MOTIVIC AND QUANTUM INVARIANCE UNDER STRATIFIED MUKAI FLOPS

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Abstract

For stratified Mukai flops of type $A_{n,k}, D_{2k+1}$ and $E_{6,I}$, it is shown that the fiber product induces isomorphisms on Chow motives.

In contrast to (standard) Mukai flops, the cup product is generally not preserved. For $A_{n,2}$, D_5 and $E_{6,I}$ flops, quantum corrections are found through degeneration/deformation to ordinary flops.

1. Introduction

1.1. Backgroumd. Two smooth projective varieties (over $\mathbb{C}$) are K *equivalent* if there are birational morphisms $\phi : Y \to X$ and $\phi' : Y \to X'$ such that $\phi^* K_X = \phi'^* K_{X'}$. This basic equivalence relation had caught considerable attention in recent years through its appearance in minimal model theory and crepant resolutions, as well as other related fields.

The conjectural behavior of K equivalence has been formulated in [**W**]. A *canonical correspondence* $\mathcal{F} \in A^*(X \times X')$ should exist and gives an isomorphism of Chow motives $[X] \cong [X']$. Under $\mathcal{F}$, X and X' should have isomorphic *B-models* (complex moduli with Hodge theory on it) as well as *A-models* (quantum cohomology ring up to analytic continuations over the extended Kähler moduli space).

Basic examples of K equivalence are *flops* (with exceptional loci Z, Z', S)

$$(X, Z) \xrightarrow{\quad f \quad} (X', Z') .$$
$$\psi \searrow \qquad \swarrow \psi'$$
$$(\bar{X}, S)$$

Among them the *ordinary flops* had been studied in [**LLW1**], where the equivalence of motives and A-models was proved. In that case $\mathcal{F}$ is the graph closure $\mathcal{F}_0 := \bar{\Gamma}_f$. In general, $\mathcal{F}$ must contain *degenerate correspondences*.

The typical examples are *Mukai flops*. They had been extensively studied in the literature in hyper-Kähler geometry. Over a general base S, they had also been studied in [**LLW1**], where the invariance of Gromov-Witten theory was proved. In that case $\mathcal{F} = X \times_{\bar{X}} X' = \mathcal{F}_0 + \mathcal{F}_1$ with $\mathcal{F}_1 = Z \times_S Z'$. We expect that for flops $\mathcal{F}$ should be basically $X \times_{\bar{X}} X'$.

To understand the general picture we are led to study flops with $\mathcal{F}$ consisting of many components. The *stratified Mukai flops* provide such examples. They appear naturally in the study of symplectic resolutions [**Fu**], [**Na2**] and they should play important roles in higher dimensional birational geometry. For hyper-Kähler manifolds, see for example [**Mar**].

In this paper, we study general stratified Mukai flops without any assumptions on the global structure of X and X'. By way of it, we hope to develop tools with perspective on future studies.

1.2. Stratified Mukai flops. Fix two natural numbers n, k such that $2k < n + 1$. Consider two smooth projective varieties X and X'. Let $F_k \subset F_{k-1} \subset \cdots F_1 \subset X$ and $F'_k \subset F'_{k-1} \subset \cdots F'_1 \subset X'$ be two collections of closed subvarieties. Assume that there exist two birational morphisms $X \xrightarrow{\psi} \bar{X} \xleftarrow{\psi'} X'$. The induced birational map $f : X \dashrightarrow X'$ is called *a (stratified) Mukai flop of type $A_{n,k}$* over $\bar{X}$ if the following conditions are satisfied:

(i) The map f induces an isomorphism $X \setminus F_1 \xrightarrow{\sim} X' \setminus F'_1$.

(ii) $\psi(F_j) = \psi'(F'_j) =: S_j$ for $1 \leq j \leq k$.

(iii) S_k is smooth and there exists a vector bundle V of rank $n + 1$ over it such that F_k is isomorphic to the relative Grassmanian $G_{S_k}(k, V)$ of k-planes over S_k and the restriction $\psi|_{F_k} : F_k \to S_k$ is the natural projection. Furthermore, the normal bundle $N_{F_k/X}$ is isomorphic to the relative cotangent bundle $T^*_{F_k/S_k}$. The analogue property holds for F'_k and ψ' with V replaced by its dual V^*.

(iv) If $k = 1$, we require that f is a usual Mukai flop along F_k. When $k \geq 2$, let Y (resp. Y', $\bar{Y}$) be the blow-up of X (resp. X', $\bar{X}$) along F_k (resp. F'_k, S_k). By the universal property of the blow-ups, we obtain morphisms $Y \to \bar{Y} \leftarrow Y'$. The proper transforms of F_j, F'_j give collections of subvarieties on Y, Y'. We require that the birational map $Y \dashrightarrow Y'$ is a Mukai flop of type $A_{n-2,k-1}$.

We define a *Mukai flop of type D_{2k+1}* in a similar way with the following changes: (1) one requires that S_k is simply connected; (2) the vector bundle V is of rank $4k + 2$ with a fiber-wise non-degenerate symmetric 2-form. Then the relative Grassmanians of k-dimensional isotropic subspaces of V over S_k has two components G^+_{iso} and G^-_{iso}. We require that F_k (resp. F'_k) is isomorphic to G^+_{iso} (resp. G^-_{iso}); (3) when $k = 1$, f is a usual Mukai flop.

Similarly, one can define a *Mukai flop of type* $E_{6,I}$ by taking $k = 2$ with V being an E_6-vector bundle of rank 27 over S_2 and F_2 being the relative E_6/P_1-bundle over S_2 in $\mathbb{P}(V)$. The dual variety F'_2 is given by the relative E_6/P_6-bundle in $\mathbb{P}(V^*)$, where P_1, P_6 are maximal standard parabolic subgroup in E_6 corresponding to the simple roots α_1, α_6 respectively. By [**CF**], when we blow up the smallest strata of the flop, we obtain a usual Mukai flop.

1.3. Main results. Our main objective of this work is to prove the following theorems.

Theorem 1.1. *Let* $f : X \dashrightarrow X'$ *be a Mukai flop of type* $A_{n,k}$, D_{2k+1} *or* $E_{6,I}$ *over* $\bar{X}$. *Let* $\mathcal{F}$ *be the correspondence* $X \times_{\bar{X}} X'$. *Then* X *and* X' *have isomorphic Chow motives under* $\mathcal{F}$. *Moreover,* $\mathcal{F}$ *preserves the Poincaré pairing of cohomology.*

Note that the flops of type $A_{n,1}$ and D_3, i.e., $k = 1$, are the usual Mukai flops, and in these cases the theorem has been proven in [**LLW1**]. Our proof uses an induction on k (for all n) via (iv). We shall give details of the proof for $A_{n,k}$ flops, while omitting the proof of the other two types, since the argument is essentially the same.

For $k = 1$ (i.e., the usual Mukai flops), the cohomology ring as well as the Gromov-Witten theory are also invariant under $\mathcal{F}$ [**LLW1**]. However, the general situation is more subtle:

Theorem 1.2. *When* $k \geq 2$, *the cup product is generally not preserved under* $\mathcal{F}$. *For* $A_{n,2}$, D_5 *and* $E_{6,I}$ *flops the defect is corrected by the genus zero Gromov-Witten invariants attached to the extremal ray, up to analytic continuations.*

While Theorem 1.1 is as expected, Theorem 1.2 is somehow surprising, since stratified Mukai flops are in some sense locally (holomorphically) symplectic and it is somehow expected that there are no quantum corrections for flops of these types. Indeed, stratified Mukai flops among hyper-Kähler manifolds can always be deformed into isomorphisms [**Huy**] hence there is no quantum correction. As it turns out, the key point is that for the projective local models of general stratified Mukai flops, in contrast to the case $k = 1$, we cannot deform them into isomorphisms!

1.4. Outline of the contents. In Section 2, the existence of $A_{n,k}$ flops in the projective category is proved via the cone theorem. In Section 3, a general criterion on equivalence of Chow motives *via graph closure* is established for *strictly semi-small* flops. While a given flop may not be so, generic deformations of it may sometimes be. When this works, we then restrict the graph closure of the one parameter deformation back to the central fiber to get the correspondence, which is necessarily the fiber product.

