J. DIFFERENTIAL GEOMETRY
80 (2008) 281-326

REGULARIZATION OF CURRENTS WITH MASS CONTROL AND SINGULAR MORSE INEQUALITIES

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Abstract

Let X be a compact complex, not necessarily Kähler, manifold of dimension n. We characterize the volume of any holomorphic line bundle $L \to X$ as the supremum of the Monge-Ampère masses $\int_X T_{ac}^n$ over all closed positive currents T in the first Chern class of L, where T_{ac} is the absolutely continuous part of T in its Lebesgue decomposition. This result, new in the non-Kähler context, can be seen as holomorphic Morse inequalities for the cohomology of high tensor powers of line bundles endowed with arbitrarily singular Hermitian metrics. It gives, in particular, a new bigness criterion for line bundles in terms of existence of singular Hermitian metrics satisfying positivity conditions. The proof is based on the construction of a new regularization for closed (1, 1)-currents with a control of the Monge-Ampère masses of the approximating sequence. To this end, we prove a potential-theoretic result in one complex variable and study the growth of multiplier ideal sheaves associated with increasingly singular metrics.

1. Introduction

Let $\varphi : \Omega \to \mathbb{R} \cup \{-\infty\}$ be a plurisubharmonic (psh) function on an open subset $\Omega \subset \mathbb{C}^n$, and let $z = (z_1, \dots, z_n)$ be the standard coordinates on $\mathbb{C}^n$. The Lelong number $\nu(\varphi, x)$ of φ at an arbitrary point $x \in \Omega$ is defined as the mass carried by the positive measure $dd^c\varphi \wedge (dd^c \log |z - x|)^{n-1}$ at x (see, for instance, Demailly's book [**Dem97**], Chapter III). It is a well-known result of Skoda ([**Sko72a**]) that the Lelong numbers of φ affect the local integrability of $e^{-2\varphi}$. Indeed, if $\nu(\varphi, x) < 1$, then $e^{-2\varphi}$ is integrable on some neighbourhood of x. On the contrary, if $\nu(\varphi, x) \geq n$, then $e^{-2\varphi}$ is not integrable near x. The integrability of $e^{-2\varphi}$ is unpredictable when $1 \leq \nu(\varphi, x) < n$.

Our first aim is to establish a potential-theoretic result in the case $n = 1$ where there is no unpredictability interval. Let $U \subset \mathbb{C}$ be an open set, $\varphi_0 : U \to \mathbb{R} \cup \{-\infty\}$ a subharmonic function, and $T = dd^c\varphi_0$ the associated closed positive current of bidegree $(1,1)$. The current T can be identified with the Laplacian $\Delta\varphi_0$ of φ_0 computed in the sense

Received 12/21/2006.

of distributions. It defines a positive measure $\mu = dd^c\varphi_0$ on U. In one complex variable, the mass of $dd^c\varphi_0$ at a point x coincides with the Lelong number $\nu(\varphi_0, x)$. Let $D(x_0, r) \Subset U$ be an arbitrary disc of radius $0 < r < \frac{1}{2}$, and let

$$\gamma = \int\limits_{P(x_0, 2r)} dd^c\varphi_0$$

be the mass carried by the measure $dd^c\varphi_0$ on the square $P(x_0, 2r)$ of edge $2r$ centred at x_0. Consider the decomposition:

$$\varphi_0 = N \star \Delta\varphi_0 + h_0, \quad \text{ on } \ D(x_0, r),$$

where $N(z) = \frac{1}{2\pi}\log|z|$ is the Newton kernel in one complex variable, and $h_0 = \operatorname{Re} g_0$ is a harmonic function expressed as the real part of a holomorphic function g_0.

First, we will be considering a method of neutralizing the $-\infty$-poles of φ_0 on a fixed disc in order to make the exponential $e^{-2m\varphi_0}$ integrable and to control the growth rate of its integral as $m \to +\infty$.

Theorem 1.1. *Let $\varphi_0 : U \to \mathbb{R} \cup \{-\infty\}$ be a subharmonic function on an open set $U \subset \mathbb{C}$, and let $D \subset D(x_0, r) \Subset U$ be an open subset contained in a disc of radius $0 < r < \frac{1}{2}$. Fix $0 < \delta < 1$. Then, for every $m \gg 1$, there exist finitely many points $a_1 = a_1(m), \dots, a_{N_m} = a_{N_m}(m) \in D$, such that the positive integers m_j defined as:*

$m_j = \max\{[m\nu(\varphi_0, a_j)],\ 1\}, \quad j = 1, \dots, N_m,$ *([ ] is the integer part),*

and the holomorphic function $f_m(z) = e^{m\, g_0(z)} \prod\limits_{j=1}^{N_m} (z - a_j)^{m_j}$ defined on D, satisfy the following properties:

(i) $\sum\limits_{j=1}^{N_m} m_j \leq m\gamma(1+\delta)$, *where γ is the $dd^c\varphi_0$-mass of $P(x_0, 2r)$;*

(ii) *there exists a constant $C = C(r) > 0$, independent of m, such that:*

$$|a_j - a_k| \geq \frac{C}{m^2},$$

for all a_j, a_k, such that $j \neq k$ and $\nu(\varphi_0, a_j), \nu(\varphi_0, a_k) < \frac{1-\delta}{m}$;

(iii) $\displaystyle\int_D |f_m(z)|^2 e^{-2m\varphi_0(z)}\, d\lambda(z) = o(m), \quad$ *when* $\quad m \to +\infty$,
where $d\lambda$ is the Lebesgue measure in $\mathbb{C}$.

Higher dimensional analogues of this result have yet to be found. However, the Ohsawa-Takegoshi L^2 extension theorem (see [**OT87**], [**Ohs88**]) applied on a complex line enables us to derive geometric applications of Theorem 1.1 in several complex variables. The first application is a global regularization theorem for closed almost positive (1, 1)-currents in the spirit of Demailly (see [**Dem92**]), but with an additional control on the Monge-Ampère masses of the regularizing currents. Here is the set-up.

Let T be a d-closed current of bidegree (1, 1) on a compact complex manifold X of dimension n. Assume that $T \geq \gamma$ for some real continuous (1, 1)-form γ (i. e. T is almost positive). The current T can be globally written as $T = \alpha + dd^c\varphi$, with a global C^∞ (1, 1)-form α and an almost psh potential φ on X (i.e., φ can be locally expressed as the sum of a psh function and a C^∞ function). The notation $dd^c := \frac{i}{\pi}\partial\bar\partial$ will be used in all that follows. A variant of Demailly's regularization theorem (see [**Dem92**, Proposition 3.7]) asserts that T is the weak limit of currents $T_m = \alpha + dd^c\varphi_m$ lying in the $\partial\bar\partial$-cohomology class of T and having analytic singularities. These are, by definition, singularities for which φ_m can be locally written as

$$\frac{c}{2}\log(|g_1|^2 + \cdots + |g_N|^2) + C^\infty, \tag{1}$$

with a constant $c > 0$, and holomorphic functions $g_1, \ldots, g_N$. Each T_m can be chosen to be smooth on $X \setminus V\mathcal{I}(mT)$, where $V\mathcal{I}(mT)$ is the zero variety of the multiplier ideal sheaf $\mathcal{I}(mT)$ associated with mT (defined locally as $\mathcal{I}(m\varphi)$, see (16)). Moreover, for any Hermitian metric ω on X, T_m can be chosen such that:

$$T_m \geq \gamma - \varepsilon_m\,\omega, \quad \text{for some sequence } \varepsilon_m \downarrow 0,$$

and the Lelong numbers satisfy: $\nu(T, x) - \dfrac{n}{m} \leq \nu(T_m, x) \leq \nu(T, x)$, $x \in X$.

What this theorem does not specify, however, is whether there exist regularizations $T_m \to T$ with analytic singularities having the extra property that the growth in m of the masses of the wedge-power currents T_m^k (Monge-Ampère currents) is under control. In other words, we would like to control the growth rate of the quantities:

$$\int_{X\setminus V\mathcal{I}(mT)} (T_m - \gamma + \varepsilon_m\,\omega)^k \wedge \omega^{n-k}, \quad k = 1, \ldots, n,$$

as $m \to +\infty$, where $V\mathcal{I}(mT) = \{\varphi_m = -\infty\}$ is the polar set of T_m. Using Theorem 1.1 we can modify Demailly's original construction to settle this question in the following form.

Theorem 1.2. *Let $T \geq \gamma$ be a d-closed current of bidegree* (1, 1) *on a compact complex manifold X, where γ is a continuous* (1, 1)*-form such that $d\gamma = 0$. Then, in the $\partial\bar\partial$-cohomology class of T, there exist closed* (1, 1)*-currents T_m with analytic singularities converging to T in the weak topology of currents such that each T_m is smooth on $X\setminus V\mathcal{I}(mT)$ and:*

(a) $T_m \geq \gamma - \dfrac{C}{m}\,\omega, \qquad m \in \mathbb{N}$;

(b) $\nu(T, x) - \varepsilon_m \leq \nu(T_m, x) \leq \nu(T, x), \quad x \in X, m \in \mathbb{N}$, *for some* $\varepsilon_m \downarrow 0$;

(c) $\displaystyle\lim_{m\to+\infty} \frac{1}{m} \int\limits_{X\setminus V\mathcal{I}(mT)} (T_m - \gamma + \frac{C}{m}\,\omega)^k \wedge \omega^{n-k} = 0, \qquad k = 1, \ldots, n = \dim_{\mathbb{C}} X$,

where ω *is an arbitrary Hermitian metric on* X.

This result can be used to prove a new characterization of big line bundles in terms of curvature currents. Let us briefly review a few basic facts. A holomorphic line bundle L over a compact complex manifold X of dimension n is said to be big if $\dim_{\mathbb{C}} H^0(X, L^m) \geq C\, m^n$ for some constant $C > 0$ and for all large enough $m \in \mathbb{N}$. This amounts to the global sections of L^m defining a bimeromorphic embedding of X into a projective space for $m \gg 0$. The compact manifold X is said to be Moishezon if the transcendence degree of its meromorphic function field equals $n := \dim_{\mathbb{C}} X$, or equivalently, if there exist n global meromorphic functions that are algebraically independent. A Moishezon manifold becomes projective after finitely many blow-ups with smooth centres. There is, moreover, a bimeromorphic counterpart to Kodaira's embedding theorem: a compact complex manifold X is Moishezon if and only if there exists a big line bundle $L \to X$. The asymptotic growth of the dimension of $H^0(X, L^m)$ as $m \to +\infty$ is actually measured by a birational invariant of L, the *volume*, defined as:

$$v(L) := \limsup_{m\to+\infty} \frac{n!}{m^n}\, h^0(X, L^m).$$

Clearly, L is big if and only if $v(L) > 0$. Switching now to the analytic point of view, recall that a singular Hermitian metric h on L is defined in a local trivialization $L_{|U} \simeq U \times \mathbb{C}$ as $h = e^{-\varphi}$ for some weight function $\varphi : U \to [-\infty, +\infty)$ which is only assumed to be locally integrable. In particular, the singularity set $\{x \in U, \varphi(x) = -\infty\}$ is Lebesgue negligible. The associated curvature current $T := i\Theta_h(L)$ is a closed current of bidegree $(1, 1)$ on X representing the first Chern class $c_1(L)$ of L. It is locally defined as $T = dd^c\varphi$, where φ is a local weight function of h.

Recall that an almost positive current T can be locally written in coordinates as $T = \sum\limits_{j,k} T_{j,k}\, dz_j \wedge d\bar{z}_k$ for some complex measures $T_{j,k}$. The Lebesgue decomposition of the coefficients $T_{j,k}$ into an absolutely continous part and a singular part with respect to the Lebesgue measure induces a current decomposition as $T = T_{ac} + T_{\mathrm{sing}}$. By the Radon-Nikodym theorem, the coefficients of the absolutely continous part are L^1_{loc}, and thus the exterior powers T^m_{ac} are well defined (though not necessarily closed) currents for $m = 1, \ldots, n$.

When applied to curvature currents, the current regularization Theorem 1.2 with controlled Monge-Ampère masses enables us to characterize the volume of a line bundle in terms of positive currents in $c_1(L)$. This gives in particular a bigness criterion for line bundles in terms of existence of singular Hermitian metrics satisfying positivity assumptions (and implicitly, a characterization of Moishezon manifolds).

Theorem 1.3. *Let L be a holomorphic line bundle over a compact complex manifold X. Then the volume of L is characterized as:*

$$v(L) = \sup_{T \in c_1(L),\, T \geq 0} \int_X T_{ac}^n.$$

In particular, L is big if and only if there exists a possibly singular Hermitian metric h on L whose curvature current $T := i\Theta_h(L)$ satisfies the following positivity conditions:

$$\text{(i)} \quad T \geq 0 \ \ \text{on } X; \qquad \text{(ii)} \quad \int_X T_{ac}^n > 0.$$

In the special case when the ambient manifold X is Kähler, this same result was obtained by Boucksom ([**Bou02**, Theorem 1.2]). This strengthens a previous (only sufficient) bigness criterion by Siu ([**Siu85**]) that solved affirmatively the Grauert–Riemenschneider conjecture ([**GR70**]). Bigness was guaranteed there under the extra assumption that the curvature current T (or the metric h) be C^∞. Theorem 1.3 falls into the mould of ideas originating in Demailly's holomorphic Morse inequalities ([**Dem85**]). Its proof hinges on the regularization Theorem 1.2 above, and on Bonavero's singular version of Demailly's Morse inequalities ([**Bon98**]). It strengthens a bigness criterion in [**Bon98**] which required the curvature current T to have analytic singularities. On the other hand, Ji and Shiffman ([**JS93**]) proved that L being big is equivalent to L having a singular metric whose curvature current is strictly positive on X (i.e., $\geq \varepsilon\omega$ for some small $\varepsilon > 0$). This implies, in particular, the "only if" part of the above Theorem 1.3. The thrust of the new "if" part of Theorem 1.3 is to relax the strict positivity assumption on the curvature current.

Let us finally stress that the main interest of Theorems 1.2 and 1.3 lies in X being an arbitrary compact manifold. Related results are known to exist for Kähler manifolds (e.g., [**DP04**, Theorem 0.4], [**Bou02**, Theorem 1.2]). The approach to the non-Kähler case treated here is quite different. The crux of the argument is modifying the existing procedure for regularizing (1, 1)-currents to get an effective control on the Monge-Ampère masses (Theorem 1.2). If X is Kähler, the sequence of masses in the usual Demailly regularization of currents is easily seen to be bounded by applying Stokes's theorem and using the closedness of ω (see [**Bou02**]). The situation is vastly different in the non-Kähler case where a new regularization of currents is needed with a possibly unbounded sequence of masses.

The paper runs as follows. Section 2 collects the main ingredients for the proof of Theorem 1.1. The most prominent of these is an atomization lemma for positive measures in $\mathbb{R}^2$ due to Yulmukhametov ([**Yul85**]) and Drasin ([**Dra01**]). Section 3 goes on to implement the proof of Theorem 1.1 which will be used in subsequent sections.

In order to make the ideas more transparent, we will first prove Theorem 1.2 in the special case when the Lelong numbers of the original current T are assumed to vanish everywhere. Section 4 gives the local procedure depending on an application of the Ohsawa-Takegoshi L^2 extension theorem on a complex line. This is a key idea in the paper. Section 5 presents the gluing process. This first non-trivial case retains many of the ideas of the proof stripped of technical details and without recourse to multiplier ideal sheaves (which are trivial in this case).

The next three sections achieve the proof of Theorem 1.2. Section 6 explains the new regularization technique for psh functions on domains in $\mathbb{C}^n$ based on using derivatives of holomorphic functions in a weighted Bergman space. Section 7 introduces another key idea of this paper, namely an effective control, with estimates, of how far multiplier ideal sheaves are from behaving linearly as the singularities increase. It turns out that in the case of analytic singularities the growth of multiplier ideal sheaves is almost linear. This provides a useful complement to the Demailly-Ein-Lazarsfeld subadditivity theorem. Section 8 combines ideas of the previous sections to complete the proof of Theorem 1.2.

Finally, Section 9 gives a proof of Theorem 1.3 as a geometric application of the new regularization procedure for currents. An appendix, Section 10, provides a few explanations that have fallen between the cracks of the previous sections.

Acknowledgments. The author is extremely grateful to Jean-Pierre Demailly for suggesting the problem and for numerous helpful discussions on the topic. The author would like to thank the referee for pertinent remarks and comments. A word of thanks is also due to quite a number of people who showed an interest in this work, of whom Bo Berndtsson, Sébastien Boucksom and Takeo Ohsawa are only a few.

2. Preliminaries to Theorem 1.1

In this section we clear the way to the proof of Theorem 1.1. The set-up is the one described in the Introduction. Fix $m \in \mathbb{N}^\star$ and $\delta > 0$. As the upper level set for Lelong numbers:

$$E_{1-\delta}(m\, dd^c\varphi_0) := \{x \in U;\, \nu(m\,\varphi_0,\, x) \geq 1-\delta\}$$

is analytic of dimension 0, its intersection with a relatively compact subset is finite. Let

$$E_{1-\delta}(m\, dd^c\varphi_0) \cap D := \{a_1, \ldots, a_{p(m)}\}.$$

As $dd^c\psi_m = m\, dd^c\varphi_0 - \sum_{j=1}^{p(m)} [m\,\nu(\varphi_0,\, a_j)]\, \delta_{a_j} \geq 0$ as currents, the function

$$\psi_m(z) := m\,\varphi_0(z) - \sum_{j=1}^{p(m)} [m\,\nu(\varphi_0,\, a_j)] \,\log|z - a_j|$$

is still subharmonic and $\nu(\psi_m,\, a_j) = m\,\nu(\varphi_0,\, a_j) - [m\,\nu(\varphi_0,\, a_j)]$ for every j.

To ensure the integrability property (iii) in Theorem 1.1, we must introduce a factor $(z - a_j)^{m_j}$ in the definition of the function f_m being constructed for every point a_j where $m_j := [m\,\nu(\varphi_0,\, a_j)] \geq 1$. To find the other points, we can assume in all that follows, after replacing $m\varphi_0$ with ψ_m, that:

$$(2) \qquad\qquad m\,\nu(\varphi_0,\, x) < 1, \quad \text{for all} \quad x \in D.$$

Once the point masses of the current $m\, dd^c\varphi_0$ have been brought down below 1, we still have to neutralize any diffuse mass scattered over the domain D that could prevent the integral of $|f_m|^2\, e^{-2m\varphi_0}$ from having the desired slow growth in m. The following lemma gives an upper bound for $e^{-2\varphi_0}$ in terms of the mass of the associated current $dd^c\varphi_0$.

