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ON THE UNIRULEDNESS OF STABLE BASE LOCI

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Abstract

We discuss the uniruledness of various base loci of linear systems related to the canonical divisor. In particular we prove that the stable base locus of the canonical divisor of a smooth projective variety of general type is covered by rational curves.

1. Introduction

The aim of this paper is to propose a way to study the stable base loci and its variants for divisors on smooth projective varieties. The *stable base locus* of a divisor D is the Zariski-closed subset $\mathrm{SBs}\,(D) = \bigcap_{m\geq 1} \mathrm{Bs}\,|mD|$, where $\mathrm{Bs}\,|mD|$ denotes the base locus of the linear system $|mD|$. The *ample locus* is the Zariski-open subset $\mathrm{Amp}\,(D)$, where the linear system $|mD|$ gives an embedding around every point of $\mathrm{Amp}\,(D)$ for a sufficiently large m. The *non-ample locus* $\mathrm{NAmp}\,(D)$ is the complement of $\mathrm{Amp}\,(D)$. This can be written as $\mathrm{NAmp}\,(D) = \bigcap \mathrm{SBs}\,(mD - A)$ for any fixed ample divisor A, where the intersection is taken over all positive integers m and the intersection does not depend on the choice of A. The *non-nef locus* is similary defined by $\mathrm{NNef}\,(D) := \bigcup \mathrm{SBs}\,(mD + A)$. The non-nef locus $\mathrm{NNef}\,(D)$ is empty if and only if D is nef. Nonetheless, this terminology is misleading, because we are not saying that $x \in \mathrm{NNef}\,(D)$ if and only if there exists a curve $C \ni x$ with $D \cdot C < 0$. It is not known whether $\mathrm{NNef}\,(D)$ is Zariski-closed, but it is at most a countable union of irreducible subvarieties. We obtain immediately that $\mathrm{NNef}\,(D) \subset \mathrm{SBs}\,(D) \subset \mathrm{NAmp}\,(D)$. The main application is as follows:

Theorem 1.1. *Let X be a smooth projective variety of general type. Then every irreducible component of* (i) $\mathrm{SBs}\,(K_X)$, (ii) $\mathrm{NAmp}\,(K_X)$, *or* (iii) $\mathrm{NNef}\,(K_X)$ *of the canonical divisor K_X is uniruled.*

Theorem 1.2. *Let X be a smooth projective variety with a numerically trivial canonical divisor, and let L be a big divisor on X. Then every irreducible component of* (i) $\mathrm{SBs}\,(L)$, (ii) $\mathrm{NAmp}\,(L)$, *or* (iii) $\mathrm{NNef}\,(L)$ *is uniruled.*

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Theorem 1.3. *Let X be a smooth projective variety with a big anti-canonical divisor $-K_X$. Then every irreducible component of* (i) $\mathrm{SBs}\,(-K_X)$*, or* (ii) $\mathrm{NAmp}\,(-K_X)$*, which is not contained in* $\mathrm{NNef}\,(-K_X)$*, is uniruled.*

In the last theorem, we note that some irreducible component of $\mathrm{NNef}\,(-K_X)$ can be non-uniruled in general (see Example 6.4). We will also discuss refinements of these theorems, and the uniruledness of subvarieties defined by asymptotic multiplier ideal sheaves, for example by $\mathcal{J}(c \cdot \|K_X\|)$ (see §6).

In case when the relevant big divisor in above theorems (K_X, L and $-K_X$ respectively) is moreover nef, then the divisor is semi-ample by Kawamata-Shokurov's base point freeness theorem ([**13**, §3-1]). Hence we are only concerned with the non-ample locus. In that case, the uniruledness is already known by Kawamata as a special case of [**11**, Theorem 2].

This paper is motivated by a conjecture of Ueno (see [**19**, p. 372]), which predicts that every divisorial component of $\mathrm{SBs}\,(K_X)$ in a case when K_X is big has negative Kodaira dimension. According to the minimal model program ([**13**]), Ueno's conjecture can be rephrased as the uniruledness of the divisorial components of $\mathrm{SBs}\,(K_X)$. Talking about the uniruledness, it is known that a smooth projective variety is uniruled if and only if the canonical divisor is not pseudo-effective. This is a consequence of a numerical criterion of uniruledness due to Miyaoka and Mori [**15**], and of a numerical characterization of pseudo-effectivity of divisors due to Boucksom, Demailly, Paun and Peternell [**3**]. We are also motivated by related results by Wilson [**19**, 3.3] [**20**, 2.3], Kawamata [**11**], Huybrechts [**9**, 5.2] (see [**2**, §4]), and by Boucksom [**2**, Proposition 4.7] and so on.

As for the technical side, the key to this paper is an extension statement from [**18**, §4] (§3). Let us explain it very briefly. Let L be a big divisor on X, and let V be an irreducible component of $\mathrm{SBs}\,(L)$. After taking a multiple of L, we obtain a decomposition $\ell L \sim_{\mathbb{Q}} A + D$ into an ample $\mathbb{Q}$-divisor A and an effective $\mathbb{Q}$-divisor D such that V is a "maximal" log-canonical center for the pair (X, D) (§4). By Kawamata's subadjunction theorem [**12**], $(K_X + D)|_V$ dominates K_V, and hence $(K_X + \ell L)|_V$ dominates $K_V + A|_V$. Then the extension statement [**18**, §4] shows that a subsystem of $|m(K_X + \ell L)|_V|$, which is something like $|m(K_V + A|_V)|$, can be extended to X for large m. Hence if K_V is pseudo-effective, we see $K_V + A|_V$ is big, and then V is not contained in $\mathrm{SBs}\,(K_X + \ell L)$. We need to consider the balance of $\mathrm{SBs}\,(L)$ and $\mathrm{SBs}\,(K_X + \ell L)$. For example, in case $L = K_X$, V is contained in both $\mathrm{SBs}\,(L)$ and $\mathrm{SBs}\,(K_X + \ell L)$. Then this concludes that K_V can not be pseudo-effective, and hence V is uniruled by [**15**] and [**3**]. For this possible application of the extension statement, we are inspired by a paper by Hacon and M[c]Kernan [**8**]. They apply their extension statement [**7**, 3.17] to the study of the loci where "$-K_X$ is relatively big".

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2. Stable Base Locus and Asymptotic Invariant

We recall some basic notions, fix some notations, and also make a remark on a structure of non-ample loci. We work over the complex number field.

2.1. Stable base locus and its variant.

(1) We refer [**5**, §1] and [**17**, III, V §1] for general properties of $\mathrm{SBs}\,(D), \mathrm{NAmp}\,(D)$ and $\mathrm{NNef}\,(D)$. In [**5**], these are denoted by $\mathbf{B}(D) = \mathrm{SBs}\,(D), \mathbf{B}_+(D) = \mathrm{NAmp}\,(D)$ and $\mathbf{B}_-(D) = \mathrm{NNef}\,(D)$. These base loci can be defined not only for integral divisors, but also any $\mathbb{Q}$-divisors. (We will not use these for $\mathbb{R}$-divisors.) Since $\mathrm{NNef}\,(D)$ might be a countable union of irreducible subvarieties ([**5**, 1.19]), we might say that an irreducible subvariety V is an irreducible component of $\mathrm{NNef}\,(D)$, if V is maximal among all irreducible subvarieties contained in $\mathrm{NNef}\,(D)$.

(2) We make a remark on non-ample loci. To state a result, we need to prepare some notations. Let L be a big divisor on a smooth projective variety X, and let m be a positive integer such that $\mathrm{Bs}\,|mL| \neq X$. Let $\Phi_m = \Phi_{|mL|} : X \dashrightarrow \mathbb{P}^N$ be the rational map associated to $|mL|$, and denote by $Y \subset \mathbb{P}^N$ the Zariski closure of the image $\Phi_m(X \setminus \mathrm{Bs}\,|mL|)$. We take a birational morphism $\mu : X' \longrightarrow X$ from a smooth projective variety X' such that μ is biregular over $X \setminus \mathrm{Bs}\,|mL|$, and $\mu^*(|mL|) = |L'| + E$ with a base point free linear system $|L'|$ and with the fixed component E. We have an induced morphism $\Phi'_m : X' \longrightarrow Y \subset \mathbb{P}^N$ such that $\Phi'_m = \Phi_m \circ \mu$ on $\mu^{-1}(X \setminus \mathrm{Bs}\,|mL|)$ and ${\Phi'_m}^*\mathcal{O}_{\mathbb{P}^N}(1) = \mathcal{O}_{X'}(L')$. We take the Stein factorization of $\Phi'_m : X' \longrightarrow Y$ into $\Psi_m : X' \longrightarrow Y'$ and $\nu : Y' \longrightarrow Y$ for a normal projective variety Y' with an ample invertible sheaf $\nu^*\mathcal{O}_{\mathbb{P}^N}(1)$. We set

$$
\begin{aligned}
S_m &= \{x \in X \setminus \mathrm{Bs}\,|mL|;\ \dim \Psi_m^{-1}(\Psi_m(\mu^{-1}(x))) > 0\},\\
\Sigma_m &= \mathrm{Bs}\,|mL| \cup S_m.
\end{aligned}
$$

These sets do not depend on the choice of $\mu : X' \longrightarrow X$, in fact $S_m = \{x \in X \setminus \mathrm{Bs}\,|mL|;\ \dim_x \Phi_m^{-1}(\Phi_m(x)) > 0\}$. In this setting, we recall the following classically known result:

Lemma 2.1. *Assume $\Sigma_m \neq X$. Then for any given divisor G on X, one has $\mathrm{Bs}\,|kL - G| \subset \Sigma_m$ for every large integer k. In particular, by taking G to be ample, one has $\mathrm{NAmp}\,(L) \subset \Sigma_m$.*

See, for example, [**4**, 7.2 (ii)] for the proof. The statement is not exactly the same as [**4**, 7.2 (ii)]. However, the proof goes through without any essential changes, by passing to the Stein factorization as above.