It is thus crucial to study deformations of flops. Global deformations are usually obstructed, so instead we study in Section 4 the *deformations of projective local models* of $A_{n,k}$ flops. While open local models can be deformed into isomorphisms, the projective local models cannot be deformed into isomorphisms in general but only be deformed into certain $A^*_{n-2,k-1}$ flops. These flops, which we call *symmetric stratified ordinary flops*, do not seem to be studied before in the literature. Nevertheless, this deformation is good enough for applying the equivalence criterion of motives.

To handle global situations, we consider *degenerations to the normal cone* to reduce problems on $A_{n,k}$ flops to problems on $A_{n-2,k-1}$ flops and on local models of $A_{n,k}$ flops. This is carried out for correspondences in Section 5. This makes inductive argument work since local models are already well handled. We also carry out this for *cup product* by proving an *orthogonal decomposition* under degenerations to the normal cone. This in particular applies to the Poincaré pairing and completes the proof of Theorem 1.1.

In Section 6 we prove Theorem 1.2 for $A_{n,2}$ flops. We apply the *degeneration formula* for GW invariants [**LR**], [**Li**] to split the *absolute* GW invariants into the *relative* ones. After degenerations, the flop is split into *two simpler flops*; one is a Mukai flop and another one can be deformed into an ordinary $\mathbb{P}^{n-2}$ flop. It turns out that each *GW invariant attached to the extremal ray* must go to one of these two factors completely, for the former the extremal invariants indeed vanish. For the latter we use a recent result on ordinary flop with general base [**LLW2**] to achieve the quantum corrections up to analytic continuations. This then completes the proof.

At the end we compare Theorem 1.2 with the hyper-Kähler case, where the ring structure is preserved and there are no non-trivial Gromov-Witten invariants. When f is not standard Mukai, all these may fail without the global hyper-Kähler condition. A careful comparison of the degeneration analysis in this case with the local model case leads to some new topological constraint on hyper-Kähler manifolds (c.f. Proposition 6.4).

2. Existence of (twisted) $A_{n,k}$ flops

Given $k, n \in \mathbb{N}$, $2k < n+1$, a flopping contraction $\psi : (X, F) \to$ $(\bar{X}, S)$ is of type $A_{n,k}$ if it admits the following inductive structure: There is a filtration $F = F_1 \supset \cdots \supset F_k$ with induced filtration $S = S_1 \supset \cdots \supset S_k$, $S_j := \psi(F_j)$ such that $\psi_{S_k} : F_k \cong G_{S_k}(k, V) \to S_k$ is a $G(k, n+1)$ bundle for some vector bundle $V \to S_k$ of rank $n+1$ with

$$N_{F_k/X} \cong T^*_{F_k/S_k} \otimes \psi^*_{S_k} L_k \quad \text{for some} \quad L_k \in \operatorname{Pic} S_k.$$

Moreover, the blow-up maps ϕ, $\bar{\phi}$ fit into a cartesian diagram

$$
\begin{array}{ccc}
Y = \operatorname{Bl}_{F_k} X \supset E & & \\
\phi \downarrow & \searrow^{\bar{\psi}} & \\
X & & \bar{Y} = \operatorname{Bl}_{S_k} \bar{X} \supset \bar{E} \\
& \searrow_{\psi} & \downarrow^{\bar{\phi}} \\
& & \bar{X}
\end{array}
$$

such that the induced contractions $\bar{\psi} : (Y, \tilde{F}) \to (\bar{Y}, \tilde{S})$ with filtrations $\tilde{F} = \tilde{F}_1 \supset \cdots \supset \tilde{F}_{k-1}$, $\tilde{F}_j := \phi_*^{-1}(F_j)$, $\tilde{S} = \tilde{S}_1 \supset \cdots \supset \tilde{S}_{k-1}$, $\tilde{S}_j = \bar{\phi}_*^{-1}(S_j)$, $1 \leq j \leq k-1$ are of type $A_{n-2,k-1}$. Here we use the convention that an $A_{n,0}$ contraction is an isomorphism. By definition, $A_{n,1}$ contractions are twisted Mukai contractions.

The main results of this paper are all concerned with the (untwisted) stratified Mukai flops, namely $L_k \cong \mathcal{O}_{S_k}$. The starting basic existence theorem of flops does, however, hold for the twisted case too.
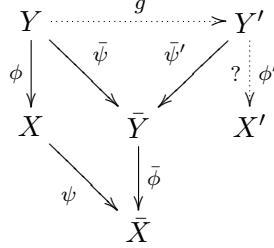
Proposition 2.1. *Given any $A_{n,k}$ contraction ψ, the corresponding $A_{n,k}$ flop*

$$
\begin{array}{ccc}
(X, F) & \xrightarrow{\quad f \quad} & (X', F') \\
\psi \searrow & & \swarrow \psi' \\
& (\bar{X}, S) &
\end{array}
$$

exists with ψ' being an $A_{n,k}$ contraction.

Proof. We construct the flop by induction on k. The case $k = 1$ has been done in [**LLW1**], Section 6, so we let $k \geq 2$ and $n+1 > 2k$. By

induction we have a diagram

where $g : (Y, \tilde{F}) \dashrightarrow (Y', \tilde{F}')$ is an $A_{n-2,k-1}$ flop and $\bar{\psi}' : (Y', \tilde{F}') \to (\bar{Y}, \tilde{S})$ is an $A_{n-2,k-1}$ contraction.

Let $C \subset E$ be a ϕ-exceptional curve and $C' = g_*C$ be its proper transform in $E' = g_*E$. We shall construct a blow-down map $\phi' : Y' \to X'$ for C'. Let γ (resp. γ') be the flopping curve for $\bar{\psi}$ (resp. $\bar{\psi}'$).

Since the Poincaré pairing is trivially preserved by the graph correspondence $\mathcal{F}_0$ of g in the divisor/curve level, and $\mathcal{F}_0C = C' + a\gamma'$ for some $a \in \mathbb{N}$ (in fact $a = 1$), we compute

$$(K_{Y'}.C') = (K_{Y'}.\mathcal{F}_0C) = (K_Y.C) < 0.$$

To show that C' is an Mori (negative) extremal curve, it is thus sufficient to find a supporting divisor for it.

Let $\bar{L}$ be a supporting divisor for $\bar{C} = \bar{\psi}(C)$ in $\bar{Y}$. Then $\bar{\psi}'^*\bar{L}$ is a supporting divisor for the extremal face spanned by C' and γ'. The idea is to perturb it to make it positive along γ', while keeping it vanishing along C'.

Let D be a supporting divisor for C in Y with $\lambda := (D.\gamma) > 0$. Let $D' = g_*D = \mathcal{F}_0D$. Since $\mathcal{F}_0\gamma = -\gamma'$, we compute

$$(D'.\gamma') = -(D, \gamma) = -\lambda < 0,$$

$$(D'.C') = (D'.\mathcal{F}_0C) - a(D', \gamma') = (D.C) + a\lambda = a\lambda.$$

Let H' be a supporting divisor for γ' in Y' with $c' := (H'.C') > 0$. Then

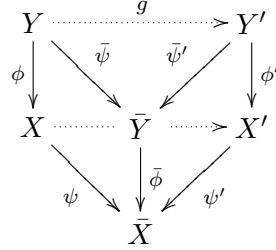$$W := a\lambda H' - c'D'$$

has the property that $(W.\gamma') > 0$ and $(W.C') = 0$.

Now for m large enough, the perturbation

$$L' := m\bar{\psi}'^*\bar{L} + W$$

is a supporting divisor for C'. Indeed, L' takes the same values as W on γ' and C', while $(L'.\beta') > 0$ for other curve classes β' in Y'. That is, L' is big and nef which vanishes precisely on the ray $\mathbb{Z}^+[C']$.

