The Research on Circle Family and Sphere Family

The Original Topic:

The Research and Generalizations on Several Kinds of Partition Problems in Combinatorics

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【Abstract】

A circle family is a group of separate or tangent circles in the plane. In this paper, we study how many parts at most a plane can be divided by several circle families if the circles in a same family must be separate (resp. if the circles can be tangent). We also study the necessary conditions for the intersection of two circle families. Then we primarily discuss the similar problems in higher dimensional space and in the end, raise some conjectures.

【Key words】

Circle Family; Structure Graph; Sphere Family; Generalized Inversion

【Changes】

1. Part 5 'Some Conjectures and Unsolved Problems' has been rewritten.

2. Lemma 4.2 has been restated.

3. Some small mistakes have been corrected.

1 The Definitions and Preliminaries

To begin with, we introduce some newly definitions and related preliminaries.

Definition 1.1 Circle Family A ***circle family of the first kind*** is a group of separate circles; A ***circle family of the second kind*** is a group of separate or tangent circles. The ***capacity of a circle family*** is the number of circles in a circle family, the ***intersection of circle families*** means there are several circle families and any two circles in different circle families intersect.

Definition 1.2 Compaction If the capacity of a circle family is no less than 3, and it intersects with another circle family with capacity 2, we call such a circle family ***compact***.

Definition 1.3 *n*-connected If the capacity of a circle family is 2, and it intersects with another circle family with capacity n, we call such a circle family ***n-connected***.

Definition 1.4 *n*-dimensional sphere The ***n-dimensional sphere*** is the set of all points that has a fixed distance r to a fixed point O in the n-dimensional space. The fixed point O is called the ***centre*** of the sphere, and the fixed value r is called the ***radius*** of the sphere.

Definition 1.5 *n*-dimensional spheroid The ***n-dimensional spheroid*** is the set of all points that has a distance no more than a fixed value r to a fixed point O in the n-dimensional space. The fixed point O is called the ***centre*** and the fixed value r is called the ***radius*** of the spheroid, respectively.

Definition 1.6 *n*-dimensional sphere family An ***n-dimensional sphere family*** is a group of separate spheres in n-dimensional space. The number of spheres is defined as the ***capacity*** of the sphere family.

Definition 1.7 Intersect For two n-dimensional spheres, we define that they ***intersect*** if and only if they have common points (if a circle is completely inside another circle in the plane, they're also called intersecting); for two n-dimensional sphere families, we define that they ***intersect*** if and only if any two spheres intersect in different families.

Definition 1.8 *n*-dimensional plane The set of points in n-dimensional space which satisfy: they′re the end points of all the vectors that start from a fixed point and are vertical to a fixed vector.

Definition 1.9 *n*-dimensional intersecting line The intersection set of two n-dimensional planes which are not parallel.

Definition 1.10 Generalized Inversion Define O is the centre of a fixed n-dimensional spheroid with radius r. The transformation in n-dimensional space satisfies: any point A (except O) and its corresponding point A' by the transformation satisfy $\overrightarrow{OA}\cdot\overrightarrow{OA'}=r^2$ and O, A, A' are collinear. Define such transformation as ***Generalized Inversion***.

Preliminary 1.11 Planar Graph[1] A graph is ***planar*** if and only if it can be drawn in the plane such that every two curves are either disjoint or meet only at a common endpoint.

Preliminary 1.12 Euler′ s Formula[1] For a connected graph in the plane or on the spherical surface, it satisfies $f-e+v=2$, where f, e and v are the numbers of faces, edges and vertices of the graph, respectively.

2 The Division Problem of Circle Families in the Euclidean Plane

During the time we think about how many parts at most can the 3-dimensional space be divided by several convex polyhedrons, we introduce the definition of circle family. When a convex polyhedron intersects with several other convex polyhedrons, it forms several group of closed figures on the surface, and every two closed figures in the same group do not intersect. We want to calculate how many parts at most can the surface of the convex polyhedron (it can be regarded as the surface of a spheroid) be divided by several convex polyhedrons, and the equivalent problem is how many parts at most can a 3-dimensional sphere be divided by several circle families on it. According to the close property of a spherical surface, we can easily arrange the position of the circles such that every two circles in different circle families intersect. However, in the Euclidean plane the case is not so simple .The problem we′ re interested in is that how many parts at most can the Euclidean plane (we call it 'plane' for short) be divided by several circle families.

Firstly we prove the following proposition.

Proposition 2.1 If several circle families (at least 2) can at most divide the plane into *S* parts, and they get *T* points of intersection or tangency, then

$$\mathrm{S}=\mathrm{T}+2 \tag{1}$$

Proof: As we′ re considering at least two circle families, we can assume that any three circles don′ t have a common point and every circle at least intersects with another circle. For such figure we have

$$f-e+v=2 \tag{2}$$

Since no three circles intersect at a common point, the degree of each vertice is 4, hence $2e=4v$, namely $e=2v$. Combining this with (2) we get $f=v+2$, thus

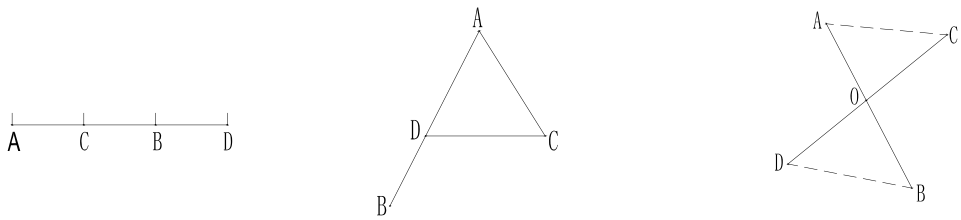
$S=f_{\max}=v_{\max}+2=T+2$. ■

Proposition 2.1 shows that we just need to know how many points of intersection or tangency can several circle families get at most, and for the circle family of first kind, the most number of pairs of intersecting circles should be considered. We define it as $f(x_1,x_2,\cdots,x_m)$, and $f(k,l)$ for the situation $m=2$. It′ s clear that $f(k,l)\le kl$, and the equality holds if and only if the two circle families intersect, the further discussion of this issue will be placed in the third part of the thesis. In order to get the expression of $f(x_1, x_2, \cdots, x_m)$, we introduce the definition of ***structure graph***.

For two circle families in the plane, let the centers of the circles be the vertices of the graph, respectively. If two circles intersect, we draw a line segment between the two vertices which correspond the two circles; for a circle family of the second kind, let the centers of the circles be the vertices of the graph, respectively. Two vertices are connected by a line segment if and only if the two circles that correspond to them are tangent. Such a graph is called the ***Structure Graph*** of the circle family (resp. circle families). We have found the following lemma about the structure graph.