By definition, $S_m = \Sigma_m \setminus \mathrm{Bs}\,|mL|$ has no isolated points, but it has a "non-trivial" fiber structure. More precisely we can show the following, by a simple geometric argument.

Lemma 2.2. *Let L be a big divisor on a smooth projective variety X. Then*

(1) $\mathrm{NAmp}\,(L) \setminus \mathrm{SBs}\,(L)$ *has no isolated points.*
(2) *Let V be an irreducible subvariety of positive dimension of X. Assume that $V \not\subset \mathrm{Bs}\,|L|$ and that the restriction of the rational map $\Phi_{|L|} : X \dashrightarrow \mathbb{P}^\ell$ on V gives a generically finite map. Then V is not an irreducible component of* $\mathrm{NAmp}\,(L)$.

As examples show, such V in (2) can be contained in $\mathrm{NAmp}\,(L)$.

Proof. (0) We recall that $\mathrm{NAmp}\,(L) = \bigcap \mathrm{Supp}\, E$, where the intersection is taken over all decompositions $D = A + E$ into an ample $\mathbb{Q}$-divisor A and an effective $\mathbb{Q}$-divisor E ([**5**, 1.2]). By the Noetherian property, the intersection is in fact a finite intersection. Thus we can take a large integer m such that $\mathrm{Bs}\,|mL| = \mathrm{SBs}\,(L)$, and the rational map $\Phi_m = \Phi_{|mL|} : X \dashrightarrow \mathbb{P}^N$ gives an embedding on $\mathrm{Amp}\,(L) = X \setminus \mathrm{NAmp}\,(L)$. Associated to this $\Phi_m : X \dashrightarrow \mathbb{P}^N$, we have the subsets S_m and Σ_m of X defined as above. Since Φ_m gives an embedding of $\mathrm{Amp}\,(L)$, we see $\Sigma_m \subset \mathrm{NAmp}\,(L)$. Combining with Lemma 2.1, we have $\mathrm{NAmp}\,(L) = \Sigma_m$. In particular $S_m = \mathrm{NAmp}\,(L) \setminus \mathrm{SBs}\,(L)$. We use this setting to show our assertions.

(1) follows from the fact that S_m has no isolated points.

(2) Let $V_0 \subset V$ be a non-empty Zariski open subset such that $V_0 \cap \mathrm{Bs}\,|mL| = \emptyset$, and the indeced morphism $\Phi_m|_{V_0} : V_0 \longrightarrow \Phi_m(V_0)$ is finite. By our assumption, we can find such V_0. For every $x_0 \in V_0$, we have $\dim(\Phi_m^{-1}(\Phi_m(x_0)) \cap V_0) = 0$.

Assume that $V \subset \mathrm{NAmp}\,(L) = \Sigma_m = \mathrm{Bs}\,|mL| \cup S_m$. Then we see $V_0 \subset S_m$. Moreover, for every $x_0 \in V_0$, if we regard it as $x_0 \in S_m$, we see that x_0 is not isolated in $\Phi_m^{-1}(\Phi_m(x_0))$. Hence V is not an irreducible component of $\Sigma_m = \mathrm{NAmp}\,(L)$. q.e.d.

Remark 2.3. It is known that for any divisor D on a smooth projective variety, $\mathrm{SBs}\,(D)$ and $\mathrm{NAmp}\,(D)$ have no isolated points ([**6**, 1.1]). This is based on a result of Zariski, whose proof is rather algebraic (see [**1**, 9.17] for a proof).

2.2. Asymptotic invariant. We recall a classical asymptotic numerical invariant of divisors. We will refer its modern treatment to [**5**, §2], [**17**, III §§1-2] (see also [**2**, §3]). Unless otherwise stated, we will discuss on a smooth projective variety X.

(1) Let V be an irreducible subvariety of X. For a big divisor D on X, we define

$$\sigma_V(D) = \lim_{m\to\infty} \frac{\mathrm{mult}_V|mD|}{m}.$$

Here $\mathrm{mult}_V|mD|$ is the multiplicity of a general member of $|mD|$ along V. We can see the limit in fact exists ([**5**, 2.2] [**17**, III §1.a]). This $\sigma_V(D)$ can be defined for any big $\mathbb{Q}$-divisor D by the homogeneity $\sigma_V(D) = \sigma_V(mD)/m$ for a large and divisible m. This is called the *asymptotic order of vanishing* of D along V.

(2) We can extend the asymptotic invariant for any pseudo-effective $\mathbb{Q}$-divisors. For a pseudo-effective $\mathbb{Q}$-divisor D, we define $\sigma_V(D) = \lim_{\varepsilon\to 0}\sigma_V(D+\varepsilon A)$, where A is any fixed ample divisor and $\varepsilon > 0$ are rational numbers. The limit exists and does not depend on the choice of ample divisors A. We have a subadditivity: $\sigma_V(D_1+D_2) \leq \sigma_V(D_1)+\sigma_V(D_2)$ for pseudo-effective $\mathbb{Q}$-divisors D_1 and D_2 ([**5**, 2.4], [**17**, III.1.1]).

(3) We reformulate [**5**, 2.8], [**17**, III.2.3(2), V.1.5], in the following way to fit our purposes. The main statement is the equivalence of (i) and (ii). Others follow from definitions and this main statement.

Lemma 2.4. *Let D be a pseudo-effective $\mathbb{Q}$-divisor, and let V be an irreducible subvariety of X.*

(1) *Let H be an ample $\mathbb{Q}$-divisor. Then the following four conditions are equivalent:*
  (i) $V \subset \mathrm{NNef}(D)$.
  (ii) $\sigma_V(D) > 0$.
  (iii) $V \subset \mathrm{NNef}(tD+H)$ *for every large rational number t.*
  (iv) $V \subset \mathrm{SBs}(tD+H)$ *for every large rational number t.*

(2) *The following four conditions are equivalent:*
  (i) $V \not\subset \mathrm{NNef}(D)$.
  (ii) $\sigma_V(D) = 0$.
  (iii) $V \not\subset \mathrm{NNef}(D+A)$ *for any ample $\mathbb{Q}$-divisor A.*
  (iv) $V \not\subset \mathrm{SBs}(D+A)$ *for any ample $\mathbb{Q}$-divisor A.*

Let us observe the following lemma as a corollary.

Lemma 2.5. *Let D be a pseudo-effective divisor, and let V be an irreducible subvariety such that $V \subset \mathrm{NNef}(D)$. Let H be an ample divisor, and consider a real number*

$$t_0 = \sup\{0 \leq t \in \mathbb{Q};\ \sigma_V(tD+H) = 0\}.$$

Then

(1) $0 < t_0 < +\infty$.
(2) *Let $t \geq 0$ be a rational number. Then $V \not\subset \mathrm{NAmp}(tD+H)$ if and only if $t < t_0$. In particular, for a rational number $t < t_0$, one*

has $tD + H \sim_{\mathbb{Q}} A + E$ *for an ample* $\mathbb{Q}$*-divisor* A *and an effective* $\mathbb{Q}$*-divisor* E *with* $V \not\subset \operatorname{Supp} E$ ([**5**, 1.2]).

Proof. (1) Since $tD + H$ is ample for a sufficiently small $t > 0$, it follows that $t_0 > 0$. On the other hand, by Lemma 2.4, $\sigma_V(D) > 0$ is equivalent to $t_0 < +\infty$.

(2) Assume $V \not\subset \operatorname{NAmp}(tD + H)$. Then we have $tD + H \sim_{\mathbb{Q}} A + E$ for an ample $\mathbb{Q}$-divisor A and an effective $\mathbb{Q}$-divisor E with $V \not\subset \operatorname{Supp} E$ ([**5**, 1.2]). We take a small rational number $\varepsilon > 0$ so that $A - \varepsilon H$ is still ample. Then $tD + (1-\varepsilon)H \sim_{\mathbb{Q}} (A - \varepsilon H) + E$, and in particular $V \not\subset \operatorname{NAmp}((1-\varepsilon)^{-1}tD + H)$. Hence $t < (1-\varepsilon)^{-1}t \le t_0$.

We take a rational number s so that $t < s < t_0$. We see $\sigma_V(tD + \frac{t}{s}H) = 0$. By Lemma 2.4, we have $V \not\subset \operatorname{SBs}((tD+\frac{t}{s}H)+\varepsilon H)$ for any $0 < \varepsilon < \frac{t}{s}$, and hence there exists an effective $\mathbb{Q}$-divisor $E \sim_{\mathbb{Q}} (tD+\frac{t}{s}H)+\varepsilon H$ such that $V \not\subset \operatorname{Supp} E$. Then $tD + H \sim_{\mathbb{Q}} (1 - \frac{t}{s} - \varepsilon)H + E$, and hence $V \not\subset \operatorname{NAmp}(tD + H)$. q.e.d.