By the (relative) cone theorem applying to $\bar{\phi} \circ \bar{\psi}' : Y' \to \bar{X}$, we complete the diagram and achieve the flop $f : X \dashrightarrow X'$:

$$
\begin{array}{ccc}
Y & \xrightarrow{g} & Y' \\
\phi\downarrow & \bar{\psi}\searrow \quad \bar{\psi}'\swarrow & \downarrow\phi' \\
X & \bar{Y} \dashrightarrow & X' \\
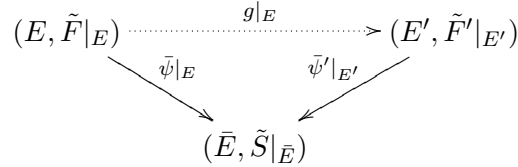\psi\searrow & \downarrow\bar{\phi} \swarrow\psi' & \\
& \bar{X} &
\end{array}
$$

It remains to show that the contraction $\psi' : X' \to \bar{X}$ is of type $A_{n,k}$. By construction, it amounts to analyze the local structure of $F'_k := \phi'(E')$. Since the flop f is unique and local with respect to $\bar{X}$, it is enough to determine its structure in a neighborhood of S_k. This can be achieved by explicit constructions.

Suppose that $F_k = G_{S_k}(k, V)$. We consider the pair of spaces $(\tilde{X}', \tilde{F}'_k)$ defined by duality. Namely $\tilde{F}'_k := G_{S_k}(k, V^*)$ and

$$\tilde{X}' \text{ is the total space of } \quad T^*_{\tilde{F}_k/S_k} \otimes \psi^*_{S_k} L_k.$$

It is well-known that, in a neighborhood of S_k, $X \dashrightarrow X'_0$ is an $A_{n,k}$ flop. Thus the local structure of (X', F'_k) must agree with $(\tilde{X}', \tilde{F}'_k)$. The proof is complete. $\hspace{4cm}$ q.e.d.

Remark 2.2. In the definition of $A_{n,k}$ contractions, the restriction to exceptional divisors $\bar{\psi}|_E : (E, \tilde{F}|_E) \to (\bar{E}, \tilde{S}|_{\bar{E}})$ is also an $A_{n-2,k-1}$ contraction. Moreover, in the proposition the restriction

$$
\begin{array}{ccc}
(E, \tilde{F}|_E) & \xrightarrow{g|_E} & (E', \tilde{F}'|_{E'}) \\
\bar{\psi}|_E\searrow & & \swarrow\bar{\psi}'|_{E'} \\
& (\bar{E}, \tilde{S}|_{\bar{E}}) &
\end{array}
$$

is also an $A_{n-2,k-1}$ flop.

3. Equivalence criteria of motives

Let $X \xrightarrow{\psi} \bar{X} \xleftarrow{\psi'} X'$ be two projective resolutions of a quasi-projective normal variety $\bar{X}$, and $f : X \dashrightarrow X'$ the induced birational map. Consider the graph closure Γ of f and $X \xleftarrow{\phi} \Gamma \xrightarrow{\phi'} X'$, the two graph projections. Then we obtain a morphism between Chow groups:

$$\mathcal{F} := \phi'_* \phi^* : A^*(X) \to A^*(X').$$

For any i, we will consider the closed subvariety

$$E_i = \{x \in X \mid \dim_x \psi^{-1}(\psi(x)) \geq i\}.$$

In a similar way we define the subvariety E_i' on X'. By Zariski's main theorem, ψ is an isomorphism over $X \setminus E_1$, and thus $\psi(E_1) = \psi'(E_1') = \bar{X}_{\text{sing}}$.

The following criterion generalizes the one for ordinary flops in [**LLW1**]:

Proposition 3.1. *If for any irreducible component D, D' of E_i and E_i' respectively, we have*

$$2 \operatorname{codim} D > \operatorname{codim} \psi(D), \quad and \;\; 2 \operatorname{codim} D' > \operatorname{codim} \psi'(D'),$$

then $\mathcal{F}$ is an isomorphism on Chow groups which preserves the Poincaré pairing on cohomology groups.

Moreover, the correspondence $[\Gamma]$ induces an isomorphism between Chow motives: $[X] \simeq [X']$.

Proof. For any smooth T, $f \times \operatorname{id}_T : X \times T \dashrightarrow X' \times T$ is also a birational map with the same condition. Thus, by the identity principle we only need to prove the equivalence of Chow groups under $\mathcal{F}$.

For any $\alpha \in A_k(X)$, up to replacing by an equivalent cycle, we may assume that α intersects $E := \sum_{i \geq 1} E_i$ properly. Then we have $\mathcal{F}\alpha = \alpha'$, where α' is the proper transform of α under f. If we denote by $\tilde{\alpha}$ the proper transform of α in $A^*(\Gamma)$, then we have

$$\phi'^* \alpha' = \tilde{\alpha} + \sum_C a_C F_C,$$

where F_C are some irreducible k-dimensional subvariety in Γ and $a_C \in \mathbb{Z}$.

For any C, note that $\phi'(F_C)$ is contained in the support of $\alpha' \cap E_1'$. As $\psi'(\alpha' \cap E_1') = \psi(\alpha) \cap \bar{X}_{\text{sing}} = \psi(\alpha \cap E_1)$, F_C is contained in $\phi^{-1}\psi^{-1}(B_C)$, where $B_C := \psi\phi(F_C) \subset \psi(\alpha \cap E_1)$. Take the largest i such that there exists an irreducible component D of E_i with $B_C \subset \psi(\alpha \cap D)$. For a general point $s \in B_C$, we denote by $F_{C,s}$ its fiber by the map $\psi \circ \phi$. Then we have

$$\dim F_{C,s} \geq \dim F_C - \dim B_C \geq \dim F_C - \dim(\alpha \cap D) = \operatorname{codim} D.$$

By our assumption, we have $\operatorname{codim} D > \dim D - \dim \psi(D)$, the latter being the dimension of a general fiber of $\psi^{-1}(B_C) \to B_C$. Thus the general fiber of the map $\phi|_{F_C}$ has positive dimension, which gives that $\phi_*(F_C) = 0$. This gives that $\mathcal{F}' \circ \mathcal{F} = \operatorname{Id}$, where $\mathcal{F}' = \phi_* \phi'^*$. A similar argument then shows that $\mathcal{F} \circ \mathcal{F}' = \operatorname{Id}$, and thus $\mathcal{F}$ and $\mathcal{F}'$ are isomorphisms.

Since F_C has positive fiber dimension in both ϕ and ϕ' directions, the statement on Poincaré pairing follows easily as in [**LLW1**], Corollary 2.3. q.e.d.

Now consider two (holomorphic) symplectic resolutions: $X \xrightarrow{\psi} \bar{X} \xleftarrow{\psi'} X'$. A conjecture in [**FN**] asserts that ψ and ψ' are deformation equivalent, i.e., there exist deformations of ψ and ψ' over $\mathbb{C}$: $\mathfrak{X} \xrightarrow{\Psi} \bar{\mathfrak{X}} \xleftarrow{\Psi'} \mathfrak{X}'$, such that for any $t \neq 0$, the morphisms Ψ_t, Ψ'_t are isomorphisms. This conjecture has been proved in various situations, such as nilpotent orbit closures of classical type [**FN**] [**Na2**], or when W is projective [**Na1**].

Assume this conjecture and consider the birational map $\mathfrak{F} : \mathfrak{X} \dashrightarrow \mathfrak{X}'$. Recall that every symplectic resolution is automatically semi-small by the work of Kaledin [**Ka**] and Namikawa [**Na1**]. We obtain that the deformed resolutions Ψ and Ψ' satisfy the condition of the precedent proposition. As a consequence, we obtain:

Theorem 3.2. *Consider two symplectic resolutions $X \xrightarrow{\psi} \bar{X} \xleftarrow{\psi'} X'$. Suppose that they are deformation equivalent (say, given by $\mathfrak{F} : \mathfrak{X} \dashrightarrow \mathfrak{X}'$). If we denote by Γ the graph of $\mathfrak{F}$ and Γ_0 its central fiber, then the correspondence $[\Gamma_0]$ induces an isomorphism of motives $[X] \simeq [X']$ that preserves also the Poincaré pairing.*

4. Deformations of local models

From now on all the stratified Mukai flops are untwisted.

