

GENUS ZERO GOPAKUMAR-VAFA INVARIANTS OF CONTRACTIBLE CURVES

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Abstract

A version of the Donaldson-Thomas invariants of a Calabi-Yau threefold is proposed as a conjectural mathematical definition of the Gopakumar-Vafa invariants. These invariants have a local version, which is verified to satisfy the required properties for contractible curves. This provides a new viewpoint on the computation of the local Gromov-Witten invariants of contractible curves by Bryan, Leung, and the author.

1. Introduction

Let X be a Calabi-Yau threefold, $\beta \in H_2(X, \mathbf{Z})$, and g a nonnegative integer. The Gopakumar-Vafa invariants n_β^g were first introduced as an integer-valued index arising from D-branes and M2-branes wrapping holomorphic curves in string theory and M-theory [6]. They are claimed to satisfy the remarkable identity

$$(1) \quad \sum_{\beta, g} N_\beta^g q^\beta \lambda^{2g-2} = \sum_{\beta, g, m} \frac{n_\beta^g}{m} \left(2 \sin \left(\frac{m\lambda}{2} \right) \right)^{2g-2} q^{m\beta},$$

where the N_β^g are the Gromov-Witten invariants of X . The coefficient of λ^{-2} in (1) yields the identity

$$(2) \quad N_\beta^0 = \sum_{k|\beta} \frac{n_{\beta/k}^0}{k^3}.$$

There have been several attempts to provide a mathematical definition, including [7, 9] but there is still no general direct mathematical definition which passes all known tests. However, the invariants can be defined recursively in terms of the Gromov-Witten invariants via (1). When defined in this way, it is only a priori clear that the Gopakumar-Vafa invariants are rational. The *integrality conjecture* asserts that the Gopakumar-Vafa invariants are integers.

In this paper, we consider a variant of the Donaldson-Thomas invariants of Calabi-Yau threefolds, and conjecture that they satisfy the

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condition (2) required to be the genus 0 Gopakumar-Vafa invariants. For this reason, we will call them the *DT-GV invariants*. As evidence, we show that a local version of these invariants satisfies the desired property for local contractible curves.

Li and Wu have performed a similar analysis for local K3 fibred Calabi-Yau threefolds in [11].

This paper is organized as follows. In Section 2 we review the definition of the Donaldson-Thomas invariants and show that they apply in the situation considered here (Proposition 2.1). Conjecture 2.3 asserts that these are the genus 0 Gopakumar-Vafa invariants. In Section 3, we review the results of [3] on the local Gromov-Witten theory of contractible curves and their implication for the associated Gopakumar-Vafa invariants. We then define the local DT-GV invariants and compute them directly for contractible curves (Proposition 3.3). As a corollary, they satisfy the genus 0 properties required of the Gopakumar-Vafa invariants (Corollary 3.4).

In this paper, we understand the Calabi-Yau condition to mean that K_X is trivial. We also refer to the holomorphic Casson invariants of [19] and their extensions as *Donaldson-Thomas invariants*. Note that we are using this terminology more broadly than it was used in [12], where it only referred to the holomorphic Casson invariants of ideal sheaves.

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2. Definition of the Invariants

In this section, we discuss certain Donaldson-Thomas invariants of Calabi-Yau threefolds X and conjecture that they are the genus zero Gopakumar-Vafa invariants of X .

The Donaldson-Thomas moduli space considered here was already studied in [7], where it was used to give a different proposal for the Gopakumar-Vafa invariants of X . It was conjectured that the genus 0 invariants introduced there coincide with the Donaldson-Thomas invariants studied in this paper. Here, we are adopting a different viewpoint by focusing on the genus 0 invariants, and treating the Donaldson-Thomas invariants as the proposal for the Gopakumar-Vafa invariants.

We start by reviewing the construction of Simpson's moduli space of semistable sheaves of pure dimension 1 [18]. Another useful reference for this section is [8].

