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DETERMINANTS OF ZEROTH ORDER OPERATORS

Leonid Friedlander & Victor Guillemin

Abstract

For compact Riemannian manifolds all of whose geodesics are closed (aka Zoll manifolds) one can define the determinant of a zeroth order pseudodifferential operator by mimicking Szego's definition of this determinant for the operator: multiplication by a bounded function, on the Hilbert space of square-integrable functions on the circle. In this paper we prove that the non-local contribution to this determinant can be computed in terms of a much simpler "zeta-regularized" determinant.

1. Introduction

In this paper we will compare two techniques for defining regularized determinants of zeroth order pseudodifferential operators and show that, modulo local terms, they give the same answer. To illustrate these two techniques let f be a C^∞ function on the circle with $f-1 \approx 0$. Szegő proves that if P_n is the orthogonal projection to the space spanned by $e^{ik\theta}$, $-n \le k \le n$, then for n large

$$\log\det P_n M_f P_n = 2n\widehat{\log f}(0) + \sum k\widehat{\log f}(k)\widehat{\log f}(-k) + O\big(n^{-\infty}\big) \tag{1.1}$$

where M_f is the operator of multiplying by f and $\widehat{\log f}(k)$ is the k-th Fourier coefficient of $\log f$. Hence by subtracting off the "counterterm" $2n\widehat{\log f}(0)$ one gets for the Szegő-regularized determinant of M_f:

$$\log\det M_f = \sum k\widehat{\log f}(k)\widehat{\log f}(-k). \tag{1.2}$$

An alternative way of regularizing this determinant is by zeta function techniques. Namely, let $Q^z : L^2(S^1) \to L^2(S^1)$ be the operator

$$Q^z e^{in\theta} = \begin{cases} |n|^z e^{in\theta}, & \text{if } n \neq 0; \\ 0, & \text{if } n = 0. \end{cases}$$

Then

$$\operatorname{trace}(\log M_f)Q^z = \operatorname{trace} M_{\log f} Q^z = 2\widehat{\log f}(0)\sum_{n=1}^{\infty} n^z = 2\widehat{\log f}(0)\zeta(-z).$$

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Since $\zeta(z)$ is regular at $z = 0$ and $\zeta(0) = -1/2$, zeta-function regularization gives one, for the regularized "log det" of M_f

$$\text{trace} \log M_f = -\widehat{\log f}(0), \tag{1.3}$$

i.e., the zeta-regularized determinant of M_f is proportional to the counterterm one had to subtract off in order to obtain the Szegő-regularized determinant of M_f.

This does not bode well for comparing these two methods of regularization in a more general setting; however, the right hand sides in (1.2) and (1.3) are local expressions of the symbol of M_f, and for both these methods of regularization the non-local contributions are zero. In the paper we will show that if one replaces M_f by a zeroth order pseudodifferential operator, B, then (1.2) and (1.3) are non-symbolic (i.e., non-local) functions of B; however, their difference is symbolic. In other words, modulo local terms, they give the same answer.

This is a special case of a more general result about "Zoll operators". Let X^d be a compact manifold and $Q : C^\infty(X) \to C^\infty(X)$ a self-adjoint first order elliptic pseudodifferential operator. Q is a *Zoll* operator if the bicharacteristic flow on $T^*X \setminus X$ generated by its symbol is periodic of period 2π. (To simplify the statements of some of the results below we will strengthen this assumption and assume the bicharacteristic flow strictly periodic of period 2π: if the initial point of a bicharacteristic is (x, ξ), the bicharacteristic returns for the first time to (x, ξ) at $t = 2\pi$.) If Q is a Zoll operator, the operator

$$W = -\frac{1}{2\pi i} \log \exp 2\pi i Q,$$

with $0 < \operatorname{Im} \log z \le 2\pi$, is a zeroth order pseudodifferential operator, and the spectrum of the operator $Q + W$ consists of positive integers. We will henceforth subsume this property into the definition of "Zoll", and assume $\operatorname{spec} Q = \mathbb{Z}_+$. (The standard example of a Zoll operator is the operator

$$\left(\Delta_{S^d} + \left(\frac{d-1}{2}\right)^2\right)^{1/2} - \frac{d-1}{2};$$

however, there are a lot of non-standard examples as well. See, for instance [**CV**].)