Lemma 2.2 The structure graph of any two circle families or a circle family of the second kind is planar.

Proof: Suppose the contrary, then there exist two edges *AB* and *CD* of the graph that intersect at a point different the vertices of the graph, see the following figures:



(I) (II) (III)

Without loss of generality, assume that the radii of the circles *A, B, C, D* are *a, b, c,* and *d,* respectively. We further assume that circles *A, C* and circles *B, D* are in the same family, respectively. On one hand, the definition of circle family shows $|AC| \ge a+c$, $|BD| \ge b+d$, hence $|AC|+|BD| \ge a+b+c+d$. On the other hand, by the definition of the structure graph, circles *A, B* and circles *C, D* are intersecting or tangent, respectively. Hence $|AB| \le a+b$, $|CD| \le c+d$, $|AB|+|CD| \le a+b+c+d$.

Synthesize the two sizes, the inequality $|AC|+|BD| \ge |AB|+|CD|$ holds.

But in (I), it is clear that $|AC|+|BD| < |AB|+|CD|$; in (II) (III) (II is the situation *D*=*O* in III),

$$|AB|+|CD| = |AO|+|OC|+|BO|+|OD| > |AC|+|BD|,$$

which is a contradiction. Therefore our assumption was wrong and Lemma 2.2 is proved. ■

Theorem 2.3 *m* circle families of the first kind in the plane with capacities $x_1, x_2, \cdots, x_m (x_i \in N, x_i \ge 2, 1 \le i \le m)$ can divide the plane into

$$2 + 4(m-1)(x_1 + x_2 + \cdots + x_m - m)$$

parts at most.

Proof: We firstly discuss the situation of m=2. By Lemma 2.2, the structure graph of these two circle families is planar. We further assume that it is connected. By Euler's Formula and $2e \geq 4f$ in bipartite graph, we have $e \leq 2v-4$. As every edge of the graph corresponds to a pair of intersecting circles, we have

$$f(k,l) = e_{\max} \leq 2(k+l)-4$$

For the general problem, every two circle families with capacities $x_i, x_j (i \neq j)$ can at most add $2(x_i + x_j) - 4$ pairs of intersecting circles to the whole number of pairs of intersecting circles, hence

$$f(x_1, x_2, \cdots, x_m) \leq \sum_{1 \leq i < j \leq m} 2(x_i + x_j) - 4 = 2(m-1)(x_1 + x_2 + \cdots + x_m - m) \qquad (3)$$

Now we construct a figure to show that the equality can hold. We temporarily assume that the circles in a same family can be externally tangent (then we can decrease the radii of the circles properly to make them separate from each other). Draw m lines $l_i (i = 1, 2, \cdots, m)$ passing a fixed point P in the plane. In each line l_i we take two points M_i, N_i different from P which satisfy P lies in the line segment $M_i N_i$. Then draw a circle with centre M_i and radius $M_i P$ and another circle with centre N_i and radius $N_i P$ ($i = 1, 2, \cdots m$). It is clear that $\odot M_i$ and $\odot N_i$ are tangent, and for $1 \leq i \neq j \leq m$, $\odot M_i$ and $\odot M_j$ intersect at P, and M_i, M_j, P are not collinear points, hence $\odot M_i$ and $\odot M_i$ are not tangent, namely they intersect. Similarly, $\odot M_i$ and $\odot N_j$, $\odot N_i$ and $\odot N_j$ intersect, respectively, for $1 \leq i \neq j \leq m$. Let $\odot M_i$ and $\odot N_i$ be two circles of the i^{th} circle family ($i = 1, 2, \cdots m$), they have already produced $2m(m-1)$ pairs of intersecting circles, hence it suffices to get

$$2(m-1)(x_1 + x_2 + \cdots + x_m - 2m) = 2(m-1)\left[(x_1 - 2) + (x_2 - 2) + \cdots + (x_m - 2)\right]$$

more pairs of intersecting circles.

Note that the number in the bracket is the number of circles we have to add, hence it suffices to insure that every time we add a circle, we add $2(m-1)$ pairs of intersecting circles as well, and the original pairs of intersecting circles maintain. As

$\odot N_1$ intersects with every circle except $\odot M_1$, we can decrease the radius of $\odot N_1$ by a sufficiently small positive real number ε_1 such that $\odot N_1$ still intersects with every circle except $\odot M_1$. Assume that the decreased circle $\odot N_1$ intersects with line l_1 at Q, draw $\odot N_{13}$ with diameter PQ, and add it to the first circle family (it is clear that $\odot N_{13}$ is externally tangent to $\odot N_1$ and $\odot M_1$). As P is on the circumference of $\odot \mathrm{N}_{13}$, and all the centers of the circles are not in l_1 except M_1, N_1, hence $\odot N_{13}$ intersects with all the circles that are not in the same family, and we get $2(m-1)$ pairs of intersecting circles while the original pairs of intersecting circles maintain. Similarly, decrease the radius of $\odot N_{13}$ by a sufficiently small positive real number ε_2 such that all the circles intersecting with $\odot N_{13}$ still intersect with it. Assume that the decreased circle $\odot N_{13}$ intersects with l_1 at R. Draw a circle N_{14} with diameter PR and add it to the first circle family, then $\odot N_{14}$ intersects with $2(m-1)$ circles which pass the point P in other circle families. Repeat such operations, every time we add a new circle to the figure, we maintain the pairs of intersecting circles, and insure that there are always $2m$ circles passing the point P, one of which is a newly added one. Continuing in this way we can add all the circles which satisfy the conditions. Hence we have

$$2m(m-1)+2(m-1)\left[(x_1-2)+(x_2-2)+\cdots+(x_m-2)\right]=2(m-1)(x_1+x_2+\cdots+x_m-m)$$

pairs of intersecting circles and the equality in (3) can hold. Thus by Proposition 2.1, the m circle families can divide the plane into

$$2+2f(x_1,x_2,\cdots,x_m)=2+4(m-1)(x_1+x_2+\cdots+x_m-m)$$

parts at most. ∎

As for the second kind of circle family, circles in a same family can be tangent, thus the parts of division will increase.

Proposition 2.4 A circle family of the second kind with capacity v can at most get $3(v-2)$ points of tangency.