Lemma 2.1. *With the notation in the introduction, if* $\gamma := \int_D dd^c\varphi_0$, *the following estimate holds:*

$$e^{-2(\varphi_0(z) - h_0(z))} \leq \frac{1}{\int\limits_D dd^c\varphi_0} \int\limits_D \frac{1}{|\zeta - z|^{2\gamma}}\, dd^c\varphi_0(\zeta),$$

for all $z \in D$.

Proof. Let $d\mu(\zeta) := \gamma^{-1}\, dd^c\varphi_0(\zeta)$ be a probability measure on D. For all $z \in D$, we have:

$$(\varphi_0 - h_0)(z) = \int_D \log|\zeta - z|\, dd^c\varphi_0(\zeta),$$

or, equivalently,

$$-(\varphi_0 - h_0)(z) = \int_D \gamma \log|\zeta - z|^{-1} d\mu(\zeta), \quad z \in D.$$

Now, Jensen's convexity inequality entails:

$$e^{-2(\varphi_0 - h_0)(z)} \leq \int_D e^{2\gamma \log|\zeta - z|^{-1}} d\mu(\zeta) = \gamma^{-1} \int_D \frac{1}{|\zeta - z|^{2\gamma}} dd^c\varphi_0(\zeta),$$

which proves the lemma. q.e.d.

Applying Lemma 2.1 to the function $m\,\varphi_0$, we get:

$$e^{-2m(\varphi_0(z)-h_0(z))} \leq \frac{1}{\int\limits_D dd^c\varphi_0} \int\limits_D \frac{1}{|\zeta - z|^{2m\gamma}}\, dd^c\varphi_0(\zeta), \quad z \in D.$$

The right-hand term above may not be integrable as a function of z when $m\gamma > 1$. To get around this, we will cut D into pieces in such a way that each piece has a mass < 1 for the measure $m\,dd^c\varphi_0$ and the number of pieces does not exceed $m\gamma(1+\delta)$. We will subsequently choose a point in each piece, intuitively its "center", and will define f_m as a holomorphic function on D whose only zeroes are these points. This will be shown to satisfy the conditions in Theorem 1.1.

Cutting D into pieces relies on the following lemma due to Yulmukhametov ([**Yul85**]) and, in a generalized form, to Drasin ([**Dra01**], Theorem 2.1]). It describes an atomization procedure for arbitrary positive measures μ in one complex variable. This is the main technical ingredient in the proof of Theorem 1.1.

Lemma 2.2 ([**Yul85**], [**Dra01**]). *Let μ be a positive measure supported in a square $R \subset \mathbb{R}^2$ with sides parallel to the coordinate axes. Suppose $\mu(R) = N > 1$, $N \in \mathbb{Z}$. Then, there exist a family of closed rectangles $(R_j)_{1\leq j\leq N}$ with sides parallel to the coordinate axes, and a family of positive measures $(\mu_j)_{1\leq j\leq N}$, such that:*

(a) $\mu = \sum\limits_{j=1}^{N} \mu_j$, $\mu_j(\mathbb{R}^2) = 1$ *and* $\operatorname{Supp}\mu_j \subset R_j$ *for all* $j = 1, \ldots, N$;

(b) $R = \bigcup\limits_{j=1}^{N} R_j = \bigcup\limits_{j=1}^{N} \operatorname{Supp}\mu_j$;

(c) *the interiors of the convex hulls of the supports* $\operatorname{Supp}\mu_j$ *of the* μ_j*'s are mutually disjoint;*

(d) *the ratio of the sides of each rectangle R_j lies in the interval* $[\frac{1}{3}, 3]$ (*i.e., R_j is an "almost square" in the terminology of* [**Dra01**]);

(e) *each point in $\mathbb{R}^2$ belongs to the interior of at most four distinct rectangles R_j;*

(f) *each* $\operatorname{Supp}\mu_j$ *is contained in a rectangle $P_j \subset R_j$, and the distance between the centres of any two distinct rectangles P_j is $\geq \frac{C}{N^2}$, where $C > 0$ is the side of the square R.*

Idea of proof (*according to* [**Dra01**]). Yulmukhametov originally proved this result (see [**Yul85**]) for absolutely continuous measures μ. The generalization to the case of arbitrary measures is due to Drasin ([**Dra01**]). We summarize here the ideas of Drasin's proof. Conclusion (f) was not explicitly stated, but it can be easily inferred from the proof given there. The first idea is to reduce the problem to the case of a measure μ satisfying $\mu(p) < 1$ at every point $p \in R$. This is done by subtracting from the original measure μ the integer part $[\mu(p)]$ of each point mass $\mu(p) > 1$. We may also assume, after a possible rotation of the coordinate system of $\mathbb{R}^2$, that for every line L parallel to one of the coordinate axes, there exists at most one point $p \in L$ such that $\mu(p) > 0$ while $\mu(L \setminus p) = 0$.

After these reductions, the key step is to prove that if an almost square R contains the support of a measure μ satisfying these properties, then there exist almost squares R_0 and R_1 and a decomposition $\mu = \mu_0 + \mu_1$ such that $\operatorname{Supp}\mu_j \subset R_j$, $j = 0, 1$, which satisfies conclusions $(b) - (d)$ of the lemma. The masses $\mu_j(R_j)$ are integers. If $\mu_j(R_j) > 1$, we repeat this procedure to obtain almost squares $R_{j,0}$, $R_{j,1}$ and a decomposition $\mu_j = \mu_{j,0} + \mu_{j,1}$. By repeatedly applying this procedure we get almost squares R_I and measures μ_I, indexed over multi-indices $I = i_1, \ldots, i_l$ made up of digits 0 and 1. The procedure terminates when all masses $\mu_I(R_I) = 1$. A technical lemma then yields conclusion (e) and thus clinches the proof of this result. We refer for details to Drasin ([**Dra01**, §2, pp. 165–171]). q.e.d.

3. Proof of Theorem 1.1

Building on preliminaries in the previous section, we will now complete the proof of Theorem 1.1. The notation and set-up are unchanged. Let R be the square of edge $2r$ centered at x_0. It contains $D(x_0, r)$ and implicitly D. By hypothesis (2), we may assume that $m\,\nu(\varphi_0, x) < 1$ for all $x \in D$. The positive measure $\mu := dd^c\varphi_0$ has mass γ on R. Fix $0 < \delta < 1$ and, for $m \gg 1$, choose an integer N_m such that:

$$\frac{2}{2-\delta}\, m\,\gamma < N_m \leq m\,\gamma\,(1+\delta). \tag{3}$$

Such an integer exists if m is so large that $m\,\gamma\,(1+\delta-\frac{2}{2-\delta}) = m\,\gamma\,\frac{\delta(1-\delta)}{2-\delta} > 1$. We now apply the atomization Lemma 2.2 to the measure $\frac{N_m}{\gamma}\,\mu = \frac{N_m}{\gamma}\,dd^c\varphi_0$ of total mass $N := N_m$ on R. We get a covering of R by closed rectangles:

$$R = \bigcup_{j=1}^{N_m} R_j(m),$$

and a decomposition of measures $\frac{N_m}{\gamma}\,\mu = \sum_{j=1}^{N_m} \nu_{m,\,j}$ such that every $R_j := R_j(m)$ is an almost square containing the support of $\nu_{m,\,j}$, and $\nu_{m,\,j}(R_j) = 1$.

By part (f) of Lemma 2.2, there is a family of possibly smaller rectangles $P_j(m) \subset R_j(m)$, $j = 1, \ldots, N_m$, still covering R such that each $P_j(m)$ contains the support of $\nu_{m,\,j}$. Their main feature is the following: if $a_j = a_j(m)$ is the center of $P_j(m)$, the mutual distances of the a_j's are $\geq C/N_m^2$ with $C > 0$ independent of m. Unlike the $R_j(m)$'s, we do not know whether the $P_j(m)$'s are almost squares.

We will prove that the integer N_m and the points a_j satisfy the conclusions of Theorem 1.1. As by Hypothesis (2) we have

$$m_j = \max\{[m\,\nu(\varphi_0,\, a_j)],\, 1\} = 1, \quad \text{for } j = 1, \ldots,\, N_m,$$

we get: $\sum\limits_{j=1}^{N_m} m_j = N_m \leq m\gamma(1+\delta)$, which is conclusion (i) of Theorem 1.1. The lower bound on the mutual distances of the points a_j and the choice of $N_m = O(m)$ (cf. (3)) ensure that the points a_j satisfy the conclusion (ii) of Theorem 1.1. It remains to check that for the holomorphic function:

$$f_m(z) := e^{m\,g_0(z)} \prod_{j=1}^{N_m} (z - a_j), \qquad z \in D,$$

the integral $\int\limits_D |f_m|^2\, e^{-2m\varphi_0}\, d\lambda$ has the desired slow growth in m as $m \to +\infty$. Since

$$\int\limits_D |f_m|^2\, e^{-2m\varphi_0}\, d\lambda \leq \sum_{j=1}^{N_m} \int\limits_{R_j} |f_m|^2\, e^{-2m\varphi_0}\, d\lambda,$$

the analysis is reduced to finding a convenient upper bound for each integral on R_j. Fix $j \in \{1, \ldots, N_m\}$. Since $a_j \in R_j$, we see that R_j is contained in the disc $D_j := D(a_j,\, r_j)$, where r_j is $\sqrt{2}$ times the longest edge of R_j. Conclusion (e) of Lemma 2.2 implies that the sum of the Euclidian areas of the R_j's is bounded above by four times the area of the square R of edge $2r$. As the R_j's are almost squares, it follows that there is a constant $C_1(r) > 0$, depending only on r, such that

$$\text{(e}'\text{)} \qquad \sum_{j=1}^{N_m} r_j^2 \leq C_1(r), \quad \text{for all } m \gg 1.$$

Lemma 2.1, when applied to the function $m\,\varphi_0$ on $D_j = D(a_j,\, r_j)$, gives:

$$\begin{aligned} &|f_m(z)|^2\, e^{-2m\varphi_0(z)} \\ &= \prod_{k=1}^{N_m} |z - a_k|^2\, e^{-2m(\varphi_0(z) - h_0(z))} \\ &\leq (2r)^{2(N_m-1)}\, |z - a_j|^2\, \frac{1}{\int\limits_{D_j} dd^c\varphi_0} \int\limits_{D_j} \frac{1}{|\zeta - z|^{2\frac{m\gamma}{N_m}}}\, dd^c\varphi_0(\zeta), \end{aligned}$$

for all $z \in D_j$. We have used the obvious upper bound $|z - a_k|^2 \leq (2r)^2$, for all $k \neq j$. When integrating above with respect to $z \in D_j$, Fubini's theorem yields:

$$\int\limits_{D_j} |f_m(z)|^2 \, e^{-2m\varphi_0(z)} d\lambda(z) \tag{4}$$
$$\leq \frac{(2r)^{2(N_m-1)}}{\int\limits_{D_j} dd^c\varphi_0} \int\limits_{D_j} \left(\int\limits_{D_j} \frac{|z-a_j|^2}{|z-\zeta|^{2\frac{m\gamma}{N_m}}} d\lambda(z) \right) dd^c\varphi_0(\zeta).$$

Let us now concentrate on the integral in z on the right-hand side. We get the following estimate for every $\zeta \in D_j$:

$$\int\limits_{D_j} \frac{|z-a_j|^2}{|z-\zeta|^{2\frac{m\gamma}{N_m}}} d\lambda(z) \tag{5}$$
$$= \int\limits_{D(a_j,r_j)} \frac{|z-a_j|^2}{|(z-a_j)-(\zeta-a_j)|^{2\frac{m\gamma}{N_m}}} d\lambda(z-a_j)$$
$$\leq 4\pi(|\zeta-a_j|+r_j)^{2(1-\frac{m\gamma}{N_m})} \left(\frac{(|\zeta-a_j|+r_j)^2}{2(2-\frac{m\gamma}{N_m})} + \frac{|\zeta-a_j|^2}{2(1-\frac{m\gamma}{N_m})} \right).$$

Indeed, if we make the change of variable $x = z - a_j$ and set $\zeta - a_j = a$, we are reduced to estimating the integral:

$$\int\limits_{D(0,r)} \frac{|x|^2}{|x-a|^\tau} d\lambda(x),$$

where we have set $r_j := r$ and $\tau := 2\frac{m\gamma}{N_m}$ to simplify the notation. By the choice (3) of N_m, we have: $0 < \tau < 2$. The change of variable $x - a = y$, followed by a switch to polar coordinates with $|y| = \rho$, implies:

$$\int\limits_{D(0,r)} \frac{|x|^2}{|x-a|^\tau} d\lambda(x) = \int\limits_{D(-a,r)} \frac{|y+a|^2}{|y|^\tau} d\lambda(y) \leq \int\limits_{D(-a,r)} \frac{(|y|+|a|)^2}{|y|^\tau} d\lambda(y)$$
$$\leq 2 \int\limits_{D(-a,r)} \frac{|y|^2+|a|^2}{|y|^\tau} d\lambda(y)$$
$$= 2 \int\limits_{D(-a,r)} |y|^{2-\tau} d\lambda(y) + 2|a|^2 \int\limits_{D(-a,r)} |y|^{-\tau} d\lambda(y)$$
$$\leq 2\pi \left(2\int_0^{|a|+r} \rho^{2-\tau} \rho \, d\rho + 2|a|^2 \int_0^{|a|+r} \rho^{-\tau} \rho \, d\rho \right)$$
$$= 4\pi \, (|a|+r)^{2-\tau} \left(\frac{(|a|+r)^2}{4-\tau} + \frac{|a|^2}{2-\tau} \right).$$

For $r = r_j$, this gives the estimate (5). Now relations (4) and (5) imply:

$$\int\limits_{D_j} |f_m(z)|^2 \, e^{-2m\varphi_0(z)} d\lambda(z)$$
$$\leq \frac{4\pi}{\int_{D_j} dd^c\varphi_0} (2r)^{2(N_m-1)}$$
$$\cdot \int_{D_j} (|\zeta - a_j| + r_j)^{2(1-\frac{m\gamma}{N_m})} \left(\frac{(|\zeta - a_j| + r_j)^2}{2(2-\frac{m\gamma}{N_m})} + \frac{|\zeta - a_j|^2}{2(1-\frac{m\gamma}{N_m})} \right) dd^c\varphi_0(\zeta).$$

Let us now shift to polar coordinates with $|\zeta - a_j| = \rho$. This gives $dd^c\varphi_0(\zeta) = dn(\rho)$, where $n(\rho) = \int_{D(a_j,\, \rho)} dd^c\varphi_0$, for all $\rho \geq 0$. Since D_j is assumed to be $D(a_j,\, r_j)$, we get:

$$\int\limits_{D_j} |f_m(z)|^2 \, e^{-2m\varphi_0(z)} d\lambda(z)$$
$$\leq C(r,\, r_j) \int_0^{r_j} (\rho + r_j)^{2(1-\frac{m\gamma}{N_m})} \left(\frac{(\rho + r_j)^2}{2(2-\frac{m\gamma}{N_m})} + \frac{\rho^2}{2(1-\frac{m\gamma}{N_m})} \right) n'(\rho) \, d\rho,$$

where $C(r,\, r_j) = \dfrac{8\pi^2}{\int_{D_j} dd^c\varphi_0} (2r)^{2(N_m-1)}$. The last expression can be successively written as:

$$\frac{C(r,\, r_j)}{2(2-\frac{m\gamma}{N_m})} \int_0^{r_j} (\rho + r_j)^{2(2-\frac{m\gamma}{N_m})} n'(\rho) \, d\rho$$
$$+ \frac{C(r,\, r_j)}{2(1-\frac{m\gamma}{N_m})} \int_0^{r_j} \rho^2 (\rho + r_j)^{2(1-\frac{m\gamma}{N_m})} n'(\rho) \, d\rho$$
$$= \frac{C(r,\, r_j)}{2(2-\frac{m\gamma}{N_m})} \Bigg(n(r_j)(2r_j)^{2(2-\frac{m\gamma}{N_m})}$$
$$- 2\left(2 - \frac{m\gamma}{N_m}\right) \int_0^{r_j} n(\rho)(\rho + r_j)^{3-2\frac{m\gamma}{N_m}} \, d\rho \Bigg)$$
$$+ \frac{C(r,\, r_j)}{2(1-\frac{m\gamma}{N_m})} \Bigg(n(r_j) r_j^2 (2r_j)^{2(1-\frac{m\gamma}{N_m})}$$
$$- \int_0^{r_j} n(\rho) \left[2\rho(\rho + r_j)^{2(1-\frac{m\gamma}{N_m})} + 2\left(1 - \frac{m\gamma}{N_m}\right)(\rho + r_j)^{1-2\frac{m\gamma}{N_m}} \rho^2 \right] d\rho \Bigg).$$

The terms appearing above with a " $-$" sign are all negative since $1 - \frac{m\gamma}{N_m} > 0$ (and implicitly $2 - \frac{m\gamma}{N_m} > 0$). Therefore, they can be ignored. We thus get the following upper estimate:

$$\int_{D_j} |f_m(z)|^2\, e^{-2m\varphi_0(z)} d\lambda(z)$$
$$\leq C(r,\, r_j)\, n(r_j) \left(\frac{(2r_j)^{2(2-\frac{m\gamma}{N_m})}}{2(2-\frac{m\gamma}{N_m})} + \frac{r_j^2 (2r_j)^{2(1-\frac{m\gamma}{N_m})}}{2(1-\frac{m\gamma}{N_m})} \right).$$

Since $n(r_j) = \int_{D_j} dd^c\varphi_0$, the previous upper bound and the formula of $C(r,\, r_j)$ show that:

$$\int_{D_j} |f_m(z)|^2\, e^{-2m\varphi_0(z)} d\lambda(z) \leq C\left(r,\, \frac{m\gamma}{N_m}\right) \cdot r_j^{2(2-\frac{m\gamma}{N_m})}, \tag{6}$$

where the constant $C(r,\, \frac{m\gamma}{N_m})$ is given by the formula:

$$C\left(r,\, \frac{m\gamma}{N_m}\right) = 8\pi^2\, (2r)^{2(N_m-1)} \left(\frac{2^{2(2-\frac{m\gamma}{N_m})}}{2(2-\frac{m\gamma}{N_m})} + \frac{2^{2(1-\frac{m\gamma}{N_m})}}{2(1-\frac{m\gamma}{N_m})} \right).$$

Since estimate (6) holds for all indices $j \in \{1, \dots,\, N_m\}$, we get, after summing over j, that:

$$\int_D |f_m(z)|^2\, e^{-2m\varphi_0(z)}\, d\lambda(z) \leq C\big(r,\, \frac{m\gamma}{N_m}\big) \sum_{j=1}^{N_m} r_j^{2(2-\frac{m\gamma}{N_m})}.$$

The choice of N_m was made in such a way that $1-\delta < \frac{1}{1+\delta} \leq \frac{m\gamma}{N_m} < 1-\frac{\delta}{2}$ (cf. (3)), which implies:

$$\frac{\delta}{2} < 1 - \frac{m\gamma}{N_m} < \delta \quad \text{and} \quad 1+\frac{\delta}{2} < 2 - \frac{m\gamma}{N_m} < 1+\delta.$$

Since $0 < 2r < 1$, there exists a constant $C_2(r) > 0$ depending only on r, such that $C(r,\, \frac{m\gamma}{N_m}) \leq C_2(r)$, for all $m \in \mathbb{N}$. Since $r_j \leq 2r < 1$, we have:

$$r_j^{2(2-\frac{m\gamma}{N_m})} < r_j^2, \quad \text{for} \quad 2(2-\frac{m\gamma}{N_m}) > 2.$$

Thus, estimate (e') (inferred above from (e) of Lemma 2.2) implies:

$$\int_D |f_m(z)|^2\, e^{-2m\varphi_0(z)}\, d\lambda(z) \leq C(r), \quad \forall m \gg 0,$$

where $C(r) = C_1(r)\, C_2(r) > 0$ is a constant depending only on the radius r of the disc D on which we are working. This yields conclusion (iii) of Theorem 1.1 and completes its proof. q.e.d.