Let (X, L) be a polarized Calabi-Yau threefold, and let $\beta \in H_2(X, \mathbf{Z})$. We consider the Simpson moduli space $S(X, \beta)$ of sheaves of \mathcal{O}_X modules

$$(3) \quad \{F \text{ of pure dim } 1 \mid \text{ch}_2(F) = \beta, \chi(F) = 1, F \text{ semistable wrt } L\}.$$

For the convenience of the reader, we recall the notion of stability adapted to the present situation. For a sheaf F of dimension 1, its *Hilbert polynomial* is

$$P_F(n) = (L \cdot \text{ch}_2(F))n + \chi(F),$$

and its *reduced Hilbert polynomial* is

$$(4) \quad p_F(n) = \frac{P_F(n)}{L \cdot \text{ch}_2(F)}.$$

A sheaf F of pure dimension 1 is (*semi*)*stable* if for all nontrivial proper subsheaves $G \subset F$ we have $p_G(n) < p_F(n)$ (resp. $p_G(n) \leq p_F(n)$ for semistable). By (4), the semistability condition is equivalent to

$$(5) \quad \frac{\chi(G)}{L \cdot \text{ch}_2(G)} \leq \frac{\chi(F)}{L \cdot \text{ch}_2(F)}$$

for all subsheaves G of F . For a subscheme $Z \subset X$ we usually write $P_Z(n)$ (resp. $p_Z(n)$) in place of $P_{\mathcal{O}_Z}(n)$ (resp. $p_{\mathcal{O}_Z}(n)$).

Note that any $F \in S(X, \beta)$ is necessarily stable. To see this, observe that for G a subsheaf of F , (5) reads $\chi(G)/(L \cdot \text{ch}_2(G)) \leq 1/(L \cdot \beta)$. But since $L \cdot \text{ch}_2(G) \leq L \cdot \beta$, equality can only hold if $\chi(G) = 1$ and $L \cdot \text{ch}_2(G) = L \cdot \beta$. But in that case, we must have $G = F$, and we conclude that F is stable.

Thus, $S(X, \beta)$ is a projective variety.

Proposition 2.1. *There exists a canonically determined perfect obstruction theory on $S(X, \beta)$ of virtual dimension 0.*

Proof. The construction of the virtual fundamental class in [19, Theorem 3.30] goes through with little modification (i.e., the positive rank assumption made in [19] is not needed for the current application). We only have to show that $\dim \text{Ext}^3(F, F)$ is independent of $F \in S(X, \beta)$. But by Serre duality,

$$\text{Ext}^3(F, F) \simeq (\text{Hom}(F, F))^*.$$

Stability implies simplicity, so that $\text{Hom}(F, F) \simeq \mathbf{C}$ is given by the scalar multiplication maps. This immediately implies that $\dim \text{Ext}^3(F, F) = 1$ for all $F \in S(X, \beta)$, so we have a canonical perfect obstruction theory.

The virtual dimension is $\dim \text{Ext}^1(F, F) - \dim \text{Ext}^2(F, F)$, which vanishes by Serre duality. Alternatively, this is a symmetric obstruction theory so that the expected dimension is 0 [1]. q.e.d.

Definition 2.2. The *DT-GV invariants* are the Donaldson-Thomas invariants $n_\beta(X) = \deg[S(X, \beta)]^{\text{vir}} \in \mathbf{Z}$.

The argument in [19] applies without modification to show that the $n_\beta(X)$ are deformation invariants of the polarized Calabi-Yau X .

We conjecture that the $n_\beta(X)$ are the genus 0 Gopakumar-Vafa invariants. More precisely, let $N_\beta(X)$ be the genus 0 Gromov-Witten invariants of X .

Conjecture 2.3.

$$N_\beta(X) = \sum_{k|\beta} \frac{n_{\beta/k}(X)}{k^3}.$$

Remark. We can similarly define a virtual fundamental class for more general smooth threefolds X whenever $\dim \text{Ext}^3(F, F)$ is independent of $F \in S(X, \beta)$. For example, this condition holds for any β and L if $-K_X$ is very ample, and in this case it can be computed by a simple modification of the computation in [13, Lemma 1] that the virtual dimension is $D = \int_\beta c_1(X)$. Note that this coincides with the virtual dimensions of Gromov-Witten theory and Donaldson-Thomas theory [13]. This leads to a conjectural definition of the integer-valued invariants whose existence was conjectured in [15]. Details will be given elsewhere.

3. Contractible curves

In this section, we define a local version of the DT-GV invariants and compute them in the case of a small neighborhood of a contractible \mathbf{P}^1 . We then verify Conjecture 2.3 in this situation.

We begin by recalling the geometry of contractible curves and their local Gromov-Witten theory.