Proof: Every point of tangency corresponds to an edge of the structure graph of the circle family. Since the structure graph is planar, by Euler's Formula and $2e \ge 3f$ we have $e \le 3(v-2)$, hence the number of points of tangency is no more than

$3(v-2)$. ∎

Theorem 2.5 *m* circle families of the second kind in the plane with capacities $x_1, x_2, \cdots, x_m (x_i \in N, x_i \geq 2, 1 \leq i \leq m)$ can divide the plane into at most

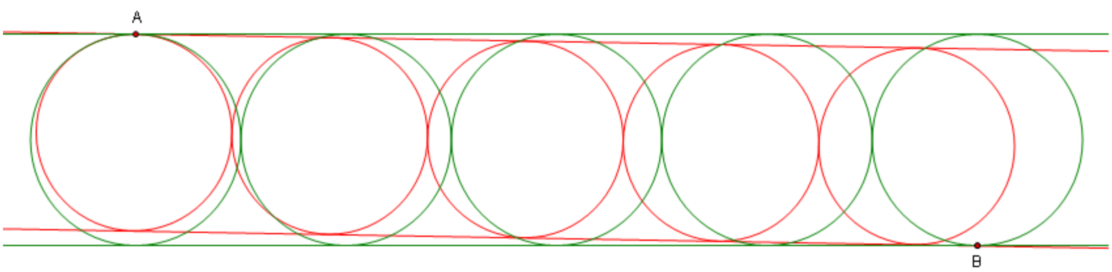
$$(4m-1)(x_1 + x_2 + \cdots + x_m - m) - 3m + 2 \tag{4}$$

parts. When the upper bound is attained, the number of points of tangency and the number of pairs of intersecting circles attain the maximum simultaneously.

Proof: To begin with, by Propositions 2.1, 2.4 and the expression of $f(x_1, x_2, \cdots, x_m)$, it is not difficult to verify that (4) is the upper bound of the number of parts that *m* circle families of the second kind divide the plane into. Moreover, the upper bound is attained if and only if the number of points of tangency produced by each circle family and the number of pairs of intersecting circles between different circle families attain the maximum simultaneously. Now we give such a graph as follows.

The inversion is needed to consider the case of m=2.

Without loss of generality, suppose the capacities of two circle families are $k \geqslant m \geqslant 2$. Draw two parallel lines l_1, l_2 (they will form circles after the inversion). Between these two parallel lines, draw $k-2$ circles $S_1, S_2, \cdots, S_{k-2}$ with equal size (from left to right) such that they are both tangent to l_1, l_2, and S_i and S_{i-1} are tangent ($2 \leqslant i \leqslant k-2$). It can be seen that $S_1, S_2, \cdots, S_{k-2}$ and l_1, l_2 form a circle family. Suppose the point of tangency between S_1 and l_1 is A, and the point of tangency between S_{k-2} and l_2 is B. Rotate l_1 and l_2 by a small angle θ around the point A and B clockwise, then we get two parallel lines l_1', l_2' which intersect with $S_1, S_2, \cdots, S_{k-2}$. Draw a circle T_1 which is tangent to l_1' at A and l_2', and then we can draw circles $T_2, T_3, \cdots, T_{m-2}$ such that T_i is tangent to l_1', l_2' and T_{i-1} ($2 \leq i \leq m-2$). These $m-2$ circles and l_1', l_2' form another circle family. (See the following graph, one circle family is colored green, while the other is painted red.)



It is easy to see that there are $3(k-2)$ points of tangency (including the one obtained from l_1,l_2 at infinity. It will be the inversion center after the inversion.) produced by the first circle family, and there are $3(m-2)$ points of tangency produced by the second circle family. When $\theta \to 0$, $T_i \to S_i$. For sufficiently small θ, T_i intersects with S_i and a circle which is tangent to S_i in the first circle family. (Especially, by the fact that l_1 passes through the point A on T_1 but is not tangent to T_1, we have T_1 intersects with l_1.) In addition, $l_1^{'},l_2^{'}$ intersect with all circles in the first circle family. Then there are $2(k+m-2)$ pairs of intersecting circles. Hence two circle families constructed above satisfy that the number of points of tangency and the number of pairs of intersecting circles attain the maximum simultaneously. Note that there exist four lines. Choose a interior point *P* from the intersection of the unbound region formed by l_1,l_2,S_1 and the unbound region formed by $l_1^{'},l_2^{'},T_1$. Now let *P* be the center and any positive real number be radius, then the inversion of the two circle families constructed above is what we need (Because the center lies outside all circles, the tangent (resp. intersecting) circles are still tangent (resp. intersecting) after the inversion). Finally, note that the number of circles pass through the inversion of A (resp. the center) are greater than two. In order to insure any three circles share no common vertices, we can shift these two circle families along sufficiently small vectors with different directions and maintain the number of pairs of intersecting circles. After these operations, we obtain the desired result.

Obviously, rearrange the circle families according to the number of circles from largest to smallest, and do the construction before the inversion as above (namely, rotate l_1 and l_2 by different angles around the point A and B clockwise, and for each pair of parallel lines we obtain, draw some circles which are tangent to them, where one of these circles passes through point A). Then after the inversion we can shift each of these circle families along sufficiently small vectors with different directions and maintain the number of pairs of intersecting circles to insure any three circles share no common vertices. Therefore, we get some circle families such that the number of points of tangency and the number of pairs of intersecting circles attain the maximum simultaneously. This completes our construction. ∎

Notice that if the radius of each circle in the construction of Theorem 2.5 decreases by a sufficiently small positive real number, we will get another maximal division scheme for the circle families of the first kind.

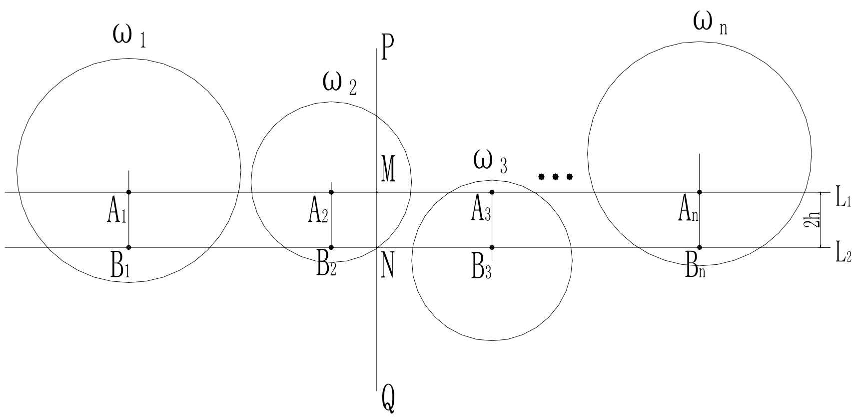
3 Intersecting Circle Families

If there exist two intersecting circle families with capacities *k*, *l*, respectively, then $kl = f(k,l) = 2(k+l)-4$. Thus either *k*=2 or *l*=2. In fact this is also the sufficient condition (see Corollary 3.2). Furthermore, the following two theorems discuss the geometric properties that two intersecting circle families should satisfy.

Theorem 3.1 A circle family is compact if and only if there exists a line which intersects with every circle in the family.