Let π_k be the orthogonal projection of $L^2(X)$ onto the k-th eigenspace of Q and let $P_n = \pi_1 + \cdots + \pi_n$. If $B : L^2(X) \to L^2(X)$ is a zeroth order pseudodifferential operator and $I - B$ is small then by a theorem of Guillemin and Okikiolu [**GO**]

$$\log \det P_n B P_n \sim b + \sum_{k=d, k \neq 0}^{-\infty} b_k n^k + b_0 \log n \tag{1.4}$$

and, as above, one can define the Szegő regularized determinant of B to be e^b. On the other hand, the expression

$$\operatorname{trace}\log BQ^z \tag{1.5}$$

is a meromorphic function in z with simple poles at $z = -d + k$, $k = 0, 1, \ldots$, and one can define the zeta function regularization of $\log\det B$ to be the finite part of this function at $z = 0$.

In Section 2 we will compare these two definitions and show that, as above, they differ by an expression that is local in B and only involves integrals of terms in the symbolic expansion of B of degree $\geq -d$. Then in Section 3 we will examine zeta regularization in more detail, allowing the "regularizer" Q to be any positive definite self-adjoint first order elliptic pseudodifferential operator (i.e., not necessarily a Zoll operator as above) and prove a number of results about the "$\log\det$":

$$w_Q(B) = (\text{f.p.})_{z=0}\operatorname{trace}(\log B)Q^z \tag{1.6}$$

for zeroth order pseudodifferential operators, B. For instance we will show that the variation, δw_Q, of this functional is local and that if Q and Q' are two regularizers, $w_Q(B) - w_{Q'}(B)$ is local. (In other words, modulo local terms, the regularization of $\log\det B$ defined by (1.6) is independent of the choice of Q.) We will also compute the multiplicative anomaly of the regularized $\log\det B_1B_2$ defined by (1.6) and show that it is also given by expressions which are local in the symbols of B_1 and B_2.

2. Szegő regularized determinants

We will give a brief sketch of how (1.4) was derived in [**GO**] and show how the zeroth order term in this expression is related to (1.6). Letting $B = I - A$ the left hand side of (1.4) becomes

$$-\sum_{k=1}^{\infty}\frac{1}{r}\operatorname{trace}(P_nAP_n)^r, \tag{2.1}$$

so to study the asymptotic behavior of (1.4) it suffices to study the asymptotic behavior as n tends to infinity of each of the summands in (2.1). To do this we will decompose the operator A into its "Fourier coefficients" as in the example discussed in Section 1. More explicitly, let $U(t) = \exp(itQ)$ and let

$$A_k = \frac{1}{2\pi}\int_0^{2\pi} e^{ikt}U(-t)AU(t)dt. \tag{2.2}$$

By Egorov's theorem the A_k's are zeroth order pseudodifferential operators, and the sum

$$A = \sum_{k=-\infty}^{\infty} A_k$$

is the "Fourier series" of A. It is shown in [**GO**] that this series converges and that the operator norms of A_k's are rapidly decreasing in k as k tends to infinity. Hence for deriving asymptotic expansions for the summands in (2.1) we can assume that

$$A = \sum_{k=-N}^{N} A_k, \quad N \text{ large.} \tag{2.3}$$

Also, since $U(t) = \sum e^{\mathrm{int}} \pi_n$,

$$A_k = \frac{1}{2\pi} \sum_{m,n} \int_0^{2\pi} e^{ikt} e^{i(n-m)t} \pi_m A \pi_n dt = \sum_n \pi_{n+k} A \pi_n. \tag{2.4}$$

