have given genus and given braid index.

1. Introduction

In this paper, we will show the following result on the skein (or HOMFLY) polynomial [**F+**].

Theorem 1. *The set*

$$\{P_K \,:\, \operatorname{span}_l P_K \leq b, \, \tilde{g}(K) \leq g\}$$

is finite for any natural numbers g and b.

Here $\operatorname{span}_l P_K$ is the span of the HOMFLY polynomial $P_K = P(K)$ of a knot K in the (non-Alexander) variable l, that is, the difference between its minimal and maximal degree in l, $\min \deg_l(P_K)$ and $\max \deg_l(P_K)$. By $\tilde{g}(K)$ we denote the weak genus of K [**St**].

The main application of this theorem is to exhibit large families of knots to have arbitrarily unsharp Bennequin inequality for any of their realizations as topological knot types of a Legendrian knot, which simplifies and extends the main result of Kanda [**K**] and its alternative proofs given by Fuchs–Tabachnikov [**FT**] and Dasbach-Mangum [**DM**].

Corollary 1. *Bennequin's inequality becomes arbitrarily unsharp on any sequence of (Legendrian embeddings of distinct) properly obversed (mirrored) homogeneous knots.*

The main tool we use for the proof of Theorem 1 is the result of [**St**] and an analysis of the skein (HOMFLY) polynomial [**F+**]. The proof of Corollary 1 uses the inequality, known from work of Tabachnikov [**T**, **FT**], relating the Thurston–Bennequin and Maslov (rotation) number of Legendrian knots to the minimal degree of their skein polynomial. This inequality suggests that one should look at knots behaving "nicely" with respect to their skein polynomial. The homogeneous knots introduced by Cromwell in [**Cr**] are, in some sense, the largest class of such knots. These are the knots having homogeneous diagrams, that is, diagrams containing in each connected component (block) of the complement of their Seifert (circle) picture only crossings of the same sign. This class contains the classes of alternating and positive/ negative knots.

The other application we give of Theorem 1 is also related to Bennequin's paper [**Be**] and the work of Birman and Menasco building on it. In their paper [**BM**], its referee made the observation that there are only finitely many knots given genus and given braid index (Theorem 2). This fact came as a bi-product of the work of the authors on braid foliations introduced in Bennequin's paper and is based on rather deep theory. Here we will use our work in [**St**] to give a simple, because entirely combinatorial, proof of a generalization of this result for homogeneous knots. In fact, we show that the lower bound for the braid index coming from the inequality of Franks–Williams [**FW**] and Morton [**Mo**] gets arbitrarily large for homogeneous knots of given genus (Corollary 3). The result of [**BM**] for homogeneous knots is then a formal consequence of ours.

Corollary 2. *There are only finitely many homogeneous knots K of given genus $g(K)$ and given braid index $b(K)$.*

We should remark that the corollary will straightforwardly generalize to links. The arguments we will give apply for links of any given (fixed) number of components, and clearly a link of braid index n has at most n components.

Since this paper was originally written (summer 2000), more work was done on the subject, including by Etnyre, Honda, Ng, and in particular Plamenevskaya [**Pl**]. A recent survey can be found in [**Et**].

2. Knot-theoretic preliminaries

The *skein* (*HOMFLY*) *polynomial* [1] P is a Laurent polynomial in two variables l and m of oriented knots and links, and can be defined by being 1 on the unknot and the (skein) relation

(1) $$l^{-1} P(\times) + l\, P(\times) = -m\, P()().$$

This convention uses the variables of Lickorish and Millett [**LM**], but differs from theirs by the interchange of l and l^{-1}.

Let $[Y]_{t^a} = [Y]_a$ denote the *coefficient* of t^a in a polynomial $Y \in \mathbb{Z}[t^{\pm 1}]$. For $Y \neq 0$, let $\mathcal{C}_Y = \{a \in \mathbb{Z} : [Y]_a \neq 0\}$ and define

$$\min \deg Y = \min \mathcal{C}_Y, \quad \max \deg Y = \max \mathcal{C}_Y, \qquad \text{and}$$
$$\operatorname{span} Y = \max \deg Y - \min \deg Y$$

to be the *minimal* and *maximal degree* and *span* (or *breadth*) of Y, respectively.