Proof : Suppose that two separate circles $\odot O_1$, $\odot O_2$ intersect with the given circle family, then there exists a line *l* that separates $\odot O_1$ and $\odot O_2$. Because all circles in the given circle family intersect with both $\odot O_1$ and $\odot O_2$, they have common points with the two half planes excluding line *l* to both sides of it. Hence all these circles intersect with *l*. That's the proof of necessity.

Now assume that there exists a line l_1 which intersects with all the circles in the given family, then there exists another line l_2 parallel to and sufficiently close to l_1 which also intersects with all the circles in the given family. Let 2*h* (*h*>0) be the distance from l_1 to l_2. For $i = 1, 2, \cdots, n$ where *n* is the capacity of the circle family, let A_i, B_i be the feet of perpendiculars from the centers of the circles in the family to l_1 and l_2, respectively. We further assume that A_1, B_1, B_n, A_n are the vertices of the convex hull of these feet of perpendiculars. Because l_1, l_2 intersect with all the circles, A_i and B_i are located inside the i^{th} circle (denoted as ω_i) for $i = 1, 2, \cdots, n$.



Assume that $|A_1A_n| = |B_1B_n| = 2d\,(d > 0)$ and the perpendicular bisector intersect with A_1A_n, B_1B_n at point *M*, *N*, respectively. Let *P*, *Q* be two points on the prolongation of NM and MN, respectively, such that $|PM|=|NQ|=l$ where l is a undetermined parameter. Draw a circle at center P with radius $|PA_1|=|PA_n|$. Draw another circle at center Q with radius $|QB_1|=|QB_n|$. Since $\odot P$ contains A_1 and A_n, it also contains points A_i ($i=1,2,\cdots,n$) on line segment A_1A_n by its convexity. Similarly, $\odot Q$ contains $B_1,\cdots,B_n$. Since

$$|PQ|-|PA_1|-|QB_1|=(2l+2h)-2\sqrt{d^2+l^2}=2(h-\frac{d^2}{\sqrt{d^2+l^2}+l})>2(h-\frac{d^2}{2l})$$

and h>0, d>0, substitute with $l=\frac{d^2}{h}$ it follows that $|PQ|-|PA_1|-|QB_1|>2\cdot\frac{h}{2}>0$. Hence $|PQ|>|PA_1|+|QB_1|$ and $\odot P$ is separate from $\odot Q$. Combining this with $A_i\in\omega_i, B_i\in\omega_i$ $(i=1,2,\cdots n)$ and $A_i\in\odot P, B_i\in\odot Q$, it shows that there exists a circle family, namely $\odot P$ and $\odot Q$, which intersects with $\omega_1,\cdots,\omega_n$. That's the proof of sufficiency. ∎

Corollary 3.2 For any positive integer $k\geq 2$, there exist two intersecting circle families with capacities 2, k, respectively.

Proof: It suffices to draw k separate circles that intersect with a fixed line, since by Theorem 3.1, there exists a circle family with capacity 2 that intersects with them. ∎

Theorem 3.3 Suppose that two separate circles $\odot A$ and $\odot B$ in the plane are k-connected($k\geqslant 3$), then $|AB|<m+n+2\sqrt{mn}\tan\theta$, where $\theta=\frac{\pi}{4\left\lceil\frac{k-1}{2}\right\rceil}$ and m,n are the radii of $\odot A$ and $\odot B$, respectively. As k tends to infinity, these two circles tend to contact.

Proof: Assume that $\odot O_1$、$\odot O_2$、⋯、$\odot O_k$ are k separate circles, which intersect with $\odot A$ and $\odot B$. Apply homothetic transformation to $\odot O_1$ at center O_1 such that $\odot O_1$ becomes tangent to one of $\odot A$ and $\odot B$ while it intersects the other. Then with this point of tangency as the center, apply homothetic transformation again to $\odot O_1$ such that $\odot O_1$ is tangent to both $\odot A$ and $\odot B$. Note that the area $\odot O_1$ covers doesn't expand, we therefore assume that $\odot O_1$、$\odot O_2$、⋯、$\odot O_k$ are all tangent to both $\odot A$ and $\odot B$.

Let r_i be the radius of $\odot O_i$ (i=1, 2, …, k). Let the midpoint of AB be the origin and line AB be the X-axis, build up the XY-coordinate system. Assume that $|AB| = 2c, c > 0, A(-c,0), B(c,0), m \ge n$ and $a = \dfrac{m-n}{2}$.

If a>0, O_i will lie on the right half of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ (x>0), where $b^2 = c^2 - a^2$. So we can write $O_i(a\sec\theta_i, b\tan\theta_i)$ where $\theta_i \in (-\dfrac{\pi}{2}, \dfrac{\pi}{2})$ for $i = 1, 2, \cdots, k$. If a=0, then $a\sec\theta_i = 0 = x_i$ (x_i is the X-coordinate of O_i), hence the expression above for O_i still works.

By Pigeonhole Principle, at least $t = \left[\dfrac{k+1}{2}\right]$ terms among $\theta_1, \theta_2, \cdots, \theta_k$ have the same sign. Assume without loss of generality that $0 \le \theta_1 < \theta_2 < \cdots < \theta_t < \dfrac{\pi}{2}$. Simple calculating shows

$$r_i = \frac{c}{a}x_i - \frac{m+n}{2} = c\sec\theta_i - \frac{m+n}{2} (1 \le i \le k)$$

Since $|O_iO_{i+1}| > r_i + r_{i+1}$,we have

$$(a\sec\theta_i - a\sec\theta_{i+1})^2 + (b\tan\theta_i - b\tan\theta_{i+1})^2 > (c\sec\theta_i + c\sec\theta_{i+1} - m - n)^2$$

It follows that

$$2(c^2 - a^2)\cos(\theta_{i+1} - \theta_i) + 2(a^2 + c^2) < 2c(m+n)(\cos\theta_i + \cos\theta_{i+1}) - (m+n)^2\cos\theta_i\cos\theta_{i+1} \tag{5}$$

For $i = 1, 2, \cdots, t-1$, let $\varphi_i = \theta_{i+1} - \theta_i \in (0, \dfrac{\pi}{2})$ and define

$$f_i(x) = 2c(\cos x + \cos(x + \varphi_i)) - (m+n)\cos x\cos(x + \varphi_i), 0 \le x < \frac{\pi}{2} - \varphi_i$$

Then $f_i'(x) = -2c(\sin x + \sin(x + \varphi_i)) + (m+n)\sin(2x + \varphi_i)$

$$= 2\sin(x + \frac{\varphi_i}{2})\left[(m+n)\cos(x + \frac{\varphi_i}{2}) - 2c\cos\frac{\varphi_i}{2}\right]$$