4. Special case of local regularization with mass control

In this section we use Theorem 1.1 combined with the Ohsawa-Takegoshi L^2 extension theorem (see [**OT87**], [**Ohs88**]) to introduce a new local approximation procedure for psh functions with zero Lelong numbers. The main new outcome is an additional control of the Monge-Ampère masses. This can be seen as a local version of Theorem 1.2 under the extra assumption that all the Lelong numbers vanish.

Let φ be a psh function on a bounded pseudoconvex open set $\Omega \subset \mathbb{C}^n$. A well-known result of Demailly (cf. [**Dem92**, Proposition 3.1]) asserts that φ can be approximated pointwise and in $L^1_{\mathrm{loc}}(\Omega)$ topology by psh functions φ_m with analytic singularities (see definition (1) in the introduction), constructed as:

$$\varphi_m = \frac{1}{2m}\,\log\sum_{j=0}^{+\infty}|\sigma_{m,\,j}|^2, \tag{7}$$

where $(\sigma_{m,\,j})_{j\in\mathbb{N}}$ is an arbitrary orthonormal basis of the Hilbert space $\mathcal{H}_\Omega(m\varphi)$ of holomorphic functions f on Ω such that $|f|^2\,e^{-2m\varphi}$ is integrable on Ω. They even satisfy the estimates:

$$\varphi(z) - \frac{C_1}{m} \leq \varphi_m(z) \leq \sup_{|\zeta - z|<r}\varphi(\zeta) + \frac{1}{m}\,\log\frac{C_2}{r^n}, \tag{8}$$

for every $z\in\Omega$ and every $r < d(z,\,\partial\Omega)$. In particular, the sequence $dd^c\varphi_m$ converges to $dd^c\varphi$ in the weak topology of currents, and the corresponding Lelong numbers satisfy:

$$\nu(\varphi,\,x) - \frac{n}{m} \leq \nu(\varphi_m,\,x) \leq \nu(\varphi,\,x), \quad x\in\Omega. \tag{9}$$

For analytic singularities, the Lelong number $\nu(\varphi_m,\,x)$ at an arbitrary point x equals $\frac{1}{m}\min\limits_{j\geq 0}\mathrm{ord}_x\sigma_{m,\,j}$, where ord_x is the vanishing order at x. The sequence $(\varphi_m)_{m\in\mathbb{N}}$ defined in (7) has come to be referred to as the Demailly approximation of φ.

Let us now suppose that φ has zero Lelong numbers everywhere (see [**Dem97**, Chapter III] for a comprehensive discussion of Lelong numbers). In other words,

$$\nu(\varphi,\,x) := \liminf_{z\to x}\frac{\varphi(z)}{\log|z-x|} = 0, \qquad \text{for every } x\in\Omega.$$

Psh functions φ for which there are points x such that $\varphi(x) = -\infty$ and $\nu(\varphi,\,x) = 0$ do exist! For instance, $\varphi(z) := -\sqrt{-\log|z|}$ has an isolated singularity with a zero Lelong number at the origin. These singularities, very different to analytic ones, are usually hard to grasp as the familiar tools at hand intended to deal with singularities, viz. multiplier ideal sheaves and Lelong numbers, are trivial at such points.

We can alter the Demailly approximation to get the following Monge-Ampère mass control.

Theorem 4.1. *Let φ be a psh function on a bounded pseudoconvex open set $\Omega \subset \mathbb{C}^n$. Suppose, furthermore, that φ has a zero Lelong number at every point $x \in \Omega$. Define the sequence of smooth psh functions $(\psi_m)_{m\in\mathbb{N}}$ on Ω as:*

$$\psi_m := \frac{1}{2m} \log \left(\sum_{j=0}^{+\infty} |\sigma_{m,j}|^2 + \sum_{j=0}^{+\infty} \left| \frac{\partial \sigma_{m,j}}{\partial z_1} \right|^2 + \cdots + \sum_{j=0}^{+\infty} \left| \frac{\partial \sigma_{m,j}}{\partial z_n} \right|^2 \right),$$

where $(\sigma_{m,j})_{j\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}_\Omega(m\varphi)$, and $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}$ are the first order partial derivarives with respect to the standard coordinates $z = (z_1, \ldots, z_n)$ on $\mathbb{C}^n$. Then $dd^c\psi_m$ converges to $dd^c\varphi$ in the weak topology of currents as $m \to +\infty$, and for any relatively compact open subset $B \Subset \Omega$ we have:

$$\int_B (dd^c\psi_m)^k \wedge \beta^{n-k} \leq C\, (\log m)^k, \quad k = 1, \ldots, n,$$

where β is the standard Kähler form on $\mathbb{C}^n$, and $C > 0$ is a constant independent of m.

Proof. The sum $\sum |\sigma_{m,j}(z)|^2$ is the square of the norm of the evaluation linear map $f \mapsto f(z)$ on $\mathcal{H}_\Omega(m\varphi)$. For any $k = 1, \ldots, n$, the analogous sum with $\sigma_{m,j}$ replaced by $\partial\sigma_{m,j}/\partial z_k$ is the square of the norm of the evaluation linear map $f \mapsto \partial f/\partial z_k(z)$. As φ is locally bounded above, the L^2 topology of $\mathcal{H}_\Omega(m\varphi)$ is stronger than the topology of uniform convergence on compact subsets of Ω. Hence, all the series defining ψ_m converge uniformly on compact subsets of Ω and their sums are real analytic. We can easily infer from Demailly's estimate (8) combined with Parseval's formula that:

$$\varphi(z) - \frac{C_1}{m} \leq \psi_m(z) \leq \sup_{|\zeta - z| < 2r} \varphi(\zeta) - \frac{1}{m} \log r + \frac{1}{m} \log \frac{C_3}{r^n}, \tag{10}$$

for every $z \in \Omega$ and every $r < \frac{1}{2} d(z, \partial\Omega)$. This means that ψ_m still converges to φ pointwise and in $L^1_{loc}(\Omega)$ topology, and thus $dd^c\psi_m$ converges to $dd^c\varphi$ in the weak topology of currents. Moreover, as the Lelong numbers of φ are assumed to vanish at every point, and as, thanks to (9),

$$\nu(\psi_m, x) \leq \nu(\varphi_m, x) \leq \nu(\varphi, x), \quad x \in \Omega,$$

for every m, each ψ_m has zero Lelong numbers everywhere. This means that the $\sigma_{m,j}$'s and their first order derivatives have no common zeroes, and therefore ψ_m is C^∞ on Ω. (Actually φ_m is also C^∞).

Our aim is to control the Monge-Ampère masses of the new regularizing smooth forms $dd^c\psi_m$ on a given open set $B \Subset \Omega$. To this end, we can apply the Chern-Levine-Nirenberg inequalities (see [**CLN69**] or [**Dem97**, Chapter III, p. 168]) to get:

$$\int_B (dd^c\psi_m)^k \wedge \beta^{n-k} \leq C\,(\sup_{\tilde{B}} |\psi_m|)^k, \quad k = 1, \ldots, n,$$

where $\tilde{B} \Subset \Omega$ is an arbitrary relatively compact open subset containing $\bar{B}$, and $C > 0$ is a constant depending only on B and $\tilde{B}$. Note that $\sup_{\tilde{B}} |\psi_m| < +\infty$ since ψ_m is smooth. We are thus reduced to accounting for the following:

Claim 4.2. There is a constant $C > 0$ independent of m such that:

$$\sup_{\tilde{B}} |\psi_m| \leq C \log m, \quad \text{for every } m.$$

The upper bound for ψ_m given in (10) is clearly sufficient for our purposes. The delicate point in estimating $|\psi_m|$ is finding a finite lower bound (possibly greatly negative) for ψ_m. If $\bar{B}_m(1)$ is the closed unit ball of $\mathcal{H}_\Omega(m\varphi)$, it is easy to see, expressing the norms of the evaluation linear maps

$$\mathcal{H}_\Omega(m\varphi) \ni f \mapsto \frac{\partial f}{\partial z_k}(z) \in \mathbb{C}, \quad k = 1, \ldots, n,$$

at a given point $z \in \Omega$ in two ways and the fact that a sum of supremums dominates the supremum of the sum, that:

(11)

$$\psi_m(z) \geq \sup_{F_m \in \bar{B}_m(1)} \frac{1}{2m} \log\left(|F_m(z)|^2 + \left|\frac{\partial F_m}{\partial z_1}(z)\right|^2 + \cdots + \left|\frac{\partial F_m}{\partial z_n}(z)\right|^2\right),$$

for every $z \in \Omega$. Now fix $x \in \Omega$. To find a uniform lower bound for $\psi_m(x)$, we need to produce an element $F_m \in \bar{B}_m(1)$ for which we can uniformly estimate below one of the first order partial derivatives at x. The Lelong number of φ at x is known to be equal to the Lelong number at x of the restriction $\varphi_{|L}$ to almost every complex line L passing through x (see [**Siu74**]). Choose such a line L and coordinates $z = (z_1, \ldots, z_n)$ centred at x such that $L = \{z_2 = \cdots = z_n = 0\}$. Consider, as in the introduction, the decompostion:

$$\varphi_{|L} = N \star \Delta\varphi_{|L} + h, \quad \text{on } \Omega \cap L,$$

where N is the one-dimensional Newton kernel, and $h = \operatorname{Re} g$ is a harmonic function equal to the real part of a holomorphic function g. Theorem 1.1 gives the existence of a holomorphic function f_m on $\Omega \cap L$ such that:

$$f_m(z_1) = e^{mg(z_1)} \prod_{j=1}^{N_m} (z_1 - a_{m,j}), \quad z_1 \in \Omega \cap L,$$

with $N_m \leq C_0\, m$, for a constant $C_0 > 0$ independent of m, and

$$C_m := \int\limits_{\Omega\cap L} |f_m|^2\, e^{-2m\varphi}\, dV_L = o(m),$$

where dV_L is the volume form on L. As the Lelong numbers of φ (and implicitly those of $m\varphi$) are assumed to be zero, the restriction of $e^{-2m\varphi}$ to some complex line L is locally integrable on $\Omega \cap L$. By Theorem 1.1 (ii) the points $a_{m,\,j}$ can be chosen such that $|a_{m,\,j} - a_{m,\,k}| \geq \frac{C_1}{m^2}$ for every $j \neq k$, with some constant $C_1 > 0$ independent of m and L.

The Ohsawa-Takegoshi L^2 extension theorem (cf. [**Ohs88**, Corollary 2, p. 266]) can now be applied to get a holomorphic extension $F_m \in \mathcal{H}_\Omega(m\varphi)$ of f_m from the line $\Omega \cap L$ to Ω, satisfying the estimate:

$$\int\limits_{\Omega} |F_m|^2\, e^{-2m\varphi}\, dV_n \leq C \int\limits_{\Omega\cap L} |f_m|^2\, e^{-2m\varphi}\, dV_L = C\, C_m,$$

for a constant $C > 0$ depending only on Ω and n. Thus the function $\frac{F_m}{\sqrt{C\, C_m}}$ belongs to the unit ball $\bar{B}_m(1)$ of the Hilbert space $\mathcal{H}_\Omega(m\varphi)$. As $\frac{\partial F_m}{\partial z_1}(z_1) = f_m'(z_1)$ at every point $z_1 \in \Omega \cap L$, estimate (11) implies the following lower bound for ψ_m:

$$\psi_m(z_1) \geq \frac{1}{m} \log|f_m'(z_1)| - \frac{1}{2m}\log(C\, C_m), \quad z_1 \in \tilde{B} \cap L.$$

Recall that we are interested in estimating ψ_m below at the origin x of the local coordinate system $(z_1, \dots, z_n)$. This is trivial if the points $a_{m,j}$ do not contain the origin. If $x = a_{m,\,j}$ for some j, we get:

$$\begin{aligned}\psi_m(a_{m,j}) &\geq h(a_{m,j}) + \frac{1}{m}\sum_{k\neq j} \log|a_{m,\,k} - a_{m,\,j}| - \frac{1}{2m}\log(C\, C_m)\\ &\geq h(a_{m,j}) + \frac{N_m - 1}{m}\, \log\frac{C_1}{m^2} - \frac{1}{2m}\log(C\, C_m).\end{aligned}$$

Since h is C^∞ (for it is harmonic), it is locally bounded (by constants independent of L). Therefore, there exists a constant $C_2 > 0$ independent of m and L such that $\psi_m \geq -C_2 \log m$ on $\tilde{B} \cap L$ for every m. In particular, $\psi_m(x) \geq -C_2 \log m$. This proves Claim 4.2 and completes the proof of Theorem 4.1. q.e.d.

Remark 4.3. Theorem 4.1 constructs a regularization of currents for which the Monge-Ampère masses have an at most slow (logarithmic) growth. It is worth stressing that the sequence of these masses may not be bounded above. To see this, suppose φ is C^∞ in the complement of an analytic set $V \subset \Omega$. If $(\varphi_m)_{m\in\mathbb{N}}$ is the Demailly approximation of φ, it is shown in [**DPS01**, pp. 701-702] that the sequence $(\varphi_{2^m} + 2^{-m})_{m\in\mathbb{N}}$ is decreasing using an effective version of the subadditivity property of multiplier ideal sheaves. The same proof shows that the corresponding sequence $(\psi_{2^m}+2^{-m})_{m\in\mathbb{N}}$ in the new regularization defined in the previous theorem is also decreasing. Then the C^∞ (1, 1)-forms $(dd^c\psi_{2^m})^k$ are well-defined on $\Omega\setminus V$ and converge in the weak topology of currents to $(dd^c\varphi)^k$ for every $k=1,\dots,n$ (see [**Dem97**, Chapter III, Theorem 3.7]). If $K\Subset B\setminus V$ and $0\leq\chi\leq 1$ is a C^∞ function with compact support in $B\setminus V$ such that $\chi\equiv 1$ on K, then:

$$\int_{B\setminus V}\chi(dd^c\psi_{2^m})^k\wedge\beta^{n-k}\leq\int_{B\setminus V}(dd^c\psi_{2^m})^k\wedge\beta^{n-k},\quad m\in\mathbb{N},$$

and taking $\liminf\limits_{m\to+\infty}$, the weak convergence implies:

$$\int_{B\setminus V}\chi(dd^c\varphi)^k\wedge\beta^{n-k}\leq\liminf_{m\to+\infty}\int_{B\setminus V}(dd^c\psi_{2^m})^k\wedge\beta^{n-k},\quad k=1,\dots,n.$$

Clearly $\int_K(dd^c\varphi)^k\wedge\beta^{n-k}\leq\int_{B\setminus V}\chi(dd^c\varphi)^k\wedge\beta^{n-k}$, and letting $K\Subset B\setminus V$ increase, we get:

$$\int_{B\setminus V}(dd^c\varphi)^k\wedge\beta^{n-k}\leq\liminf_{m\to+\infty}\int_{B\setminus V}(dd^c\psi_{2^m})^k\wedge\beta^{n-k},\quad k=1,\dots,n.$$

Now, there are examples of psh functions φ for which the Monge-Ampère mass on the left-hand side above is infinite for $k=n$ (see Kiselman's example in [**Kis84**, pp. 141–143] of a φ with zero Lelong numbers, or the Shiffman-Taylor example in [**Siu75**, pp. 451–453]). Thus the last inequality shows that for such functions the sequence of Monge-Ampère masses associated with the above regularization is unbounded.

5. Special case of global regularization with mass control

In this section we patch together the local regularizations constructed in the previous section to prove Theorem 1.2 under the extra assumption that the original current T has vanishing Lelong numbers everywhere. For the sake of simplicity we assume X to be compact. The result actually holds for any manifold X that can be covered by finitely many coordinate patches on which the local regularization Theorem 4.1 can be applied.

Theorem 5.1. *Let $T\geq\gamma$ be a d-closed current of bidegree* (1, 1) *on a compact complex manifold X, where γ is a continuous* (1, 1)*-form such that $d\gamma=0$. Assume T has a zero Lelong number at every point in X. Then, there exist C^1* (1, 1)*-forms T_m in the same $\partial\bar\partial$-cohomology class as T which converge to T in the weak topology of currents and satisfy:*

(a) $T_m\geq\gamma-\dfrac{C}{m}\,\omega$*;*

(b) $\int\limits_X(T_m-\gamma+\frac{C}{m}\,\omega)^q\wedge\omega^{n-q}\leq C\,(\log m)^q,\qquad q=1,\dots,n=\dim_{\mathbb{C}}X,$

for a fixed Hermitian metric ω on X and some $C>0$ independent of m.

Proof. As the patching procedure is essentially well-known (see, for instance, [**Dem92**, Section 3]), we will only point out the new aspects.