Plugging (2.3) into the rth summand of (2.1) and replacing each term in the product by the sum (2.4) one gets:

$$\operatorname{trace}(P_n A P_n)^r = \sum_{j_1+\cdots+j_r=0} \operatorname{trace} \sum_{k+\sigma(j)\leq n} \pi_k A_{j_r} \cdots A_{j_1} \pi_k \tag{2.5}$$

where $j = (j_1, \ldots, j_r)$,

$$\sigma(j) = \max(0, j_1, j_1 + j_2, \ldots, j_1 + \cdots + j_r), \tag{2.6}$$

and the number of summands in j is finite. We will use the notation $A_j = A_{j_r} \cdots A_{j_1}$. The asymptotics of each of the summands in (2.5) can be read off from a theorem of Colin de Verdiere [**CV**] which says that

$$\operatorname{trace} \pi_n A_j \pi_n \sim \sum c_l(A_j) n^l. \tag{2.7}$$

Moreover, Colin's theorem asserts that the terms on the right are local functionals of A and are given explicitly by the non-abelian residues

$$c_l(A_j) = \operatorname{res} Q^{-(l+1)} A_j. \tag{2.8}$$

Finally, by plugging (2.8) into (2.7) we obtain an asymptotic expansion

$$\operatorname{trace}(P_n A P_n)^r \sim a_r + \sum_{k=d, k\neq 0}^{-\infty} a_{r,k} n^k + a_{r,0} \log n \tag{2.9}$$

in which all terms except the constatnt term, a_r, are local functions of A.

The same argument can also be used to compute $\operatorname{trace}(P_n A P_n)^r Q^z$. Namely, by (2.5),

$$\operatorname{trace}(P_n A P_n)^r Q^z = \sum_{j_1+\cdots+j_r=0} \operatorname{trace} \sum_{k+\sigma(j)\leq n} \pi_k A_{j_r} \cdots A_{j_1} \pi_k k^z, \tag{2.10}$$

and by combining this with (2.7) we will prove

Theorem 2.1. *For $z \neq -d+k$, $k = 0, 1, 2, \ldots$, there is an asymptotic expansion*

$$\operatorname{trace}(P_n A P_n)^r Q^z \sim a_r(z) + \sum_{k=d}^{-\infty} a_{r,k}(z) n^{k+z}. \tag{2.11}$$

Moreover, the coefficients in this expansion depend meromorphically on z and, except for $a_r(z)$, are symbolic functions of A. In addition, $a_{r,k}(z)$ has a simple pole at $z = -k$ and is holomorphic elsewhere, and $a_r(z)$ is meromorphic with simple poles at $z = -d+k$, $k = 0, 1, 2, \ldots$.

Proof. The j-th summand above is equal to

$$\operatorname{trace} \sum_{k=1}^{n-\sigma(j)} \big(\pi_k A_j \pi_k\big) k^z$$

and by (2.7)

$$\operatorname{trace} \sum_{k=1}^{n-\sigma(j)} \big(\pi_k A_j \pi_k\big) k^z \sim b(z) + \sum_{l=d}^{-\infty} c_{l-1}\big(A_j\big) \sum_{k=1}^{n-\sigma(j)} k^{l-1+z}.$$

By a theorem of Hardy (see [**Ha**], §13.10, p. 338),

$$\sum_{k=1}^{m} k^{l-1+z} \sim C(-z-l+1) + \frac{m^{l+z}-1}{l+z} + \frac{m^{l+z-1}}{2} + \sum_{p=1}^{\infty} (-1)^p (-z-l+1)^{(2p-2)} \frac{B_p}{(2p)!} m^{l+z-2p}$$

where $C(s) = \zeta(s) - 1/(s-1)$, B_p is the p-th Beroulli number, and $s^{(r)} = s(s+1)\cdots(s+r)$. Plugging this (with $m = n - \sigma(j)$) into (2.10) we get an expression of the form (2.11) where the coefficients are holomorphic in z and $a_{r,k}(z)$ is holomorphic except of $z = -k$ where it has a simple pole. Moreover, if $\mathrm{Re} z < -d$ one can take the limit of both sides of (2.11) as n tends to infinity to obtain

$$\operatorname{trace} A^r Q^z = a_r(z), \tag{2.12}$$

and since $\operatorname{trace} A^r Q^z$ is meromorphic with simple poles at $z = -d+k$, $k = 0, 1, 2, \ldots$, the same is true of $a_r(z)$. q.e.d.