Since $0 < \dfrac{\varphi_i}{2} \le x + \dfrac{\varphi_i}{2} < \dfrac{\pi}{2} - \dfrac{\varphi_i}{2} < \dfrac{\pi}{2}$ when $0 \le x < \dfrac{\pi}{2} - \varphi_i$, we have

$$\sin(x + \frac{\varphi_i}{2}) > 0, 0 < \cos(x + \frac{\varphi_i}{2}) \le \cos\frac{\varphi_i}{2}$$

Because $\odot A$ and $\odot B$ are two separate circles, we have $2c = |AB| > m+n > 0$, hence $(m+n)\cos(x+\frac{\varphi_i}{2}) - 2c\cos\frac{\varphi_i}{2} < 0$ and $f_i'(x) < 0$. So $f_i(x)$ is monotone decreasing when $x \in \left[0, \frac{\pi}{2} - \varphi_i\right)$, implying that $f_i(x) \le f_i(0)$. Combining this with (5) it follows that

$$2(c^2 - a^2)\cos\varphi_i + 2(a^2 + c^2) < (m+n) f_i(\theta_i) \le (m+n) f_i(0) = 2c(m+n)(1+\cos\varphi_i) - (m+n)^2\cos\varphi_i$$

which is equivalent to

$$2(1+\cos\varphi_i)c^2 - 2(m+n)(1+\cos\varphi_i)c + 2a^2(1-\cos\varphi_i) + (m+n)^2\cos\varphi_i < 0$$

Hence

$$c < \frac{m+n}{2} + \frac{\sqrt{(m+n)^2(1+\cos\varphi_i)^2 - 4a^2(1-\cos\varphi_i)(1+\cos\varphi_i) - 2(m+n)^2\cos\varphi_i(1+\cos\varphi_i)}}{2(1+\cos\varphi_i)}$$

$$= \frac{m+n}{2} + \frac{1}{2(1+\cos\varphi_i)}\sqrt{\left[(m+n)^2 - 4(\frac{m-n}{2})^2\right](1+\cos\varphi_i)(1-\cos\varphi_i)}$$

$$= \frac{m+n}{2} + \sqrt{mn}\cdot\frac{\sin\varphi_i}{1+\cos\varphi_i} = \frac{m+n}{2} + \sqrt{mn}\cdot\tan\frac{\varphi_i}{2}, i = 1, 2, \cdots, t-1$$

Because $\varphi_1 + \varphi_2 + \cdots + \varphi_{t-1} = \theta_t - \theta_1 < \frac{\pi}{2}$, there exists φ_i that is less than

$$\frac{\pi}{2(t-1)} = \frac{\pi}{2\left[\frac{k-1}{2}\right]} = 2\theta$$

For this φ_i, $2c < m+n+2\sqrt{mn}\tan\frac{\varphi_i}{2} < m+n+2\sqrt{mn}\tan\theta$.

When $k \to +\infty$, $\theta \to 0$. Let L_k be the supremum of $|AB|$, by the Squeeze Rule we have $\lim_{k\to+\infty} L_k = m+n$. This completes the proof of Theorem 3.3. ∎

4 Primary Research on Sphere Family

As we generalize the idea of circle family in the plane to sphere family in higher dimensional space and consider the similar problems, we obtain the following results.

Lemma 4.1 Suppose *P*, *Q* are two spheres in the *n*-dimensional space that do not intersect, and point *O* lies on *P*. By the generalized inversion at center *O* with any positive radius, P is transformed into an n-dimensional plane p while Q becomes an n-dimensional sphere q that doesn't meet O. Besides, sphere q and point O are located to the same side of plane p.

Proof: Let M be the center of Q. Assume that line OM intersects sphere Q at A and B, which are antipodal points to each other. Let A', B' be the inverse points of A, B, respectively. For any point R different from O on sphere Q, we have $|MA|=|MB|=|MR|$, hence $\angle ARB=90°$. Let R' be the inverse point of R. By definition $\overrightarrow{OR}\cdot\overrightarrow{OR'}=\overrightarrow{OA}\cdot\overrightarrow{OA'}=\overrightarrow{OB}\cdot\overrightarrow{OB'}$, so $\Delta ORA \backsim \Delta OA'R'$, $\Delta ORB \backsim \Delta OB'R'$. Hence $\angle A'R'B'=\angle OR'A'-\angle OR'B'=\angle OAR-\angle OBR=\angle ARB=90°$, which means R' lies on the sphere with diameter $A'B'$. Therefore the image of Q by the inversion is an n-dimensional sphere. Similarly, we obtain that P is transformed into an n-dimensional plane.

Note that the image of spheroid (not sphere this time) P by the inversion doesn't contain point O, it must be the very side of plane p that doesn't contain O. Since spheroid Q doesn't intersect with P, its image by the inversion doesn't intersect with the image of P by the inversion, so sphere q and point O are located to the same side of plane p. ∎

Lemma 4.2 Suppose P, Q are two intersecting spheroids in the n-dimensional space, and p, q are two n-dimensional planes. Plane p is separate from Q and intersects with P while q is separate from P and intersects with Q. Plane p and q intersect at intersecting line k. Then p, q divide the space into four parts and exactly one of them contains the intersection of p and q. Let Ω denote this part. Its opposite part is separate from P, Q. Suppose plane r contains intersecting line k, passes through region Ω and intersects with both P and Q, then

$$P\cap Q\cap r\neq\varnothing.$$

Proof: Build up the rectangular coordinate system in the n-dimensional space. Then the equation of an n-dimensional plane can be written in terms of

$$F(x_1,x_2,\cdots,x_n)=a_1x_1+a_2x_2+\cdots+a_nx_n+b=0.$$

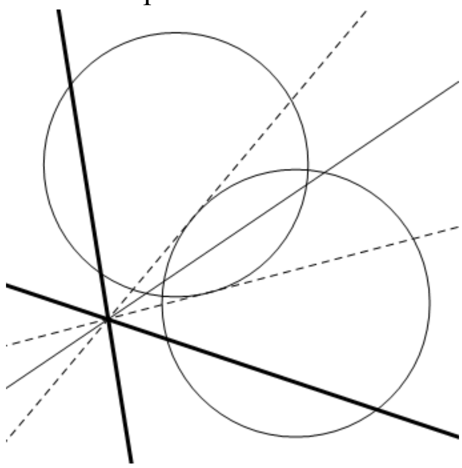
And each side of the plane corresponds n-element arrays satisfying $F>0$ and $F<0$, respectively. As for two planes p, q, each value of their corresponding polynomials F, G has 2 possible signs on different arrays, resulting 4 different outcomes in total. Hence the space is divided into 4 parts. Since p is separate from Q and q is separate from P, the signs of F, G on the region containing the intersection of P, Q are uniquely determined. Hence there is exactly one such region. And because its opposite region also has opposite signs of F, G to region Ω, it is separate from both P and Q. That's the former part of the lemma, now we prove the latter part.