We always have $T = \alpha + dd^c\varphi \geq \gamma$ globally on X, with some C^∞ (1, 1)-form α and some almost psh function φ on X. The set-up is the one described in the introduction. After possibly replacing T with $T-\alpha$ and γ with $\gamma - \alpha$, we can assume $T = dd^c\varphi \geq \gamma$. Let us fix $\delta > 0$, and four finite coverings of X by concentric coordinate balls $(B_j^{(3)})_j$, $(B'_j)_j$, $(B''_j)_j$ and $(B_j)_j$ of radii $\frac{\delta}{2}$, δ, $\frac{3}{2}\delta$, and respectively 2δ. Since $d\gamma = 0$, γ is locally exact and we can assume that, for every j, $\gamma = dd^c h_j$ on B_j for some C^1 function h_j. The function:

$$\psi_j := \varphi - h_j,$$

is psh on B_j for every j. As, by assumption, ψ_j has a zero Lelong number at every point in B_j, Theorem 4.1 can be applied to each ψ_j on B_j to get smooth approximations:

$$\psi_{j,m} := \frac{1}{2m} \log\Bigg(\sum_{l=0}^{+\infty} |\sigma_{j,m,l}|^2 + \sum_{r=1}^{n}\sum_{l=0}^{+\infty}\left|\frac{\partial\sigma_{j,m,l}}{\partial z_r}\right|^2\Bigg),$$

with an arbitrary orthonormal basis $(\sigma_{j,m,l})_{l\in\mathbb{N}}$ of the Hilbert space $\mathcal{H}_{B_j}(m\psi_j)$ (see notation in the previous section). Then $\varphi_{j,m} := \psi_{j,m} + h_j$ converges pointwise and in L^1_{loc} topology to φ as $m \to +\infty$ on B_j, and these local approximations can be glued together into a global approximation of φ defined as:

$$\varphi_m(z) := \sup_{B''_j \ni z}\Bigg(\varphi_{j,m}(z) + \frac{C_1(\delta)}{m}\,(\delta^2 - |z^j|^2)\Bigg),$$

with a constant $C_1(\delta) > 0$ depending only on δ which will be specified below, and a local holomorphic coordinate system z^j centred at the centre of B_j. The currents $T_m := dd^c\varphi_m$ satisfy the conclusions of Theorem 5.1 if the following patching condition holds:

$$\varphi_{j,m}(z) + \frac{C_1(\delta)}{m}\,(\delta^2 - |z^j|^2) \leq \varphi_{k,m}(z) + \frac{C_1(\delta)}{m}\,(\delta^2 - |z^k|^2), \tag{12}$$

for $z \in (\overline{B}''_j \setminus B'_j) \cap B_k^{(3)}$. One can then prove the existence of a constant $C_1(\delta) > 0$ satisfying this patching condition by means of Hörmander's L^2 estimates ([**Hor65**]). One need only estimate the difference $\psi_{j,m} - \psi_{k,m}$ on $B''_j \cap B''_k$ and show that $\varphi_{j,m} - \varphi_{k,m}$ is uniformly bounded above on $B''_j \cap B''_k$ by $O(\frac{1}{m})$ as $m \to +\infty$. Now, for every fixed $z \in B_j$, the norms of the linear maps $f \mapsto f(z)$ and $f \mapsto \frac{\partial f}{\partial z_r}(z)$, $r = 1, \ldots, n$, defined on the Hilbert space $\mathcal{H}_{B_j}(m\psi_j)$, can be expressed in terms of an orthonormal basis, and we get:

$$\frac{1}{2m}\log\sup_{f\in\overline{B}_{j,m}}\Bigg(|f(z)|^2 + \sum_{r=1}^{n}\left|\frac{\partial f}{\partial z_r}(z)\right|^2\Bigg) \leq \psi_{j,m}(z)$$

$$\leq \frac{1}{2m} \log \left((n+1) \sup_{f \in \overline{B}_{j,m}} \left(|f(z)|^2 + \sum_{r=1}^{n} \left| \frac{\partial f}{\partial z_r}(z) \right|^2 \right) \right),$$

where $\overline{B}_{j,m}$ is the unit ball of $\mathcal{H}_{B_j}(m\psi_j)$. We also have the analogous relations for $\psi_{k,m}$ on B_k. This means that to compare $\psi_{j,m}$ and $\psi_{k,m}$ at a fixed point $x_0 \in B''_j \cap B''_k$, it is enough to show that for every holomorphic function f_j on B_j such that $\int_{B_j} |f_j|^2 e^{-2m\psi_j} = 1$, there exists a holomorphic function f_k on B_k having an L^2-norm under control and satisfying:

$$f_k(x_0) = f_j(x_0), \quad \text{and} \quad \frac{\partial f_k}{\partial z_r}(x_0) = \frac{\partial f_j}{\partial z_r}(x_0), \quad \text{for } r = 1, \dots, n.$$

This is done using Hörmander's L^2 estimates ([**Hor65**]). Let θ be a cut-off function supported in a neighborhood of x_0 such that $\theta \equiv 1$ near x_0, and solve the equation

$$\bar{\partial} g = \bar{\partial}(\theta f_j) \tag{13}$$

on B_k with a weight containing the term $2(n+1)\log|z - x_0|$ which forces the solution g to vanish to order at least 2 at x_0. Specifically, if h_{jk} is a holomorphic function on $B_j \cup B_k$ such that $h_j - h_k = \operatorname{Re} h_{jk}$ on $B_j \cap B_k$, we can find a solution g to the above equation on B_k satisfying Hörmander's L^2 estimates with the strictly psh weight:

$$2m(\psi_k - \operatorname{Re} h_{jk}) + 2(n+1)\log|z - x_0| + |z - x_0|^2. \tag{14}$$

Now set $f_k := \theta f_j - g$, which is easily seen to satisfy the requirements. The precise estimate of the solution g gives the uniform upper estimate of $\varphi_{j,m} - \varphi_{k,m}$ on $B''_j \cap B''_k$ by $O(\frac{1}{m})$, which implies the existence of a constant $C_1(\delta) > 0$ satisfying the patching condition (12). The details are left to the reader.

The loss of positivity incurred in T_m with respect to the original T can be seen to be at most $\frac{C}{m}$ as in [**Pop04**] thanks to the form γ being closed. This proves (a). That the approximating currents $T_m := dd^c\varphi_m$ constructed through this patching procedure satisfy the condition (b) on Monge-Ampère masses follows from the local Theorem 4.1 proved in the previous section. Theorem 5.1 is thus proved. q.e.d.

6. Modified regularizations in the general case

Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex open set, and let φ be a strictly psh function on Ω such that $i\partial\bar{\partial}\varphi \geq C_0\,\beta$ for some constant $C_0 > 0$ and the standard Kähler form β on $\mathbb{C}^n$. Our goal is to construct regularizing psh functions with analytic singularities approximating φ for which the Monge-Ampère masses can be controlled. The overall idea is to modify Demailly's regularization (7) by adding derivatives of the functions $\sigma_{m,j}, j \in \mathbb{N}$, which form an orthonormal basis of the Hilbert space:

(15)

$$\mathcal{H}_\Omega(m\varphi) := \left\{ f \in \mathcal{O}(\Omega);\ \int_\Omega |f|^2 e^{-2m\varphi} dV_n < +\infty \right\}, \quad dV_n := \frac{\beta^n}{n!}.$$

We shall repeatedly denote D^α the derivation operator with respect to any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ of length $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and [ ] the integer part. Unlike Section 4, where it was enough to derive to order one as φ was assumed to have zero Lelong numbers everywhere, we need to derive more in the general case. Yet we cannot afford to derive too much, and have to accept singularities in the regularizing functions as shown below.

Lemma 6.1. *For every $\delta > 0$, the psh functions with analytic singularities defined as:*

$$\varphi_m^\delta = \frac{1}{2m} \log \sum_{j=0}^{+\infty} \sum_{|\alpha|=0}^{[\delta m]} \left| \frac{D^\alpha \sigma_{m,j}}{\alpha!} \right|^2, \quad m \in \mathbb{N},$$

satisfy the estimates below for constants $C_1, C_3 > 0$ independent of m and φ:

$$\varphi(z) - \frac{C_1}{m} \le \varphi_m^\delta(z) \le \sup_{|\zeta - z| < 2r} \varphi(\zeta) - \left(\frac{[\delta m]}{m} + \frac{n}{m} \right) \log r + \frac{1}{m} \log C_3,$$

at every point $z \in \Omega$ and for every $0 \le r < \min\{\frac{1}{2} d(z, \partial\Omega), 1\}$. In particular, if we choose $\delta_m := C\,\varepsilon_m$ with $\varepsilon_m \downarrow 0$ and $C > 0$ a constant independent of m, $\varphi_m^{\delta_m}$ converges pointwise and in L^1_{loc} topology to φ when $m \to +\infty$.

Proof. The lower bound follows from the lower estimate in (8) since $\varphi_m^\delta \ge \varphi_m$. To get the upper bound, we apply Parseval's formula to each function $\sigma_{m,j}$ on the sphere $S(z, r)$ and then sum over j to get:

$$\begin{aligned} \frac{C}{r^{2n-1}} \int_{S(z,r)} \sum_{j=0}^{+\infty} |\sigma_{m,j}(\zeta)|^2 \, d\sigma(\zeta) &= \sum_{j=0}^{+\infty} \sum_{\alpha \in \mathbb{N}^n} \left| \frac{D^\alpha \sigma_{m,j}}{\alpha!}(z) \right|^2 r^{2|\alpha|} \\ &\ge r^{2[\delta m]} \sum_{j=0}^{+\infty} \sum_{|\alpha|=0}^{[\delta m]} \left| \frac{D^\alpha \sigma_{m,j}}{\alpha!}(z) \right|^2. \end{aligned}$$

It is now enough to take $\frac{1}{2m}$ log on both sides and to use the upper estimate in (8) to conclude. q.e.d.

In other words, we still have an approximation of φ if we derive the $\sigma_{m,j}$'s up to order $\le o(m)$ (e.g. $[C\varepsilon_m m]$ for some $\varepsilon_m \downarrow 0$, even when $[C\varepsilon_m m] \to +\infty$). But we would lose control on the upper bound of

φ_m^δ if we derived up to removing all the common zeroes of the $\sigma_{m,\,j}$'s (e.g., up to order $[Am]$ with $A := \sup_{x\in\Omega} \nu(\varphi,\, x)$ assumed to be finite after possibly shrinking Ω).

Besides taking derivatives, we will further modify Demailly's regularizing functions (7). A crucial role will be played by multiplier ideal sheaves $\mathcal{I}(m\varphi) \subset \mathcal{O}_\Omega$ associated with psh functions. They are defined, for every $m \in \mathbb{N}$, as (cf. [**Nad90**], [**Dem93**]):
(16)

$$\mathcal{I}(m\varphi)_x := \{f \in \mathcal{O}_{\Omega,\,x},\ |f|^2\, e^{-2m\varphi} \ \text{ is Lebesgue-integrable near } \ x\},$$

at every $x \in \Omega$. It is a well-known result of Nadel ([**Nad90**], see also [**Dem93**]) that, for every $m \in \mathbb{N}$, the multiplier ideal sheaf $\mathcal{I}(m\varphi)$ is coherent and is generated as an $\mathcal{O}_\Omega$-module by an arbitrary orthonormal basis $(\sigma_{m,\,j})_{j\in\mathbb{N}}$ of the Hilbert space $\mathcal{H}_\Omega(m\varphi)$. By coherence, the restriction of $\mathcal{I}(m\varphi)$ to every compact subset has only finitely many generators $(\sigma_{m,\,j})_{1\leq j\leq N_m}$. When $m \to +\infty$, this local finite generation property was made effective in [**Pop06**, Theorem 1.1] in the following way. It was shown there that for any given relatively compact open subset $B \Subset \Omega$, there exist a subset $B_0 \Subset B$ and $m_0 = m_0(C_0) \in \mathbb{N}$ such that for every $m \geq m_0$ one can find an orthonormal basis $(\sigma_{m,\,j})_{j\in\mathbb{N}}$ of $\mathcal{H}_\Omega(m\varphi)$ and finitely many elements $\sigma_{m,\,1}, \ldots, \sigma_{m,\,N_m}$ in it with the following property. Every local section $g \in \mathcal{H}_B(m\varphi)$ admits a decomposition:

$$g(z) = \sum_{j=1}^{N_m} h_{m,\,j}(z)\, \sigma_{m,\,j}(z), \qquad z \in B_0, \tag{17}$$

with some holomorphic functions $h_{m,\,j}$ on B_0 satisfying:

$$\sup_{B_0} \sum_{j=1}^{N_m} |h_{m,\,j}|^2 \leq C\, N_m \int_B |g|^2\, e^{-2m\varphi} < +\infty, \tag{18}$$

for a constant $C > 0$ depending only on n, r, and the diameter of Ω. Furthermore, in the case of a φ with analytic singularities, the growth of the number N_m of local generators needed is at most polynomial of degree n as $m \to +\infty$, namely

$$\lim_{m\to+\infty} \frac{n!}{m^n}\, N_m = \frac{2^n}{\pi^n} \int_B (i\partial\bar\partial\varphi)^n_{ac} < +\infty, \tag{19}$$

where $(i\partial\bar\partial\varphi)_{ac}$ denotes the absolutely continuous part of $i\partial\bar\partial\varphi$ in the Lebesgue decomposition of its measure coefficients into an absolutely continuous and singular part with respect to the Lebesgue measure. An immediate corollary of formula (17) and estimate (18) is the following comparison relation:

$$\sum_{j=0}^{+\infty}|\sigma_{m,\,j}|^2 \leq C\,N_m\sum_{j=1}^{N_m}|\sigma_{m,\,j}|^2 \quad \text{on} \;\; B_0. \tag{20}$$

Building on these results from [**Pop06**], we can define a new regularization of φ using only finitely many generators of $\mathcal{I}(m\varphi)_{|B_0}$ to modify the definition in the previous Lemma 6.1.

Lemma 6.2. *The psh functions with analytic singularities:*

$$\psi_m = \frac{1}{2m}\log\sum_{j=1}^{N_m}\sum_{|\alpha|=0}^{[C\varepsilon_m m]}\left|\frac{D^\alpha\sigma_{m,\,j}}{\alpha!}\right|^2, \quad m\in\mathbb{N},$$

satisfy the estimates below for constants $C_1, C_3>0$ independent of m and φ:

$$\begin{aligned}\varphi(z)-\frac{\log N_m + C_1}{2m} &\leq \psi_m(z)\\ &\leq \sup_{|\zeta-z|<2r}\varphi(\zeta)-\frac{[C\varepsilon_m m]+n}{m}\,\log r+\frac{1}{m}\log C_3,\end{aligned}$$

at every point $z\in B_0$ and for every $0\leq r<\min\{\frac{1}{2}\,\mathrm{dist}\,(z,\,\partial\Omega),\,1\}$. In particular, if $\varepsilon_m\downarrow 0$ and $C>0$ is a constant independent of m, ψ_m converges pointwise on B_0 and in $L^1_{\mathrm{loc}}(B_0)$ topology to φ when $m\to+\infty$.

Proof. It follows immediately from the above considerations. q.e.d.

This regularization is not yet satisfactory from the point of view of the Monge-Ampère masses as ψ_m still has singularities and the Chern-Levine-Nirenberg inequalities cannot be applied as such (see Section 4 where they were applied to smooth psh functions). Building on ideas in this section, we will construct a new regularization of φ in subsequent sections for which we can control the Monge-Ampère masses.

7. Additivity defect of multiplier ideal sheaves

Throughout this section we shall suppose that φ is a psh function with analytic singularities on $\Omega\Subset\mathbb{C}^n$ of the form:

$$\varphi=\frac{c}{2}\,\log(|g_1|^2+\cdots+|g_N|^2)+v, \tag{21}$$

for some (possibly infinitely many) holomorphic functions $g_1,\ldots,g_N\in\mathcal{O}(\Omega)$, some constant $c>0$, and some C^∞ function v on Ω. We also assume that $i\partial\bar\partial\varphi\geq C_0\,\beta$ for some constant $C_0>0$ and the standard Kähler form β on $\mathbb{C}^n$. The results in this section will be subsequently applied to φ_p in place of φ, where $(\varphi_p)_{p\in\mathbb{N}}$ are the Demailly regularizations (7) of an arbitrary psh function φ. We work with general functions of the form (21) in this section for the sake of generality.

Fix now a relatively compact pseudoconvex open subset $B \Subset \Omega$. Our aim is to find better regularizations of φ for which we can control the Monge-Ampère masses. In so doing, we will still use the ideal sheaves $\mathcal{I}(m\varphi)$, but the main idea underlying the argument is to take $m = m_0 q$ with $m \gg m_0$ (though $m_0 \to +\infty$) and to reduce the study of $\mathcal{I}(m\varphi)$ to the study of $\mathcal{I}(m_0\varphi)$. We shall then get a grip on $\mathcal{I}(m\varphi)$ as being "almost" equal to $\mathcal{I}(m_0\varphi)^q$ up to an error that we are able to estimate. This is made precise in the following.

Proposition 7.1. *For any $\varepsilon > 0$, any $m_0 \geq \frac{n+2}{c\varepsilon}$, any $q \in \mathbb{N}$, and any $B \Subset \Omega$, the following inclusions of multiplier ideal sheaves hold:*

$$\mathcal{I}(m_0(1+\varepsilon)\varphi)^q_{|B} \subset \mathcal{I}(m_0 q\varphi)_{|B} \subset \mathcal{I}(m_0\varphi)^q_{|B}. \tag{22}$$

The right-hand inclusion actually holds on Ω for every m_0 and is the subadditivity property of multiplier ideal sheaves proved by Demailly, Ein and Lazarsfeld in [**DEL00**]. It relies on the Ohsawa-Takegoshi L^2 extension theorem ([**OT87**]). The left-hand inclusion was proved in [**Pop06**] using Skoda's L^2 division theorem ([**Sko72b**]). It can be seen as measuring the extent to which multiplier ideal sheaves fail to have an additive growth. These inclusions can be given effective versions, with estimates, that we are now undertaking to make explicit.