Similarly one defines for $Y \in \mathbb{Z}[x_1, \ldots, x_n]$ the coefficient $[Y]_X$ for some monomial X in the x_i. For a multi-variable polynomial the coefficient may be taken with respect only to some variables, and is a polynomial in the remaining variables, for example $[Y]_{x_1^k} \in \mathbb{Z}[x_2, \ldots, x_n]$. (Thus it must be clear in which variables X is meant to be a monomial. For example, for $X = x_1^k \in \mathbb{Z}[x_1]$, the coefficient $[Y]_X = [Y]_{x_1^k} \in \mathbb{Z}[x_2, \ldots, x_n]$ is *not* the same as when regarding $X = x_1^k = x_1^k x_2^0 \in \mathbb{Z}[x_1, x_2]$ and taking $[Y]_X = [Y]_{x_1^k x_2^0} \in \mathbb{Z}[x_3, \ldots, x_n]$.)

We call the three diagram fragments in (1) from left to right a *positive* crossing, a *negative* crossing and a *smoothed out* crossing (in the skein sense). The smoothing out of each crossing in a diagram D leaves a collection of disjoint circles called *Seifert circles*. We write $c(D)$ for the number of crossings of D and $s(D)$ for the number of its Seifert circles.

The *weak* (or canonical) *genus* of K [**St**] is the minimal genus of all its diagrams, where the genus $g(D)$ of a diagram D is defined to be the genus of the surface, obtained by applying the Seifert algorithm to this diagram:

$$\tilde{g}(K) = \min \left\{ g(D) = \frac{c(D) - s(D) + 1}{2} \ : \ D \text{ is a diagram of } K \right\}.$$

The *genus* $g(K)$ of K is the minimal genus of all Seifert surfaces of K (not necessarily coming from Seifert's algorithm on diagrams of K). The *slice genus* $g_s(K)$ of K is the minimal genus of all smoothly embedded surfaces $S \subset B^4$ with $\partial S = K \subset S^3 = \partial B^4$. Clearly $g_s(K) \leq g(K) \leq \tilde{g}(K)$. For many knots, $g(K)$ and $\tilde{g}(K)$ coincide, in particular knots up to 10 crossings and homogeneous knots.

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[1]Further names I have seen in the literature are: 2-variable Jones, Jones-Conway, LYMFHO, FLYPMOTH, HOMFLYPT, LYMPHTOFU, ...

The *braid index* $b(K)$ of K is the minimal number of strings of a braid having K as its braid closure. See [**BM**, **FW**, **Mo**].

Recall that a knot K is *homogeneous* if it has a diagram D containing in each connected component of the complement (in $\mathbb{R}^2$) of the Seifert circles of D (called *block* in [**Cr**, §1]) only crossings of the same sign (that is, only positive or only negative ones). This notion was introduced in [**Cr**] as a generalization of the notion of alternating and positive knots.

3. On the genus and braid index of homogeneous knots

The main tool we use for the proof of Theorem 1 is the result of [**St**]. For this we recall the following two moves on diagrams. The first one is the *flype* [**MT**] (with all possible strand orientations allowed):
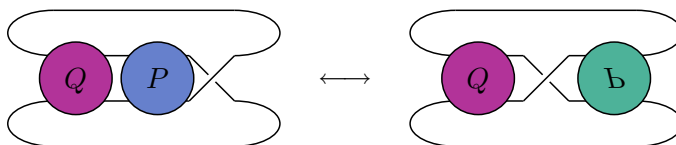
The second move is called a $\bar{t}_2$ *move*. It consists in an application of an antiparallel (full) twist at a crossing:

$$(2) \qquad\qquad \text{} \;.$$
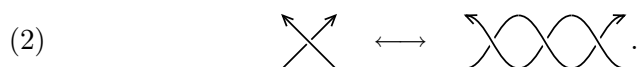
Both moves are understood up to mirroring. (So a $\bar{t}_2$ move can also replace a positive crossing by 3 such.) A diagram is *reduced* if it has no nugatory crossings.

Theorem 2 ([**St**]). *Reduced knot diagrams of given genus, modulo crossing changes, decompose into finitely many equivalence classes under flypes and $\bar{t}_2$ moves.*

This theorem allows us to define for every natural number g an integer d_g as follows (see [**St**] for more details). Call two crossings in a knot diagram $\sim$-*equivalent* (or simply equivalent) if there is a sequence of flypes making them to form a clasp ⟩⟨, in which the strands are reversely oriented. One checks that this is an equivalence relation. Then d_g is the maximal number of equivalence classes of crossings of diagrams of genus g. The theorem ensures that d_g is finite. It follows from the work of Menasco and Thistlethwaite [**MT**] that d_g can be expressed more self-containedly as

$$d_g = 1 + \sup\left\{ i \in \mathbb{R} \;:\; \limsup_{n\to\infty} \frac{a_{n,g}}{n^i} > 0 \right\},$$

where $a_{n,g}$ is the number of alternating knots of n crossings and genus g. (Note that it is not *a priori* clear, whether the supremum on the

right is integral, or even finite.) A while after this paper was originally written, it was determined that $d_g = 6g - 3$ [**SV**].