Intersecting line k divides each of plane p, q into two half planes. For the two half planes that enclose Ω, we still call them p, q. Since Ω's opposite region intersects with neither P nor Q, we only need to consider the half plane of r (we still call it r) in Ω.

Because p intersect with P while q doesn't, there exists a half plane m with boundary k which is tangent to spheroid P at point M. Similarly, there exists another half plane n with boundary k which is tangent to spheroid Q at point N. Obviously, the region enclosed by half planes m, n (denoted as Σ) contains the intersection of P, Q and r is in Σ definitely. As half planes m, n, r have the same boundary k, line segment MN will intersect with r. Let R be the point of intersection, then $R \in r$. Since an n-dimensional spheroid is convex, we have $R \in P \cap Q$, hence

$$R \in P \cap Q \cap r \quad \Rightarrow \quad P \cap Q \cap r \neq \varnothing \qquad \blacksquare$$

The following picture is an explanation of Lemma 4.2 in the plane.



According to our analysis in Part 3, there exists one of two intersecting circle families that has the capacity 2. Generalize this conclusion we have

Theorem 4.3 If two sphere families with capacities a, b, respectively, intersect in the n-dimensional space, then

$$\min\{a,b\} \le n$$

Proof: We will prove this statement by induction on n.

For case n=1, we need to prove that $\min\{a,b\} \leqslant 1$ when there exist two intersecting line segment families with capacities a, b in the number axis. It is obvious that this argument is true.

We now assume that we have proven the statement for (n−1)-dimensional space and consider the case for n-dimensional space.

If there exists a spheroid P which contains another spheroid Q, then they are in different families. Since spheroids in the same family as P are separate from P, they are also separate from Q. Hence spheroid P itself constitutes a sphere family, $\min\{a,b\}=1 \leqslant n$ as desired. Consequently, we may assume that no spheroid contains any other spheroid and that a, b are both greater than 1.

Let P, Q be two spheroids which belong to different families. By our assumption, their surfaces intersect. Let O be a point on their intersection. Exert the generalized inversion at center O with any positive radius. By Lemma 4.1, P, Q

becomes two n-dimensional planes p, q while other spheroids become two new sphere families P_1, P_2, $\cdots$, P_{a-1} and Q_1, Q_2, $\cdots$, Q_{b-1}. Assume that p, q intersect at intersecting line r. By inversive property, Q_1, Q_2, $\cdots$, Q_{b-1} intersect with plane p and are separate from plane q. By Lemma 4.1, they are to the same side of plane q as point O. Thus their intersection with p is located in the same half plane of p divided by intersecting line r. Let p denote the half plane as well. Similarly, P_1, P_2, $\cdots$, P_{a-1} intersect with plane q and are separate from plane p, and their intersection with q is located in the same half plane of q divided by intersecting line r.

Since half plane p intersect with Q_1, Q_2, $\cdots$, Q_{b-1} while half plane q is separate from them, there exists a half plane R' with boundary r between p and q which is tangent to one of Q_1, Q_2, $\cdots$, Q_{b-1} (assume that it is Q_j) and intersects with the other $b-2$ spheres. Assume that there exists a P_i among P_1, P_2, $\cdots$, P_{a-1} that doesn't intersect with R'. Since P_i intersects with q and is separate from p while Q_j intersects with p and is separate from q, these two spheroids (namely P_i and Q_j) is located to different sides of R', implying that they will not intersect, a contradiction. Therefore our assumption was wrong and R' intersects with P_1, P_2, $\cdots$, P_{a-1}. Now rotate the half plane R' by a sufficiently small angle to obtain a new half plane R that intersects with all of P_1, P_2, $\cdots$, P_{a-1} and Q_1, Q_2, $\cdots$, Q_{b-1}.

Consider the intersections of R and each of P_1, P_2, $\cdots$, P_{a-1} and Q_1, Q_2, $\cdots$, Q_{b-1}, which constitute two $(n-1)$-dimensional sphere families with capacities $a-1$, $b-1$, respectively. By Lemma 4.2,

$$P_i \cap Q_j \cap R \neq \varnothing \Leftrightarrow (P_i \cap R) \cap (Q_j \cap R) \neq \varnothing \qquad \forall 1 \leq i \leq a-1, 1 \leq j \leq b-1$$

So these two sphere families intersect. Using the induction hypothesis we have $\min\{a-1, b-1\} \leq n-1$. Hence $\min\{a,b\} \leqslant n$. Therefore we finish our inductive step and our proof is completed. ■

Theorem 4.4 Define $a(n,\theta)$ as the maximum number of points on the surface of an n-dimensional unit ball such that the spherical distance between any pair of them is greater than or equal to θ. Suppose a sphere family with capacity 3 intersects with another sphere family in the n-dimensional space. Let M_n denote the maximum capacity of the latter sphere family. Then $M_n = a(n-1, \frac{\pi}{3})$ or $M_n = a(n-1, \frac{\pi}{3}) - 1$. Which value M_n equals to depends on whether the points are arranged compactly.

Proof: For a group of separate spheres, choose one of them, increase its radius such that it is tangent to exactly one of the other spheres and is separate from the left ones, and then exert generalized inversion with the point of tangency as center and with any positive radius. Such operation is called ***tangent expansion***. Thus, if two n-dimensional sphere families intersect and the capacity of one of them is 3, then by the tangent expansion these three spheres become two parallel n-dimensional plane A, B and an n-dimensional sphere O in between, while the other sphere family becomes a sphere family which intersect with *A*, *B* and *O*. Obviously their radii are all greater than that of sphere *O*.

For convenience we assume that the two parallel planes *A* and *B* are both tangent to sphere *O* and that each sphere of the other family is tangent to at least one of *A*, *B* (after applying homothetic transformation to each of them) because such operation doesn't increase M_n. Let ∂ denote the plane parallel to *A*, *B* which passes the center of sphere *O*. Assume that O_i is the center of sphere in the other family and P_i is its projection on ∂ $(1 \le i \le M_n)$. We claim that

$$P_iP_j > \max\left\{OP_i, OP_j\right\}(1 \le i < j \le M_n) \tag{6}$$

Build up the rectangular coordinate system with *O* as the origin in the *n*-dimensional space. Assume that $O_i(x_1, x_2, \cdots, x_n), O_j(y_1, y_{2,} \cdots, y_n)$, then plane ∂ is an (*n*−1)-dimensional subspace that passes point *O* and the coordinates of P_i, P_j in this subspace are $(x_1, x_2, \cdots, x_{n-1})$ and $(y_1, y_{2,} \cdots, y_{n-1})$, respectively. In addition, $O_iP_i = |x_n|, O_jP_j = |y_n|$. Assume that 1, R_i, R_j are the radii of sphere *O*, O_i, O_j, respectively. Since sphere O_i is tangent to at least one of *A* and *B*, it follows that

$$R_i = |x_n| + 1, OO_i = \sqrt{x_n^2 + OP_i^2} \le 1 + R_i = |x_n| + 2$$

Hence $4 + 4|x_n| \ge OP_i^2$. Similar reasoning shows $4 + 4|y_n| \ge OP_j^2$. Besides, we have

$$O_iO_j^{\ 2} \le P_iP_j^2 + (|x_n| + |y_n|)^2$$

And $$O_iO_j^2 > (R_i + R_j)^2 = (|x_n| + |y_n| + 2)^2 = (|x_n| + |y_n|)^2 + 4(|x_n| + |y_n|) + 4$$