Consistent with the notation in (15), consider the Hilbert spaces:

$$\mathcal{H}_\Omega(m_0(1+\varepsilon)\,\varphi), \quad \mathcal{H}_\Omega(m_0 q\,\varphi), \quad \text{and} \quad \mathcal{H}_\Omega(m_0\,\varphi),$$

and respective orthonormal bases:

$$(\sigma_{m_0(1+\varepsilon),\,j})_{j\in\mathbb{N}}, \quad (\sigma_{m_0 q,\,j})_{j\in\mathbb{N}}, \quad \text{and} \quad (\sigma_{m_0,\,j})_{j\in\mathbb{N}}.$$

The multiplier ideal sheaves in (22) are generated as $\mathcal{O}_\Omega$-modules by the above Hilbert space orthonormal bases respectively. Since they are coherent, their restrictions to the relatively compact subset B are finitely generated. Possibly after reordering, we can therefore assume that they are generated as follows:

$$\mathcal{I}(m_0(1+\varepsilon)\,\varphi)_{|B} = (\sigma_{m_0(1+\varepsilon),\,j})_{1\leq j\leq N_{m_0(1+\varepsilon)}},$$

$$\mathcal{I}(m_0 q\,\varphi)_{|B} = (\sigma_{m_0 q,\,j})_{1\leq j\leq N_{m_0 q}}, \quad \mathcal{I}(m_0\,\varphi)_{|B} = (\sigma_{m_0,\,j})_{1\leq j\leq N_{m_0}}.$$

The results of [**Pop06**], summarized in the previous section as (17)–(20), will be made an essential use of in all that follows. We start by stating the following effective versions of the left-hand (cf. (a)) and right-hand (cf. (b)) inclusions in (22) when derivatives are also taken into account.

Proposition 7.2.

(a) *Let φ be a psh function on $\Omega \Subset \mathbb{C}^n$ of the form* (21). *Given $B \Subset \Omega$, there exist an open subset $B_0 \Subset B$ and an orthonormal basis $(\sigma_{m_0q,\,j})_{j\in\mathbb{N}}$ of $\mathcal{H}_\Omega(m_0q\,\varphi)$ such that $\sigma_{m_0q,\,j}$, $j = 1, \ldots, N_{m_0q}$, generate the $\mathcal{O}_\Omega$-module $\mathcal{I}(m_0q\,\varphi)$ on B_0 and the following estimate holds at every $z \in B_0$ for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and every orthonormal basis $(\sigma_{m_0(1+\varepsilon),\,j})_{j\in\mathbb{N}}$ of $\mathcal{H}_\Omega(m_0(1+\varepsilon)\varphi)$:*

$$\begin{aligned}(23)\qquad &\sum_{j_1,\ldots,j_q=0}^{+\infty} |D^\alpha(\sigma_{m_0(1+\varepsilon),\,j_1} \ldots \sigma_{m_0(1+\varepsilon),\,j_q})|^2 \\ &\leq C_\alpha\, N_{m_0q}^2\, C_{m_0}^q \sum_{j=1}^{N_{m_0q}} \sum_{\beta\leq\alpha} |D^\beta \sigma_{m_0q,\,j}|^2,\end{aligned}$$

for any $q \in \mathbb{N}$, any $0 < \varepsilon << 1$, and any $m_0 \geq \frac{n+2}{c\,\varepsilon}$. The constants:

$$0 < C_\alpha \leq O\Bigg(|\alpha|^2 \sum_{\beta\leq\alpha} \binom{\alpha}{\beta}^2\Bigg),$$
$$C_{m_0} := C_n(m_0\,c\,(1+\varepsilon) - n)\,(\sup_B e^{\varphi})^{2m_0\varepsilon}$$

are such that C_α depends only on α, n, B_0, B, and $C_n > 0$ depends only on n, B, Ω;

(b) *For any orthonormal bases of $\mathcal{H}_\Omega(m_0q\varphi)$ and $\mathcal{H}_\Omega(m_0\varphi)$, we have:*

$$(24)\quad \sum_{j=0}^{+\infty} |D^\alpha \sigma_{m_0q,\,j}|^2 \leq C_n^{q-1} \sum_{j_1,\ldots,j_q=0}^{+\infty} |D^\alpha(\sigma_{m_0,\,j_1} \ldots \sigma_{m_0,\,j_q})|^2 \quad \text{on } \Omega,$$

for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, with a constant $C_n > 0$ depending only on n and Ω.

Proof.

(b) Estimate (24) follows from the effective version of the subadditivity property of multiplier ideal sheaves established in [**DEL00**] (see also [**DPS01**, Proof of Theorem 2.2.1.]). Indeed, for every $f \in \mathcal{O}(\Omega)$ satisfying

$$\int_\Omega |f|^2\, e^{-2q\,(m_0\,\varphi)}\, dV_n = 1,$$

Step 3 in the proof of Theorem 2.2.1. in [**DPS01**], when applied in this context, shows by means of the Ohsawa-Takegoshi L^2 extension theorem applied $q-1$ times that, at every point $z \in \Omega$, there is a decomposition:

$$(25) \qquad f(z) = \sum_{j_1,\dots,j_q=0}^{+\infty} c_{j_1,\dots,j_q}\, \sigma_{m_0,\,j_1}(z)\dots\sigma_{m_0,\,j_q}(z),$$

$$\sum_{j-1,\dots,j_q=0}^{+\infty} |c_{j_1,\dots,j_q}|^2 \leq C_n^{q-1},$$

with scalar coefficients $c_{j_1,\dots,j_q} \in \mathbb{C}$ satisfying the above estimate for some constant $C_n > 0$ depending only on n and Ω. Applying D^α we get:

$$D^\alpha f(z) = \sum_{j_1,\dots,j_q=0}^{+\infty} c_{j_1,\dots,j_q}\, D^\alpha(\sigma_{m_0,\,j_1}\dots\sigma_{m_0,\,j_q})(z), \qquad z \in \Omega.$$

The Cauchy-Schwarz inequality and estimate (25) of the scalar coefficients $c_{j_1,\dots,j_q}$ give:

$$|D^\alpha f(z)|^2 \leq C_n^{q-1} \sum_{j_1,\dots,j_q=0}^{+\infty} |D^\alpha(\sigma_{m_0,\,j_1}\dots\sigma_{m_0,\,j_q})(z)|^2, \quad z \in \Omega.$$

Taking the supremum over all f in the unit sphere of the Hilbert space $\mathcal{H}_\Omega(m_0 q\varphi)$ gives the desired estimate.

(a) We first briefly recall the use made of Skoda's L^2 division theorem in [**Pop06**, Theorem 4.1] to obtain an effective version of a superadditivity result on multiplier ideal sheaves ([**Pop06**, Theorem 1.2]) corresponding in the present context to the left-hand inclusion in (22). Let $f \in \mathcal{O}(\Omega)$ be an arbitrary element in the unit sphere of the Hilbert space $\mathcal{H}_\Omega(m_0(1+\varepsilon)\,\varphi)$. Combined with assumption (21), this means that:

$$1 = \int_\Omega |f|^2\, e^{-2m_0(1+\varepsilon)\,\varphi}\, dV_n = \int_\Omega \frac{|f|^2}{\left(\sum\limits_{j=0}^{N} |g_j|^2\right)^{m_0 c(1+\varepsilon)}}\, e^{-2m_0(1+\varepsilon)v}\, dV_n.$$

Choose $m_0 \geq \frac{n+2}{c\,\varepsilon}$. We can apply Skoda's L^2 division theorem ([**Sko72b**]) stated as Theorem 10.1 in Appendix 10 to write f as a linear combination with holomorphic coefficients of products of $s := [m_0 c(1+\varepsilon)] - (n+1)$ functions among the g_j's. Namely, for all multi-indices $L = (l_1, \dots, l_s) \in \{1, \dots, N\}^s$, there exist holomorphic functions h_L on Ω such that:

$$f = \sum_L h_L g^L \quad \text{on} \quad \Omega, \qquad \text{with} \quad g^L = g_{l_1}\dots g_{l_s},$$

with precise L^2 estimates on the coefficients h_L (see Theorem 10.1). Combined with the Cauchy-Schwarz inequality for $|f|^2$ and the submean value inequality for each $|h_L|^2$ on $B \Subset \Omega$, these L^2 estimates yield:

$$|f|^2 \leq C'_{m_0}\,(\sum_{j=0}^{N}|g_j|^2)^{[m_0c(1+\varepsilon)]-(n+1)} \quad \text{on } B,$$

with a constant $C'_{m_0} > 0$ whose dependence on m_0 is explicit. Thus:

$$|f|^2\,e^{-2m_0\varphi} \leq C'_{m_0}\,(\sum_{j=0}^{N}|g_j|^2)^{[m_0c(1+\varepsilon)]-(n+1)-m_0c} \quad \text{on } B,$$

and the crucial fact is that the exponent $[m_0c(1+\varepsilon)]-(n+1)-m_0c$ is non-negative by the choice of $m_0 \geq \frac{n+2}{c\,\varepsilon}$. Therefore, the right-hand term above is bounded on B and thus the initial L^2 condition satisfied by f on Ω leads to an L^∞ property on B for a slightly less singular weight (i.e., without $(1+\varepsilon)$ in the exponent). The explicit bound we finally get is:

$$|f|^2\,e^{-2m_0\,\varphi} \leq C_n\,(m_0c(1+\varepsilon)-n)\,(\sup_B e^{\varphi})^{2m_0\varepsilon} := C_{m_0},$$

on $B \Subset \Omega$, where $C_n > 0$ is a constant depending only on n and the diameters of B and Ω. The details can be found in [**Pop06**] (taking in Theorem 4.1. obtained there $c = 1+\varepsilon$, $1-\delta = \frac{1}{1+\varepsilon}$, $m = m_0$). This readily implies that for any q functions $f_1,\dots,f_q$ in the unit sphere of $\mathcal{H}_\Omega(m_0(1+\varepsilon)\,\varphi)$ we have:

$$|f_1\dots f_q|^2\,e^{-2m_0q\,\varphi} \leq C_{m_0}^q \quad \text{on } B,$$

and in particular $f_1\dots f_q$ is a section on B of the ideal sheaf $\mathcal{I}(m_0q\,\varphi)$ with L^2 norm

$$\int_B |f_1\dots f_q|^2\,e^{-2m_0q\,\varphi}\,dV_n \leq \mathrm{Vol}(B)\,C_{m_0}^q, \tag{26}$$

where $\mathrm{Vol}(B)$ stands for the Lebesgue measure of B. We will now apply the effective local finite generation theorem for multiplier ideal sheaves (Theorem 1.1 in [**Pop06**]). That theorem gives the existence of an open subset $B'_0 \Subset B$ and of an orthonormal basis $(\sigma_{m_0q,\,j})_{j\in\mathbb{N}}$ of $\mathcal{H}_\Omega(m_0q\,\varphi)$ such that finitely many elements of this basis, say $\sigma_{m_0q,\,j}$, $1 \leq j \leq N_{m_0q}$, generate $\mathcal{I}(m_0q\,\varphi)$ on B'_0 in such a way that estimates analogous to (17)–(20) hold. When applied to $f_1\dots f_q$ regarded as a section on B of $\mathcal{I}(m_0q\varphi)$ with local generators $\sigma_{m_0q,\,j}, 1 \leq j \leq N_{m_0q}$, (17) reads:

$$f_1(z)\dots f_q(z) = \sum_{j=1}^{N_{m_0q}} h^{(j)}_{j_1,\dots,j_q}(z)\,\sigma_{m_0q,\,j}(z), \qquad z \in B'_0,\ m_0, q \gg 1 \tag{27}$$

with holomorphic coefficients $h^{(j)}_{j_1,\dots,j_q}$ estimated as (cf.(18), (26)):

$$\begin{aligned}\sup_{B_0'}\sum_{j=1}^{N_{m_0q}}|h^{(j)}_{j_1,\dots,j_q}|^2 &\leq C\,N_{m_0q}\int\limits_B |f_1\dots f_q|^2\,e^{-2m_0q\,\varphi}\\ &\leq C\,N_{m_0q}\,\mathrm{Vol}(B)\,C^q_{m_0}, \qquad m_0,q\gg 1.\end{aligned}\tag{28}$$

Taking derivatives in identity (27) we get:

$$D^\alpha(f_1\dots f_q)=\sum_{j=1}^{N_{m_0q}}\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}D^{\alpha-\beta}h^{(j)}_{j_1,\dots,j_q}\,D^\beta\sigma_{m_0q,\,j}\quad\text{on } B_0'.\tag{29}$$

Applying twice the Cauchy-Schwarz inequality we get at every point in B_0':
(30)

$$|D^\alpha(f_1\dots f_q)|^2\leq C'_\alpha\sum_{\beta\leq\alpha}\bigg(\sum_{j=1}^{N_{m_0q}}|D^{\alpha-\beta}h^{(j)}_{j_1,\dots,j_q}|^2\bigg)\bigg(\sum_{j=1}^{N_{m_0q}}|D^\beta\sigma_{m_0q,\,j}|^2\bigg),$$

where $C'_\alpha:=\sum\limits_{\beta\leq\alpha}\binom{\alpha}{\beta}^2$. The supremum of the term on the left in the above estimate (30) over all $f_1,\dots,f_q$ ranging over the unit sphere of $\mathcal{H}_\Omega(m_0(1+\varepsilon)\,\varphi)$ equals at every point in Ω the value of the left-hand term in the inequality we intend to prove. The estimate of the coefficients $h^{(j)}_{j_1,\dots,j_q}$ obtained in (28), combined with the Cauchy inequalities estimating $|D^\beta h^{(j)}_{j_1,\dots,j_q}|^2$ above on any $B_0\Subset B_0'$ in terms of $\sup\limits_{B_0'}|h^{(j)}_{j_1,\dots,j_q}|^2$, gives the desired estimate on B_0 . q.e.d.

The main ingredient in the regularization process to be described in the next section will be the following estimate on derivatives relying on Theorem 1.1 combined with the Ohsawa-Takegoshi L^2 extension theorem applied on a complex line. This strategy was already used back in Section 4 when the Lelong numbers of φ were assumed to vanish. The extra difficulty in the general case stems from the fact that in Theorem 1.1 the distances between the points a_j where the Lelong numbers of φ are $\geq\frac{1}{m}$ cannot be estimated when $m\to+\infty$ (they may decrease arbitrarily fast to 0). As the derivatives of the functions f_m constructed in Theorem 1.1 depend on these distances, it follows that we cannot control the growth of these derivatives as $m\to+\infty$ if this procedure is applied to $m\varphi$ to produce sections of $\mathcal{I}(m\varphi)$. The solution we propose to this problem is to choose $m=m_0q$ and to apply this procedure to $m_0(1+\varepsilon)\varphi$ instead in order to construct sections $g_{m_0(1+\varepsilon)}$ of $\mathcal{I}(m_0(1+\varepsilon)\varphi)$. Then the left-hand inclusion in (22) of Proposition 7.1 ensures that $g^q_{m_0(1+\varepsilon)}$ is a section of $\mathcal{I}(m_0q\varphi)=\mathcal{I}(m\varphi)$ and we have a complete control on it and its derivatives by means of the effective estimates of Proposition

7.2. The points a_j featuring in the definition of $g^q_{m_0(1+\varepsilon)}$ are now the same as those of $g_{m_0(1+\varepsilon)}$ (only the exponents m_j are multiplied by q, see notation in Theorem 1.1). Thus their minimum mutual distance δ_{m_0} depends only on m_0 (though in an uncontrollable fashion). To obtain a control of the Monge-Ampère masses in the next section, we shall choose $m = m_0 q$ with $q = q(m_0)$ sufficiently large to neutralize the growth of $\delta_{m_0} \downarrow 0$. The main interest of the next proposition is that it gives a lower bound independent of q.

Proposition 7.3. *Let φ be any psh function on $\Omega \Subset \mathbb{C}^n$ and let $\nu(x) = \nu(\varphi, x)$ denote the Lelong number of φ at any $x \in \Omega$. Then, for every $m_0, q \in \mathbb{N}$ and every $0 < \varepsilon < 1$, any orthonormal basis $(\sigma_{m_0(1+\varepsilon),\, j})_{j\in\mathbb{N}}$ of $\mathcal{H}_\Omega(m_0(1+\varepsilon)\varphi)$ satisfies the following estimate at every $x \in \Omega$:*

$$\frac{1}{2m_0 q} \log \sum_{j_1,\dots,j_q=0}^{+\infty} \sum_{|\alpha|=0}^{d_{m_0 q}(x)} |D^\alpha(\sigma_{m_0(1+\varepsilon),\, j_1} \dots \sigma_{m_0(1+\varepsilon),\, j_q})(x)|^2$$
$$\geq C_0 \log \delta_{m_0},$$

where $d_{m_0 q}(x) := \max\{[m_0 q \nu(x)(1+\varepsilon)],\, 1\}$ and $0 < C_0 \leq C_\Omega \int_\Omega dd^c\varphi \wedge \beta^{n-1}$ with $C_\Omega > 0$ depending only on Ω. The constant $\delta_{m_0} > 0$ depends only on m_0 in a way that we cannot control.

Proof. Let $x \in \Omega$ be an arbitrary point. The case $\nu(x) = 0$ was settled in Section 4. Assume that $\nu(x) > 0$. For every multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, the linear map of evaluation at x:

$$\mathcal{H}_\Omega(m_0(1+\varepsilon)\varphi)\hat{\otimes} \dots \hat{\otimes}\mathcal{H}_\Omega(m_0(1+\varepsilon)\varphi) \ni u \longmapsto D^\alpha u(x) \in \mathbb{C}$$

of the derivatives of elements u in the completed tensor product of the Hilbert space $\mathcal{H}_\Omega(m_0(1+\varepsilon)\varphi)$ by itself q times defines a continuous linear map whose squared norm can be expressed in two ways as:

$$\sum_{j_1,\dots,j_q=0}^{+\infty} |D^\alpha(\sigma_{m_0(1+\varepsilon),\, j_1} \dots \sigma_{m_0(1+\varepsilon),\, j_q})(x)|^2 = \sup_u |D^\alpha u(x)|^2,$$

since $(\sigma_{m_0(1+\varepsilon),\, j_1} \dots \sigma_{m_0(1+\varepsilon),\, j_q})_{j_1,\dots,j_q\in\mathbb{N}}$ defines an orthonormal basis in the completed tensor product Hilbert space, and the supremum on the right is taken over all u in the unit ball of this space. To find the desired lower bound for the term on the right side, we shall produce an element in the unit ball of the completed tensor product Hilbert space $\mathcal{H}_\Omega(m_0(1+\varepsilon)\varphi)\hat{\otimes} \dots \hat{\otimes}\mathcal{H}_\Omega(m_0(1+\varepsilon)\varphi)$ (q times) for which one of the partial derivatives up to order $[A\, m_0(1+\varepsilon)q]$ can be estimated below in absolute value at x. This element will be chosen as the q^{th} power of an element in the unit ball of $\mathcal{H}_\Omega(m_0(1+\varepsilon)\varphi)$.