4.1. Deformations of open local models. Let S be a smooth variety and $V \to S$ a vector bundle of rank $n + 1$. The relative Grassmanian bundle of k-planes in V is denoted by $\psi : F := G_S(k, V) \to S$. Let T be the universal sub-bundle of rank k on F and Q the universal quotient bundle of rank $n+1-k$. As is well-known, the relative cotangent bundle $T^*_{F/S}$ is isomorphic to $\mathrm{Hom}_F(Q, T)$. Thus, it is natural to construct deformations of $T^*_{F/S}$ inside the endomorphism bundle $\mathrm{End}_F \psi^* V = \psi^* \mathrm{End}_S V$.

Consider the vector bundle $\mathcal{E}$ over F defined as follows: For $x \in F$,

$$\mathcal{E}_x := \{(p, t) \in \mathrm{End}\, V_{\psi(x)} \times \mathbb{C} \mid \mathrm{Im}\, p \subset T_x,\ p|_{T_x} = t\, \mathrm{Id}_{T_x}\}.$$

We have an inclusion $T^*_{F/S,x} = \mathrm{Hom}(Q_x, T_x) \to \mathcal{E}_x$ which sends $q \in \mathrm{Hom}(Q_x, T_x)$ to $(\tilde{q}, 0) \in \mathcal{E}_x$, where $\tilde{q}$ is the composition

$$\tilde{q} : V_{\psi(x)} \to Q_x \xrightarrow{q} T_x \hookrightarrow V_{\psi(x)}.$$

The projection to the second factor $\pi : \mathcal{E} \to \mathbb{C}$ is then an one-dimensional deformation of $\pi^{-1}(0) = T^*_{F/S}$.

Equivalently, the Euler sequence $0 \to T \to \psi^* V \to Q \to 0$ leads to

$$0 \to T^*_{F/S} = \mathrm{Hom}_F(Q, T) \to \mathrm{Hom}_F(\psi^* V, T) \to \mathrm{End}_F T \to 0.$$

The deformation is simply the inverse image of $\mathbb{C}\, \mathrm{Id}_F T \cong \mathbb{C}$.

The projection to the first factor, followed by ψ_*

$$
\begin{array}{ccc}
\mathcal{E} \longrightarrow \psi^* \operatorname{End}_S V & \xrightarrow{\ \psi_* \ } & \operatorname{End}_S V \\
\searrow \qquad \downarrow & & \downarrow \\
F & \xrightarrow{\ \psi \ } & S
\end{array}
$$

gives rise to a map $\mathcal{E} \to \operatorname{End}_S V$, which is a birational morphism onto its image $\bar{\mathcal{E}}$. Indeed, $\Psi : \mathcal{E} \to \bar{\mathcal{E}}$ is isomorphic over the loci with rank $p = k$. In particular, it is isomorphic outside $\pi^{-1}(0)$. For any $s \in S$,

$$
\bar{\mathcal{E}}_s := \overline{\{\, p \in \operatorname{End} V_s \mid \operatorname{rank} p = k \text{ and } p^2 = tp \text{ for some } t \in \mathbb{C} \,\}}
$$

is the *cone of scaled projectors* with rank at most k. Thus, $\pi = \bar{\pi} \circ \Psi$, where

$$
\bar{\pi} : \bar{\mathcal{E}} \to \mathbb{C} \quad \text{via} \quad \phi \mapsto \frac{1}{k} \operatorname{Tr} \phi.
$$

For $t \neq 0$, $\Psi_t : \mathcal{E}_t \xrightarrow{\sim} \bar{\mathcal{E}}_t$. For $t = 0$, $\psi := \Psi_0 : T^*_{F/S} = \mathcal{E}_0 \to \bar{\mathcal{E}}_0$ is the open local model of an $A_{n,k}$ contraction.

We do a similar construction for the dual bundle $V^* \to S$. Under the canonical isomorphism $\operatorname{End}_S V \simeq \operatorname{End}_S V^*$, we see that $\bar{\mathcal{E}}$ is identified with $\bar{\mathcal{E}}' = \bar{\mathcal{E}}(V^*)$. Thus we get a birational map $\mathfrak{F} : \mathcal{E} \dashrightarrow \mathcal{E}'$ over $\bar{\mathcal{E}}$. This proves

Proposition 4.1. *The birational map $\mathfrak{F}$ over $\mathbb{C}$:*

deforms the birational map ($A_{n,k}$ flop) $f : T^*_{F/S} \dashrightarrow T^*_{F'/S}$ into isomorphisms.

Let Γ be the graph closure of $\mathfrak{F} : \mathcal{E} \dashrightarrow \mathcal{E}'$ and Γ_0 be its central fiber. By Proposition 3.1, the map $\Gamma_* : A^*(\mathcal{E}) \to A^*(\mathcal{E}')$ is an isomorphism. Since $\Gamma \to \mathcal{E} \times_{\bar{\mathcal{E}}} \mathcal{E}'$ is birational, $(\mathcal{E} \times_{\bar{\mathcal{E}}} \mathcal{E}')_* : A^*(\mathcal{E}) \to A^*(\mathcal{E}')$ is again an isomorphism. It follows that its central fiber $\mathcal{F}_{open} := T^*_{F/S} \times_{\bar{\mathcal{E}}_0} T^*_{F'/S}$ induces an isomorphism $A^*(T^*_{F/S}) \to A^*(T^*_{F'/S})$.

Consider the fiber product

$$
\mathcal{F}_{\mathrm{loc}} := \mathbb{P}_F(T^*_{F/S} \oplus \mathcal{O}) \times_{\mathbb{P}_S(\bar{\mathcal{E}}_0 \times \mathbb{C})} \mathbb{P}_{F'}(T^*_{F'/S} \oplus \mathcal{O})
$$

and $\mathcal{F}_\infty := \mathbb{P}_F(T^*_{F/S}) \times_{\mathbb{P}_S(\bar{\mathcal{E}}_0)} \mathbb{P}_{F'}(T^*_{F'/S})$. Note that the push-forward map $A_*(\mathbb{P}_F(T^*_{F/S})) \to A_*(\mathbb{P}_F(T^*_{F/S} \oplus \mathcal{O}))$ is injective.

Proposition 4.2. *We have the following commutative diagrams with exact horizontal rows (induced by the localization formula in Chow groups):*

$$0 \longrightarrow A_*(\mathbb{P}(T^*_{F/S})) \longrightarrow A_*(\mathbb{P}(T^*_{F/S} \oplus \mathcal{O})) \longrightarrow A_*(T^*_{F/S}) \longrightarrow 0$$

$$\mathfrak{F}_\infty \downarrow \qquad\qquad \mathfrak{F}_{\mathrm{loc}} \downarrow \qquad\qquad \mathfrak{F}_{\mathrm{open}} \downarrow$$

$$0 \longrightarrow A_*(\mathbb{P}(T^*_{F'/S})) \longrightarrow A_*(\mathbb{P}(T^*_{F'/S} \oplus \mathcal{O})) \longrightarrow A_*(T^*_{F'/S}) \longrightarrow 0$$

Thus if $\mathfrak{F}_\infty$ is an isomorphism, so is $\mathfrak{F}_{\mathrm{loc}}$.

Note that $\mathbb{P}_F(T^*_{F/S}) \dashrightarrow \mathbb{P}_{F'}(T^*_{F'/S})$ is a stratified Mukai flop of type $A_{n-2,k-1}$. This allows us to perform an inductive argument later.

4.2. Deformations of projective local models. Consider the rational map $\pi : \mathbb{P}(\mathcal{E} \oplus \mathcal{O}) \dashrightarrow \mathbb{P}^1$ which extends the map $\pi : \mathcal{E} \to \mathbb{C}$ and maps $\mathbb{P}(\mathcal{E})$ to ∞. The map π is undefined exactly along $E := \mathbb{P}(\mathcal{E}_0)$. Blow-up $\mathbb{P}(\mathcal{E} \oplus \mathcal{O})$ along E resolves the map π, and thus we obtain $\hat{\pi} : \mathfrak{X} := \mathrm{Bl}_E \mathbb{P}(\mathcal{E} \oplus \mathcal{O}) \to \mathbb{P}^1$.