If we rewrite the right hand side of (2.11) in the form

$$a_r(z) - a_{r,0}(z) + \sum_{k=d, k\neq 0}^{-\infty} a_{r,k}(z) n^{k+z} + z a_{r,0}(z) \frac{n^z - 1}{z}$$

and let z tend to zero we recapture (2.9) with $a_{r,k} = a_{r,k}(0)$ for $k \neq 0$, $a_{r,0} = \mathrm{Res}_{z=0} a_{r,0}(z)$, and, by (2.12),

$$a_r = (\text{f.p.})_{z=0} \operatorname{trace} A^r Q^z - (\text{f.p.})_{z=0} a_{r,0}(z). \tag{2.13}$$

However, $(\text{f.p.})_{z=0}a_{r,0}(z)$ is a local function of A depending only on the first d terms in its asymptotic expansion; hence the same is true of $a_r - (\text{f.p.})_{z=0}\text{trace}A^rQ^z$.

Finally, by applying this argument to each summand in the series

$$\text{trace}\log(P_nBP_n)Q^z = -\sum_{r=1}^{\infty}\frac{1}{r}\text{trace}\big(P_nAP_n\big)^rQ^z,$$

we conclude that the constant term, b, in the expansion (1.4) differs from the zeta regularized "log det" of B

$$(\text{f.p.})_{z=0}\text{trace}(\log B)Q^z$$

by a term which is local in B and only depends on the first d terms in its symbolic expansion.

3. Zeta regularized determinants

In this section we relax assumptions on a zeroth order pseudodifferential operator B and on a regularizer Q. We will assume that the spectrum of B lies in a domain D of the complex plane where the logarithm is defined; Γ is the boundary of D oriented counterclockwise. Then $\log B$ is defined by the formula

$$\log B = \frac{1}{2\pi i}\int_\Gamma (\lambda I - B)^{-1}d\lambda,$$

and it is a zeroth order PDO. A regularizer Q will be a positive elliptic PDO of order 1. The zeta regularized "log det" of B is defined by the formula (1.6). To compare regularizations of "log det" of B for two different regularizers, Q and Q', we compute their difference:

$$\begin{aligned} w_Q(B) - w_{Q'}(B) &= (\text{f.p.})_{z=0}\text{trace}\log B\cdot(Q^z - (Q')^z)\\ &= \text{res}_{z=0}\text{trace}\log B\frac{Q^z-(Q')^z}{z}\\ &= \text{res}\big(\log B(\log Q - \log Q')\big). \end{aligned}$$

Notice that $\log B(\log Q - \log Q')$ is a zeroth order pseudodifferential operator. The last formula shows that $w_Q(B) - w_{Q'}(B)$ is a local quantity; it depends on the first $d+1$ terms in the symbolic expansions of B, Q, and Q'.

In the remaining part of this section we will be computing the multiplicative anomalies for the "log det", namely, $w_Q(AB) - w_Q(BA)$ and $w_Q(AB) - w_Q(A) - w_Q(B)$. We will show that both are local quantities and in the case when $d = 2$ we will obtain explicit formulas for them that involve principal symbols of the operators A, B, and Q. Clearly, locality of $w_Q(AB) - w_Q(A) - w_Q(B)$ implies locality of $w_Q(AB) - w_Q(BA)$. However, we present both calculations, since the latter is simpler. The main tool for computing multiplicative anomalies is the variational formula for "log det". Let δA be a variation of an operator and let

$$\sigma_Q(A,\delta A)=\delta w_Q(A)-\frac{1}{2}(\text{f.p.})_{z=0}\text{trace}\{\delta A A^{-1}+A^{-1}\delta A\}Q^z.$$

Proposition 3.1. *$\sigma_Q(A,\delta A)$ is a local quantity that depends on $d-1$ terms in the symbolic expansions of A, δA, and Q. If $d=2$ then*

$$\sigma_Q(A,\delta A)=\frac{1}{6}\text{res}(\delta\log a\{\log a,\{\log a,\log q\}\}), \tag{3.1}$$

where $a(x,\xi)$ is the principal symbol of A, $q(x,\xi)$ is the principal symbol of Q, $\{\cdot,\cdot\}$ is the Poisson bracket, and res *is the symbolic residue* (*see* [**Gu**]).