2.3. Multiplier ideal. We recall the notion of multiplier ideal sheaves and singularities of pairs. We refer to [**14**, Chapters 9, 11] for the basics on these topics.

For a real number α, we let $\llcorner \alpha \lrcorner$ be the largest integer which is less than or equal to α, and let $\ulcorner \alpha \urcorner$ be the smallest integer which is greater than or equal to α. We also use the notation $\llcorner B \lrcorner$ and $\ulcorner B \urcorner$ for $\mathbb{R}$-divisors B on smooth varieties.

In the rest of this subsection, we let X be a smooth variety, D be an effective $\mathbb{Q}$-divisor, and let L be a divisor on X. Associated to a coherent ideal sheaf $\mathcal{J} \subset \mathcal{O}_X$, we denote by $V\mathcal{J} = \operatorname{Supp} \mathcal{O}_X/\mathcal{J}$ the co-support of $\mathcal{J}$.

(1) [**14**, 9.2.1]. Let $\mu : X' \longrightarrow X$ be a log-resolution of D, namely $\mu : X' \longrightarrow X$ is a projective birational morphism from a smooth variety X' such that $\operatorname{Supp}(\mu^*D + \operatorname{Exc}(\mu))$ is a divisor with simple normal crossing. Here $\operatorname{Exc}(\mu)$ denotes the sum of the exceptional divisors. Then the *multiplier ideal sheaf* of D is defined to be $\mathcal{J}(D) = \mathcal{J}(X, D) = \mu_*\mathcal{O}_{X'}(K_{X'/X} - \llcorner \mu^*D \lrcorner) \subset \mathcal{O}_X$.

(2) The pair (X, D) is said to have only *Kawamata log-terminal singularities, klt* for short (resp. *log-canonical singularities, lc* for short), if $\mathcal{J}(X, D) = \mathcal{O}_X$ (resp. $\mathcal{J}(X, (1-\varepsilon)D) = \mathcal{O}_X$ for all rational numbers $0 < \varepsilon < 1$). The pair (X, D) is said to be *klt* (resp. *lc*) *at* $x \in X$, if $(U, D|_U)$ is klt (resp. lc) for some Zariski open neighbourhood U of x.

(3) We set $\operatorname{Nklt}(X, D) = V\mathcal{J}(X, D) \subset X$ with the reduced structure, and call it the *non-klt locus* of (X, D). An irreducible component W of $\operatorname{Nklt}(X, D)$ is called a *maximal lc center* for (X, D) if there exists a Zariski open subset $U \subset X$ such that $W \cap U \neq \emptyset$ and $(U, D|_U)$ is lc.

(4) [**14**, 9.2.10]. Let $V \subset H^0(X, \mathcal{O}_X(L))$ be a non-zero vector subspace. We denote by $|V| \subset |L|$ the associated linear subsystem. Let $\mu : X' \longrightarrow X$ be a log-resolution of $|V|$ such that X' is smooth and $\mu^*|V| = |W| + F$, where F is the fixed part and $\mathrm{Supp}\,(F + \mathrm{Exc}(\mu))$ is simple normal crossing, and $W \subset H^0(X', \mathcal{O}_{X'}(\mu^*L - F))$ defines a base point free linear system. Given a rational number $c > 0$, the *multiplier ideal sheaf corresponding to* c *and* $|V|$ is defined to be $\mathcal{J}(c \cdot |V|) = \mathcal{J}(X, c \cdot |V|) = \mu_* \mathcal{O}_{X'}(K_{X'/X} - \llcorner cF \lrcorner) \subset \mathcal{O}_X$. In case $V = 0$, we set $\mathcal{J}(c \cdot |V|) = 0$ for every $c > 0$.

(5) [**14**, 11.1.2]. Assume that X is projective, and L is big. Let $c > 0$ be a rational number, and let p be a positive integer. Then $\mathcal{J}(\frac{c}{p} \cdot |pL|) \subset \mathcal{J}(\frac{c}{pk} \cdot |pkL|)$ holds for every integer $k > 0$ ([**14**, 11.1.1]). The *asymptotic multiplier ideal sheaf associated to* c *and* L, $\mathcal{J}(c \cdot \|L\|) = \mathcal{J}(X, c \cdot \|L\|) \subset \mathcal{O}_X$, is defined to be the unique maximal member among the family of ideals $\{\mathcal{J}(\frac{c}{p} \cdot |pL|)\}_{p \in \mathbb{N}}$. We set $\mathcal{J}(c \cdot \|L\|) = \mathcal{O}_X$ for $c = 0$.

Above these multiplier ideal sheaves in (1), (4), and (5) are indepenedent of the log-resolution used to construct them ([**14**, 9.2.18]).

(6) We conclude this section by noting a fundamental relation [**5**, 2.10].

Lemma 2.6. *Assume that* X *is projective, and* L *is big. Then* $\mathrm{NNef}\,(L) = \bigcup_{m \in \mathbb{N}} V\mathcal{J}(\|mL\|)$.

3. Application of Extension Theorem

The following theorem is the key extension statement from [**18**]. As we will now explain, the proof of [**18**, 4.5] in fact proves Theorem 3.1 to follow, even if the latter looks stronger. We only give an outline of the proof, and refer to the original article for details.

Theorem 3.1. *Let* X *be a smooth projective variety,* V *be a smooth irreducible subvariety of positive dimension, and let* L *be a divisor on* X*. Assume that there exists a decomposition* $L \sim_{\mathbb{Q}} A + D$ *into* (i) *an ample* $\mathbb{Q}$*-divisor* A*, and* (ii) *an effective* $\mathbb{Q}$*-divisor* D *such that* V *is a maximal lc center for the pair* (X, D).

If K_V *is pseudo-effective, the linear system* $|m(K_X + L)|$ *on* X *separates two general distinct points on* V *for every large and divisible integer* m.

Proof. We will extract the proof from that of [**18**, 4.5].

(1) The case when V is a divisor (see the proof of [**18**, 4.7]). We take a log-resolution $\mu : Y \longrightarrow X$ of D. We can write $\mu^*D = S + F$, where S is the strict transform of V, and where F is an effective $\mathbb{Q}$-divisor which is not containing S and $\mathrm{Supp}\,(S + F)$ is a simple normal crossing. Then $K_Y + \mu^*L - \llcorner F \lrcorner \sim_{\mathbb{Q}} K_Y + S + (F - \llcorner F \lrcorner) + \mu^*A$. We note that $S \not\subset \mathrm{NAmp}\,(\mu^*A)$ (namely S is in μ^*A-general position in the

terminology in [**18**, §2.4]), and that the pair $(S, (F - \llcorner F \lrcorner)|_S)$ is klt. Then [**18**, 4.1] implies that the restriction map

$$H^0(Y, \mathcal{O}_Y(m(K_Y + \mu^* L - \llcorner F \lrcorner))) \longrightarrow H^0(S, \mathcal{O}_S(m(K_S + (F - \llcorner F \lrcorner)|_S + (\mu^* A)|_S)))$$

is surjective for every $m > 0$. Since K_S is pseudo-effective and the $\mathbb{Q}$-divisor $(F - \llcorner F \lrcorner)|_S + (\mu^* A)|_S$ on S is big, the linear system $|m(K_S + (F - \llcorner F \lrcorner)|_S + (\mu^* A)|_S)|$ separates two general distinct points on S for every large and divisible integer m. Since the divisor $\llcorner F \lrcorner$ is effective and integral, we have a natural injection:

$$H^0(Y, \mathcal{O}_Y(m(K_Y + \mu^* L - \llcorner F \lrcorner))) \longrightarrow H^0(Y, \mathcal{O}_Y(m(K_Y + \mu^* L))) \cong H^0(X, \mathcal{O}_X(m(K_X + L))).$$

The last isomorphism is obtained by push-down μ_*. Noting that $\mu(\operatorname{Supp} \llcorner F \lrcorner)$ does not contain V, we see that the linear system $|m(K_X + L)|$ separates two general distinct points on V for every large and divisible integer m.

(2) The case when $\operatorname{codim} V > 1$. We note that the conclusion does not depend on a decomposition $L \sim_{\mathbb{Q}} A + D$ satisfying (i) and (ii). By taking another decomposition of L if necessary, we can assume that there exists a log-resolution $\mu : Y \longrightarrow X$ of D with only one place S of log-canonical singularities for the pair (X, D) dominating V ([**18**, 4.8]). We can write $\mu^*(K_X + D) \sim_{\mathbb{Q}} K_Y + S + F_Y$ with the properties in [**18**, 4.9]. We denote by $f = \mu|_S : S \longrightarrow V$, $F := F_Y|_S$ and $M = (K_X + D)|_V - K_V$. Then $K_S + F \sim_{\mathbb{Q}} f^*(K_V + M)$ and $(K_X + L)|_V = K_V + M + A|_V$ ([**18**, 4.12]).