3.1. Contractible curves and their local GW invariants. Suppose that $C \simeq \mathbf{P}^1$ is an analytically contractible curve in a Calabi-Yau threefold X . Then there is an analytic contraction map $f : X \rightarrow Y$ such that $f(C)$ is a point $p \in Y$ which is a normal singularity of Y , and f induces an isomorphism between $X - C$ and $Y - p$. We can and will replace X by a small neighborhood of C and let Y be a small neighborhood of p .

By a lemma of Reid [16], the generic hyperplane section through p is a surface S with an isolated rational double point at p , and the proper transform of S is a partial resolution $\bar{S} \rightarrow S$, i.e. \bar{S} has at worst rational double points, and the minimal resolution $\tilde{S} \rightarrow S$ of S factors through $\bar{S} \rightarrow S$. Note that $C \subset \bar{S}$. It follows that, shrinking X and Y if necessary, there is a map $\pi : X \rightarrow \Delta$, where $\Delta \subset \mathbf{C}$ is a disc containing the origin, such that $\bar{S} = \pi^{-1}(0)$.

The possible singularity types of S and \bar{S} were classified in [10], and there are six cases. The singularity $p \in S$ is either an A_1 , D_4 , E_6 , E_7 ,

or E_8 singularity, and there are two subcases of the E_8 case. The curve $C \subset \bar{S}$ is the exceptional curve of a partial resolution of S , associated to a particular vertex of the corresponding Dynkin diagram which we call the *distinguished vertex*. The surface \bar{S} is obtained from the minimal resolution \tilde{S} of $p \in S$ by a map $\psi : \tilde{S} \rightarrow \bar{S}$, which blows down all of the exceptional curves of $\tilde{S} \rightarrow S$ except for the one corresponding to the distinguished vertex of the Dynkin diagram.

The vertex can be uniquely specified by giving the multiplicity of that vertex in the fundamental cycle of the minimal resolution. The cases listed as a pair (singularity type, multiplicity) are

$$(A_1, 1), (D_4, 2), (E_6, 3), (E_7, 4), (E_8, 5), (E_8, 6).$$

More details are given in [10] or the appendix to this paper.

These situations can be distinguished by an invariant of Kollár called the *length* of C , which is the multiplicity of $f^{-1}(m_p)$ at the generic point of C . The length ℓ coincides with the multiplicity of the distinguished vertex in each of these cases.

In [3], it was shown that the Gromov-Witten invariants of X have a well-defined contribution arising from stable maps to X with image contained in C , and this contribution was computed.

To state the result of [3], let \bar{S} be as above and let $I = I_{C, \bar{S}}$ be the ideal sheaf of C in \bar{S} . For each i with $1 \leq i \leq \ell$, let $I^{(i)}$ be the saturation of I^i , i.e., the smallest ideal sheaf containing I^i which defines a subscheme of X of pure dimension 1 (necessarily having support C). Let $C_i \subset X$ be the subscheme of X defined by $I^{(i)}$. Then it was shown in [3] that C_i is an isolated point of the component of the Hilbert scheme of X that it is contained in. Letting n_i be the multiplicity of this point, the result is

Proposition 3.1 ([3, Theorem 1.5]). *The contribution of C to the genus 0 Gromov-Witten invariant $N_{d[C]}(X)$ is*

$$\sum_{k|d} \frac{n_{d/k}}{k^3}.$$

Remark. In [3], the local Gromov-Witten invariants were computed for every genus and were shown to be completely determined by these n_d according to (1) with $n_d^0 = n_d$ and $n_d^g = 0$ for $g > 0$.

For later use, we give here some properties of the curves C_i .

Lemma 3.2.

- (i) *The curve C_i is the unique 1 dimensional subscheme of \bar{S} supported on C without embedded points, having multiplicity i at its generic point.*
- (ii) *The curve C_ℓ coincides with the scheme-theoretic inverse image of p by f .*

- (iii) For each i with $1 \leq i \leq \ell$ we have $\chi(\mathcal{O}_{C_i}) = 1$ and $H^1(\mathcal{O}_{C_i}) = 0$.
- (iv) The sheaf \mathcal{O}_{C_i} is stable for each i with $1 \leq i \leq \ell$.