Proof. One has

$$\begin{aligned}
&\sigma_Q(A,\delta A)\\
&=\frac{1}{2\pi i}\int_\Gamma(\text{f.p.})_{z=0}\text{trace}[\log\lambda(\lambda I-A)^{-1}\delta A(\lambda I-A)^{-1}Q^z]d\lambda\\
&\quad-\frac{1}{4\pi i}\int_\Gamma(\text{f.p.})_{z=0}\text{trace}\left[\frac{1}{\lambda}\left(\delta A(\lambda I-A)^{-1}+(\lambda I-A)^{-1}\delta A\right)Q^z\right]d\lambda\\
&=\frac{1}{4\pi i}\int_\Gamma\log\lambda d\lambda(\text{f.p.})_{z=0}\text{trace}[2(\lambda I-A)^{-1}\delta A(\lambda I-A)^{-1}\\
&\quad-\delta A(\lambda I-A)^{-2}-(\lambda I-A)^{-2}\delta A]Q^z\\
&=\frac{1}{4\pi i}(\text{f.p.})_{z=0}\text{trace}\left\{\int_\Gamma\log\lambda[[(\lambda I-A)^{-1},\delta A],(\lambda I-A)^{-1}]d\lambda Q^z\right\}.
\end{aligned}$$

Notice that

$$\begin{aligned}
&\text{trace}[[(\lambda I-A)^{-1},\delta A],(\lambda I-A)^{-1}]Q^z\\
&\quad=\text{trace}[(\lambda I-A)^{-1},\delta A][(\lambda I-A)^{-1},Q^z],
\end{aligned}$$

and

$$\begin{aligned}
&(\text{f.p.})_{z=0}\text{trace}[(\lambda I-A)^{-1},\delta A][(\lambda I-A)^{-1},Q^z]\\
&=\text{res}_{z=0}\text{trace}\frac{[(\lambda I-A)^{-1},\delta A][(\lambda I-A)^{-1},Q^z]}{z}\\
&=\text{res}[(\lambda I-A)^{-1},\delta A][(\lambda I-A)^{-1},\log Q].
\end{aligned}$$

The operator on the right is of order $d-2$, so its residue depends on $d-1$ terms in the symbolic expansions of A, δA, and Q. For the variation of "log det" we obtain:

$$\sigma_Q(A,\delta A)=\frac{1}{4\pi i}\int_\Gamma\log\lambda\,\text{res}[(\lambda I-A)^{-1},\delta A][(\lambda I-A)^{-1},\log Q]d\lambda. \tag{3.2}$$

In the case $d = 2$,

$$\begin{aligned}&(\text{f.p.})_{z=0}\text{trace}[(\lambda I - A)^{-1}, \delta A][(\lambda I - A)^{-1}, Q^z]\\&= -\text{res}(\{(\lambda - a)^{-1}, \delta a\}\{(\lambda - a)^{-1}, \log q\})\\&= -\text{res}((\lambda - a)^{-4}\{a, \delta a\}\{a, \log q\}),\end{aligned}$$

and

$$\begin{aligned}\sigma_Q(A, \delta A) &= -\frac{1}{6}\text{res}(a^{-3}\{a, \delta a\}\{a, \log q\})\\&= \frac{1}{6}\text{res}(\{a^{-1}, \delta a\}\{\log a, \log q\})\\&= -\frac{1}{6}\text{res}(\delta a\{a^{-1}, \{\log a, \log q\}\})\\&= \frac{1}{6}\text{res}(\delta \log a\{\log a, \{\log a, \log q\}\}).\end{aligned}$$

q.e.d.