We apply a flattening technique for f ([**18**, 4.14]), and then we have a birational morphism $\tau : V' \longrightarrow V$ (resp. $\tau' : S' \longrightarrow S$) from a smooth projective variety V' (resp. S'), and a morphism $f' : S' \longrightarrow V'$ which is compatible with other morphisms, with certain properties ([**18**, 4.15]). We obtain the following commutative diagram ([**18**, 4.16]):

$$\begin{array}{ccccc} S' & \xrightarrow{\tau'} & S & \subset & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mu} \\ V' & \xrightarrow[\tau]{} & V & \subset & X \end{array}$$

Let us denote by $j_V : V \longrightarrow X$ (resp. $j_S : S \longrightarrow Y$) the inclusion, and $A_{V'} = \tau^*(A|_V)$. We set $F' = \tau'^* F - K_{S'/S}$ and $M' = \tau^* M - K_{V'/V}$. Then $K_{S'} + F' \sim_{\mathbb{Q}} f'^*(K_{V'} + M')$ and $K_{V'} + M' + A_{V'} = \tau^*(K_V + M + A|_V) = \tau^*((K_X + L)|_V)$.

We apply Kawamata's positivity result [**12**, Theorem 2] for the fiber space $f' : S' \longrightarrow V'$ with the f'-$\mathbb{Q}$-trivial log-canonical divisor $K_{S'} + F'$, and then we have a $\mathbb{Q}$-divisor Δ' on V' such that $M' - \Delta'$ is nef on V' ([**18**, 4.17]). Since $K_{V'}$ is pseudo-effective by our assumption here and since $A_{V'}$ is (nef and) big, the linear system $|m(K_{V'} + M' - \Delta' + A_{V'})|$ separates two general distinct points on V' for every large and divisible integer m. Applying the extension theorem [**18**, 4.1], we have an injection

$$f'^{*}H^0(V', \mathcal{O}_{V'}(m(K_{V'} + M' - \Delta' + A_{V'}))) \\ \longrightarrow (\mu \circ j_S \circ \tau')^{*} H^0(X, \mathcal{O}_X(m(K_X + L)))$$

for every large and divisible integer m ([**18**, 4.19(2)]). This injection is given by a composition of multiplications by effective divisors with respect to τ' or μ, by disregarding the effects of effective exceptional divisors, and by restricting sections on Y to S, namely the extension from S to Y ([**18**, 4.1] or [**18**, 4.11(1)]). Since the linear system $|m(K_{V'} + M' - \Delta' + A_{V'})|$ separates two general distinct points on V' for every large and divisible integer m, so does the induced linear system $\tau^{*} j_V^{*} |m(K_X + L)|$ on V'. q.e.d.

A variety X is said to be *uniruled* if there exists a dominant rational map $Y \times \mathbb{P}^1 \dashrightarrow X$ from a product of $\mathbb{P}^1$ and a variety Y of $\dim Y = \dim X - 1$. By definition, a uniruled variety has positive dimension. We quote a uniruledness criterion in a birational setting.

Theorem 3.2 ([**15**], [**3**]). *A proper algebraic variety X is uniruled, if and only if there exists a smooth projective model X' whose canonical divisor $K_{X'}$ is not pseudo-effective.*

Using this criterion, we will use Theorem 3.1 in the following form.

Corollary 3.3. *Let X be a smooth projective variety, V be an irreducible subvariety, and let L be a divisor on X. Assume that there exists a decomposition $L \sim_{\mathbb{Q}} A + D$ into* (i) *an ample $\mathbb{Q}$-divisor A, and* (ii) *an effective $\mathbb{Q}$-divisor D such that V is a maximal lc center for the pair (X, D). Then*

(1) *V is uniruled, provided $V \subset \mathrm{SBs}\,(K_X + L)$.*
(2) *V is uniruled, provided that $K_X + L$ is big and that V is an irreducible component of* $\mathrm{NAmp}\,(K_X + L)$.

Proof. We shall prove by contradiction. Namely we shall claim that

(1) if V is not uniruled, then $V \not\subset \mathrm{SBs}\,(K_X + L)$, and that

(2) if V is not uniruled and if $K_X + L$ is big, then V is not an irreducible component of $\mathrm{NAmp}\,(K_X + L)$.

(0) We start with a remark in the case of $\dim V = 0$. The point V is not uniruled. By (ii), the point V is isolated in the non-klt locus $\mathrm{Nklt}\,(X, D)$. Then Nadel's vanishing $H^1(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(X, D)) = 0$ (see for example [**14**, 9.4.8]) implies that $V \notin \mathrm{Bs}\,|K_X + L|$, and hence $V \notin \mathrm{SBs}\,(K_X + L)$. We now assume that $K_X + L$ is big. We have either $V \notin \mathrm{NAmp}\,(K_X + L)$ or $V \in \mathrm{NAmp}\,(K_X + L) \setminus \mathrm{SBs}\,(K_X + L)$. By

taking into account Lemma 2.2 (1), the point V is not an irreducible component of $\mathrm{NAmp}\,(K_X+L)$ in any way.

Hereafter we consider the case when $\dim V > 0$. We consider the following change of models. Let $\mu : X' \longrightarrow X$ be an embedded resolution of V, and let $V' \subset X'$ be the strict transform of V. We see that the $\mathbb{Q}$-divisor μ^*A is nef and big, and $V' \not\subset \mathrm{NAmp}\,(\mu^*A)$, and that V' is a maximal lc center for the pair (X', μ^*D). By an equivalent definition of non-ample loci ([**5**, 1.2]), we have a decomposition $\mu^*A \sim_{\mathbb{Q}} A' + D_0$ into an ample $\mathbb{Q}$-divisor A' and an effective $\mathbb{Q}$-divisor D_0 with $V' \not\subset \mathrm{Supp}\, D_0$. Hence we have a decomposition $\mu^*L \sim_{\mathbb{Q}} A' + D'$ with $D' = D_0 + \mu^*D$ such that V' is a maximal lc center for the pair (X', D').

(1) We see that V is not uniruled if and only if $K_{V'}$ is pseudo-effective (by Theorem 3.2), and that $V \not\subset \mathrm{SBs}\,(K_X+L)$ if and only if $V' \not\subset \mathrm{SBs}\,(K_{X'}+\mu^*L)$. Then applying Theorem 3.1 on X' with the smooth model V' and the decomposition $\mu^*L \sim_{\mathbb{Q}} A' + D'$, we obtain (1).

(2) Here we assume that V is not uniruled and that $K_X + L$ is big. We also see that V is an irreducible component of $\mathrm{NAmp}\,(K_X+L)$ if and only if V' is an irreducible component of $\mathrm{NAmp}\,(K_{X'}+\mu^*L)$. Hence as in the proof of (1), we may assume that V is smooth, and K_V is pseudo-effective by Theorem 3.2. Since $\dim V > 0$ and K_V is pseudo-effective, by Theorem 3.1, we can take a positive integer m such that the linear system $|m(K_X+L)|$ on X separates two general distinct points on V. Then by Lemma 2.2 (2), V is not an irreducible component of $\mathrm{NAmp}\,(K_X+L)$. q.e.d.

4. Decomposition of Big Divisor

According to Corollary 3.3, a special decomposition of a big divisor concludes a property of the stable base locus, or the non-ample locus of the adjoint divisor. Here we construct such decompositions as a preliminary step, which can be seen as a refinement of the so-called Kodaira's lemma. We stress that to find an effective $\mathbb{Q}$-divisor D such that a given subvarity V is a maximal lc center for the pair (X, D) is not enough. A complementary ample part A in $L \sim_{\mathbb{Q}} A + D$ is needed. We would like to state our result in a slightly general form than we will need later in this paper. This is because it becomes more and more important to control log-canonical centers with extra ample parts, as we can see in Fujita type conjecture on adjoint bundles, the extensions of pluricanonical forms [**7**] [**18**], a recent paper by Hacon and M$^{\mathrm{c}}$Kernan on the existence of flips, and so on.

In this section, we let X be an n-dimensional smooth projective variety, and let L be a big $\mathbb{Q}$-divisor on X.

Lemma 4.1. *Let V be an irreducible subvariety of X which is contained in* $\mathrm{NNef}(L)$, *and let ε be a positive constant. Then there exists an effective $\mathbb{Q}$-divisor D such that $D \sim_{\mathbb{Q}} L$, $\mathrm{mult}_x(D) < \varepsilon$ for any $x \in X \setminus \mathrm{NNef}(L)$, and $\sigma_V(L) \leq \mathrm{mult}_V(D) < \sigma_V(L) + \varepsilon$.*

Proof. We take a positive integer m_0 such that $m_0 L$ becomes integral. We set $L' = m_0 L$. We take a positive integer p so large that $\frac{n}{\varepsilon p} < 1$ and $\mathrm{mult}_V |pL'|/p < \sigma_V(L') + \varepsilon$. By [**14**, 11.1.1], we can moreover assume that $\mathcal{J}(\frac{n}{\varepsilon p}|pL'|) = \mathcal{J}(\frac{n}{\varepsilon}\|L'\|)$ holds. We take a general member $D'_p \in |pL'|$. Then $\sigma_V(L') \leq \mathrm{mult}_V(D'_p)/p < \sigma_V(L') + \varepsilon$, and $\mathcal{J}(\frac{n}{\varepsilon p}D'_p) = \mathcal{J}(\frac{n}{\varepsilon p}|pL'|)$ holds by [**14**, 9.2.26]. We set $D' := D'_p/p \sim_{\mathbb{Q}} L'$.