Call a diagram D' a *generator* if it cannot be reduced further in crossing number by a sequence flypes and $\bar{t}_2$ moves. An alternative formulation is that all equivalence classes of crossings of D' should have at most 2 crossings. So the theorem says that for a given genus there are finitely many generators.

Proof of Theorem 1. It is an easy observation that a flype commutes with the crossing number *reducing* direction of the $\bar{t}_2$ move (2). This means that any sequence of flypes and $\bar{t}_2$ moves that takes a generator diagram D' to D can be replaced by a sequence of flypes, followed by a sequence of $\bar{t}_2$ moves.

Now there in only a finite number of diagrams differing by flypes from a given one. It follows then from Theorem 2 that we can w.l.o.g. ignore flypes and consider only one equivalence class $\mathcal{D}$ of diagrams of genus g modulo the move (2). It is enough to show that there exist a finite number of skein polynomials for knots in $\mathcal{D}$ with given span.

Let D' be a smallest crossing number diagram in $\mathcal{D}$. Any other diagram D can be obtained from D' by possibly switching crossings and crossing number augmenting $\bar{t}_2$ moves.

We will now argue from the skein relation for the HOMFLY polynomial that for a knot diagram D in $\mathcal{D}$ its polynomial $P(D) = P_D$ is of the form

$$(3) \qquad (l^2 + 1)^{d_g} P(D) = \sum_{i=1}^{n_{\mathcal{D}}} (-l^2)^{t_i(D)} L_{i,\mathcal{D}},$$

where the number $n_{\mathcal{D}}$ and the polynomials $L_{i,\mathcal{D}} \in \mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ depend on $\mathcal{D}$ only, and the only numbers depending on D are the t_i. (The reader may compare to [**St2**, proof of Theorem 3.1] for the case of the Jones polynomial, which is analogous.) An easy consequence of this skein relation (1) is the relation

$$(4) \qquad P_{2n+1} = \frac{(-l^2)^n - 1}{l + l^{-1}} m \, P_\infty + (-l^2)^n P_1 \,,$$

where L_{2n+1} is the the link diagram obtained by $n \, \bar{t}_2$ moves at a positive crossing p in the link diagram L_1 (we call the tangle in L_{2n+1} containing the $2n + 1$ crossings so obtained a "twist box"), L_∞ is L_1 with p smoothed out, and P_i is the polynomial of L_i. (Compare also to the formula (8) of [**St2**], but note the misprint in that formula: the second term on the right must be multiplied by t.) If p is negative, replace $-l^2$ by $-l^{-2}$ in (4).

To obtain (3), iterate (4) over the at most d_g twist boxes of a genus g diagram. The factor $(l^2 + 1)^{d_g}$ is used to get disposed of the denominators. The $t_i(D)$ become some linear combinations of the number of

twists that take D' to D, where the twists at each twist box may or may not enter into the various $t_i(D)$ — and if they do, then with sign given by the sign (positive/negative) of the crossings of D in that twist box.

From (3) we obtain for a diagram D of genus g

(5) $$\big| \big[(l^2 + 1)^{d_g} P(D) \big]_{l^p m^q} \big| \le C_g$$

for some constant C_g depending on g only.

Morton showed in [**Mo**] that $[P(D)]_{l^p m^q} = 0$ if $q > 2g(D)$, and, as well-known, the same is true for $q < 0$ (we assume that D is a *knot* diagram). Furthermore, it follows from the identity $P_D(l, -l - l^{-1}) = 1$ that $[P(D)]_{l^p m^q} \ne 0$ for some $|p| \le 2g(D)$. Thus, if $\operatorname{span}_l P_D \le b$ and $g(D) \le g$, then $[P_D]_{l^p m^q} = 0$ for $|p| + q > C'_{b,g}$ with $C'_{b,g}$ depending only on b and g, and hence the same is true for $(l^2 + 1)^{d_g} P_D$. But we already saw that $(l^2 + 1)^{d_g} P_D$ has uniformly bounded coefficients (5), so that

$$\{ (l^2 + 1)^{d_g} P_D \; : \; \operatorname{span}_l P_D \le b, \; g(D) \le g \}$$

is finite. From this the theorem follows because multiplication with $(l^2 + 1)^{d_g}$ is injective (the polynomial ring is an integrality domain).
                                                                        q.e.d.