Combining these two inequalities to yield

$$P_iP_j^2 > 4(|x_n| + |y_n|) + 4 \ge \max\left\{4|x_n| + 4, 4|y_n| + 4\right\} \ge \max\left\{OP_i^2, OP_j^2\right\}$$

Hence the inequality (6) holds. Then P_iP_j is the longest side of ΔOP_iP_j, implying that $\angle P_iOP_j > \frac{\pi}{3}$, which follows that

$$M_n \le a(n-1, \frac{\pi}{3})$$

If the point arrangement corresponding to $a(n-1, \frac{\pi}{3})$ is compact, the upper bound won't be attained. Hence

$$M_n = a(n-1,\frac{\pi}{3})-1$$

Such as in the case n=3.

If the point arrangement corresponding to $a(n-1,\frac{\pi}{3})$ is not compact, the points can be rearranged such that the flare angle of any pair of them to the point O is greater than $\frac{\pi}{3}$, thus the upper bound is attained, that is,

$$M_n = a(n-1,\frac{\pi}{3}),$$

such as the '12 balls problem' in the case n=4.[5] ∎

5 Some Conjectures and Unsolved Problems

On one hand, for the division problem of two sphere families in higher dimensional space, we propose some conjectures as follows. Although these conjectures still need to be proved strictly, they point out our research direction.

Definition 5.1 For integers k, m≥2, define $F_n(k,m)$ as the maximum number of pairs of intersecting spheres produced by two sphere families with capacities k, m in n-dimensional space and $P_n(k,m)$ as the maximum number of pairs of intersecting spheres produced by two unit ball families (that is, sphere families which consist of unit balls only) with capacities k, m in n-dimensional space.

Definition 5.2 For two sphere families in n-dimensional space, let the centers of the spheres be the vertices of a graph, two vertices are connected if and only if their corresponding spheres intersect. Such a graph is called the ***structure graph*** of two sphere families. A simple graph is called ***available*** if and only if there exist several separate or tangent spheres corresponding to the vertices of the graph, respectively, and two spheres are tangent if and only if their corresponding vertices are connected in the graph.

Conjecture 5.3 The structure graph of any two sphere families in n-dimensional space(n⩾2) is available.

Reasons: We begin with the case n=1. The structure graph of any two line segment families is available if no line segment contains any other line segment (we don't need such restriction in higher dimensional space, since that if sphere A contains sphere B, we can replace B with a sufficient small sphere which intersects with A). In fact, for any two intersecting line segments, we may choose a common endpoint for them in their intersection and we will be done. So the conjecture is true for some cases for n=1(We consider them best cases, since we are calculating the maximum number of pairs of intersecting line segments).

As for n=2, by Lemma 2.2, the structure graph of two circle families is planar, and by Circle Packing Theorem (also known as the Koebe-Andreev-Thurston Theorem) [2], it is available. Hence the conjecture is also true for n=2.

Moreover, if we continuously shift the spheres or change their sizes until they all get separate, there should be a situation in between where all pairs of intersecting spheres become pairs of tangent spheres. Consequently we think that the conjecture holds. ∎

This conjecture helps us to simplify the configuration of sphere families (see the following corollary whose proof is based on Conjecture 5.3). It also makes a little step in proving Conjecture 5.5.

Corollary 5.4 $K_{4,4}-e$ is not the structure graph of any two sphere families in 3-dimensional space, where $K_{4,4}-e$ denotes the graph obtained from $K_{4,4}$ by deleting an edge.

Proof: For the sake of contradiction we assume that $K_{4,4}-e$ is the structure graph of two sphere families. Then by Conjecture 5.3, it is available, that is, there exist two sphere families $\alpha=\{A,B,C,D\}$ and $\beta=\{E,F,G,H\}$ such that any two spheres in different families are tangent except for D and H. By increasing the radii of A, B, C, D and decreasing the radii of E, F, G, H by an identical positive real number, all the pairs of tangent spheres maintain and we can further assume that there exist two tangent spheres in family α. Essentially there are two cases.

Case 1 A and B are tangent. Exert the generalized inversion with the point of tangency as the center, and A, B become two parallel planes. Since E, F, G, H are all tangent to A, B, they have a same radius (assume that it is 1) and their centers are in a same plane ∂. Since C is tangent to E, F, G, H and D is tangent to E, F, G, we have

$$|CE|=|CF|=|CG|=|CH|,\ |DE|=|DF|=|DG|,$$

And $CD\perp$plane ∂. Assume that line CD intersects with plane ∂ at P, $|CP|=h$, and r is the radius of C. We have

$$|PE|=|PF|=|PG|=|PH|=\sqrt{(1+r)^2-h^2}$$

But since $|EF|,|FG|,|GH|,|HE|,|EG|$ and $|HF|$ are all greater than 2 while

$$\min\{\angle XPY\,|\,X,Y\in\beta,X\neq Y\}\leq 90°$$

We have $\sqrt{(1+r)^2-h^2}<\sqrt{2}$ or $(1+r+h)(1+r-h)<2$ .Combining this with $r+h<1$ (since C doesn't intersect with plane A, B) we have $r>h$, hence $P\in C$. Similarly $P\in D$, and therefore $C\bigcap D\neq\varnothing$, which contradicts the definition of sphere family.

Case 2 A and D are tangent. Exert the generalized inversion with the point of tangency as the center, and A, D become two parallel planes. Note that E, F, G are tangent to A, B, C, D, it follows that any sphere tangent to sphere B, C and plane D (resp. A) is also tangent to plane A (resp. D). Hence sphere H is tangent to all of A, B, C, D and the structure graph of such sphere families is $K_{4,4}$. However, this is impossible by Theorem 4.3(n=3).

Overall we show that our assumption was wrong and Corollary 5.4 is proved.