Since E is a divisor of the central fiber $\mathbb{P}(\mathcal{E} \oplus \mathcal{O})_0 = \mathbb{P}(T^*_{F/S} \oplus \mathcal{O})$, we have $\hat{\pi}^{-1}(0) \simeq \mathbb{P}(T^*_{F/S} \oplus \mathcal{O})$. When $t \neq 0$, $\mathfrak{X}_t \simeq \mathbb{P}(\mathcal{E})$, which is the compactification of $\mathcal{E}_t$ by $E \cong \mathbb{P}(\mathcal{E}_0) = \mathbb{P}(T^*_{F/S})$. This gives a deformation of $\mathbb{P}(T^*_{F/S} \oplus \mathcal{O})$ over $\mathbb{P}^1$ with other fibers isomorphic to $\mathbb{P}(\mathcal{E})$.

We do a similar construction on the dual side, which gives a deformation of $\mathbb{P}(T^*_{F'/S} \oplus \mathcal{O})$ by $\hat{\pi}' : \mathfrak{X}' \to \mathbb{P}^1$. We get also an induced birational map $\mathfrak{F} : \mathfrak{X} \dashrightarrow \mathfrak{X}'$ over $\bar{\mathfrak{X}}$ extending $\mathfrak{F} : \mathcal{E} \dashrightarrow \mathcal{E}'$ over $\bar{\mathcal{E}}$.

The flop $\mathfrak{F}_t : \mathfrak{X}_t \dashrightarrow \mathfrak{X}'_t$ for $t \neq 0$ has the property that there are smooth divisors $E \subset \mathfrak{X}_t$ and $E' \subset \mathfrak{X}'_t$ such that (i) the exceptional loci $Z \subset \mathfrak{X}_t$ (resp. $Z' \subset \mathfrak{X}'_t$) are contained in E (resp. E'), (ii) $\mathfrak{F}_t|_E : E \dashrightarrow E'$ is a stratified Mukai flop. We call such flops *symmetric stratified ordinary flops* if furthermore (iii) $N_{E/\mathfrak{X}_t}|_{\mathbb{P}^1} \cong \mathcal{O}$ and $N_{E'/\mathfrak{X}'_t}|_{\mathbb{P}^1} \cong \mathcal{O}$ along the flopping extremal rays.

If $\mathfrak{F}_t|_E$ is of type A, D or E, then we say $\mathfrak{F}_t$ is of type A^*, D^* or E^* respectively. Notice that (symmetric) stratified ordinary flops of type $A^*_{m,1}$ are precisely ordinary $\mathbb{P}^m$ flops with $V' = V^*$, which explains the choice of terminology.

Proposition 4.3. *The birational map $\mathfrak{F}$ over $\mathbb{P}^1$:*

deforms the $A_{n,k}$ flop $f = \mathfrak{F}_0 : \mathbb{P}(T^*_{F/S} \oplus \mathcal{O}) \dashrightarrow \mathbb{P}(T^*_{F'/S} \oplus \mathcal{O})$ into $A^*_{n-2,k-1}$ flops $\mathfrak{F}_t : \mathbb{P}(\mathcal{E}) \dashrightarrow \mathbb{P}(\mathcal{E}')$ for $t \neq 0$.

Proof. It remains to check condition (iii), which is equivalent to that $(E.C) = 0$ for any flopping curve $C \cong \mathbb{P}^1$. Since $(E.C)$ is independent of $t \in \mathbb{P}^1$ we may compute it at $t = 0$. As a projective bundle $\rho : \mathfrak{X}_0 \to F$ it is clear that

$$K_{\mathfrak{X}_0} = -(2 \dim F/S + 2)E + \rho^* K_{\mathfrak{X}_0}|_F.$$

Since $(K_{\mathfrak{X}_0}.C) = 0$ by the definition of flops, we get $(E.C) = 0$ as well.
                                                                    q.e.d.

Clearly for $t \neq 0$, the map $\mathfrak{F}_t : \mathbb{P}(\mathcal{E}) \dashrightarrow \mathbb{P}(\mathcal{E}')$ is an isomorphism only for the case of ordinary Mukai flop, i.e., $k = 1$.

Remark 4.4. For $A_{n,2}$ flops, the deformed flop $\mathfrak{F}_t$ is a family of ordinary flop, which has defect of cup product by [**LLW1**]. As the classical cohomology ring is invariant under deformations, the fiber product of f does not preserve the ring structure. This implies that we cannot deform a projective local stratified Mukai flop of type $A_{n,2}$ into isomorphisms, which is a crucial difference to usual Mukai flops. Thus, there exist quantum corrections even in this local case.

Corollary 4.5. *For projective local model of $A_{n,k}$ flops $X \xrightarrow{\psi} \bar{X} \xleftarrow{\psi'} X'$, the correspondence defined by fiber product $\mathfrak{F} = X \times_{\bar{X}} X'$ induces isomorphism of Chow motives $[X] \cong [X']$ which preserves also the Poincaré pairing.*

Proof. By definition, the $A_{n,k}$ contraction satisfies

$$2 \operatorname{codim} D = \operatorname{codim} \psi(D)$$

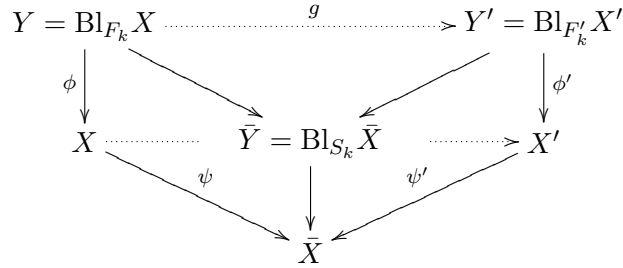for each irreducible component D of E_i. The deformation

$$\mathfrak{X} \xrightarrow{\Psi} \bar{\mathfrak{X}} \xleftarrow{\Psi'} \mathfrak{X}'$$

constructed by Proposition 4.3 is not isomorphic on general fibers; instead, it gives $A^*_{n-2,k-1}$ flops. Thus, the additional deformation dimension makes it satisfy the assumption of Proposition 3.1. The result follows by noticing that the graph closure restricts to $\mathfrak{F}$ on the central fiber.   q.e.d.

Remark 4.6. Proposition 4.3 suggests certain inductive structure on $A_{n,k}$ flops. It will become more useful (e.g., for the discussion of global $A_{n,k}$ flops or Gromov-Witten theory) after we develop detailed analysis on correspondences.

5. Degeneration of correspondences

5.1. Setup of degeneration. Let $f : X \dashrightarrow X'$ be a stratified Mukai flop, say of type $A_{n,k}$ with $2k < n+1$. The aim of the following theorem is to show that the degeneration to normal cone for (X, F_k) and (X', F_k') splits the correspondence $\mathcal{F}^f$ defined by $X \times_{\bar{X}} X'$ into the one $\mathcal{F}^g$ defined by $Y \times_{\bar{Y}} Y'$ of type $A_{n-2,k-1}$ and its version $\mathcal{F}^f_{\mathrm{loc}}$ on projectivized local models relative to F_k and $F_{k'}$. Conversely, we may define $\mathcal{F}$ inductively by gluing these two parts. Here is the blow-up diagram

with $g : Y \dashrightarrow Y'$ being the induced $A_{n-2,k-1}$ flop. To save the notations we use the same symbol $\mathcal{F}$ for $\mathcal{F}^f$ and its local models as well if no confusion is likely to arise.

We consider degenerations to the normal cone $W \to \mathbb{A}^1$ of X, where W is the blow-up of $X \times \mathbb{A}^1$ along $F_k \times \{0\}$. Similarly we get $W' \to \mathbb{A}^1$ for X'.