Corollary 3. *There are only finitely many homogeneous knots K of given genus $g(K)$ and given value of* $\operatorname{span}_l P_K$.

We should point out that (trivially) a given knot K may have infinitely many (reduced) diagrams D of given genus, though it is an interesting question whether actually infinitely many D with $g(D) = \tilde{g}(K)$ can exist. Furthermore, even infinitely many different knots may have diagrams of given genus with the same HOMFLY polynomial [**Ka**]. This, not unexpectedly, shows that the combinatorial approach has its limits.

Proof of Corollary 3. Combine Theorem 1 with the facts that for a homogeneous knot K we have $g(K) = \tilde{g}(K)$ [**Cr**, Corollary 4.1], and that there are only finitely many homogeneous knots already of given Alexander polynomial [**St**, Corollary 3.5].   q.e.d.

Finally, as the inequality $\max \deg_m P(K) \le 2\tilde{g}(K)$ of Morton [**Mo**] is known to be sharp in very many cases, we are led to conjecture a stronger statement/property.

Conjecture 1. The set

$$\{ P_K \; : \; K \text{ knot, } \operatorname{span}_l P_K \le b, \; \max \deg_m P_K \le g \}$$

is finite for any natural numbers g and b.

4. The HOMFLY polynomial and Bennequin's inequality for Legendrian knots

A *contact structure* on a smooth 3-manifold is a 1-form α with $\alpha \wedge d\alpha \neq 0$ (which is equivalent to the non-integrability of the plane distribution defined by $\ker \alpha$). In the following we consider the 1-form $\alpha = dx + y\,dz$ on $\mathbb{R}^3(x, y, z)$, called the *standard contact space*. A *Legendrian knot* is a smooth embedding $\mathcal{K} : S^1 \to \mathbb{R}^3$ with $\alpha\left(\frac{\partial \mathcal{K}}{\partial t}\right) \equiv 0$. Each such knot has its underlying topological knot type $K = [\mathcal{K}]$ and two fundamental invariants in contact geometry known as the *Thurston–Bennequin number* $tb(\mathcal{K})$ and *Maslov index* $\mu(\mathcal{K})$. (See [**FT**, **Fe**] for an excellent introductory account on this subject.)

Definition 1. The Thurston–Bennequin number $tb(\mathcal{K})$ of a Legendrian knot $\mathcal{K}$ in the standard contact space is the linking number of $\mathcal{K}$ with $\mathcal{K}'$, where $\mathcal{K}'$ is obtained from $\mathcal{K}$ by a push-forward along a vector field transverse to the (hyperplanes of the) contact structure.

The Maslov (rotation) index $\mu(\mathcal{K})$ of $\mathcal{K}$ is the degree of the map

$$t \in S^1 \mapsto \frac{\mathrm{pr}\, \frac{\partial \mathcal{K}}{\partial t}(t)}{\left|\mathrm{pr}\, \frac{\partial \mathcal{K}}{\partial t}(t)\right|} \in S^1 \,,$$

where $\mathrm{pr} : \mathbb{R}^3 \to \mathbb{R}^2 \simeq \mathbb{C}$ is the projection $(x, y, z) \mapsto (y, z)$.

Both invariants $tb(\mathcal{K})$ and $\mu(\mathcal{K})$ can be interpreted in terms of a regular diagram of the (topological) knot $[\mathcal{K}]$, and thus it was recently realized that the theory of polynomial invariants of knots and links in $\mathbb{R}^3$, developed after Jones [**J**], can be applied in the context of Legendrian knots to give inequalities for tb and μ. In particular we have the inequality

$$(6) \qquad tb(\mathcal{K}) + |\mu(\mathcal{K})| \leq \min\deg_l P([\mathcal{K}]) - 1\,.$$

This follows from the work of Morton [**Mo**] and Franks–Williams [**FW**], and was translated to the Legendrian knot context by Tabachnikov and Fuchs [**FT**]. See also [**T**, **CGM**, **GH**, **Fe**].