Proof. Away from the singularities of \bar{S} , a scheme supported on C with no embedded points and multiplicity i coincides with the scheme defined by I^i , so we are reduced to a local question near the singularities of \bar{S} . We let A be the local ring of \bar{S} at a singular point, and we abuse notation slightly by again denoting by I the prime ideal of A corresponding to C . Letting J be an ideal of A with $J \subset I$ and A/J of pure dimension 1, we see that J is I -primary. Then, by localizing at I , we conclude that if J_I is an ideal of A_I of multiplicity i then J is the i^{th} symbolic power of I , which again by primary decomposition is the saturation of I^i . This proves (i).

Note that the scheme $f^{-1}(p)$ is contained in \bar{S} . So (ii) will follow from (i) if we can show that $f^{-1}(p)$ has no embedded points. But a curve with embedded points has nonconstant regular functions; pulling back via ψ^* gives a nonconstant regular function on the exceptional scheme of $\tilde{S} \rightarrow S$, a contradiction.

Assertion (iii) is trivial for $i = 1$, since $C_1 = C \simeq \mathbf{P}^1$. For $i > 1$ we use the short exact sequences

$$(6) \quad 0 \rightarrow I_{C_j}/I_{C_{j+1}} \rightarrow \mathcal{O}_{C_{j+1}} \rightarrow \mathcal{O}_{C_j} \rightarrow 0.$$

Now we compute that $I_{C_j}/I_{C_{j+1}} \simeq \mathcal{O}_C(-1)$ for $1 \leq j \leq \ell - 1$. To see this, first note that $I_{C_j}/I_{C_{j+1}}$ is a torsion-free sheaf of rank 1 on C , hence locally free. We compute its degree by the method in [14]. Let $\pi : \tilde{S} \rightarrow \bar{S}$ be the contraction map used to produce the partial resolution, and let $\tilde{C} \simeq C$ be the proper transform of C via ψ . There is a map

$$(7) \quad g : \psi^*(I_{C_j}/I_{C_{j+1}}) \rightarrow I_{\tilde{C}}^j/I_{\tilde{C}}^{j+1} \simeq \mathcal{O}_C(2j).$$

The map g is an isomorphism away from the singularities of \bar{S} . So we can find the degree of $(I_{C_j}/I_{C_{j+1}})$ by explicit computation of the order of vanishing of g at the singular points of \bar{S} . Doing this explicitly in each case, using the local equations in [14], we compute that the degree is -1 . Some details of the computation are given in the Appendix.

The first assertion of (iii) now follows from (6). The second assertion follows since p is a rational singularity.

To prove assertion (iv), we let Z be a nontrivial proper subscheme of C_i . We have to show that the reduced Hilbert polynomials satisfy $p_{I_{Z,C_i}}(n) < p_{C_i}(n)$.

First we show this inequality for each $Z = C_j$ with $j < i$. We have

$$P_{I_{C_j,C_i}}(n) = P_{C_i}(n) - P_{C_j}(n) = (i - j)(L \cdot C)n$$

by Lemma 3.2. Then the relevant reduced Hilbert polynomial is given by $p_{I_{C_j, C_i}}(n) = n$ so that

$$(8) \quad p_{I_{C_j, C_i}}(n) < p_{C_i}(n) = n + \frac{1}{i(L \cdot C)},$$

as required.

If Z is 0 dimensional, then $\text{ch}_2(I_Z) = \text{ch}_2(\mathcal{O}_{C_i})$, so we just have to show that $\chi_{I_Z}(n) < \chi_{C_i}(n)$. This follows immediately from $\chi_{I_Z}(n) = \chi_{C_i}(n) - \chi(\mathcal{O}_Z)$.

If Z has length j at the generic point of C , then $Z \subset C_j$ and there is a short exact sequence

$$0 \rightarrow I_{Z, C_i} \rightarrow I_{C_j, C_i} \rightarrow \mathcal{O}_Y \rightarrow 0$$

for some zero-dimensional subscheme $Y \subset C$. The desired inequality

$$p_{I_{Z, C_i}}(n) < p_{C_i}(n)$$

follows from $\text{ch}_2(I_Z) = \text{ch}_2(C_j)$ and $\chi(I_{Z, C_i}) = \chi(I_{C_j, C_i}) - \chi(\mathcal{O}_Y) \leq \chi(I_{C_j, C_i})$, which implies $p_{I_{Z, C_i}}(n) \leq p_{I_{C_j, C_i}}(n)$ together with (8). q.e.d.

Remark. An alternative proof of $\chi(\mathcal{O}_{C_i}) = 1$ can be given using the rationality of the singularity, which implies that $h^1(\mathcal{O}_{C_i}) = 0$ and the use of ψ^* to show that $h^0(\mathcal{O}_{C_i}) = 1$.