Let L be a complex line through x such that the restriction of φ to L has the same Lelong number at x as φ. This is the case for almost all lines passing through x ([**Siu74**]). After possibly changing coordinates, we can assume that $x = 0$ and $L = \{z_2 = \cdots = z_n = 0\}$. As in the Introduction, we have a decomposition:

$$\varphi_{|L} = N \star \Delta\varphi_{|L} + \operatorname{Re} g \quad \text{ on } \Omega \cap L,$$

for some holomorphic function g and the one-dimensional Newton kernel N. Theorem 1.1 applied to $(1+\varepsilon)\,\varphi_{|L}$ gives the existence, for every $m_0 \in \mathbb{N}$, of a holomorphic function of one variable $g_{m_0(1+\varepsilon)}$ on $\Omega \cap L$ of the form:

$$g_{m_0(1+\varepsilon)}(z_1) = e^{m_0(1+\varepsilon)\,g(z_1)} \prod_{j=1}^{N_{m_0}} (z - a_j)^{m_j}, \qquad z_1 \in \Omega \cap L,$$

such that $\sum\limits_{j=1}^{N_{m_0}} m_j \leq C_0\,(1+\varepsilon) m_0$ with a constant $C_0 = \int\limits_{\Omega\cap L} dd^c \varphi_{|L} > 0$ independent of m_0. It further satisfies:

$$C_{m_0} := \int_{\Omega\cap L} |g_{m_0(1+\varepsilon)}|^2\, e^{-2m_0(1+\varepsilon)\varphi}\, dV_L = o(m_0),$$

where dV_L is the volume form on L. We can now apply the Ohsawa-Takegoshi L^2 extension theorem ([**Ohs88**, Corollary 2, p. 266]) to get a holomorphic extension $G_{m_0(1+\varepsilon)} \in \mathcal{H}_\Omega(m_0(1+\varepsilon)\varphi)$ of $g_{m_0(1+\varepsilon)}$ from $\Omega \cap L$ to Ω which satisfies the L^2 estimate:

$$\begin{aligned} &\int\limits_{\Omega} |G_{m_0(1+\varepsilon)}|^2\, e^{-2m_0(1+\varepsilon)\varphi}\, dV_n \\ &\leq C_n \int\limits_{\Omega\cap L} |g_{m_0(1+\varepsilon)}|^2\, e^{-2m_0(1+\varepsilon)\varphi}\, dV_L = C_n\, C_{m_0}, \end{aligned}$$

with a constant $C_n > 0$ depending only on n. Thus, $\frac{1}{(C\,C_{m_0})^{1/2}}\, G_{m_0(1+\varepsilon)}$ belongs to the unit ball of $\mathcal{H}_\Omega(m_0(1+\varepsilon)\varphi)$. Then the holomorphic function defined on Ω as:

$$F_{m_0 q} := \frac{1}{(C_n\, C_{m_0})^{q/2}}\, G^q_{m_0(1+\varepsilon)}$$

belongs to the unit ball of the completed tensor product Hilbert space $\mathcal{H}_\Omega(m_0(1+\varepsilon)\varphi)\hat{\otimes}\ldots\hat{\otimes}\mathcal{H}_\Omega(m_0(1+\varepsilon)\varphi)$ (q times). Let $\delta_{m_0} := \min\limits_{j\neq k} |a_j - a_k| > 0$. It is enough to get a lower bound for our expression at the points a_k. Suppose, for example, that $x = a_k$ for some k. As, by construction, the derivatives $D^{(l,0\ldots,0)} G_{m_0(1+\varepsilon)}(x)$ in the direction of the line L coincide on L with the derivatives $g^{(l)}_{m_0(1+\varepsilon)}(x)$, $l \in \mathbb{N}$, we get, for an appropriate $l \le d_{m_0 q}(x)$:

$$\left|F^{(l)}_{m_0 q}(x)\right| \ge \frac{e^{q m_0 (1+\varepsilon)\, Re\, g(z_1)}}{(C_n\, C_{m_0})^{q/2}}\ \delta_{m_0}^{q\left(\sum_{j=1}^{N_{m_0}} m_j\right)}$$
$$\ge \frac{e^{q m_0 (1+\varepsilon)\, Re\, g(z_1)}}{(C_n\, C_{m_0})^{q/2}}\ \delta_{m_0}^{C_0(1+\varepsilon) q m_0}.$$

Then $\frac{1}{2m_0 q}\ \log|F^{(l)}_{m_0 q}(x)| \ge C_0(1+\varepsilon)\ \log\delta_{m_0} + O(1)$, and the stated lower bound follows after we absorb $(1+\varepsilon)$ in C_0 and the $O(1)$ term in the first term. The estimate of the constant C_0 follows from the following averaging argument. Suppose, for simplicity, that $\Omega = B(0,\, r)$ for some $r > 0$. Then:

$$\int\limits_{L\in\mathbb{P}^{n-1}} \left(\ \int\limits_{B(0,\,r)\cap L} dd^c\varphi_{|L}\right) dv(L) = \int\limits_{B(0,\,r)} dd^c\varphi \wedge \int\limits_{L\in\mathbb{P}^{n-1}} [L]\, dv(L)$$
$$= \int\limits_{B(0,\,r)} dd^c\varphi \wedge (dd^c \log|z|)^{n-1} = \frac{1}{r^{2(n-1)}} \int\limits_{B(0,\,r)} dd^c\varphi \wedge \beta^{n-1},$$

where $[L]$ denotes the $(n-1,\, n-1)$-current of integration on the line L, dv denotes the unique unitary invariant measure of total mass 1 on projective space, the second equality above follows from Crofton's formula (see e.g., [**Dem97**, Chapter III, p. 196]), and the third equality is a well-known formula of Lelong (see e.g., [**Dem97**, Chapter III, Formula 5.5.]). In particular, for L in a subset of $\mathbb{P}^{n-1}$ of positive dv-measure, $C_0 = \int\limits_{B(0,\,r)\cap L} dd^c\varphi_{|L} \le \frac{2}{r^{2(n-1)}} \int\limits_{B(0,\,r)} dd^c\varphi \wedge \beta^{n-1}$, and it is enough to choose such a line L to have the stated estimate on C_0. q.e.d.

The estimates obtained in Proposition 7.2 (a) and in Proposition 7.3 combine to achieve the main purpose of this section: a lower estimate for derivatives in the following sense. Given pseudoconvex open sets $B \Subset \Omega \Subset \mathbb{C}^n$ and a psh function φ on Ω of the form (21), let $\nu := \sup\limits_{x\in B} \nu(\varphi,\, x)$. As the case $\nu = 0$ was treated in Section 4, suppose now that $\nu > 0$. If $(\sigma_{m,\,j})_{j\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}_\Omega(m\varphi)$ such that finitely many elements $(\sigma_{m,\,j})_{1\le j\le N_m}$ generate $\mathcal{I}(m\varphi)$ effectively on some $B_0 \Subset B$ with properties (17)—(20) satisfied, we obtain a lower estimate on B_0 for the psh functions:

$$u_m := \frac{1}{2m}\ \log \sum_{j=1}^{N_m} \sum_{|\alpha|=0}^{[m\nu(1+\varepsilon)]} \left|\frac{D^\alpha \sigma_{m,\,j}}{\alpha!}\right|^2 \tag{31}$$

as $m \to +\infty$ in the following form summing up the discussion in this section.

Corollary 7.4. *For all $q \in \mathbb{N}$, $0 < \varepsilon << 1$, and $m_0 \geq \frac{n+2}{c\varepsilon}$, there exists an orthonormal basis $(\sigma_{m_0q,\,j})_{j\in\mathbb{N}}$ of $\mathcal{H}_\Omega(m_0q\varphi)$ satisfying, for $m = m_0q$, the estimate:*

$$u_m := \frac{1}{2m}\,\log\sum_{j=1}^{N_m}\sum_{|\alpha|=0}^{[m\nu(1+\varepsilon)]}\left|\frac{D^\alpha\sigma_{m,\,j}}{\alpha!}\right|^2 \geq C_0\log\delta_{m_0} - A_m \quad \textit{on} \quad B_0.$$

Crucially, δ_{m_0} depends only on m_0 (though in an uncontrollable way) and not directly on m. The other constants satisfy: $0 \leq C_0 \leq C_\Omega \int\limits_\Omega dd^c\varphi \wedge\beta^{n-1}$, $0 \leq A_m \leq \frac{1}{m}\,\log(C_n\,(m\nu)^n\,N_m^2\,C_{m_0}^q)$, with $C_{m_0} = C_n\,(m_0c(1+\varepsilon)-n)(\sup\limits_B e^\varphi)^{2m_0\varepsilon}$, and C_n depending only on n, B_0, B, Ω.

Proof. This is done in two steps. First, apply Proposition 7.2 (a) for every $\alpha \in \mathbb{N}^n$, sum over $0 \leq |\alpha| \leq [m(1+\varepsilon)\nu]$, and take $\frac{1}{2m}\log$ to get the following lower estimate on $u_m = u_{m_0q}$ for any orthonormal basis $(\sigma_{m_0(1+\varepsilon),\,j})_{j\in\mathbb{N}}$ of $\mathcal{H}_\Omega(m_0(1+\varepsilon)\varphi)$:

$$u_m \geq \frac{1}{2m}\,\log\sum_{j_1,\dots,j_q=0}^{+\infty}\;\sum_{|\alpha|=0}^{[m_0q\nu(1+\varepsilon)]}\left|\frac{D^\alpha(\sigma_{m_0(1+\varepsilon),\,j_1}\cdots\sigma_{m_0(1+\varepsilon),\,j_q})}{\alpha!}\right|^2 - A_m$$

on B_0. To estimate A_m we notice, with the notation of Proposition 7.2(a), that: $\sum\limits_{|\alpha|=0}^{[m(1+\varepsilon)\nu]} C_\alpha \leq C'_n\,m^n(1+\varepsilon)^n\nu^n \leq 2^nC'_n\,(m\nu)^n$, for some $C'_n > 0$. Second, apply Proposition 7.3 to get the desired lower bound for the term appearing on the right above. q.e.d.

The above functions u_m modify Demailly's regularization of φ (cf. (7)) by taking into account derivatives up to order $[m(1+\varepsilon)\nu]$ and only finitely many $\sigma_{m,\,j}$'s. The upshot is that the functions u_m can be estimated below by finite constants. This is a major improvement of the situation described in Lemma 6.2 where the lower bound for ψ_m in terms of φ was unsatisfactory owing to the $-\infty$ poles of φ. The present functions u_m are thus candidates to defining better regularizations of φ. However, the shortcoming of such a definition would be that u_m may not converge to φ as $m \to +\infty$ owing to the derivation order $[m(1+\varepsilon)\nu]$ being too large (the limit of $[m(1+\varepsilon)\nu]/m$ is > 0 instead of zero as required by Lemma 6.1 for convergence).

We will overcome this obstacle in the next section by applying this procedure after we have removed enough singularities to lower substantially the necessary derivation order.

8. Regularization with mass control: the general case

Let φ be an arbitrary psh function on a bounded pseudoconvex domain $\Omega \Subset \mathbb{C}^n$ such that $i\partial\bar\partial\varphi \geq C_0\,\beta$ for some $C_0 > 0$. As usual, β

is the standard Kähler form on $\mathbb{C}^n$. For every fixed $p \in \mathbb{N}$, we will approximate Demailly's regularizing function φ_p (cf. (7)):

$$\varphi_p = \frac{1}{2p}\,\log\sum_{j=0}^{+\infty}|\sigma_{p,\,j}|^2, \quad (\sigma_{p,\,j})_{j\in\mathbb{N}} \text{ an orthonormal basis of } \mathcal{H}_\Omega(p\varphi),$$

by a sequence of psh functions $(\psi_{m,\,p})_{m\in\mathbb{N}}$ with analytic singularities for which we can control the Monge-Ampère masses. In the last step of the proof, the desired approximation of the original φ will be obtained by letting $p \to +\infty$. The results of the previous Section 7 will now be applied to φ_p (which is of the form (21) with $c = \frac{1}{p}$) in place of φ. The notation is the same as in Section 7 with an extra p in the indices (e.g., $(\sigma_{m_0(1+\varepsilon),\,p,\,j})_{j\in\mathbb{N}}$ denotes an orthonormal basis of $\mathcal{H}_\Omega(m_0(1+\varepsilon)\varphi_p)$, $N_{m_0q,\,p}$ is the number of local generators of $\mathcal{I}(m_0q\varphi_p)$, etc).

The m^{th} regularization $\psi_{m,\,p}$ of φ_p will be constructed using the ideal sheaf $\mathcal{I}(m\varphi_p)$. As already explained, the main idea is to write $m = m_0q$ with $q \gg m_0$ and to make full use of the inclusions (22) of Proposition 7.1 which show intuitively that $\mathcal{I}(m\varphi_p)$ and $\mathcal{I}(m_0\varphi_p)^q$ are "almost" equal when $m_0 \to +\infty$ (and $\varepsilon \downarrow 0$ with $m_0 \geq p\,\frac{n+2}{\varepsilon}$). Considering a log-resolution of the ideal sheaf $\mathcal{I}(m_0\varphi_p)$, we shall "almost" get a log-resolution of $\mathcal{I}(m\varphi_p)$ with $m = m_0\,q \gg m_0$ up to a small loss which can be controlled explicitly in terms of $0 < \varepsilon \ll 1$ by means of the estimates of Propositions 7.2 and 7.3. The advantage of this method over considering right away a log-resolution of $\mathcal{I}(m\varphi_p)$ is that the blow-up does not depend explicitly on m but only on m_0. Thus, the mass estimates of the m^{th} regularization $\psi_{m,\,p}$ will only depend on m_0 (and p) and can be neutralized by choosing $m = m_0q$ with $q = q(m_0)$ large enough. This idea ties in with the explanation given before Proposition 7.3.

Let $\tilde\Omega = \tilde\Omega_{m_0}$ be a smooth variety and let $\mu = \mu_{m_0} : \tilde\Omega \to \Omega$ be a proper modification (i.e., a holomorphic bimeromorphic map) such that

$$\mu^\star\mathcal{I}(m_0(1+\varepsilon)\varphi_p) = \mathcal{O}(-E_{m_0(1+\varepsilon),\,p}), \qquad \mu^\star\mathcal{I}(m_0\varphi_p) = \mathcal{O}(-E_{m_0,\,p}),$$

where $E_{m_0(1+\varepsilon),\,p}, E_{m_0,\,p}$ are effective normal crossing divisors on $\tilde\Omega$. If $V\,\mathcal{I}(m_0(1+\varepsilon)\varphi_p)$ denotes the zero variety of $\mathcal{I}(m_0(1+\varepsilon)\varphi_p)$, the restriction

$$\mu : \tilde\Omega\setminus E_{m_0(1+\varepsilon),\,p} \longrightarrow \Omega\setminus V\,\mathcal{I}(m_0(1+\varepsilon)\varphi_p)$$

defines a biholomorphism. The inclusion $\mu^\star\mathcal{I}(m_0(1+\varepsilon)\varphi_p) \subset \mu^\star\mathcal{I}(m_0\varphi_p)$ also implies the existence of an effective divisor $D_{m_0(1+\varepsilon),\,p}$ such that

$$E_{m_0(1+\varepsilon),\,p} = E_{m_0,\,p} + D_{m_0(1+\varepsilon),\,p}.$$

Let $(\tilde U_l)_{1\leq l\leq N}$ be a finite collection of open coordinate balls covering $\tilde\Omega$ such that the restrictions of $\mu^\star\mathcal{I}(m_0(1+\varepsilon)\varphi_p)$ and $\mu^\star\mathcal{I}(m_0\varphi_p)$ to each of these have a unique generator. Let $(U_l)_{1\leq l\leq N}$ be a corresponding collection of open balls covering Ω such that $\tilde U_l\setminus E_{m_0(1+\varepsilon),\,p} = \mu^{-1}(U_l\setminus$

$V\,\mathcal{I}(m_0(1+\varepsilon)\varphi_p))$. The modification μ is a simultaneous log-resolution of the ideal sheaves $\mathcal{I}(m_0(1+\varepsilon)\varphi_p)$ and $\mathcal{I}(m_0\varphi_p)$ whose existence follows from Hironaka's theorem on the resolution of singularities.

In the sequel, we shall need the following comparison lemma in passing from the global to the local picture on coordinate patches and vice-versa.

Lemma 8.1. *Let φ be a psh function on a bounded pseudoconvex open set $\Omega \subset \mathbb{C}^n$, and let $B \Subset \Omega$ be a relatively compact open subset. For every $m, p \in \mathbb{N}$, the expressions analogous to the Bergman kernels associated with the weight $m\varphi$ on Ω and respectively B which take into account derivatives up to order p:*

$$B^{(p)}_{m\varphi,\,\Omega} := \sum_{k=0}^{+\infty}\sum_{|\alpha|=0}^{p} |D^\alpha \sigma_{m,\,k}|^2, \qquad B^{(p)}_{m\varphi,\,B} := \sum_{k=0}^{+\infty}\sum_{|\alpha|=0}^{p} |D^\alpha \mu_{m,\,k}|^2,$$

defined by orthonormal bases $(\sigma_{m,\,k})_{k\in\mathbb{N}}$ and $(\mu_{m,\,k})_{k\in\mathbb{N}}$ of the Hilbert spaces $\mathcal{H}_\Omega(m\varphi)$ and respectively $\mathcal{H}_B(m\varphi)$, can be compared as:

$$B^{(p)}_{m\varphi,\,\Omega} \leq B^{(p)}_{m\varphi,\,B} \leq C_{n,\,d,\,r}\,(d/r)^p\,B^{(p)}_{m\varphi,\,\Omega} \quad \text{on any } B_0 \Subset B \Subset \Omega,$$

where $C_{n,\,d,\,r} > 0$ is a constant depending only on n, the diameter d of Ω, and the distance $r > 0$ between the boundaries of B_0 and B.

Proof. It is by a standard application of Hörmander's L^2 estimates ([**Hor65**]) and it is given in Appendix 10. q.e.d.