We let $x \notin \mathrm{NNef}(L') = \mathrm{NNef}(L)$. We have $\mathcal{J}(\frac{n}{\varepsilon}\|L'\|)_x = \mathcal{O}_{X,x}$ by Lemma 2.6. Hence $\mathcal{J}(\frac{n}{\varepsilon}D') = \mathcal{O}_{X,x}$. We have $\mathrm{mult}_x(\frac{n}{\varepsilon}D') < n$ ([**14**, 9.3.2]), and hence $\mathrm{mult}_x(D') < \varepsilon$. Then we can see that $D := D'/m_0 \sim_{\mathbb{Q}} L$ satisfies all the properties stated in the lemma. q.e.d.

Lemma 4.2. *Let ε be a positive constant. Assume that there exist a subset $T \subset X$ and an effective $\mathbb{Q}$-divisor D such that $D \sim_{\mathbb{Q}} L$ and $\mathrm{mult}_x(D) < \varepsilon$ for any $x \in X \setminus T$. Let B be a divisor on X. Then there exists a decomposition $L \sim_{\mathbb{Q}} bB + G$ with a rational number $b > 0$, and with an effective $\mathbb{Q}$-divisor G such that $\mathrm{mult}_x(G) < 2\varepsilon$ for any $x \in X \setminus T$.*

Proof. Since L is big, by Kodaira's lemma, there exists a positive integer m_0 such that $m_0 L \sim B + E$ for some effective divisor E. We take a large integer m_1 such that $m_0 < m_1$ and $\max_{x \in X} \mathrm{mult}_x E < \varepsilon m_1$. Then $L = \frac{m_0}{m_1}L + (1 - \frac{m_0}{m_1})L \sim_{\mathbb{Q}} \frac{1}{m_1}B + \frac{1}{m_1}E + (1 - \frac{m_0}{m_1})D$. We set $b = 1/m_1$ and $G = \frac{1}{m_1}E + (1 - \frac{m_0}{m_1})D$. Then $\mathrm{mult}_x(G) < 2\varepsilon$ for any $x \in X \setminus T$. q.e.d.

Proposition 4.3. *Let V be an irreducible component of* (i) $\mathrm{SBs}(L)$ (*respectively* (ii) $\mathrm{NAmp}(L)$ *and* (iii) $\mathrm{NNef}(L)$). *Let ε be a number with $0 < \varepsilon < 1$.*

Then there exist a rational number $\alpha > 0$, and a decomposition $\alpha L \sim_{\mathbb{Q}} A + D$ into an ample $\mathbb{Q}$-divisor A and an effective $\mathbb{Q}$-divisor D such that V is a maximal lc center for the pair (X, D), and that $\mathrm{mult}_x(D) < \varepsilon$ for any $x \in X$ outside (i) $\mathrm{SBs}(L)$ (*respectively* (ii) $\mathrm{NAmp}(L)$ *and* (iii) $\mathrm{NNef}(L)$). *In case* (iii), *α can be taken so that*

$$\frac{1}{\sigma_V(L)} - \varepsilon < \alpha \leq \frac{\mathrm{codim}\, V}{\sigma_V(L)}.$$

Proof. Let H be an ample divisor on X. We denote by $d = \dim V$.

Proof of (iii). We first consider the case (iii). We note $\sigma_V(L) > 0$.

(1) Let $X' \longrightarrow X$ be the blowing-up of X along V. We take a modification $Y \longrightarrow X'$ from a smooth projective variety Y so that the induced morphism $\mu : Y \longrightarrow X$ is isomorphic over $X \setminus V$. We denote by $E_V \subset Y$ the strict transform of the exceptional divisor of $X' \longrightarrow X$.

(2) We will divide into three substeps.

(2.1) We take positive numbers $\varepsilon_1 < \varepsilon$, and then ε_2 so that

$$\frac{1}{\sigma_V(L)} - \varepsilon < \frac{n-d}{n-d+\varepsilon_1} \cdot \frac{1}{\sigma_V(L)+\varepsilon_1},$$
$$\varepsilon_2 < \frac{\varepsilon_1}{2n} \min\left\{\sigma_V(L), \sigma_V(L)^{-1}\right\}.$$

By Lemma 4.1, there exists an effective $\mathbb{Q}$-divisor F such that $F \sim_{\mathbb{Q}} L$, $\mathrm{mult}_x(F) < \varepsilon_2$ for any $x \in X \setminus \mathrm{NNef}\,(L)$, and $\sigma_V(L) \leq \mathrm{mult}_V(F) < \sigma_V(L)(1+\varepsilon_2) < \sigma_V(L) + \varepsilon_1$.

(2.2) We take a large and divisible integer m such that mF becomes integral, $\mathrm{Bs}\,|mL| = \mathrm{SBs}\,(L)$, the associated map $\Phi_{|mL|}$ is birational onto its image, and that $\sigma_V(L) \leq \mathrm{mult}_V|mL|/m \leq \mathrm{mult}_V(F)$. We denote $r = r(m) = \mathrm{mult}_V|mL| > 0$.

(2.3) We have a big $\mathbb{Q}$-divisor $M := (n-d)r^{-1}(m\mu^*L - rE_V)$, and an effective $\mathbb{Q}$-divisor $F_Y := (n-d)r^{-1}(m\mu^*F - rE_V)$ on Y with $F_Y \sim_{\mathbb{Q}} M$. We see $\mathrm{mult}_{E_V}(F_Y) = (n-d)(\frac{m}{r}\mathrm{mult}_V(F) - 1) < n(\sigma_V(L)^{-1}\sigma_V(L)(1+\varepsilon_2) - 1) < \varepsilon_1/2$. In particular, there exists a non-empty Zariski open subset $E_V^0 \subset E_V$ such that $\mathrm{mult}_y(F_Y) < \varepsilon_1/2$ for any $y \in E_V^0$. For every $y \notin \mu^{-1}(\mathrm{NNef}\,(L))$, we have $\mathrm{mult}_y(F_Y) = (n-d)\frac{m}{r}\mathrm{mult}_{\mu(y)}(F) < n\sigma_V(L)^{-1}\varepsilon_2 < \varepsilon_1/2$. In summary, $F_Y \sim_{\mathbb{Q}} M$ and $\mathrm{mult}_y(F_Y) < \varepsilon_1/2$ for any $y \in (Y \setminus \mu^{-1}(\mathrm{NNef}\,(L))) \cup E_V^0$.

(3) We apply Lemma 4.2 for the big $\mathbb{Q}$-divisor M on Y with the divisor $(B =)\mu^*H$. We obtain a decomposition $M \sim_{\mathbb{Q}} h\mu^*H + G$ with a rational number $h > 0$, and with an effective $\mathbb{Q}$-divisor G on Y such that $\mathrm{mult}_y(G) < \varepsilon_1$ for any $y \in (Y \setminus \mu^{-1}(\mathrm{NNef}\,(L))) \cup E_V^0$.

(4) We can push down the effective $\mathbb{Q}$-divisor $(n-d)E_V + G \sim_{\mathbb{Q}} \mu^*((n-d)(m/r)L - hH)$, namely we have an effective $\mathbb{Q}$-divisor D_0 on X such that

$$\mu^*D_0 = (n-d)E_V + G \text{ and that } (n-d)(m/r)L \sim_{\mathbb{Q}} hH + D_0.$$

Since $\mathrm{mult}_V D_0 = \mathrm{mult}_{E_V}((n-d)E_V + G) \geq \mathrm{codim}\,V$, the pair (X, D_0) is not klt along V ([**14**, 9.3.5]). On the other hand, $\mathrm{mult}_x(D_0) = \mathrm{mult}_{\mu^{-1}(x)}(G) < \varepsilon_1 < \varepsilon < 1$ for any $x \in X \setminus \mathrm{NNef}\,(L)$, and in particular the pair (X, D_0) is klt on $X \setminus \mathrm{NNef}\,(L)$. Then, since V is an irreducible component of $\mathrm{NNef}\,(L)$, there exists a rational number $0 < \delta \leq 1$ such that V is a maximal lc center for the pair $(X, \delta D_0)$. The rationality of δ follows from the rationality of log-canonical thresholds ([**14**, 9.3.12, 9.3.16]). Thus we can take $\alpha = \delta(n-d)m/r, A = \delta hH$ and $D = \delta D_0$.

(5) Let us discuss the bounds for α. Since $(X, \delta D_0)$ is not klt along V, it follows that $\mathrm{mult}_V(\delta D_0) \geq 1$ ([**14**, 9.5.13]). By our construction in (3), we have $\mathrm{mult}_V D_0 = \mathrm{mult}_{E_V}((n-d)E_V + G) < n-d+\varepsilon_1$. These two inequalities show that $\delta > 1/(n-d+\varepsilon_1)$. Then the upper and the lower

bounds for $\alpha = \delta(n-d)m/r$ follow from $\sigma_V(L) \le r/m < \sigma_V(L) + \varepsilon_1$, $1/(n-d+\varepsilon_1) < \delta \le 1$ and the first property of ε_1 in (2.1).

Proof of (i). We next consider the case (i). If $V \subset \mathrm{NNef}\,(L)$, our assertion is a special case of (iii). Hence we may assume $V \not\subset \mathrm{NNef}\,(L)$. We start with the same (1) as in the case (iii) above, and continue as follows.