Note that the central fiber

$$W_0 := Y_1 \cup Y_2 = Y \cup X_{\mathrm{loc}}, \qquad W_0' := Y_1' \cup Y_2' = Y' \cup X_{\mathrm{loc}}',$$

where $X_{\mathrm{loc}} = \mathbb{P}(T^*_{F_k/S} \oplus \mathcal{O})$ and $X_{\mathrm{loc}}' = \mathbb{P}(T^*_{F_k'/S} \oplus \mathcal{O})$. The intersections $E := Y \cap X_{\mathrm{loc}}$ and $E' := Y' \cap X_{\mathrm{loc}}'$ are isomorphic respectively to $\mathbb{P}(T^*_{F_k/S})$ and $\mathbb{P}(T^*_{F_k'/S})$. The map $f : X \dashrightarrow X'$ induces the Mukai flop of the same type for local models: $f : X_{\mathrm{loc}} \dashrightarrow X_{\mathrm{loc}}'$ and Mukai flop $g : Y \dashrightarrow Y'$ of type $A_{n-2,k-1}$. Let $p : X_{\mathrm{loc}} \to F_k$ be the projection and similarly we get p'.

5.2. Correspondences. A *lifting* of an element $a \in A^*(X)$ is a couple (a_1, a_2) with $a_1 \in A^*(Y)$ and $a_2 \in A^*(X_{\mathrm{loc}})$ such that $\phi_* a_1 + p_* a_2 = a$ and $a_1|_E = a_2|_E$. Similarly, one defines the lifting of an element in $A^*(X')$.

Theorem 5.1. *Let $a \in A^*(X)$ with (a_1, a_2) and (a_1', a_2') be liftings of a and $\mathcal{F}a$ respectively. Then*

$$\mathcal{F}a_1 = a_1' \Longleftrightarrow \mathcal{F}a_2 = a_2'.$$

Moreover, it is always possible to pick such liftings.

It is instructive to re-examine the Mukai case $(k = 1)$ first. In this case $Y = Y'$ and f is an isomorphism outside the blow-up loci $Z = F_1$

and $Z' = F_1'$. Let us denote $\mathcal{F} = \mathcal{F}_0 + \mathcal{F}_1$ with $\mathcal{F}_0 = [\bar{\Gamma}_f] = \phi_*'\phi^*$ and $\mathcal{F}_1$ the degenerate correspondence $[Z \times_S Z']$.

By Lemma 4.2 in [**LLW1**], it is enough to prove the result for any single choice of $a_1 = a_1'$. Consider the standard liftings

$$a(0) = (a_1, a_2) = (\phi^*a, p^*(a|_Z)),$$

$$(\mathcal{F}a)(0) = (\phi'^*\mathcal{F}a, p'^*(\mathcal{F}a|_{Z'})).$$

Since $\phi'^*\mathcal{F}a = \phi^*a + \lambda$ with λ supported on $E' = E$, we may select lifting (a_1', a_2') with $a_1' = a_1$. In that case,

$$(\mathcal{F}a_2 - a_2')|_{E'} = a_2|_E - a_2'|_{E'} = a_1|_E - a_1'|_E = 0$$

by the compatibility constraint on E and the fact that $\mathcal{F}$ restricts to an isomorphism on E. Hence $\mathcal{F}a_2 - a_2' = \iota_*(z')$, for some $z' \in A^*(Z')$, where $\iota : Z' \to X'_{\mathrm{loc}}$ is the natural inclusion.

To prove that $z' = 0$, consider

$$z' = p_*'\iota_*z' = p_*'\mathcal{F}a_2 - p_*'a_2'.$$

By substituting $\phi_*'a_1' + p_*'a_2' = \mathcal{F}a$, $a_1' = \phi^*a$ and $a_2 = p^*(a|_Z)$, we get

$$p_*'a_2' = \mathcal{F}a - \phi_*'\phi^*a = \mathcal{F}_1a.$$

Let q, q' be the projections of $Z \times_S Z'$ to the two factors and

$$j : X_{\mathrm{loc}} \times_{\bar{X}_{\mathrm{loc}}} X'_{\mathrm{loc}} \to Z \times_S Z'$$

the natural morphism. Then

$$z' = p_*'\mathcal{F}_{\mathrm{loc}}p^*(a|_Z) - \mathcal{F}_1a = q_*'j_*j^*q^*(a|_Z) - q_*'q^*(a|_Z).$$

Note that there exists a unique irreducible component in $X_{\mathrm{loc}} \times_{\bar{X}_{\mathrm{loc}}} X'_{\mathrm{loc}}$ birational to $Z \times_S Z'$ via j, so $j_*j^* = \mathrm{Id}$, which gives $z' = 0$.

Now we proceed for general $A_{n,k}$ flops. It is enough to prove the result for any single choice of a_1 and a_1', since other choices differ from this one by elements supported on E and E' where the theorem holds by induction on k. To make $a_1' = \mathcal{F}a_1$, notice that g is an isomorphism outside $\tilde{F}_1 = \phi_*^{-1}(F_1)$ and $\tilde{F}_1' = \phi_*'^{-1}(F_1')$ but we may adjust the standard lifting $\phi'^*\mathcal{F}^f a$ only by elements lying over F_k', namely classes in $E' = \mathbb{P}(T^*_{F_k'/S_k})$.

The following simple observation resolves this as well as later difficulties. Recall that $\mathcal{F} = \sum_j \mathcal{F}_j$ with $\mathcal{F}_j = [F_j \times_{S_j} F_j']$.

Lemma 5.2. *We have decomposition of correspondences:*

$$\mathcal{F}^f = \phi_*'\mathcal{F}^g\phi^* + \mathcal{F}_k^f.$$

In particular, $\phi'^*\mathcal{F}^f = \mathcal{F}^g\phi^*$ modulo $A^*(E')$.

Proof. This follows from the definition and the base change property of fiber product.   q.e.d.

Thus, we may pick

$$a_1 = \phi^* a, \qquad a_1' = \mathcal{F}^g (\phi^* a) = \mathcal{F} a_1.$$

Then

$$(\mathcal{F} a_2 - a_2')|_{E'} = \mathcal{F}(a_2|_E) - a_2'|_{E'}$$
$$= \mathcal{F}(a_1|_E) - a_1'|_{E'} = (\mathcal{F} a_1 - a_1')|_{E'} = 0$$

and so $\mathcal{F} a_2 - a_2' = \iota_*(z')$, for some $z' \in A^*(F_k')$, where $\iota : Z' \to X_{\mathrm{loc}}'$ is the natural inclusion.

To prove that $z' = 0$, consider

$$z' = p_*' \iota_* z' = p_*' \mathcal{F} a_2 - p_*' a_2'.$$

By substituting $\phi_*' a_1' + p_*' a_2' = \mathcal{F} a$, $a_1' = \mathcal{F} \phi^* a$ and $a_2 = p^*(a|_{F_k})$, we get

$$p_*' a_2' = \mathcal{F} a - \phi_*' \mathcal{F} \phi^* a = \mathcal{F}_k a$$

by the above lemma.

Let q, q' be the projections of $F_k \times_{S_k} F_k'$ to the two factors and

$$j : X_{\mathrm{loc}} \times_{\bar{X}_{\mathrm{loc}}} X_{\mathrm{loc}}' \to F_k \times_{S_k} F_k'$$

the natural morphism. Then

$$z' = p_*' \mathcal{F}_{\mathrm{loc}} p^*(a|_{F_k}) - \mathcal{F}_k a = q_*' j_* j^* q^*(a|_{F_k}) - q_*' q^*(a|_{F_k}).$$

Note that there exists a unique irreducible component in $X_{\mathrm{loc}} \times_{\bar{X}_{\mathrm{loc}}} X_{\mathrm{loc}}'$ birational to $F_k \times_{S_k} F_k'$ via j, so $j_* j^* = \mathrm{Id}$, which gives $z' = 0$. The proof is complete.

5.3. Cup product and the Poincaré pairing. Besides correspondences, we also need to understand the effect on the Poincaré pairing under degeneration. We will in fact degenerate classical cup product and this works for any degenerations to normal cones $W \to X \times \mathbb{A}^1$ with respect to $Z \subset X$. Let $W_0 = Y_1 \cup Y_2$, where $\phi : Y_1 = Y \to X$ is the blow up along Z, $p : Y_2 = \tilde{E} = \mathbb{P}_Z(N_{Z/X} \oplus \mathcal{O}) \to Z$ is the local model, and $Y_1 \cap Y_2 = E$ is the ϕ exceptional divisor. Let $i_1 : E \hookrightarrow Y_1$, $i_2 : E \hookrightarrow Y_2$.