3.2. Computation of the DT-GV invariants. Let L be a polarization of X . Let F be a stable sheaf of pure dimension 1 supported on C with $\text{ch}_2(F) = d[C]$ and $\chi(F) = 1$. The rigidity lemma of [4] implies that every deformation of F is supported on C (not necessarily with the reduced structure). Thus $S(X, d[C])$ has a connected component $S_C(X, d[C])$ consisting of sheaves supported on C . We can therefore define the contribution $n_d(X)$ of C to our genus 0 invariants as the degree of the part of $[S(X, d[C])]^{\text{vir}}$ supported on $S_C(X, d[C])$.

Comparing Proposition 3.1 with Conjecture 2.3, it is clear what needs to be proven, and in fact:

Proposition 3.3. $n_d(X) = n_d$.

Note in particular that the invariants $n_d(X)$ are independent of the polarization chosen.

Proof. We have an exact sequence

$$(9) \quad 0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{\bar{S}} \rightarrow 0,$$

since $I_{\bar{S}, X} = \pi^*(I_{0, \Delta})$, which implies that $I_{\bar{S}, X}$ is trivial.

Let $F \in S_C(X, d[C])$. Tensoring with (9), we get an exact sequence

$$(10) \quad F \rightarrow F \rightarrow F \otimes \mathcal{O}_{\bar{S}} \rightarrow 0.$$

By the stability of F , the first map in (10) must be a scalar multiplication, which is either 0 or an isomorphism. But in the latter case, we

would conclude that $F \otimes \mathcal{O}_{\bar{S}} = 0$, which is impossible. Thus the first map is 0, and we conclude that $F \simeq F \otimes \mathcal{O}_{\bar{S}}$, i.e. that the scheme-theoretic support of F is contained in \bar{S} . Since the intersection of all surfaces $S \subset Y$ containing p is the reduced point p , it follows that the support of F is contained in the intersection of all the possible surfaces \bar{S} , which is the scheme-theoretic inverse image $f^{-1}(p)$, i.e., the scheme C_ℓ by Lemma 3.2.

Next we show that F must be isomorphic to one of the sheaves \mathcal{O}_{C_i} for $1 \leq i \leq \ell$. The argument is similar to the argument in [7]. By Lemma 3.2, we already know that $\mathcal{O}_{C_i} \in S(X, i[C])$ for $1 \leq i \leq \ell$.

Since $\chi(\mathcal{O}_F) = 1$ and $h^2(\mathcal{O}_F) = 0$, it follows that F has a section, i.e., we have a map $s : \mathcal{O}_X \rightarrow F$. Since F is an \mathcal{O}_{C_ℓ} -module, the kernel of s is the ideal sheaf of a subscheme $Z \subset C_\ell$. Since F has pure dimension 1, it follows that Z has pure dimension 1 as well. Lemma 3.2 implies that $Z \simeq C_i$ for some i between 1 and ℓ and we have an injection $\mathcal{O}_{C_i} \hookrightarrow F$.

The Hilbert polynomial of F is $P_F(n) = d(L \cdot [C])n + 1$, and the Hilbert polynomial of C_i is $P_{C_i}(n) = i(L \cdot [C])n + 1$. By stability, we conclude that $d \leq i$, while $i \leq d$ since \mathcal{O}_{C_i} is a subsheaf of F . It follows that $i = d$, whence $\mathcal{O}_{C_i} \hookrightarrow F$ is an isomorphism.

We have shown that $S(X, d[C])$ consists of the single sheaf \mathcal{O}_{C_d} if $d \leq \ell$ and is zero if $d > \ell$, so we may as well assume that $d \leq \ell$. Next we identify $S(X, d[C])$ with a component of the Hilbert scheme.

Let A be a local ring and let \mathcal{F} be a coherent sheaf on $X \times \text{Spec}(A)$, flat over $\text{Spec}(A)$. Let $\rho : X \times \text{Spec}(A) \rightarrow \text{Spec}(A)$ be the projection. We have $H^1(\mathcal{O}_{C_d}) = 0$ and $h^0(\mathcal{O}_{C_d}) = 1$ by Lemma 3.2, so by standard base change results, $\rho_*(\mathcal{F})$ is invertible, hence free of rank 1 on $\text{Spec}(A)$. The adjoint map to any isomorphism $\mathcal{O}_{\text{Spec}(A)} \rightarrow \rho_*(\mathcal{F})$ gives a map

$$(11) \quad \mathcal{O}_{X \times \text{Spec}(A)} \rightarrow \mathcal{F},$$

necessarily a surjection, so that \mathcal{F} can be identified with a flat family of subschemes of X . The remaining details are straightforward and left to the reader.