(2) We take a large and divisible integer m such that $\mathrm{Bs}\,|mL| = \mathrm{SBs}\,(L)$, the associated map $\Phi_{|mL|}$ is birational onto its image, and $\mathrm{mult}_V|mL|/m < \varepsilon/2$. We denote $r = r(m) = \mathrm{mult}_V|mL| > 0$. We note that $\mu^{-1}(\mathrm{Bs}\,|mL|) = \mathrm{Bs}\,|m\mu^*L| = \mathrm{Bs}\,|m\mu^*L - rE_V| \cup E_V$, $E_V \not\subset \mathrm{Bs}\,|m\mu^*L - rE_V|$, and in particular $\mu^{-1}(\mathrm{SBs}\,(L)) \supset \mathrm{Bs}\,|m\mu^*L - rE_V|$. We also note that a $\mathbb{Q}$-divisor $M := (n-d)r^{-1}(m\mu^*L - rE_V)$ on Y is big.

(3) We apply Lemma 4.1 and 4.2 for the big $\mathbb{Q}$-divisor M on Y with the divisor $(B =)\mu^*H$. We obtain a decomposition $M \sim_{\mathbb{Q}} h\mu^*H + G$ with a rational number $h > 0$, and with an effective $\mathbb{Q}$-divisor G on Y such that $\mathrm{mult}_y(G) < \varepsilon$ for any $y \in Y \setminus \mathrm{NNef}\,(m\mu^*L - rE_V)$. In particular $\mathrm{mult}_{E_V}(G) < \varepsilon$. Since $\mathrm{NNef}\,(m\mu^*L - rE_V) \subset \mathrm{Bs}\,|m\mu^*L - rE_V| \subset \mu^{-1}(\mathrm{SBs}\,(L))$, we also have $\mathrm{mult}_y(G) < \varepsilon$ for any $y \in \mu^{-1}(X \setminus \mathrm{SBs}\,(L))$.

(4) The next step to find α and $\alpha L \sim_{\mathbb{Q}} A + D$ is parallel to that in the case (iii) above.

Proof of (ii). We finally consider case (ii). We note by [**5**, 1.3] that there exists a positive integer k such that $kL - H$ is big and $\mathrm{SBs}\,(kL - H) = \mathrm{NAmp}\,(L)$. Then the irreducible component V of $\mathrm{NAmp}\,(L)$ is an irreducible component of $\mathrm{SBs}\,(kL - H)$. By case (i), there exist a rational number $\alpha_0 > 0$ and a decomposition $\alpha_0(kL - H) \sim_{\mathbb{Q}} A_0 + D_0$ into an ample $\mathbb{Q}$-divisor A_0, and an effective $\mathbb{Q}$-divisor D_0 such that V is a maximal lc center for the pair (X, D_0), and that $\mathrm{mult}_x(D_0) < \varepsilon$ for any $x \notin \mathrm{SBs}\,(kL - H) = \mathrm{NAmp}\,(L)$. Then $\alpha_0 kL \sim_{\mathbb{Q}} (A_0 + \alpha_0 H) + D_0$ is a desired decomposition. q.e.d.

Lemma 4.4. *Assume that L is integral (and big), and let $0 \le d < c$ be rational numbers (then $\mathcal{J}(c\cdot\|L\|) \subseteq \mathcal{J}(d\cdot\|L\|)$ by* [**14**, 11.1.7]*). Assume that an irreducible component V of $V\mathcal{J}(c\cdot\|L\|)$ is not contained in $V\mathcal{J}(d\cdot\|L\|)$. Then there exist a rational number α with $d < \alpha \le c$, and an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} \alpha L$ such that V is a maximal lc center for the pair (X, D).*

Proof. We take a sufficiently large integer $p > c$ such that $\mathcal{J}(c\cdot\|L\|) = \mathcal{J}(\frac{c}{p}\cdot|pL|)$ and $\mathcal{J}(d\cdot\|L\|) = \mathcal{J}(\frac{d}{p}\cdot|pL|)$ ([**14**, 11.1.4]). By [**14**, 9.2.26], for a general member $D_p \in |pL|$, we have $\mathcal{J}(\frac{c}{p}\cdot|pL|) = \mathcal{J}(\frac{c}{p}D_p)$ and $\mathcal{J}(\frac{d}{p}\cdot|pL|) = \mathcal{J}(\frac{d}{p}D_p)$. We consider a real number $t_0 = \inf\{0 < t \in \mathbb{Q};\ V \subset V\mathcal{J}(\frac{t}{p}D_p)\}$; the log-canonical threshold along V. By our assumption, it follows $d < t_0 \leq c$. The infimum is in fact minimum, and t_0 is a rational number ([**14**, 9.3.12, 9.3.16]). Hence we can take as $\alpha = t_0$ and $D = \frac{t_0}{p} D_p$. q.e.d.

5. Uniruledness I: Non-Ample Locus and Stable Base Locus

We will give several uniruledness criteria for subvarieties. We will consider them devided into two cases. The first case is that of a subvariety V which appears as a component of $\mathrm{SBs}\,(L)$, or $\mathrm{NAmp}\,(L)$ of some big divisor L with vanishing asymptotic invariant, i.e., $\sigma_V(L) = 0$. The second case is that of $\sigma_V(L) > 0$, namely $V \subset \mathrm{NNef}\,(L)$. The former will be discussed here, and the latter will be discussed in the next section. In this section, we will also prove the theorems stated in the introduction.

We let X be a smooth projective variety.

5.1. Non-ample locus other than stable base locus.

Proposition 5.1. *Let L be a big divisor on X. Let V be a subvariety of X such that*

(i) *V is an irreducible component of* $\mathrm{NAmp}\,(L)$*,*
(ii) *$V \not\subset \mathrm{SBs}\,(L)$, and*
(iii) *V is an irreducible component of* $\mathrm{NAmp}\,(K_X + mL)$ *for every large integer m.*

Then V is uniruled.

Proof. By Proposition 4.3 (ii), there exist a rational number $\alpha > 0$, and a decomposition $\alpha L \sim_{\mathbb{Q}} A + D$ into an ample $\mathbb{Q}$-divisor A and an effective $\mathbb{Q}$-divisor D such that V is a maximal lc center for the pair (X, D). We take a large integer m so that $m > \alpha$, $K_X + mL$ is big, and that V is an irreducible component of $\mathrm{NAmp}\,(K_X + mL)$. Since $V \not\subset \mathrm{SBs}\,(L)$, there exists an effective $\mathbb{Q}$-divisor $E \sim_{\mathbb{Q}} (m - \alpha)L$ with $V \not\subset \mathrm{Supp}\, E$. Then we obtain a decomposition $mL = \alpha L + (m - \alpha)L \sim_{\mathbb{Q}} A + D + E$ so that A is ample, and that V is a maximal lc center for the pair $(X, D+E)$. Then our assertion follows from Corollary 3.3 (2). q.e.d.

5.2. Stable base locus other than non-nef locus.

Proposition 5.2. *Let L be a big divisor on X. Let V be a subvariety of X such that*

(i) *V is an irreducible component of* $\mathrm{SBs}\,(L)$*,*
(ii) *$V \not\subset \mathrm{NNef}\,(L)$, and*
(iii) *$V \subset \mathrm{SBs}\,(K_X + mL)$ for every large integer m.*

Then V is uniruled.

Proof. By Proposition 4.3 (i), there exist a rational number $\alpha > 0$, and a decomposition $\alpha L \sim_{\mathbb{Q}} A + D$ into an ample $\mathbb{Q}$-divisor A and an effective $\mathbb{Q}$-divisor D such that V is a maximal lc center for the pair (X, D). We take a large integer m so that $m > \alpha$, and that $V \subset \mathrm{SBs}\,(K_X + mL)$. Since $\sigma_V((m-\alpha)L) = 0$ and A is ample, we have $V \not\subset \mathrm{SBs}\,((m-\alpha)L + 2^{-1}A)$ by Lemma 2.4. Hence we can take an effective $\mathbb{Q}$-divisor $E \sim_{\mathbb{Q}} (m-\alpha)L + 2^{-1}A$ with $V \not\subset \mathrm{Supp}\,E$. Then we obtain a decomposition $mL \sim_{\mathbb{Q}} A + D + (m-\alpha)L \sim_{\mathbb{Q}} 2^{-1}A + D + E$ so that $2^{-1}A$ is ample, and that V is a maximal lc center for the pair $(X, D+E)$. Then our assertion follows from Corollary 3.3 (1). q.e.d.

Remark 5.3. As we saw in the statement and in the proof above, there are two technical issues to applying Corollary 3.3:

(i) the rational number α is not necessarily integral, and
(ii) the balance of L and K_X.

When we deal with non-nef loci in the next section, another issue will come into the picture, that is $\sigma_V(L) > 0$.

5.3. Proof of theorem. Let us give the proof of the theorems stated in the introduction. The part (iii) of Theorem 1.1 and 1.2 are special cases of Proposition 6.1 and 6.2 below. By taking for granted part (iii) of Theorem 1.1 and 1.2, let us show (i) and (ii) of Theorem 1.1, 1.2 and 1.3.

Proof of Theorem 1.1. Assume that K_X is big.