Lemma 5.3. *Let $a, b \in H^*(X)$. Then for any lifting (a_1, a_2) of a and any lifting (b_1, b_2) of b, the pair $(a_1 b_1, a_2 b_2)$ is a lifting of ab.*

In particular, if a, b are of complementary degree, then we have an orthogonal splitting of the Poincaré pairing: $(a.b)_X = (a_1.b_1)_{Y_1} + (a_2.b_2)_{Y_2}$.

Proof. We compute

$$a.b = \phi_* a_1.b + p_* a_2.b = \phi_*(a_1.\phi^* b) + p_*(a_2.p^*(b|_Z)).$$

Since $a_1.\phi^* b|_E = a_2.p^*(b|_Z)|_E$, $(a_1 b_1, a_2 b_2)$ is a lifting of ab for the special lifting $(b_1, b_2) = (\phi^* b, p^*(b|_Z))$ of b. By [**LLW1**], Lemma 4.2, any other lifting of b is of the form $(b_1 + i_{1*} e, b_2 - i_{2*} e)$ for some class e in E.

Since $i_1^* a_1.e = i_2^* a_2.e$ is a class $e' \in H^*(E)$, the correction terms are

$$a_1.i_{1*}e = i_{1*}(i_1^* a_1.e) = i_{1*}e', \quad -a_2.i_{2*}e = -i_{2*}(i_2^* a_2.e) = -i_{2*}e'.$$

The lemma then follows from

$$i_1^*(i_{1*}e') = e'.c_1(N_{E/Y_1}) = -e'.c_1(N_{E/Y_2}) = i_2^*(-i_{2*}e')$$

and $\phi_* i_{1*}e' - p_* i_{2*}e' = (\phi|_E)_* e' - (\phi|_E)_* e' = 0.$ \hfill q.e.d.

Theorem 5.4 (= Theorem 1.1). *For $A_{n,k}$ flops, $\mathcal{F}$ induces an isomorphism on Chow motives and the Poincaré pairing.*

Proof. If $f : X \dashrightarrow X'$ is an $A_{n,k}$ flop, $f \times \mathrm{id} : X \times T \dashrightarrow X' \times T$ is also an $A_{n,k}$ flop. Thus, by the identity principle, to prove $[X] \cong [X']$ we only need to prove the equivalence of Chow groups under $\mathcal{F}$ for any $A_{n,k}$ flop.

We prove this for all n with $n + 1 > 2k$ by induction on k. We start with $k = 0$, which is trivial.

Given $k \geq 1$, by Theorem 5.1, the equivalence of Chow groups is reduced to the $A_{n-2,k-1}$ case and the local $A_{n,k}$ case. The former is true by induction. The later follows from Corollary 4.5 directly.

The same procedure proves the isomorphism of Poincare pairings by using Theorem 5.1, Lemma 5.3, and Corollary 4.5. \hfill q.e.d.

For cohomology rings we need to proceed carefully. In order to run induction on k, using Theorem 5.1, Lemma 5.3 we must first consider the local $A_{n,k}$ case. By Remark 4.4, for $k = 2$, the classical cup product is *not* preserved by the correspondence $\mathcal{F}$! This is analyzed in the next section.

6. Quantum corrections

6.1. The proof of Theorem 1.2. We now prove the invariance of big quantum product attached to the extremal rays, up to analytic continuations, under $A_{n,2}$ flops.

As in the precedent section, we consider degenerations to the normal cone $W \to \mathbb{A}^1$ of X and $W' \to \mathbb{A}^1$ of X'. Note that the map $f : X \dashrightarrow X'$ induces the Mukai flop of the same type for local models: $f : X_{\mathrm{loc}} \dashrightarrow X'_{\mathrm{loc}}$ and Mukai flop $g : Y \dashrightarrow Y'$ of type $A_{n-2,1}$.

By the degeneration formula (for the algebraic version used here, c.f. [**Li**]), any Gromov-Witten invariant on X splits into products of relative invariants of (Y, E) and (X_{loc}, E). Let $a \in H^*(X)^{\oplus m}$ with lifting (a_1, a_2):

$$\langle a \rangle_{g,m,\beta}^X = \sum_{\eta \in \Omega(g,\beta)} C_\eta \left(\langle a_1 \rangle_{\Gamma_1}^{\bullet(Y,E)} . \langle a_2 \rangle_{\Gamma_2}^{\bullet(X_{\mathrm{loc}},E)} \right)^{E^\rho}.$$

Here ρ is the number of gluing points (in E) and $\Gamma_1 \cup \Gamma_2$ forms a connected graph. Thus $\rho = 0$ if and only if one of the Γ_i is empty.

The relative invariants take values in $H^*(E^\rho)$ and the formula is in terms of the Poincaré pairing of E^ρ.

We apply it to X' as well and get:

$$\langle \mathcal{F}a \rangle_{g,m,\mathcal{F}\beta}^{X'} = \sum_{\eta' \in \Omega(g,\mathcal{F}\beta)} C_{\eta'} \left(\langle a_1' \rangle_{\Gamma_1'}^{\bullet(Y',E')} . \langle a_2' \rangle_{\Gamma_2'}^{\bullet(X_{\mathrm{loc}}',E')} \right)^{E'^\rho}.$$

There is a one-to-one correspondence between admissible triples $\eta = (\Gamma_1, \Gamma_2, I_\rho)$ and $\eta' = (\Gamma_1', \Gamma_2', I_{\rho'})$ via $\eta' := \mathcal{F}\eta$. The combinatorial structure is kept the same, while the curve classes are related by $\mathcal{F}$. We do still need the cohomology class splitting on X and X' to be compatible.

By Theorem 5.1 we may split the cohomology classes $a \in H^*(X)^{\oplus m}$ into (a_1, a_2) with $a_i \in H^*(Y_i)^{\oplus m}$ and $\mathcal{F}a \in H^*(X')^{\oplus m}$ into (a_1', a_2') with $a_i' \in H^*(Y_i')^{\oplus m}$, such that

$$\mathcal{F}a_1 = a_1', \qquad \mathcal{F}a_2 = a_2'.$$

By Theorem 5.4, the Poincaré pairing is preserved by $\mathcal{F}$ under stratified Mukai flops $E \dashrightarrow E'$; the same holds true for $E^\rho \dashrightarrow E'^\rho$ by $\mathcal{F}^\rho$, which for simplicity is still denoted by $\mathcal{F}$. Thus, by the degeneration formula the problem is reduced to showing that $\mathcal{F}$ maps the relative invariants of (Y, E) and (X_{loc}, E) to the corresponding ones of (Y', E') and (X_{loc}', E').

Since we are only interested in invariants attached to the extremal ray $\beta = d\ell$, for any splitting $\beta = (\beta_1, \beta_2) = (d_1\ell, d_2\ell)$, we must have $\rho = (E.\beta_2) = d_2(E.\ell) = 0$ (since ℓ can be represented by a curve in F_2). But this implies that β is not split at all and in the degeneration formula the invariant $\langle a \rangle_{g,m,d\ell}^X$ goes to Y or X_{loc} completely:

$$\langle a \rangle_{g,m,d\ell}^X = \langle a_1 \rangle_{g,m,d\ell}^Y + \langle a_2 \rangle_{g,m,d\ell}^{X_{\mathrm{loc}}}.$$

Lemma 6.1. $\mathcal{F}$ *maps isomorphically the cup product and full Gromov-Witten theory of Y to those of Y'. Moreover, $\langle a_1 \rangle_{g,m,d\ell}^Y = 0$ for all $d \in \mathbb{N}$.*

Proof. The birational map $g : Y \dashrightarrow Y'$ is a Mukai flop of type $A_{n-2,1}$. Hence, this follows from [**LLW1**], Theorem 6.3.