The variation of $w_Q(AB) - w_Q(BA)$ with respect to A (the operator B is fixed) equals the sum of

$$\sigma_Q(AB, \delta AB) - \sigma_Q(BA, B\delta A) \tag{3.3}$$

and

(3.4)

$$\begin{aligned}&\frac{1}{2}(\text{f.p.})_{z=0}(\delta A A^{-1} + B^{-1}A^{-1}(\delta A)B - A^{-1}\delta A - B(\delta A)A^{-1}B^{-1})Q^z\\&= \frac{1}{2}\text{res}_{z=0}\left((\delta A)A^{-1}\frac{Q^z - B^{-1}Q^z B}{z} - A^{-1}\delta A\frac{Q^z - BQ^z B^{-1}}{z}\right)\\&= \frac{1}{2}\text{res}((\delta A)A^{-1}(\log Q - B^{-1}\log QB) - A^{-1}\delta A(\log Q - B\log QB^{-1}))\\&= \frac{1}{2}\text{res}((\delta A)A^{-1}B^{-1}[B, \log Q] + A^{-1}\delta A[B, \log Q]B^{-1}).\end{aligned}$$

Both the expressions (3.3) and (3.4) are local and depend on a finite number of terms in the symbolic expansions of A, B, δA, and Q. Let $A(t) = A^t$. Then the t-derivative of $w_Q(A(t)B) - w_Q(BA(t))$ is a local quantity. Clearly, $w_Q(A(0)B) - w_Q(BA(0)) = 0$; hence $w_Q(AB) - w_Q(BA)$ is a local quantity.

The above derivation is valid if there exists a domain in the complex plane where log is defined and that contains the spectrum of $A(t)B$ (and, therefore, of $BA(t)$) for all t, $0 \leq t \leq 1$. One can replace the family A^t by any family of zeroth order pseudodifferential operators that connects A with the identity. We will operate under this assumption. It is satisfied if, for example, operators A and B are close to identity or if both of them are positive.

In the case $d = 2$, the quantity (3.3) vanishes because by (3.1) it depends on the principal symbols of the operators A and B only and on the level of principal symbols they commute. Therefore,

$$\begin{aligned}\frac{d}{dt}&\big(w_Q(A^tB) - w_Q(BA^t)\big)\\ &= \frac{1}{2}\mathrm{res}(\log A(B^{-1}[B, \log Q] + [B, \log Q]B^{-1}))\\ &= \frac{1}{2}\mathrm{res}(\log A(B \log QB^{-1} - B^{-1} \log QB)),\end{aligned}$$

and

$$w_Q(AB) - w_Q(BA) = \frac{1}{2}\mathrm{res}(\log A(B \log QB^{-1} - B^{-1} \log QB)). \tag{3.5}$$

Now, we fix A and take the family $B(t) = B^t$. Let

$$g(t) = w_Q(AB^t) - w_Q(B^tA).$$

From (3.5),

$$g'(t) = \frac{1}{2}\mathrm{res}\big(\log A(B^t[\log B, \log Q]B^{-t} + B^{-t}[\log B, \log Q]B^t)\big),$$

and

$$\begin{aligned}g''(t) = &\frac{1}{2}\mathrm{res}\big(\log A(B^t[\log B, [\log B, \log Q]]B^{-t}\\ &\quad - B^{-t}[\log B, [\log B, \log Q]]B^t)\big)\\ = &-\frac{1}{2}\mathrm{res}(\log a(\{\log b, \{\log b, \log q\}\} - \{\log b, \{\log b, \log q\}\})) = 0;\end{aligned}$$

here $b(x, \xi)$ is the principal symbol of B. In the last equality, we used the fact that the non-abelian residue of an operator of order -2 on a two-dimensional manifold is the residue of the principal symbol. Hence,

$$g'(t) = g'(0) = \mathrm{res}(\log A[\log B, \log Q]).$$

Clearly, $g(0) = 0$, so

$$w_Q(AB) - w_Q(BA) = \mathrm{res}(\log A[\log B, \log Q]). \tag{3.6}$$

The expression (3.6) is of the same form as the Kravchenko–Khesin cocycle in dimension 1 [**KrKh**].