We will now concentrate attention on an arbitrary $\tilde{U}_l$ that we generically call $\tilde{U}$. Let $g_{m_0(1+\varepsilon),\,p}$ (respectively $g_{m_0,\,p}$) be the unique generator on $\tilde{U}$ of $\mu^\star\mathcal{I}(m_0(1+\varepsilon)\varphi_p)$ (respectively $\mu^\star\mathcal{I}(m_0\varphi_p)$). By pull-back, the inclusions (22) become:

$$\mu^\star\mathcal{I}(m_0(1+\varepsilon)\varphi_p)^q_{|\tilde{U}} \subset \mu^\star\mathcal{I}(m_0q\varphi_p)_{|\tilde{U}} \subset \mu^\star\mathcal{I}(m_0\varphi_p)^q_{|\tilde{U}}. \tag{32}$$

The analogous inclusions to (22) of Proposition 7.1 applied to $\varphi_p\circ\mu$ on $\tilde{U} \Subset \tilde{U}'$ in place of φ_p on $B \Subset \Omega$ read:

$$\mathcal{I}(m_0(1+\varepsilon)\varphi_p\circ\mu)^q_{|\tilde{U}} \subset \mathcal{I}(m_0q\varphi_p\circ\mu)_{|\tilde{U}} \subset \mathcal{I}(m_0\varphi_p\circ\mu)^q_{|\tilde{U}}, \tag{33}$$

with the three ideal sheaves in (33) above generated respectively on $\tilde{U}$ by:

$$(\tilde{\sigma}_{m_0(1+\varepsilon),\,p,\,j_1}\dots\tilde{\sigma}_{m_0(1+\varepsilon),\,p,\,j_q})_{j_1,\dots,j_q\in\mathbb{N}},$$
$$\text{with } \tilde{\sigma}_{m_0(1+\varepsilon),\,p,\,j} := J_\mu\,\sigma_{m_0(1+\varepsilon),\,p,\,j}\circ\mu$$
$$(\tilde{\sigma}_{m_0q,\,p,\,j})_{j\in\mathbb{N}}, \text{ with } \tilde{\sigma}_{m_0q,\,p,\,j} := J_\mu\,\sigma_{m_0q,\,p,\,j}\circ\mu,$$
$$(\tilde{\sigma}_{m_0,\,p,\,j_1}\dots\tilde{\sigma}_{m_0,\,p,\,j_q})_{j_1,\dots,j_q\in\mathbb{N}}, \text{ with } \tilde{\sigma}_{m_0,\,p,\,j} := J_\mu\,\sigma_{m_0,\,p,\,j}\circ\mu,$$

where J_μ is the Jacobian of μ. On the other hand, the ideal sheaves $\mathcal{I}(m_0(1+\varepsilon)\varphi_p\circ\mu)$ and $\mathcal{I}(m_0\varphi_p\circ\mu)$ have on $\tilde{U}$ unique generators:

$$\tilde{g}_{m_0(1+\varepsilon),\,p} := J_\mu\, g_{m_0(1+\varepsilon),\,p} \quad \text{and respectively} \quad \tilde{g}_{m_0,\,p} := J_\mu\, g_{m_0,\,p}.$$

Thus, on $\tilde{U}$, we get:

$$\tilde{\sigma}_{m_0(1+\varepsilon),\,p,\,j} = \tilde{g}_{m_0(1+\varepsilon),\,p}\,\tilde{\mu}_{m_0(1+\varepsilon),\,p,\,j}, \qquad \tilde{\sigma}_{m_0,\,p,\,j} = \tilde{g}_{m_0,\,p}\,\tilde{\mu}_{m_0,\,p,\,j},$$

with holomorphic functions $(\tilde{\mu}_{m_0(1+\varepsilon),\,p,\,j})_{j\in\mathbb{N}}$ without common zeroes and holomorphic functions $(\tilde{\mu}_{m_0,\,p,\,j})_{j\in\mathbb{N}}$ without common zeroes as well. Moreover, by the right-hand inclusion in (33) there are holomorphic functions $(\tilde{\mu}_{m_0q,\,p,\,j})_{j\in\mathbb{N}}$ on $\tilde{U}$ such that:

$$\tilde{\sigma}_{m_0q,\,p,\,j} = \tilde{g}^{q}_{m_0,\,p}\,\tilde{\mu}_{m_0q,\,p,\,j}, \qquad j\in\mathbb{N}.$$

The functions $(\tilde{\mu}_{m_0q,\,p,\,j})_{j\in\mathbb{N}}$ may still have common zeroes to the extent to which the right-hand inclusion in (33) fails to be an equality. However, the conjunction with the left-hand inclusion in (33) limits the amount of common zeroes which we are now proceeding to estimate.

By virtue of Lemma 8.1, replacing $\mathcal{H}_\Omega(m_0q\varphi_p)$ with its counterpart $\mathcal{H}_U(m_0q\varphi_p)$ defined on a smaller set $U\Subset\Omega$ can only change the Bergman kernel estimates and their analogues with derivatives by insignificant constants. We can thus suppose, without loss of generality, that $(\sigma_{m_0q,\,p,\,j})_{j\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}_U(m_0q\varphi_p)$ (and the analogous assumptions for the weights $m_0(1+\varepsilon)\varphi_p$ and $m_0\varphi_p$.) The change of variable formula shows that:

$$\begin{aligned}1 &= \int_U |\sigma_{m_0q,\,p,\,j}|^2\, e^{-2m_0q\varphi_p}\, dV = \int_{\tilde{U}} |J_\mu|^2\, |\sigma_{m_0q,\,p,\,j}\circ\mu|^2\, e^{-2m_0q\varphi_p\circ\mu}\, d\tilde{V}\\ &= \int_{\tilde{U}} |\tilde{\mu}_{m_0q,\,p,\,j}|^2\, e^{-2m_0q(\varphi_p\circ\mu-\frac{1}{m_0}\,\log|\tilde{g}_{m_0,\,p}|)}\, d\tilde{V}.\end{aligned}$$

If we denote: $$\tilde{\psi}_{m_0,\,p} := \varphi_p\circ\mu - \frac{1}{m_0}\log|\tilde{g}_{m_0,\,p}| \qquad \text{on } \tilde{U},$$

we see that $\tilde{\psi}_{m_0,\,p}$ is a psh function on $\tilde{U}$ and $(\tilde{\mu}_{m_0q,\,p,\,j})_{j\in\mathbb{N}}$ defines an orthonormal basis of $\mathcal{H}_{\tilde{U}}(m_0q\,\tilde{\psi}_{m_0,\,p})$. The same argument shows that $(\tilde{\sigma}_{m_0,\,p,\,j})_{j\in\mathbb{N}}$ defines an orthonormal basis of $\mathcal{H}_{\tilde{U}}(m_0\varphi_p\circ\mu)$ and implicitly a system of generators for the ideal sheaf $\mathcal{I}(m_0\varphi_p\circ\mu)$ on $\tilde{U}$. Demailly's inequality on Lelong numbers (9) applied to $\mathcal{H}_{\tilde{U}}(m_0\varphi_p\circ\mu)$ shows that the unique generator $\tilde{g}_{m_0,\,p}$ of $\mathcal{I}(m_0\varphi_p\circ\mu)$ (which concentrates all the common zeroes of the $\tilde{\sigma}_{m_0,\,p,\,j}$'s) satisfies:

$$\nu(\varphi_p\circ\mu,\,x) - \frac{n}{m_0} \le \frac{1}{m_0}\,\nu(\log|\tilde{g}_{m_0,\,p}|,\,x) \le \nu(\varphi_p\circ\mu,\,x), \quad x\in\tilde{U},$$

and implicitly

$$0\le\nu(\tilde{\psi}_{m_0,\,p},\,x)\le\frac{n}{m_0}, \quad x\in\tilde{U}.$$

By the same inequality (9) applied to $\mathcal{H}_{\tilde{U}}(m_0 q\,\tilde{\psi}_{m_0,\,p})$, the minimum of the vanishing orders of the $\tilde{\mu}_{m_0 q,\,p,\,j}$'s satisfies:

$$0 \leq \min_{j\in\mathbb{N}} \mathrm{ord}_x \tilde{\mu}_{m_0 q,\,p,\,j} \leq m_0 q\ \nu(\tilde{\psi}_{m_0,\,p},\,x) \leq nq, \quad x \in \tilde{U}. \tag{34}$$

Thus all the common zeroes of the $\tilde{\mu}_{m_0 q,p,j}$'s can be removed by deriving them up to order $\geq nq$. With a view to constructing regularizing functions, we set the following analogue of the functions u_m defined in (31) for which we obtained a lower bound in Corollary 7.4. While $m = m_0 q$, $\mathcal{H}_\Omega(m_0 q\varphi)$ is now replaced with $\mathcal{H}_{\tilde{U}}(m_0 q\tilde{\psi}_{m_0,p})$ having $(\tilde{\mu}_{m_0 q,p,j})_{j\in\mathbb{N}}$ as an orthonormal basis. By (34), $m\nu$ can be replaced with nq.

Definition 8.2. For every $m_0, q \in \mathbb{N}$ and $\varepsilon = p\,\frac{n+2}{m_0}$, we let $m = m_0 q$ and

$$\tilde{\psi}_{m_0 q,\,p} := \frac{1}{2m_0 q}\log\sum_{j=1}^{N_{m_0 q,\,p}}\sum_{|\alpha|=0}^{[nq(1+\varepsilon)]}\left|\frac{D^\alpha\tilde{\mu}_{m_0 q,p,j}}{\alpha!}\right|^2 + \frac{1}{m_0}\log|\tilde{g}_{m_0,p}|.$$

The advantage over the situation summed up in Corollary 7.4 is that the maximal derivation order $[nq(1+\varepsilon)]$ is now small compared with $m_0 q$ appearing in the denominator (i.e., $[nq(1+\varepsilon)]/m_0 q \to 0$ when $m_0 \to +\infty$). In view of Lemma 6.1, this is a significant improvement leading to a new regularization of φ_p.

Lemma 8.3. *The functions $\psi_{m_0 q,\,p} := \tilde{\psi}_{m_0 q,\,p}\circ\mu^{-1}$ defined on $U \setminus V\,\mathcal{I}(m_0(1+\varepsilon)\varphi_p)$ extend to psh functions on U which converge pointwise and in L^1_{loc} topology to φ_p when $m_0, q \to +\infty$. In particular, $dd^c\psi_{m_0 q,\,p}$ converges to $dd^c\varphi_p$ as currents.*

Proof. Lemma 6.2 applied to the orthonormal basis $(\tilde{\mu}_{m_0 q,\,p,\,j})_{j\in\mathbb{N}}$ of $\mathcal{H}_{\tilde{U}}(m_0 q\,\tilde{\psi}_{m_0,\,p})$ with $m = m_0 q$, $\varepsilon_m = \frac{n+2}{m_0}(1+\varepsilon)$, and $C = \frac{n}{n+2}$ gives:

$$\begin{aligned}\varphi_p\circ\mu(w) &- \frac{\log N_{m_0 q,\,p} + C_1}{2m_0 q}\\ &\leq \tilde{\psi}_{m_0 q,\,p}(w)\\ &\leq \sup_{|v-w|<2r}\left(\psi_{m_0,\,p}(v) + \frac{1}{m_0}\log|\tilde{g}_{m_0,\,p}(w)|\right)\\ &\quad - \frac{[nq(1+\varepsilon)]+n}{m_0 q}\log r + \frac{1}{m_0 q}\log C_3,\end{aligned} \tag{35}$$

for every $w \in \tilde{U}$ and every $0 \leq r < \min\{\frac{1}{2}\,\mathrm{dist}(w,\,\partial\tilde{U}),\,1\}$. As every $w \in \tilde{U}\setminus E_{m_0(1+\varepsilon),\,p}$ is the image of a unique $z \in U \setminus V\,\mathcal{I}(m_0 q(1+\varepsilon)\varphi_p)$ under μ^{-1}, the conclusion follows. q.e.d.

We can now estimate the growth of the Monge-Ampère masses of $(dd^c\psi_{m_0 q,\,p})^k$, $k = 1, \ldots, n$, regarded as well-defined smooth forms on their smooth locus $U \setminus V\,\mathcal{I}(m_0(1+\varepsilon)\varphi_p)$, as $m_0,\, q \to +\infty$.

Lemma 8.4. *For every $k = 1, \dots, n$, the following mass estimate holds:*

$$\int_{U'} (dd^c \psi_{m_0 q,\, p})^k_{ac} \wedge \beta^{n-k} \leq C_{m_0} \, (-\log \delta_{m_0})^k, \quad m_0 \gg 1,\, q \in \mathbb{N}$$

on every $U' \Subset U$, with a constant $C_{m_0} > 0$ independent of p, q. In particular, if $q = q(m_0)$ is chosen so big as to have $\frac{1}{q(m_0)}\, C_{m_0}\, (-\log \delta_{m_0})^n \to 0$ when $m_0 \to +\infty$, we get:

$$\lim_{m_0 \to +\infty} \frac{1}{m_0 \, q(m_0)} \int_{U'} (dd^c \psi_{m_0 q,\, p})^k_{ac} \wedge \beta^{n-k} = 0, \quad k = 1, \dots, n,$$

which, in view of the result expected in Theorem 1.2 (*c*) *with $m = m_0 q$, is satisfactory.*

Proof.

As the singularities of $\psi_{m_0 q,\, p}$ are analytic, the mass of $(dd^c \psi_{m_0 q,\, p})^k_{ac}$ on U' equals the mass of $(dd^c \psi_{m_0 q,\, p})^k$ outside the singular locus (i.e., on $U' \setminus V\mathcal{I}(m_0(1+\varepsilon)\varphi_p)$. Let $\tilde{U}' \Subset \tilde{U}$ be the inverse image of U' under μ. If we single out the C^∞ part of $\tilde{\psi}_{m_0 q,\, p}$ as:

$$\tilde{u}_{m_0 q,\, p} := \frac{1}{2 m_0 q} \, \log \sum_{j=1}^{N_{m_0 q,\, p}} \sum_{|\alpha|=0}^{[nq(1+\varepsilon)]} \left| \frac{D^\alpha \tilde{\mu}_{m_0 q, p, j}}{\alpha!} \right|^2, \tag{36}$$

we have:

$$\begin{aligned}\int_{U'} (dd^c \psi_{m_0 q,\, p})^k_{ac} \wedge \beta^{n-k} &= \int_{\tilde{U}' \setminus E_{m_0(1+\varepsilon),\, p}} (dd^c \tilde{\psi}_{m_0 q,\, p})^k \wedge \mu^\star \beta^{n-k} \\ &= \int_{\tilde{U}'} (dd^c \tilde{u}_{m_0 q,\, p})^k \wedge \mu^\star \beta^{n-k}.\end{aligned}$$

The latter equality above follows from the fact that $dd^c(\frac{1}{m_0} \log |\tilde{g}_{m_0,\, p}|)$ equals the current of integration on $\operatorname{div} \tilde{g}_{m_0,\, p}$ whose support is included in $E_{m_0(1+\varepsilon),\, p}$. As the mass is calculated in the complement of $E_{m_0(1+\varepsilon),\, p}$, $dd^c(\frac{1}{m_0} \log |\tilde{g}_{m_0,\, p}|)$ has no contribution. By the Chern-Levine-Nirenberg inequalities ([**CLN69**]) we get:

$$\int_{\tilde{U}'} (dd^c \tilde{u}_{m_0 q,\, p})^k \wedge \beta^{n-k} \leq C'_{m_0} \, (\sup_{\tilde{U}} |\tilde{u}_{m_0 q,\, p}|)^k, \quad k = 1, \dots, n, \tag{37}$$

with a constant $C'_{m_0} > 0$ depending only on $\tilde{U}' \Subset \tilde{U} \Subset \tilde{\Omega}_{m_0}$ (and implicitly on m_0). The supremum above is finite since $\tilde{u}_{m_0 q,\, p}$ is C^∞. Thus, controlling the Monge-Ampère masses comes down to controlling the growth of $\sup_{\tilde{U}} |\tilde{u}_{m_0 q,\, p}|$ as $m_0, q \to +\infty$.

Claim 8.5. There are constants C''_{m_0}, $A_{m_0} \geq 0$ independent of p, q such that for $q \gg m_0 \gg p$ we have:

$$|\tilde{u}_{m_0q,\,p}| \leq C''_{m_0}\,(-\log\delta_{m_0}) + A_{m_0} \quad \text{on } \tilde{U}' \Subset \tilde{U}.$$

As $(\tilde{\mu}_{m_0q,\,p,\,j})_{j\in\mathbb{N}}$ defines an orthonormal basis of $\mathcal{H}_{\tilde{U}}(m_0q\tilde{\psi}_{m_0,\,p})$, we will make use of the results in Section 7 to get the desired control. To prove the claim, we need to find upper and lower bounds for $\tilde{u}_{m_0q,\,p}$. We first settle the upper bound question. We can use the effective version of the subadditivity property of multiplier ideal sheaves involving derivatives. Proposition 7.2 (b), applied to the orthonormal bases $(\tilde{\mu}_{m_0q,\,p,\,j})_{j\in\mathbb{N}}$ and $(\tilde{\mu}_{m_0,\,p,\,j})_{j\in\mathbb{N}}$ of $\mathcal{H}_{\tilde{U}}(m_0q\tilde{\psi}_{m_0,\,p})$ and $\mathcal{H}_{\tilde{U}}(m_0\tilde{\psi}_{m_0,\,p})$, gives:

$$\begin{aligned}
&\sum_{j=0}^{+\infty}|D^\alpha\tilde{\mu}_{m_0q,\,j}|^2\\
&\leq C_n^{q-1}\sum_{j_1,\dots,j_q=0}^{+\infty}|D^\alpha(\tilde{\mu}_{m_0,\,j_1}\dots\tilde{\mu}_{m_0,\,j_q})|^2\\
&\leq C_n^{q-1}\sum_{j_1,\dots,j_q=0}^{+\infty}\Bigg|\sum_{\beta_1+\dots+\beta_q=\alpha}C^\alpha_{\beta_1,\dots,\beta_q}\,D^{\beta_1}\tilde{\mu}_{m_0,\,j_1}\dots D^{\beta_q}\tilde{\mu}_{m_0,\,j_q})\Bigg|^2\\
&\leq C_n^{q-1}\,C_\alpha\left(\sum_{j=0}^{+\infty}\sum_{\beta\leq\alpha}|D^\beta\tilde{\mu}_{m_0,\,j}|^2\right)^q,
\end{aligned}$$

for every $\alpha\in\mathbb{N}^n$, where $C_\alpha := \sum\limits_{\beta_1+\beta_q=\alpha}(C^\alpha_{\beta_1,\dots,\beta_q})^2$. This implies that:

$$\tilde{u}_{m_0q,\,p} \leq \tilde{u}_{m_0,\,p} + \frac{\log N_{m_0,\,p}}{2m_0} + \frac{C_q}{m_0q} \quad \text{on } \tilde{U},$$

where $C_q > 0$ is a constant depending only on q and n such that C_q/q is bounded as $q\to+\infty$. This means that the sequence $(\tilde{u}_{m_0q,\,p})_{m_0,\,q\in\mathbb{N}}$ is non-increasing up to constants independent of q. It is in particular bounded above if we choose $m_0 \gg p$.