(i) Let V be an irreducible component of $\mathrm{SBs}\,(K_X)$. If $V \subset \mathrm{NNef}\,(K_X)$, V is uniruled by Theorem 1.1 (iii). If $V \not\subset \mathrm{NNef}\,(K_X)$, we apply Proposition 5.2 with the big divisor $L = K_X$, and we have the uniruledness of V.

(ii) Let V be an irreducible component of $\mathrm{NAmp}\,(K_X)$. If $V \subset \mathrm{SBs}\,(K_X)$, V is uniruled by Theorem 1.1 (i). If $V \not\subset \mathrm{SBs}\,(K_X)$, we apply Proposition 5.1 with the big divisor $L = K_X$, and we have the uniruledness of V. q.e.d.

Proof of Theorem 1.2. Assume that K_X is numerically trivial. Since the non-ample locus depends only on the numerical equivalence class of the big divisor ([**5**, 1.4]), we have $\mathrm{NAmp}\,(K_X + D) = \mathrm{NAmp}\,(D)$ for any big divisor D. By the same token ([**5**, 2.7]), we also have $\sigma_V(K_X + D) = \sigma_V(D)$; in particular, $\mathrm{NNef}\,(K_X + D) = \mathrm{NNef}\,(D)$ for any divisor D. Moreover, it follows from Kawamata [**10**, Theorem 8.2] that there exists a positive integer k_0 such that $k_0 K_X \sim 0$. Hence we also have $\mathrm{SBs}\,(K_X + D) = \mathrm{SBs}\,(D)$ for any divisor D. Thanks to these facts, the proof of Theorem 1.2 (i) and (ii) are parallel to those of Theorem 1.1. q.e.d.

Proof of Theorem 1.3. Assume that $-K_X$ is big, and let V be an irreducible component of (i) $\mathrm{SBs}\,(-K_X)$, or (ii) $\mathrm{NAmp}\,(-K_X)$ such that $V \not\subset \mathrm{NNef}\,(-K_X)$. We can see easily that V is uniruled, in a similar manner to in the proof of Theorem 1.1. q.e.d.

6. Uniruledness II: Non-Nef Locus

We consider the uniruledness of non-nef loci. By a technical reason, we state our results devided into three cases. We just recall [**5**, 2.10] (Lemma 2.6) that $\mathrm{NNef}\,(L) = \bigcup_{m\in\mathbb{N}} V\mathcal{J}(\|mL\|)$ for a big divisor L.

We let X be a smooth projective variety.

Proposition 6.1.

(1) *Assume that K_X is big. For every rational number $c > 0$, every irreducible component of $V\mathcal{J}(c\cdot\|K_X\|)$ is uniruled.*

(2) *Assume that K_X is pseudo-effective. Every irreducible component of* $\mathrm{NNef}\,(K_X)$ *is uniruled.*

Proposition 6.2. *Assume that K_X is numerically trivial.*

(1) *Let L be a big divisor on X. For every rational number $c > 1$, every irreducible component of $V\mathcal{J}(c\cdot\|L\|)$, which is not contained in $V\mathcal{J}(\|L\|)$, is uniruled.*

(2) *Let L be a pseudo-effective divisor on X. Every irreducible component of* $\mathrm{NNef}\,(L)$ *is uniruled.*

Proposition 6.3.

(1) *Assume that $-K_X$ is big. For every rational number $c > 2$, every irreducible component of $V\mathcal{J}(c\cdot\|-K_X\|)$, which is not contained in $V\mathcal{J}(\|-2K_X\|)$, is uniruled.*

(2) *Assume that $-K_X$ is big. Every irreducible component of* $\mathrm{NNef}\,(-K_X)$*, which is not contained in $V\mathcal{J}(\|-K_X\|)$, is uniruled.*

(3) *Assume that $-K_X$ is pseudo-effective. Every irreducible component V of* $\mathrm{NNef}\,(-K_X)$ *with $0 < \sigma_V(-K_X) < 1$ is uniruled.*

We do not know whether the assumptions $c > 1$ in Proposition 6.2, and $c > 2$ in Proposition 6.3, are really necessary or not. On the other hand, Proposition 6.3 (2) and (3) are sharp in a sense. In fact, we have an example as follows.

Example 6.4. Let $S \subset \mathbb{P}^3$ be a cone over a smooth elliptic curve C of $\deg C = 3$. Let $\mu : X \longrightarrow S$ be the blowing-up of S at the vertex. Then X is smooth, and is a $\mathbb{P}^1$-bundle over C. Let H be the hyperplane section divisor of $S \subset \mathbb{P}^3$, and let E be the μ-exceptional divisor on X. Then $-K_X = \mu^*H + E$ is big, but not nef. We can see easily that $E = \mathrm{NNef}\,(-K_X) = \mathrm{SBs}\,(-K_X) = \mathrm{NAmp}\,(-K_X)$, $\sigma_E(-K_X) = 1$ and $V\mathcal{J}(\|-K_X\|) = E$, while $E \cong C$ is a smooth elliptic curve.

As in the proof of propositions in §5, our main task is to construct a decomposition of a certain big divisor with a special regard to the balance with the canonical divisor K_X.

For the rest of this section we fix an ample divisor H on X.

Proof of (1) *in Proposition* 6.1*,* 6.2 *and* 6.3*.* The proof of the first assertions are parallel. We will denote a big divisor L and an integer d as follows.

In 6.1, $L = K_X$ and $d = 0$.

In 6.2, $L = L$ (the one in the statement) and $d = 1$.

In 6.3, $L = -K_X$ and $d = 2$.

In each case, we have $d < c$. We take an irreducible component V of $V\mathcal{J}(c \cdot \|L\|)$, which is not contained in $V\mathcal{J}(d \cdot \|L\|)$. It follows from Lemma 2.6 that $V \subset \mathrm{NNef}\,(L)$.

Step 1: *threshold*. We consider a real number

$$t_0 = \sup\{0 \leq t \in \mathbb{Q};\ \sigma_V(tL + H) = 0\}.$$

By Lemma 2.5, we have $0 < t_0 < +\infty$.

Step 2: *lc center*. By Lemma 4.4, there exist a rational number α with $d < \alpha \leq c$, and an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} \alpha L$ such that V is a maximal lc center for the pair (X, D).

Step 3: *complementary ample*. We take large integers p and q satisfying $p > \alpha$ and $d \leq p - qt_0 < \alpha$ (in case $d = 0$, $-1 < p - qt_0 < \alpha$ is enough for our latter purpose). The existence of such p and q is verified as follows. In case $t_0 \in \mathbb{Q}$, we take $t_0 = (p - d)/q$ for large p and q. In case $t_0 \notin \mathbb{Q}$, it follows from an elementary result in Diophantine approximation theory. Then we see $0 < (p - \alpha)/q < t_0$, and hence by Lemma 2.5, we have a decomposition $(p - \alpha)L + qH \sim_{\mathbb{Q}} A + E$ into an ample $\mathbb{Q}$-divisor A, and an effective $\mathbb{Q}$-divisor E with $V \not\subset \mathrm{Supp}\, E$.

Step 4: *decomposition*. We set $M = pL + qH$ an integral big divisor on X. Then we obtain a decomposition $M = \alpha L + (p-\alpha)L + qH \sim_{\mathbb{Q}} A + D + E$ so that A is ample, and that V is a maximal lc center for the pair $(X, D + E)$.

Step 5: *balance*. Since $K_X + M = (p + 1 - d)L + qH$ and since $(p + 1 - d)/q > t_0$, we have $\sigma_V(K_X + M) = \sigma_V((p + 1 - d)L + qH) > 0$, and in particular $V \subset \mathrm{SBs}\,(K_X + M)$. Then our assertion follows from Corollary 3.3 (1). q.e.d.

Proof of (2) *in Proposition* 6.1 *and* 6.2*.* We denote a pseudo-effective divisor L as $L = K_X$ in 6.1, and $L = L$ (the one in the statement) in 6.2. We take an irreducible component V of $\mathrm{NNef}\,(L)$.

Step 1. We consider a real number $t_0 = \sup\{0 \leq t \in \mathbb{Q};\ \sigma_V(tL + H) = 0\}$. We see $0 < t_0 < +\infty$ as before.

Step 2. By Lemma 2.4, we see that there exist a positive integer m_0 such that V is an irreducible component of $\mathrm{SBs}\,(mL + H)$ for every integer $m > m_0$.

We take an integer m_1 such that $m_1 > \max\{m_0, t_0+1\}$. Then m_1L+H is big, $\sigma_V(m_1L + H) > 0$, and V is an irreducible component of $\mathrm{SBs}\,(m_1L+H)$. By Proposition 4.3, there exist a rational number $\alpha > 0$ and a decompotision $\alpha(m_1L + H) \sim_{\mathbb{Q}} A_1 + D_1$ into an ample $\mathbb{Q}$-divisor A_1, and an effective $\mathbb{Q}$-divisor D_1 such that V is a maximal lc center for the pair (X, D_1). We take a general effective $\mathbb{Q}$-divisor $D_2 \sim_{\mathbb{Q}} A_1$ and set $D = D_1 + D_2$. Then $D \sim_{\mathbb{Q}} \alpha(m_1L+H)$ and V is a maximal lc center for the pair (X, D).