On the other hand, a purely topological inequality was previously known for a while – Bennequin's inequality. In [**Be**], Bennequin proved

$$(7) \qquad tb(\mathcal{K}) + |\mu(\mathcal{K})| \leq 2g([\mathcal{K}]) - 1.$$

This inequality was later improved by Rudolph [**Ru3**], who showed

$$(8) \qquad tb(\mathcal{K}) + |\mu(\mathcal{K})| \leq 2g_s([\mathcal{K}]) - 1,$$

where $g_s(K)$ is the slice (4-ball) genus of K. This improvement used the proof of the Thom conjecture by Kronheimer and Mrowka, achieved originally by gauge theory [**KM**, **KM2**], and later much more elegantly by Seiberg–Witten invariants [**KM3**].

While the r.h.s. of (7) and (8) are invariant w.r.t. taking the mirror image, the l.h.s. are strongly sensitive, so we have

$$(9) \qquad \tau'(L) \le \tau(L) \le 2g_s(L) - 1 \le 2g(L) - 1$$

for any topological knot type L, where

$$\tau'(L) := \max\{tb(\mathcal{K}) + |\mu(\mathcal{K})| \, : \, [\mathcal{K}] = L \,\}$$

and

$$\tau(L) := \max\{tb(\mathcal{K}) + |\mu(\mathcal{K})| \, : \, [\mathcal{K}] \in \{L, !L\} \,\} \,=\, \max(\tau'(L), \tau'(!L)) \,,$$

and $!L$ is the obverse (mirror image) of L.

In [**K**], Kanda used an original argument and the theory of convex surfaces in contact manifolds developed mainly by Giroux [**Gi**] to show that the inequality $\tau' \le 2g - 1$ can get arbitrarily unsharp, i.e. $\exists \{L_i\}$: $\tau'(L_i) - 2g(L_i) \to -\infty$. (Here, and in the following, an expression of the form '$x_n \to \infty$' should abbreviate $\lim\limits_{n \to \infty} x_n = \infty$. Analogously '$x_{n_m} \to \infty$' should mean the limit for $m \to \infty$ etc.) In Kanda's paper, all L_i are alternating pretzel knots, and hence of genus 1, so that for these examples in fact we also have $\tau'(L_i) - 2g_s(L_i) \to -\infty$.

It was realized (see the remarks on [**FT**, p. 1035]) that Kanda's result admits an alternative proof using (6) (whose proof in turn is also "elementary" in a sense discussed more detailedly in [**Fe**]). Other examples (connected sums of two $(2, .)$-torus knots) were given by Dasbach and Mangum [**DM**, §4.3], for which even $\tau - 2g \to -\infty$. However, their examples do not apply for the slice version (8) of Bennequin's inequality. In [**Fe**] it was observed that Kanda's result also follows from the work of Rudolph [**Ru**, **Ru2**].

Here we give a larger series of examples of knots with $2g - \tau \to \infty$ containing as very special cases the previous ones given by Kanda and Dasbach–Mangum. These knots show that the inexactness of Bennequin's inequality is by far not an exceptional phenomenon. While our arguments also use (6) (and hence are much simpler than the original proof of Kanda), they still also apply in many cases for the slice version (8) of Bennequin's inequality. Similar reasoning works for links of any fixed number of components, but for simplicity we content ourselves only with knots.

From Theorem 1, the aforementioned application to the unsharpness of Bennequin's inequality is almost straightforward. We formulate the consequence somewhat more generally and more precisely than in the introduction.

Theorem 3. *Let $\{L_i\}$ be a sequence of knots, such that only finitely many of the L_i have the same skein polynomial. Then $2\tilde{g}(L_i) - \min(\tau'(L_i), \tau'(!L_i)) \to \infty$. If additionally $\tilde{g}(L_i) \le C$ for some constant C, then even $\min(\tau'(L_i), \tau'(!L_i)) \to -\infty$.*

The condition $g = \tilde{g}$ is very often satisfied, but unfortunately this is not always the case, as pointed out by Morton [**Mo**, Remark 2]. Worse yet, as shown in [**St2**], there cannot be any inequality of the type $\tilde{g}(K) \leq f(g(K))$ for any function $f : \mathbb{N} \to \mathbb{N}$ for a general knot K. Nevertheless, by the results mentioned in the proof of Corollary 3, any sequence of homogeneous knots satisfies $g(L_i) = \tilde{g}(L_i)$ and the condition of Theorem 3. In particular, we have

Corollary 4. *If $\{L_i\}$ are negative or achiral homogeneous knots, then* $2g(L_i) - \tau'(L_i) \to \infty$. *If L_i are negative or additionally $g(L_i) \leq C$ for some constant C, then even $2g_s(L_i) - \tau'(L_i) \to \infty$.*   q.e.d.