The degree of the virtual fundamental class can be computed by [17, Theorem 4.6]. Since $S_C(X, d[C])$ is a point, the conclusion is that it coincides with the degree of the Fulton Chern class, which in turn is equal to the multiplicity of C_d as a point of $S_C(X, d[C])$ by [5]. Alternatively, the methods of [2] apply. q.e.d.

Corollary 3.4. *The local version of Conjecture 2.3 holds for contractible curves.*

Remarks.

- (i) There is an alternative simpler proof of Proposition 3.3. The cited result of [3] was proven by showing that in a generic deformation of X , each C_i deforms to n_i isolated curves, each isomorphic to

\mathbf{P}^1 with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, then using the deformation invariance of the Gromov-Witten invariants. We can also invoke a local version of the deformation invariance of the DT-GV invariants in combination with the computation of the moduli space of stable sheaves supported on a \mathbf{P}^1 with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ done in [7]. We have not gone this route to highlight the direct computability of the DT-GV invariants.

- (ii) Since the genus g Gopakumar-Vafa invariants can be (very roughly) thought of as the “virtual number of genus g Jacobians” contained in the moduli space $S(X, \beta)$, any reasonable mathematical definition of the invariants n_β^g should produce zero for $g > 0$ whenever $S(X, \beta)$ is zero-dimensional. In particular, we should get zero for the higher genus local Gopakumar-Vafa invariants of contractible curves, as computed by Gromov-Witten theory in [3]. However, we will not single out a specific proposed definition here.

Question: For a general Calabi-Yau threefold X , are the DT-GV invariants independent of L ?

Appendix: Some computations

In this appendix, we give some of the calculations supporting the proof of part Lemma 3.2, (iii). The illustrative examples given here should suffice to allow the interested reader to carry the calculation to its completion.

First, we list the singularities of \bar{S} , which are visible from Figure 1 in [10].

p	ℓ	Sing(S)
A_1	1	none
D_4	2	A_1, A_1, A_1
E_6	3	A_2, A_2, A_1
E_7	4	A_3, A_2, A_1
E_8	5	A_4, A_3
E_8	6	A_4, A_2, A_1

We need to compute the order of vanishing of (7) at the singularities of \bar{S} .

A uniform treatment can be given for all cases except the A_4 singularity in the length 5 case. Suppose that we are in any of the other situations, with an A_k singularity. Then by the computations in the appendix to [14], we can describe \bar{S} locally as the hypersurface $xy + z^{k+1} = 0$ with $C \subset \bar{S}$ having ideal (x, z) . We can then compute each $I^{(j)}$ locally and from that the order of vanishing of the map g of (7) by looking at a generator of $I^{(j)}/I^{(j+1)}$.

We illustrate with the A_1 case. The blowup map $\psi : \tilde{S} \rightarrow \bar{S}$ can be described on an affine piece of \tilde{S} by $(u, v) \mapsto (u^2v, v, uv)$. Here $u = 0$ defines the proper transform \tilde{C} of C and $(u, v) = (0, 0)$ is the point of \tilde{C} lying over the singularity of \bar{S} , so the order of vanishing of a function is just given by the exponent of v . We get

j	$I^{(j)}$	generator	ord(g)
1	(x, z)	z	1
2	(x)	x	1
3	(x^2, xz)	xz	2
4	(x^2)	x^2	2
5	(x^3, x^2z)	x^3	3

So if, for example, p is a D_4 singularity, we have three A_1 singularities. Putting $j = 1$ in (7), we see that g vanishes simply at each singularity; hence the degree of $I/I^{(2)}$ is $2 - 3(1) = -1$ as claimed.

In the A_2 case, we similarly compute that g has vanishing orders 1, 2, 2, 3, 4 in the respective cases $j = 1, 2, 3, 4, 5$.