The variation of

$$\kappa_Q(A, B) = w_Q(AB) - w_Q(A) - w_Q(B) \tag{3.7}$$

with respect to A is the sum of

$$\sigma_Q(AB, \delta AB) - \sigma_Q(A, \delta A) \tag{3.8}$$

and

$$\begin{aligned}(3.9)\qquad &\frac{1}{2}(\text{f.p.})_{z=0}\text{trace}((\delta AB)(AB)^{-1}+(AB)^{-1}\delta(AB)\\ &-(\delta A)A^{-1}-A^{-1}\delta A)Q^z+\sigma_Q(AB,\delta AB)-\sigma_Q(A,\delta A)\\ &=\frac{1}{2}(\text{f.p.})_{z=0}\text{trace}(B^{-1}A^{-1}(\delta A)B-A^{-1}\delta A)Q^z\\ &=\frac{1}{2}(\text{f.p.})_{z=0}\text{trace}(A^{-1}(\delta A)BQ^zB^{-1}-A^{-1}\delta AQ^z)\\ &=\frac{1}{2}\text{res}_{z=0}\text{trace}\frac{A^{-1}(\delta A)BQ^zB^{-1}-A^{-1}\delta AQ^z}{z}\\ &=\frac{1}{2}\text{res}(A^{-1}(\delta A)B\log QB^{-1}-A^{-1}\delta A\log Q)\\ &=\frac{1}{2}\text{res}(A^{-1}\delta A[B,\log Q]B^{-1}).\end{aligned}$$

Both (3.8) and (3.9) are local expressions and depend on a finite number of terms in the symbolic expansions of A, B, δA, and Q. By taking a family $A(t)$ that connects A with the identity, we conclude that $\kappa_Q(A,B)$ is a local quantity ($\kappa_Q(I,B)=0$.)

We will next make these computations more explicit in the two-dimensional situation. It is covenient to deal with the symmetrized multiplicative anomaly $(\kappa_Q(A,B)+\kappa_Q(B,A))/2$. In a similar way to (3.8), (3.9), one derives

$$\begin{aligned}&\delta\kappa_Q(B,A)\\ &=-\frac{1}{2}\text{res}((\delta A)A^{-1}B^{-1}[B,\log Q])+\sigma_Q(BA,B\delta A)-\sigma_Q(A,\delta A),\end{aligned}$$

and, therefore,

$$\begin{aligned}(3.10)\qquad &\delta\frac{\kappa_Q(A,B)+\kappa_Q(B,A)}{2}\\ &=\frac{1}{4}\text{res}(A^{-1}\delta A[B,\log Q]B^{-1}-(\delta A)A^{-1}B^{-1}[B,\log Q])\\ &\quad+\frac{1}{2}\sigma_Q(AB,(\delta A)B)+\frac{1}{2}\sigma_Q(BA,B\delta A)-\sigma_Q(A,\delta A)\\ &=\frac{1}{4}\text{res}(\delta A[B,\log Q][B^{-1},A^{-1}]+\delta A[[B,\log Q],A^{-1}B^{-1}])\\ &\quad+\frac{1}{2}\sigma_Q(AB,(\delta A)B)+\frac{1}{2}\sigma_Q(BA,B\delta A)-\sigma_Q(A,\delta A).\end{aligned}$$

The first term, T_1, on the right in (3.10) equals

$$\begin{aligned}&-\frac{1}{4}\text{res}(\delta a\{b,\log q\}a^{-2}b^{-2}\{b,a\})\\ &=-\frac{1}{4}\text{res}(\delta\log a\{\log b,\log q\}\{\log b,\log a\}).\end{aligned}$$

The second term, T_2, equals

$$\begin{aligned}
&-\frac{1}{4}\mathrm{res}(\delta a\{\{b,\log q\},a^{-1}b^{-1}\}) = -\frac{1}{4}\mathrm{res}(\delta\log a\{\{b,\log q\},b^{-1}\})\\
&\quad+\frac{1}{4}\mathrm{res}(b^{-1}\delta a\{\{b,\log q\},a^{-1}\}) = \frac{1}{4}\mathrm{res}(b^{-1}\delta\log a\{\{b,\log q\},\log b\})\\
&\quad+\frac{1}{4}\mathrm{res}(b^{-1}\delta\log a\{\{b,\log q\},\log a\}).
\end{aligned}$$