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Conjecture 5.5 $F_n(k,m)=\max\{P_{n-1}(k-2,m)+2m, P_{n-1}(k,m-2)+2k\}$

If $k \geq m \geq 2$, we guess

$$P_{n-1}(k-2,m)+2m \leq P_{n-1}(k,m-2)+2k \tag{7}$$

Reasons: On one hand, there exist two unit ball families with capacities k, m−2 in the (n−1)-dimensional producing $P_{n-1}(k,m-2)$ pairs of intersecting spheres. Draw an n-dimensional unit ball at the center of each of these (n−1)-dimensional unit balls and then draw two n-dimensional planes parallel to those n-dimensional unit balls. Then we increase the radius of the unit balls in the family with capacities k by a sufficiently small positive real number and we get $P_{n-1}(k,m-2)+2k$ pairs of intersecting spheres in n-dimensional space. Similarly $P_{n-1}(k-2,m)+2m$ is also obtainable as the number of pairs of intersecting spheres. Since the inequality (7) holds for n=2, 3, we wonder whether it holds for all positive integer n.

On the other hand, it is easy to obtain

$$P_1(k,m)=\begin{cases} 2m & (k>m) \\ 2m-1 & (k=m) \end{cases}$$

Combining this with Theorem 2.3, Conjecture 5.5 holds for n=2. As for n=3, by Theorem 4.4 we have $F_3(k,3)=\min\{3k,2k+5\}$ and it is easy to obtain $P_2(k,1)=\min\{k,5\}$, implying that Conjecture 5.5 holds for n=3, m=3. Moreover, by Corollary 5.4 we have $F_3(4,4)=14$. Since $P_2(4,2)=6$, Conjecture 5.5 is also true when n=3, k=m=4.

Combining the two sides, we think that the conjecture is true. █

On the other hand, we raise some unsolved problems as follows and hope to discuss them with the readers.

① The value of $P_n(k,m)$. According to the discussion of 2-dimensional case, we think such a problem can be evaluated but does not have exact answer, and the result will change in terms of the number of circles (or spheres).

② For the problem of n-dimensional sphere family, we hope to find more maximal forbidden subgraph of the structure graph, where maximal forbidden subgraph is defined to be the graph can not be a structure graph, but the resultant graph by deleting an edge can be a structure graph. For two 3-dimensional sphere families, by Theorem 4.4 and Corollary 5.4, $K_{3,6}$ and $K_{4,4}-e$ are two maximal forbidden subgraphs. Since the set of maximal forbidden subgraphs for planar graphs is $\{K_{3,3},K_5\}$[1], we want to know that whether $\{K_{3,6},K_{4,4}-e\}$ is the maximal forbidden subgraph set of the bipartite structure graph of two sphere families.

③ Note that there are only two techniques used to handle the problem of sphere family, namely, homothetic transformation and inversion. We hope to find some new methods to solve the above problems, and discuss the division problem of m sphere families. It is worth mentioning that if we solve the division problem of two circle families, then it can be generalized to general cases smoothly (since we just need to consider the number of pairs of intersecting circles). But it is different for the case of 3-dimension. Since we shall consider the division in the intersection of some sphere families, it is not as simple as just counting the number of pairs of intersecting spheres. So such a problem is very challenging.

④ The division problem about convex polyhedrons in the space is the derivation of circle family, sphere family and their generalizations. However, this problem itself still needs to be solved.

Appendix

The title of the former edition of our paper is 'The Research and Generalizations on Several Kinds of Partition Problems in Combinatorics'. We studied some partition problems about the Catalan Number under certain constraints. We also developed the idea of circle family and sphere family. However, afterwards we were told that some of our research on the Catalan Number has already been studied by former mathematicians, so we state those result in this appendix and place our research on circle family and sphere family in the main text. Then we change the title so as to fit the current edition of our paper.

Definition 1 The triangular partition of a convex polygon is the partition that divides the polygon into most parts by several diagonals which don't intersect inside the polygon. An isolated triangle is a triangle three sides of which are the diagonals of a convex polygon in its triangular partition. Define $G_k(n)$ ($n\geqslant 3$, $k\geqslant 0$, $n, k\in Z$) as the number of different kinds of triangular partition of a convex polygon with n sides which has k isolated triangles.

Definition 2 The function $R_k(2n)(k\geq 0, n\geq 2, n, k\in Z)$ denotes the number of different ways to pair the vertices of a convex polygon with $2n$ vertices and join each pair with line segments such that the n line segments produce k points of intersection. It requires that any three diagonals of the polygon share no common point inside it.

Definition 3 Let $B_n(k)$ denote the number of different ways to divide a polygon with n sides into several k-sided polygons by its diagonals. Let $b_n(k)$ denote the number of different ways to divide the vertices of an n-sided polygon into several k-element group such that any pair of k-sided polygons formed by two such groups don't intersect. We only consider these partition problems in the case that the polygon can be divided in such ways.

Definition 4 For positive integers m, n, define a $[m,n]$ broken line as a broken line in the coordinate plane which starts from the origin and is formed by vectors $(1,m)$ or $(1,-n)$ end to end. If it has no intersection with the X-axis except its starting point and its end point, we call it a broken line with property U (short for 'up').

Theorem 1 $G_k(n)=\dfrac{n\cdot 2^{n-4-2k}C_{n-4}^{2k}C_{2k}^{k}}{(k+1)(k+2)}$, where $0\le k\le\left[\dfrac{n}{2}\right]-2$.

Theorem 2 $R_2(2n)=n\left[\dfrac{C_{2(n+3)}^{n+3}}{n+4}-\dfrac{5C_{2(n+2)}^{n+2}}{n+3}+\dfrac{6C_{2(n+1)}^{n+1}}{n+2}-\dfrac{C_{2n}^{n}}{n+1}\right]$.

Theorem 3 $B_n(k)=\dfrac{C_{\frac{(k-1)(n-2)}{k-2}+1}^{\frac{n-2}{k-2}}}{\dfrac{(k-1)(n-2)}{k-2}+1}$; $b_n(k)=\dfrac{C_{n+1}^{\frac{n}{k}}}{n+1}$.

Theorem 4 There are $\dfrac{C_{s+t}^{s}}{s+t}(s-nt)$ $[1,n]$ broken lines with property U whose end point is $P(s+t,s-nt)$, where s, $t\in N, s-nt>0$.

Theorem 5 Let $f(s,t)$ be the number of $[n,1]$ broken lines with property U whose end point is $Q(s+t,ns-t)$. Then

$$f(s,t)=C_{s+t-1}^{t}-\sum_{i=1}^{k}C_{s+t-i(n+1)}^{s-i}\frac{C_{i(n+1)-1}^{i}}{i(n+1)-1}$$

where $k=\left[\dfrac{t}{n}\right]$ and s, $t\in N, ns-t>0$.

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