Indeed this follows from Lemma 5.3 and the above degeneration formula by applying it to the Mukai flop $Y \dashrightarrow Y'$. Here we use the facts that projective local models of Mukai flops $g_{\mathrm{loc}} : Y_{\mathrm{loc}} \dashrightarrow Y_{\mathrm{loc}}'$ can be deformed into isomorphisms $g_t : Y_t \xrightarrow{\sim} Y_t'$ and that the cup product as well as the Gromov-Witten theory are both invariant under deformations. For ℓ being the extremal ray of g_{loc}, if $d\ell \sim C_t$ for $t \neq 0$ then $C_t' \cong g_t(C_t) \sim \mathcal{F}d\ell = -d\ell'$, which is impossible. Thus $\langle a_1 \rangle_{g,m,d\ell}^Y = 0$ for all $d \in \mathbb{N}$.   q.e.d.

Denote by $\langle a \rangle_f = \sum_{d=0}^\infty \langle a \rangle_{0,m,d\ell} \, q^{d\ell}$ the generating function of $g = 0$ Gromov-Witten invariants attached to the extremal ray. Then the

degeneration formula and Lemma 6.1 lead to

$$\langle a \rangle_f^X = \delta_{n3} \langle a_1 \rangle_{0,3,0}^Y + \langle a_2 \rangle_f^{X_{\mathrm{loc}}}.$$

The correspondence $\mathcal{F}$ acts on q^β by $\mathcal{F}q^\beta = q^{\mathcal{F}\beta}$. In particular, for the extremal rays ℓ and ℓ' we have $\mathcal{F}q^{d\ell} = q^{-d\ell'}$. If we regard $q^\ell = e^{-2\pi(\omega.\ell)}$ as an analytic function on $\omega \in H_{\mathbb{R}}^{1,1}(X)$, then it is known that $\langle a \rangle_f^X$ converges in the Kähler cone $\mathcal{K}_X$ of X. Under the identification $H_{\mathbb{R}}^{1,1}(X) \cong \mathcal{F}H_{\mathbb{R}}^{1,1}(X) = H_{\mathbb{R}}^{1,1}(X')$, it makes sense to compare $\mathcal{F}\langle a \rangle_f^X$ with $\langle \mathcal{F}a \rangle_f^{X'}$ as analytic functions on $\mathcal{K}_X \cup \mathcal{K}_{X'} \subset H_{\mathbb{R}}^{1,1}$ up to analytic continuations.

Lemma 6.2. *For $m \geq 3$, $\mathcal{F}\langle a_2 \rangle_f^{X_{\mathrm{loc}}} \cong \langle \mathcal{F}a_2 \rangle_f^{X'_{\mathrm{loc}}}$ up to analytic continuations.*

Proof. By Proposition 4.3, the $A_{n,2}$ flop $f : X_{\mathrm{loc}} \dashrightarrow X'_{\mathrm{loc}}$ can be projectively deformed into ordinary $\mathbb{P}^{n-2}$ flops. By the deformation invariance of Gromov-Witten theory, the lemma is reduced to the case of ordinary flops (with non-trivial base). For simple ordinary flops the invariance

$$\mathcal{F}\langle a_2 \rangle_f^{X_{\mathrm{loc}}} \cong \langle \mathcal{F}a_2 \rangle_f^{X'_{\mathrm{loc}}}$$

up to analytic continuations is proved in [**LLW1**]. It has been extended to general ordinary flops with base in [**LLW2**]. Hence, the lemma follows.   q.e.d.

Notice that the $g = 0$, $d = 0$ invariants are non-zero if and only if $m = 3$ and they are given by the cubic product. By Lemma 6.1, $\langle a_{11}, a_{12}, a_{13} \rangle_{0,3,0}^Y = \langle \mathcal{F}a_{11}, \mathcal{F}a_{12}, \mathcal{F}a_{13} \rangle_{0,3,0}^{Y'}$. From Lemma 6.2 we get $\mathcal{F}\langle a \rangle_f^X = \langle \mathcal{F}a \rangle_f^{X'}$ for $m \geq 3$.

Together with the $\mathcal{F}$ invariance of Poncaré pairing, the big quantum product attached to the extremal ray is invariant under $\mathcal{F}$. This completes the proof of Theorem 1.2 for type $A_{n,2}$. The cases of type D_5 and $E_{6,I}$ are completely similar, since the geometric picture in Proposition 4.3 is the same by [**CF**]. The proof is complete.

Remark 6.3. The degeneration formula is in terms of the Poincaré pairing of relative GW invariants. Thus *invariance of the Poincaré pairing* is crucial in our study. Indeed, the Poincaré pairing together with 3-point functions determine the (small) quantum product. So far this is *the only constraint* we have found for the correspondence $\mathcal{F}$ under K equivalence to be *canonical*.

6.2. A new topological constraint. Consider stratified Mukai flops of type $A_{n,2}$, $f : X \dashrightarrow X'$ with $i : F_2 \hookrightarrow X$, such that $\mathcal{F}$ preserves the cup product (e.g., for X and X' being hyper-Kähler manifolds). By Proposition 4.3 and Remark 4.4, there exists a defect of cup product on

X_{loc}. A priori there seems to be a contradiction. A closer look at them leads to

Proposition 6.4. *For a Mukai flop $f : X \dashrightarrow X'$ of type $A_{n,2}$, D_5 or $E_{6,I}$, if the restriction map $i^* : H^*(X, \mathbb{Q}) \to H^*(F_2, \mathbb{Q})$ is surjective then $\mathcal{F}$ does not preserve the cup product. In particular, if X is hyper-Kähler, then i^* is not surjective.*

Proof. We shall investigate the degeneration analysis on cup product for an arbitrary $A_{n,2}$ flop f as presented above. The other cases are similar.

Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$ be two elements in $H^*(X)$ with their lifting. By Lemma 5.3, $ab = (a_1 b_1, a_2 b_2)$. Then Theorem 5.1 implies that $\mathcal{F}(ab) = (\mathcal{F}(a_1 b_1), \mathcal{F}(a_2 b_2))$. By Lemma 5.3 again

$$\mathcal{F}(a)\mathcal{F}(b) = (\mathcal{F}(a_1)\mathcal{F}(b_1), \mathcal{F}(a_2)\mathcal{F}(b_2)) = (\mathcal{F}(a_1 b_1), \mathcal{F}(a_2)\mathcal{F}(b_2)),$$

where the last equality follows from Lemma 6.1 applied to $g : Y \dashrightarrow Y'$. So

$$\mathcal{F}(ab) = \mathcal{F}(a)\mathcal{F}(b) \quad \Longleftrightarrow \quad \mathcal{F}(a_2 b_2) = \mathcal{F}(a_2)\mathcal{F}(b_2).$$

That is, the invariance of cup product on $H^*(X)$ is equivalent to the invariance on elements in $H^*(X_{\text{loc}})$ which come from lifting of elements in $H^*(X)$. Indeed let $i : F_2 \hookrightarrow X$ and $p : X_{\text{loc}} \to F_2$ be the projection, and we may choose standard lifting $a_2 = p^* i^* a$. Such elements form a subring

$$\Delta_f := p^* i^* H^*(X) \subset p^* H^*(F_2) \subset H^*(X_{\text{loc}}).$$

By applying this analysis to the case $X = X_{\text{loc}} = \mathbb{P}_{F_2}(T^*_{F_2/S} \oplus \mathcal{O})$ where the cup product is not preserved under $\mathcal{F}$, we find that the defect of cup product is completely realized in the subring $p^* H^*(F_2)$ since

$$i^* H^*(X_{\text{loc}}) = H^*(F_2).$$

For general X, if $i^* : H^*(X, \mathbb{Q}) \to H^*(F_2, \mathbb{Q})$ is surjective, then $\Delta_f \otimes \mathbb{Q} = p^* H^*(F_2) \otimes \mathbb{Q}$ must contain the defect on X_{loc}; hence the ring structure on $H^*(X)$ is not preserved under $\mathcal{F}$. This completes the proof.

<div align="right">q.e.d.</div>

Example 6.5. Consider a simple $A_{4,2}$ flop on hyper-Kähler manifold X of dimension 12, where $F_2 = G(2,5)$ is of dimension 6. The divisor $c_1(G) = i^* H$ for some $H \in H^2(X, \mathbb{Q})$. But $c_2(G) \notin i^* H^4(X, \mathbb{Q})$.

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