Remark 1. Before we prove Theorem 3, we make some comments on Corollary 4.

1) Clearly for an achiral knot L we have $\tau(L) = \tau'(L)$, so that in the case that all L_i are achiral (like the examples $T_{2,n} \# T_{2,-n}$, with $T_{2,n}$ being the $(2,n)$-torus knot, given in [**DM**]) the stronger growth statement with τ' replaced by τ holds, $2g - \tau \to \infty$.

2) Contrarily, the statement $2g - \tau \to \infty$ is not true in the negative case: Tanaka [**Ta**, Theorem 2] showed that $\tau' = 2g - 1$ for positive knots. On the other hand, this means that for negative knots $2g - \tau' \to \infty$, and in fact $2g_s - \tau' \to \infty$, as by [**St4**] $g = g_s$ for positive (and hence also for negative) knots. However, we have from [**St5**] the stronger statement that $\tau' \to -\infty$, which also holds for almost negative knots (see [**St6**, §5]).

3) The conditions can be further weakened. For example, we can replace achirality by self-conjugacy of the HOMFLY polynomial (invariance under the interchange $l \leftrightarrow l^{-1}$) and 'negative' by 'k-almost negative' for any fixed number k, as the condition $g = \tilde{g}$ in Corollary 3 can in fact be weakened to $\tilde{g} \leq f(g)$ for any (fixed) function $f : \mathbb{N} \to \mathbb{N}$. However, in the latter case the assumption needs to be retained that only finitely many L_i have the same polynomial. (This was established to be automatically true for $k \leq 1$ [**St5**, **St6**], but is not known for $k \geq 2$.)

4) The fact that our collection of examples is richer than the one of Kanda can be made precise as follows: the number of all pretzel knots of at most n crossings is $O(n^3)$, while it follows from [**ES**] and [**St3**] that the number of achiral and positive knots of crossing number at most n, already among the 2-bridged ones, grows exponentially in n.

5) The boundedness condition on the genus in the achiral case is essential (at least for this method of proof), as is shown by the examples $T_{2,n} \# T_{2,-m}$ of Dasbach and Mangum, on which the skein polynomial argument fails for the slice genus.

Proof of Theorem 3. If $\tilde{g}(L_i) \leq C$, then Theorem 1 implies that $\max\deg_l P(L_i) - \min\deg_l P(L_i) = \mathrm{span}_l P(L_i) \to \infty$. Thus $\min(\min\deg_l P(L_i), \min\deg_l P(!L_i)) \to -\infty$, and the assertions follow from (6) and (8). Otherwise there is a subsequence $\{L_{i_j}\}$ with $\tilde{g}(L_{i_j}) \to \infty$. Clearly,

$$\min(\min\deg_l P(L_{i_j}), \min\deg_l P(!L_{i_j})) \leq 0,$$

so that $2\tilde{g}(L_{i_j}) - \min(\tau'(L_{i_j}), \tau'(!L_{i_j})) \to \infty$; that is, $\{L_i\}$ always has a subsequence with the asserted property. Applying the argument on any subsequence of $\{L_i\}$ gives the property on the whole $\{L_i\}$.   q.e.d.

As a final remark, there is another inequality, proved in [**CG**] and [**T**], involving the Kauffman polynomial F (in the convention of [**Kf**]),

$$(10) \qquad\qquad tb(\mathcal{K}) \leq -\max\deg_a F([\mathcal{K}]) - 1.$$

It gives generally better estimates on $tb(\mathcal{K})$, but lacks the additional term $|\mu(\mathcal{K})|$ and also a translation to the transverse knot context (see [**Fe**, remark at end of §6]). Contrarily, (6) admits a version for transverse knots as well (in which case the term $|\mu(\mathcal{K})|$ is dropped; see [**GH**] and [**FT**, Theorem 2.4]), and so our results hold in the transverse case, too. Moreover, as remarked by Ferrand in [**Fe**, §8] (see also [**Ta**, Problem, p. 3428]), the inequality

$$(11) \qquad\qquad -\max\deg_a F(K) \leq \min\deg_l P(K)$$

is not always satisfied. Among the 313,230 prime knots of at most 15 crossings tabulated in [**HT**], there are 134 knots K such that at least one of K and $!K$ fails to satisfy (11). The simplest examples are two 12 crossing knots, one of them, 12_{1584}, being quoted by Ferrand.

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Research Institute for Mathematical Sciences
Kyoto University
Kyoto 606-8502, Japan

E-mail address: stoimeno@kurims.kyoto-u.ac.jp