So if p is an E_6 singularity, we have two A_2 singularities and an A_1 singularity. For $j = 1$ we get

$$\deg \left(\frac{I}{I^{(2)}} \right) = 2 - 2(1) - 1 = -1,$$

while for $j = 2$ we get

$$\deg \left(\frac{I^{(2)}}{I^{(3)}} \right) = 4 - 2(2) - 1 = -1$$

as claimed.

The other cases are similar.

References

- [1] K. Behrend, *Donaldson-Thomas invariants via microlocal geometry*, math.AG/0507523.
- [2] K. Behrend & B. Fantechi, *Symmetric obstruction theories and Hilbert schemes of points on threefolds*, math.AG/0512556.
- [3] J. Bryan, S. Katz, & N.C. Leung, *Multiple covers and the integrality conjecture for rational curves in Calabi-Yau threefolds*, J. Algebraic Geom. **10** (2001) 549–568, MR 1905890, Zbl 1045.14019, math.AG/9911056.
- [4] H. Clemens, J. Kollár, & S. Mori, *Higher-dimensional complex geometry*, Astérisque **166** (1988), MR 1004926, Zbl 0689.14016.
- [5] W. Fulton, *Intersection Theory*, Springer-Verlag, New York-Berlin-Heidelberg, 1984, MR 1644323, Zbl 0541.14005.
- [6] R. Gopakumar & C. Vafa, *M-Theory and Topological Strings, II*, hep-th/9812127.

- [7] S. Hosono, M. Saito, & A. Takahashi, *Relative Lefschetz Action and BPS State Counting*, Internat. Math. Res. Notices **15** (2001) 783–816, MR 1849482, Zbl 1060.14017, math.AG/0105148.
- [8] D. Huybrechts & M. Lehn, *The Geometry of Moduli Spaces of Sheaves*, Asp. Math., MPI Bonn, Fried. Vieweg & Sohn, Wiesbaden, 1997, MR 1450870, Zbl 0872.14002.
- [9] S. Katz, A. Klemm, & C. Vafa, *M-Theory, Topological Strings and Spinning Black Holes*, Adv. Theor. Math. Phys. **3** (1999) 1445–1537, MR 1796683, Zbl 0985.81081, hep-th/9910181.
- [10] S. Katz & D.R. Morrison, *Gorenstein threefold singularities with small resolutions via invariant theory for Weyl groups*, J. Algebraic Geom. **1** (1992) 449–530, MR 1158626, Zbl 0788.14036.
- [11] J. Li & B. Wu, *Note on a Conjecture of Gopakumar-Vafa*, Chinese Ann. Math. **27B(2)** (2006) 219–242, MR 2243681.
- [12] D. Maulik, N. Nekrasov, A. Okounkov, & R. Pandharipande, *Gromov-Witten theory and Donaldson-Thomas theory, I*, Compos. Math. **142** (2006) 1263–1285, MR 2264664, math.AG/0312059.
- [13] D. Maulik, N. Nekrasov, A. Okounkov, & R. Pandharipande, *Gromov-Witten theory and Donaldson-Thomas theory, II*, Compos. Math. **142** (2006) 1286–1304, MR 2264665, math.AG/0406092.
- [14] D.R. Morrison, *The birational geometry of surfaces with rational double points*, Math. Ann. **271** (1985) 415–438, MR 0787190, Zbl 0539.14008.
- [15] R. Pandharipande, *Three questions in Gromov-Witten theory*, in ‘Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)’, 503–512, Higher Ed. Press, Beijing, 2002, MR 1957060, Zbl 1047.14043, math.AG/0302077.
- [16] M. Reid, *Minimal models of canonical 3-folds*, in ‘Algebraic varieties and analytic varieties’ (Tokyo, 1981), 131–180, North-Holland, Amsterdam, 1983, MR 0715649, Zbl 0558.14028.
- [17] B. Siebert, *Virtual fundamental classes, global normal cones and Fulton’s canonical classes*, in ‘Frobenius manifolds’, 341–358, Aspects Math., **E36**, Vieweg 2004, MR 2115776, Zbl 1083.14066, math.AG/0509076.
- [18] C. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety, I*, Publ. Math. IHES **79** (1994) 47–129, MR 1307297, Zbl 0891.14005.
- [19] R. Thomas, *A holomorphic Casson invariant for Calabi-Yau 3-folds, and bundles on K3 fibrations*, Jour. Differential Geom. **54** (2000) 367–438, MR 1818182, Zbl 1034.14015, math.AG/ 9806111.

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