One uses identities

$$\{\{b,\log q\},\log b\} = \{b,\{\log q,\log b\}\}$$

and

$$b^{-1}\{\{b,\log q\},\log a\} = -\{\log a,\{\log b,\log q\}\}+\{\log b,\log q\}\{\log b,\log a\}$$

to get

$$\begin{aligned}
T_2 = -\frac{1}{4}&\mathrm{res}(\delta\log a\{\log(ab),\{\log b,\log q\}\})\\
&+\frac{1}{4}\mathrm{res}(\delta\log a\{\log b,\log q\}\{\log b,\log a\})
\end{aligned}$$

and

$$T_1+T_2 = -\frac{1}{4}\mathrm{res}(\delta\log a\{\log(ab),\{\log b,\log q\}\}).$$

By (3.1),

$$\begin{aligned}
T_3 &= \frac{1}{2}\sigma_Q(AB,(\delta A)B)+\frac{1}{2}\sigma_Q(BA,B\delta A)-\sigma_Q(A,\delta A)\\
&= \frac{1}{6}\mathrm{res}(\delta\log a\{\log(ab),\{\log(ab),\log q\}\})\\
&\quad-\frac{1}{6}\mathrm{res}(\delta\log a\{\log a,\{\log a,\log q\}\})\\
&= \frac{1}{6}\mathrm{res}(\delta\log a\{\log a,\{\log b,\log q\}\})\\
&\quad+\frac{1}{6}\mathrm{res}(\delta\log a\{\log b,\{\log a,\log q\}\})\\
&\quad+\frac{1}{6}\mathrm{res}(\delta\log a\{\log b,\{\log b,\log q\}\}).
\end{aligned}$$

Finally,

$$\begin{aligned}
\delta\frac{\kappa_Q(A,B)+\kappa_Q(B,A)}{2} &= -\frac{1}{12}\mathrm{res}(\delta\log a\{\log b,\{\log b,\log q\}\})\\
&\quad-\frac{1}{12}\mathrm{res}(\delta\log a\{\log a,\{\log b,\log q\}\})\\
&\quad+\frac{1}{6}\mathrm{res}(\delta\log a\{\log b,\{\log a,\log q\}\}).
\end{aligned}$$

Consider now the family $A(t) = A^t$, the operator B being fixed. Then $\log a(t) = t \log a$, and

$$\begin{aligned}(3.11)\qquad &\frac{d}{dt}\left(\frac{\kappa_Q(A(t),B)+\kappa_Q(B,A(t))}{2}\right)\\ &= -\frac{1}{12}\operatorname{res}(\log a\{\log b, \{\log b, \log q\}\})\\ &\quad -\frac{t}{12}\operatorname{res}(\log a\{\log a, \{\log b, \log q\}\})\\ &\quad +\frac{t}{6}\operatorname{res}(\log a\{\log b, \{\log a, \log q\}\}).\end{aligned}$$

The second term on the right in (3.11) vanishes because

$$\operatorname{res}(\log a\{\log a, \{\log b, \log q\}\}) = \frac{1}{2}\operatorname{res}(\{\log^2 a, \{\log b, \log q\}\}) = 0.$$

One integrates (3.11) from 0 to 1:

$$\begin{aligned}(3.12)\qquad \frac{\kappa_Q(A,B)+\kappa_Q(B,A)}{2} &= \frac{1}{12}\operatorname{res}(\log a\{\log b, \{\log(a/b), \log q\}\})\\ &= \frac{1}{12}\operatorname{res}(\{\log a, \log b\}\{\log(a/b), \log q\}\}).\end{aligned}$$

(Note that the expression on the right in (3.12) is symmetric in (a,b), as it should be.)

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Department of Mathematics
The University of Arizona
617 N. Santa Rita Ave., Tucson, AZ 85721
E-mail address: friedlan@math.arizona.edu

Departament of Mathematics
Massachusetts Institute of Technology
77 Massachusetts Avenue, Cambridge. MA 02139
E-mail address: vwg@math.mit.edu