Step 3. We take large integers p and q satisfying $p > \max\{\alpha m_1, m_0\}$, $q > \max\{\alpha, 1\}$ and $0 \le p - qt_0 < \alpha$ (for the case of 6.1, $-1 < p - qt_0 < \alpha$ is enough for our latter purpose). These inequalities imply that $(p-\alpha m_1)/(q-\alpha) < t_0 \le p/q$. Since $(p-\alpha m_1)/(q-\alpha) < t_0$, by Lemma 2.5, we have a decomposition $(p - \alpha m_1)L + (q - \alpha)H \sim_{\mathbb{Q}} A + E$ into an ample $\mathbb{Q}$-divisor A, and an effective $\mathbb{Q}$-divisor E such that $V \not\subset \mathrm{Supp}\,E$.

Step 4. We set $M = pL + qH$ as an integral big divisor on X. Then we obtain a decomposition $M = \alpha(m_1L + H) + (p - \alpha m_1)L + (q - \alpha)H \sim_{\mathbb{Q}} A + D + E$ so that A is ample, and that V is a maximal lc center for the pair $(X, D + E)$.

We add a side remark (see Remark 6.5 below). Since $q > 1$, we have $\mathrm{SBs}\,(M) \subset \mathrm{NAmp}\,(M) \subset \mathrm{SBs}\,(pL + H)$. On the other hand, since $p > m_0$, we see that V is an irreducible component of $\mathrm{SBs}\,(pL + H)$.

Step 5. The final step depends on the canonical divisor.

Step 5 *for* 6.1. Since $K_X+M = (p+1)L+qH$ and since $(p+1)/q > t_0$, we have $\sigma_V(K_X + M) = \sigma_V((p + 1)L + qH) > 0$, and in particular $V \subset \mathrm{SBs}\,(K_X + M)$. Then our assertion follows from Corollary 3.3 (1).

Step 5 *for* 6.2. We recall the remark in the proof of Theorem 1.2 in §5.3, that $\sigma_V(K_X + M) = \sigma_V(M)$, $\mathrm{SBs}\,(K_X + M) = \mathrm{SBs}\,(M)$, and $\mathrm{NAmp}\,(K_X + M) = \mathrm{NAmp}\,(M)$.

(i) Case $t_0 \notin \mathbb{Q}$. Then $p/q > t_0$. We obtain $\sigma_V(K_X+M) = \sigma_V(M) = \sigma_V(pL + qH) > 0$, and in particular $V \subset \mathrm{SBs}\,(K_X + M)$. Then our assertion follows from Corollary 3.3 (1).

Since $\mathrm{SBs}\,(K_X + M) = \mathrm{SBs}\,(M) \subset \mathrm{SBs}\,(pL + H)$, and since V is an irreducible component of $\mathrm{SBs}\,(pL + H)$, V is an irreducible component of $\mathrm{SBs}\,(K_X + M)$. Moreover, $\sigma_V(K_X + M) > 0$ implies that V is in fact an irreducible component of $\mathrm{NNef}\,(K_X + M)$.

(ii) Case $t_0 = p/q \in \mathbb{Q}$. Our assertion will follow from Corollary 3.3 (2), if we can show that V is an irreducible component of $\mathrm{NAmp}\,(K_X + M)$. Since $\mathrm{NAmp}\,(K_X + M) = \mathrm{NAmp}\,(M) \subset \mathrm{SBs}\,(pL + H)$, and V is an irreducible component of $\mathrm{SBs}\,(pL+H)$, it is enough to show that $V \subset \mathrm{NAmp}\,(M)$. This in fact follows from Lemma 2.5. q.e.d.

Remark 6.5. If we want to study whether the subvariety V in the proofs above is contractible or not, the rationality of t_0 in Step 1 will be more important. For example in Step 5 for 6.2 (2), we can at least devide into the following three cases.

(1) $t_0 \notin \mathbb{Q}$.
(2) $t_0 = p/q \in \mathbb{Q}$, and V is an irreducible component of $\mathrm{SBs}\,(pL+qH)$, moreover $\sigma_V(pL+qH) = 0$.
(3) $t_0 = p/q \in \mathbb{Q}$, $V \not\subset \mathrm{SBs}\,(pL+qH)$, and V is an irreducible component of $\mathrm{NAmp}\,(pL+qH)$.

For the moment we cannot say anything in cases (1) and (2).

Proof of Proposition 6.3 (2) *and* (3). We denote $L = -K_X$ in both cases. We take an irreducible component V of $\mathrm{NNef}\,(L)$ satisfying the following condition: in case (2), V is not contained in $V\mathcal{J}(\|L\|)$; in case (3), $\sigma_V(L) < 1$.

Step 1. We let $t_0 = \sup\{0 \le t \in \mathbb{Q};\ \sigma_V(tL+H) = 0\}$. We see $0 < t_0 < +\infty$.

Step 2 *and* 3. By Lemma 2.4, we can take a positive integer m_0 such that V is an irreducible component of $\mathrm{NNef}\,(mL+H)$ for every integer $m > m_0$.

Step 2 *and* 3 *for* (2). By Lemma 2.6, there exists an integer $m > 1$ such that V is an irreducible component of $V\mathcal{J}(\|mL\|)$. Then by Lemma 4.4, there exist a rational number α with $1 < \alpha \le m$ and an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} \alpha L$ such that V is a maximal lc center for the pair (X, D).

We take large integers p and q satisfying $p > \max\{\alpha, m_0\}, q > 1$ and $0 \le p - qt_0 < \alpha - 1$. We see $(p+1-\alpha)/q < t_0 \le p/q$. By Lemma 2.5, we have a decomposition $(p+1-\alpha)L + qH \sim_{\mathbb{Q}} A + E$ into an ample $\mathbb{Q}$-divisor A, and an effective $\mathbb{Q}$-divisor E with $V \not\subset \mathrm{Supp}\,E$.

Step 2 *and* 3 *for* (3). Arguments will be a bit narrow. We denote $s_0 = \sigma_V(L)$, and we have $0 < s_0 < 1$. We take a number ε such that $1 < 1+\varepsilon < 1/s_0$. We take an integer m_1 such that $m_1 > m_0$, $2\mathrm{codim}\,V/s_0 < m_1\varepsilon/t_0$ and that $s_0/2 < \sigma_V(L + m_1^{-1}H) < s_0$. Then m_1L+H is big, and V is an irreducible component of $\mathrm{NNef}\,(m_1L+H)$. By Proposition 4.3 (iii), there exist a rational number $\alpha > 0$ and an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} \alpha(m_1L+H) = \alpha m_1(L + m_1^{-1}H)$ such that V is a maximal lc center for the pair (X, D). We can take α so that $1/\sigma_V(L+m_1^{-1}H) - \varepsilon' < \alpha m_1 \le \mathrm{codim}\,V/\sigma_V(L+m_1^{-1}H) (< 2\mathrm{codim}\,V/s_0)$ for any given $\varepsilon' > 0$ so that $1+\varepsilon < 1/s_0 - \varepsilon'$. Hence we can take α so that $1+\varepsilon < \alpha m_1 \le 2\mathrm{codim}\,V/s_0 (< m_1\varepsilon/t_0)$. In particular we have $\varepsilon - \alpha t_0 < \alpha m_1 - 1 - \alpha t_0$ and $\varepsilon - \alpha t_0 > 0$.

We take large integers p and q satisfying $p > \max\{\alpha m_1, m_0\}, q > \max\{\alpha, 1\}$ and $0 \le p - qt_0 < \varepsilon - \alpha t_0$. The inequalities $p - qt_0 < \varepsilon - \alpha t_0 < \alpha m_1 - 1 - \alpha t_0$ show that $(p+1-\alpha m_1)/(q-\alpha) < t_0$. Hence by Lemma

2.5, we have a decomposition $(p+1-\alpha m_1)L+(q-\alpha)H \sim_{\mathbb{Q}} A+E$ into an ample $\mathbb{Q}$-divisor A, and an effective $\mathbb{Q}$-divisor E with $V \not\subset \operatorname{Supp} E$.

Step 4. We set $M = (p+1)L + qH$. In both cases, we obtain a decomposition $M \sim_{\mathbb{Q}} A + D + E$ so that A is ample, and that V is a maximal lc center for the pair $(X, D+E)$.

Step 5. This is parallel to Step 5 for 6.2 (2). We note $K_X + M = pL + qH$.

(i) Case $t_0 \notin \mathbb{Q}$. Then $p/q > t_0$. We obtain $\sigma_V(K_X+M) = \sigma_V(pL + qH) > 0$, and in particular $V \subset \mathrm{SBs}\,(K_X + M)$. Then our assertion follows from Corollary 3.3 (1).

(ii) Case $t_0 = p/q \in \mathbb{Q}$. Our assertion will follow from Corollary 3.3 (2), if we can show that V is an irreducible component of $\mathrm{NAmp}\,(K_X + M)$. Since $\mathrm{NAmp}\,(K_X + M) = \mathrm{NAmp}\,(pL + qH) \subset \mathrm{SBs}\,(pL + H)$ by $q > 1$, and since V is an irreducible component of $\mathrm{SBs}\,(pL+H)$ by $p > m_0$, it is enough to show that $V \subset \mathrm{NAmp}\,(pL+qH)$. This again follows from Lemma 2.5. q.e.d.

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