For the much subtler problem of finding a lower bound for $\tilde{u}_{m_0q,\,p}$ (cf. (36)), we use Corollary 7.4 to get an orthonormal basis $(\tilde{\mu}_{m_0q,\,p,\,j})_{j\in\mathbb{N}}$ of $\mathcal{H}_{\tilde{U}}(m_0q\tilde{\psi}_{m_0,\,p})$ such that the derivatives up to order $[n(1+\varepsilon)q]$ satisfy:

$$\tilde{u}_{m_0q,\,p} \geq C_0^{(m_0,p)}\,\log\delta_{m_0} - A_{m_0q} \quad \text{on} \quad \tilde{U}' \Subset \tilde{U}.$$

As $\varepsilon = p\,\frac{n+2}{m_0}$ (cf. Definition 8.2), the estimates of Corollary 7.4 give, for $q \gg m_0 \gg p$:

$$A_{m_0q} \leq \frac{\log(C_nq^nN^2_{m_0q,\,p})}{m_0q} + 2(n+2)\frac{p}{m_0}\sup_{\tilde{U}'}\tilde{\psi}_{m_0,\,p} \leq A_{m_0},$$

with $C_n > 0$ depending on $\tilde{U}', \tilde{U}$ and implicitly on m_0. We also have:

$$0 \leq C_0^{(m_0,p)} \leq C_{\tilde{U}} \int_{\tilde{U}} dd^c \tilde{\psi}_{m_0,\,p} \wedge \mu^\star \beta^{n-1} \leq C_{\tilde{U}} \int_U dd^c \varphi_p \wedge \beta^{n-1}.$$

Since $dd^c\varphi_p$ converges weakly to $dd^c\varphi$ on Ω as $p \to +\infty$, the sequence $(\int_U dd^c\varphi_p \wedge \beta^{n-1})_{p\in\mathbb{N}}$ is bounded. As $\tilde{U} \Subset \tilde{\Omega}_{m_0}$, $C_{\tilde{U}}$ depends on m_0. Thus, the constants $C_0^{(m_0,p)}$ are bounded by a constant depending only on m_0 and independent of p. Moreover, δ_{m_0} depends a priori on p as the minimum mutual distance of points where $\varphi_p\circ\mu$ has large enough Lelong numbers (see the proof of Proposition 7.3 for the definition of δ_{m_0}), but it can actually be taken independent of p since the singularities of all φ_p's are among those of φ and the Lelong numbers of φ_p are within n/p of those of φ (see (9)).

This completes the proof of Claim 8.5. Lemma 8.4 follows from estimate (37) and Claim 8.5 by taking $C_{m_0} = C'_{m_0}\, C''^{\,k}_{m_0}$ and absorbing A_{m_0} in C_{m_0}. q.e.d.

Estimate (35) satisfied by $\tilde{\psi}_{m_0q,\,p}$ (cf. the proof of Lemma 8.3), combined with estimate (8) satisfied by φ_p, implies that $\psi_{m_0q,\,p}$ converges pointwise and in L^1_{loc} topology to φ as m_0, q, $p \to +\infty$. Now Lemma 8.3 defines a function $\psi_{m_0q,p}$ on each open set of the covering $(U_l)_{1\leq l\leq N}$ of Ω (take $U = U_l$). Taking $m = m_0q$ with $q \gg m_0 \gg p$, we can patch together the psh functions:

$$\psi_{m,\,p}$$

defined on various open subsets $U_l \subset \Omega$, $1 \leq l \leq N$, into regularizing functions for φ on Ω. The following is a local version of Theorem 1.2.

Proposition 8.6. *Let φ be an arbitrary psh function on a bounded pseudoconvex domain $\Omega \Subset \mathbb{C}^n$. There exists a sequence of almost psh functions $(\psi_m)_{m\in\mathbb{N}}$ with analytic singularities such that ψ_m converges pointwise on Ω and in L^1_{loc} topology to φ as $m \to +\infty$, each ψ_m is smooth on $\Omega \setminus V\mathcal{I}(m\varphi)$, and the following hold:*

(a) $dd^c\psi_m \geq -\frac{C}{m}\,\beta$ *for some constant $C > 0$ independent of m;*
(b) $\nu(\varphi, x) - \varepsilon_m \leq \nu(\psi_m, x) \leq \nu(\varphi, x)$, $\quad x \in \Omega$, $m \in \mathbb{N}$, *for some $\varepsilon_m \downarrow 0$;*
(c) $$\lim_{m\to+\infty} \frac{1}{m} \int_B \left(dd^c\psi_m + \frac{C}{m}\,\beta\right)^k_{ac} \wedge \beta^{n-k} = 0, \qquad k = 1, \dots, n,$$

for every relatively compact open subset $B \Subset \Omega$.

Proof. For any $x_0 \in \Omega$, adding $|z - x_0|^2$ to φ if necessary, we can achieve that $i\partial\bar{\partial}\varphi \geq \beta$ and thus the results of this section apply. Let $(\varphi_p)_{p\in\mathbb{N}}$ be the Demailly regularization (cf. (7)) of φ. For every $l = 1, \dots, N$, we define $\psi^{(l)}_{m_0q,\,p}$ on U_l like in Definition 8.2 and Lemma 8.3 with U in place of U_l. We can consider finite covers $(U'_l)_{1\leq l\leq N}$,

$(U_l'')_{1\leq l\leq N}$ and $(U_l)_{1\leq l\leq N}$ of Ω by concentric balls of radii δ, $\frac{3}{2}\delta$ and respectively 2δ. We then use the (essentially well-known) patching procedure recalled in Section 5 to set:

$$\psi_{m_0q,\,p}(z) := \sup_{U_l''\ni z}\left(\psi^{(l)}_{m_0q,\,p}(z) + \frac{C(\delta)}{m_0q}(\delta^2 - |z^{(l)}|^2)\right),$$

and show, by means of Hörmander's L^2 estimates, that there is a constant $C(\delta) > 0$ depending only on δ for which the patching condition (12) holds. Here $z^{(l)}$ is a coordinate system centred at the centre of $U^{(l)}$ and Hörmander's L^2 estimates are applied with the analogous weight to (14) in which $\log|z-x_0|$ has now the coefficient $n+[n(1+\varepsilon)q]$ to force the solution of the equation (13) to vanish at x_0 to order $[n(1+\varepsilon)q]$. For every m_0, we choose $q = q(m_0) \gg m_0$ like in Lemma 8.4. For $m = m_0q$, we choose an increasing sequence of integers $p_m \to +\infty$ with $p_m << m_0$ and set:

$$\psi_m := \psi_{m,\,p_m},$$

which is easily seen to satisfy the requirements thanks to the results obtained in the present section. q.e.d.

End of Proof of Theorem 1.2. Theorem 1.2 stated in the Introduction now follows by patching together functions analogous to ψ_m obtained in the above Proposition 8.6 on various open sets contained in coordinate patches covering X. The patching procedure, recalled in Section 5 and applied there and in the above proof, can be repeated. A word of explanation is in order here. This patching procedure, based on Hörmander's L^2 estimates, was used in [**Dem92**] to patch together regularizing functions of type (7) when no derivatives are involved (see Proposition 3.7. in [**Dem92**]). In order to obtain smooth regularizations, derivatives (or jets) had to be introduced and a new patching procedure, based on Skoda's L^2 estimates, was devised in [**Dem92**, Section 5] to handle this situation. It is worth stressing that this latter, more powerful patching procedure was necessary there to handle derivatives of order up to $[cm]$ in the definition of the m^{th} regularizing function. In our present case, we only derive up to order $[C\varepsilon_m m]$ with $\varepsilon_m \downarrow 0$ (see Lemma 6.2 and Definition 8.2) and the former patching procedure can be used with minor modifications as explained in Section 5 when the derivation order was the constant 1. q.e.d.

9. Singular hermitian metrics and big line bundles

We are now in a position to prove the analytic characterization of the volume of a line bundle spelt out in Theorem 1.3 as a geometric application of our current regularization Theorem 1.2 with controlled Monge-Ampère masses. As mentioned in the Introduction, the case of a non-Kähler compact manifold X is new.

We first briefly review the set-up (cf. [**Dem85**], [**Bon98**], [**Bou02**]). Let (L, h) be a holomorphic line bundle over a compact Hermitian manifold (X, ω) equipped with a possibly singular Hermitian metric h. Let $T := i\Theta_h(L)$ be the curvature current associated with h. There is a global representation of T as $T = \alpha + dd^c\varphi$ with a global C^∞ $(1, 1)$-form α on X. For every $q = 0, 1, \dots, n$, define the q-index set of T as the open subset $X(q, T)$ of X consisting of those points x such that $T_{ac}(x)$ has precisely q negative and $n - q$ positive eigenvalues. Let $X(\leq q, T) := X(0, T) \cup \dots \cup X(q, T)$. For every $m \in \mathbb{N}^\star$, consider the singular metric h^m on L^m induced by h. This means that if $h = e^{-\varphi}$ on an open subset $U \subset X$ on which L is trivial, h^m is defined as $h^m = e^{-m\varphi}$ on U. Now suppose that $T := i\Theta_h(L) \geq -C\,\omega$ for some constant $C > 0$ (i.e., T is almost positive and φ is almost psh). Then the associated multiplier ideal sheaf $\mathcal{I}(h^m)$ is the coherent subsheaf of $\mathcal{O}_X$ defined locally as $\mathcal{I}(h^m)_{|U} = \mathcal{I}(m\varphi)$ (see (16)).

Demailly's holomorphic Morse inequalities ([**Dem85**]) for smooth metrics h were generalized by Bonavero ([**Bon98**]) to the case of singular metrics h with analytic singularities in the form of the following asymptotical estimates for the cohomology group dimensions of the twisted coherent sheaves $\mathcal{O}_X(L^m) \otimes \mathcal{I}(h^m)$:

$$\sum_{j=0}^{q} (-1)^{q-j}\, h^j(X,\, \mathcal{O}_X(L^m) \otimes \mathcal{I}(h^m)) \leq \frac{m^n}{n!} \int\limits_{X(\leq q,\, T)} (-1)^q\, T_{ac}^n + o(m^n),$$

as $m \to \infty$, for all $q = 1, \dots, n$. For $q = 1$, we get:

$$\begin{aligned} &h^0(X,\, \mathcal{O}_X(L^m) \otimes \mathcal{I}(h^m)) - h^1(X,\, \mathcal{O}_X(L^m) \otimes \mathcal{I}(h^m)) \\ &\geq \frac{m^n}{n!} \int\limits_{X(\leq 1,\, T)} T_{ac}^n + o(m^n). \end{aligned}$$

As $h^0(X,\, \mathcal{O}_X(L^m)) \geq h^0(X,\, \mathcal{O}_X(L^m) \otimes \mathcal{I}(h^m))$

$$\geq h^0(X,\, \mathcal{O}_X(L^m) \otimes \mathcal{I}(h^m)) - h^1(X,\, \mathcal{O}_X(L^m) \otimes \mathcal{I}(h^m)),$$

we infer the following lower bound for the volume of L:

$$v(L) \geq \int\limits_{X(\leq 1,\, T)} T_{ac}^n, \tag{38}$$

for every almost positive closed current T with analytic singularities (if any) in $c_1(L)$.

Proof of Theorem 1.3. Clearly, it is enough to prove the equality characterizing the volume, as the bigness criterion is an immediate consequence of it. The inequality "$\leq$" bounding the volume above can be proved as in [**Bou02**] since X is Moishezon when $v(L) > 0$ and can be modified into a projective manifold. If $v(L) = 0$, the inequality "$\leq$" is obvious.

Thus proving Theorem 1.3 boils down to obtaining a lower bound for the volume of L in terms of curvature currents. In the light of the above explanations, this can be seen as singular Morse inequalities for arbitrary singularities. Let $T := i\Theta_h(L) \geq 0$ be the curvature current associated with a singular Hermitian metric h with arbitrary singularities on L. If no positive current exists in $c_1(L)$, there is nothing to prove. By Theorem 1.2, there exist regularizing currents with analytic singularities $T_m \to T$ in $c_1(L)$ such that $T_m \geq -\frac{C}{m}\,\omega$ for some constant $C > 0$ independent of m which satisfy the Monge-Ampère mass condition (c). Furthermore, Theorem 2.4 in [**Bou02**, p. 1050] asserts that we can modify our sequence $(T_m)_{m\in\mathbb{N}}$ in such a way that besides all its properties it also satisfies:

$$T_m(x) \to T_{ac}(x) \quad \text{as} \quad m \to +\infty, \quad \text{for almost every} \quad x \in X. \tag{39}$$

A word of explanation is in order here. Property (39) is achieved in [**Bou02**] by combining the potentials ψ_m of the currents T_m with those of another regularizing sequence α_m of smooth forms constructed in [**Dem82**]. For tubular neighbourhoods $W_m \Subset U_m$ of the singular locus of ψ_m, and for arbitrary sequences $\delta_m \downarrow 0$ and $C_m \uparrow +\infty$, ψ_m is replaced with $(1-\delta_m)\psi_m - C_m$ on U_m, and with a regularized maximum function of $(1-\delta_m)\psi_m - C_m$ and the smooth potential of α_m on $X \setminus W_m$. In the proof of Theorem 1.2 above, we have constructed the potentials ψ_m of T_m to be "not too small" (after removing their singularities on a modification) and hence their moduli, which control the Monge-Ampère masses by the Chern-Levine-Nirenberg inequality, to be "not too large". It is clear that this property is preserved by the regularized maximum construction of [**Bou02**] and hence so is the Monge-Ampère mass control of Theorem 1.2 (c).

As explained above, by the Morse inequalities applied to L with $T_m \in c_1(L)$ as curvature current with analytic singularities, we get (cf. (38)):

$$\begin{aligned} v(L) &\geq \int_{X(\leq 1,\, T_m)} T^n_{m,\,ac} \\ &= \int_{X(0,\, T_m)} T^n_{m,\,ac} + \int_{X(1,\, T_m)} T^n_{m,\,ac} \quad \text{for every } m \in \mathbb{N}. \end{aligned}$$

On the other hand, the proof of Proposition 3.1. in [**Bou02**, pp. 1052–53] uses the Fatou lemma to derive the following inequality from Property (39):

$$\liminf_{m\to+\infty} \int_{X(0,\,T_m)} T^n_{m,\,ac} \geq \int_{X(T,\,0)} T^n_{ac} = \int_X T^n_{ac}.$$

Thus, to prove the Morse-type inequality "$\geq$" it is enough to show that $\lim\limits_{m\to+\infty}\int\limits_{X(1,\,T_m)} T^n_{m,\,ac}=0$. Note that on the open set $X(1,\,T_m)$ we have:

$$0\leq -T^n_{m,\,ac}\leq n\,\frac{C}{m}\,\left(T_{m,\,ac}+\frac{C}{m}\,\omega\right)^{n-1}\wedge\omega.$$

It is thus enough to show that

$$\lim_{m\to+\infty}\frac{C}{m}\int\limits_X\left(T_{m,\,ac}+\frac{C}{m}\,\omega\right)^{n-1}\wedge\omega=0.$$

Since $\int\limits_X (T_{m,\,ac}+\frac{C}{m}\,\omega)^{n-1}\wedge\omega=\int\limits_{X\setminus V\mathcal{I}(mT)}(T_m+\frac{C}{m}\,\omega)^{n-1}\wedge\omega$, this is precisely the Monge-Ampère mass property obtained in Theorem 1.2 (c). The proof is thus the same as in the Kähler case settled in [**Bou02**] once we have obtained Theorem 1.2, which is new in the non-Kähler context. q.e.d.

10. Appendix: auxiliary results

For the sake of perspicuity and completeness, we spell out in this Appendix two results related to the L^2 estimates that were used in the foregoing sections. In Section 7, an essential use was made of Skoda's L^2 division theorem in the following form.

Theorem 10.1 (Skoda [**Sko72b**]). *Let φ be a psh function on a pseudoconvex open set $\Omega\subset\mathbb{C}^n$, and let $g_1,\dots,g_N$ be (possibly infinitely many) holomorphic functions on Ω. Set $r:=\min\{N-1,\,n\}$ and $|g|^2=\sum\limits_{j=1}^N|g_j|^2$. Then, for every holomorphic function f on Ω satisfying:*

$$\int\limits_\Omega|f|^2\,|g|^{-2(r+s+\alpha)}\,e^{-2\varphi}\,dV_n<+\infty,\qquad \alpha>0,\ s\in\mathbb{N}^\star,$$

there exist holomorphic functions h_L on Ω for all $L=(l_1,\dots,l_s)\in\{1,\dots,N\}^s$ such that:

$$f=\sum_L h_L\,g^L\quad\text{on }\Omega,\qquad\text{with}\quad g^L=g_{l_1}\dots g_{l_s},$$

$$\int\limits_\Omega\sum_L|h_L|^2\,|g|^{-2(r+\alpha)}\,e^{-2\varphi}\,dV_n\leq\frac{\alpha+s}{\alpha}\int\limits_\Omega|f|^2\,|g|^{-2(r+s+\alpha)}\,e^{-2\varphi}\,dV_n.$$

This statement, in which the original function f is divided by products of s functions g_l if it satisfies an appropriate L^2 condition depending on $s\in\mathbb{N}^\star$, follows straightforwardly by induction on s from Skoda's original statement given in [**Sko72b**] for $s=1$. (See e.g., [**Dem92**, Corollary A.5., p. 407] for the present statement).

The other ingredient listed in this Appendix is the proof of Lemma 8.1 used in Section 8.

Proof of Lemma 8.1. The restriction to B clearly defines an injection $\mathcal{H}_\Omega(m\varphi) \hookrightarrow \mathcal{H}_B(m\varphi)$ and the image of the unit ball of the first space is contained in the unit ball of the second space. Thus, the first inequality holds on B. For the second inequality, let $x \in B_0$ be a fixed point, and suppose, for instance, that $B_0 = B(x,\, r/2)$. Let θ be a C^∞ cut-off function such that

$$\operatorname{Supp}\theta \Subset B(x,\, r) \Subset B, \quad \theta \equiv 1 \ \text{ on } B(x,\, r/2), \quad 0 \leq \theta \leq 1 \ \text{ on } \mathbb{C}^n.$$

Let $f \in \mathcal{O}(B)$ such that $\int_B |f|^2\, e^{-2m\varphi} = 1$ is an arbitrary element in the unit sphere of $\mathcal{H}_B(m\varphi)$. We need to produce a global holomorphic function on Ω all of whose derivatives up to order p assume the same values at x as those of f. We can use Hörmander's L^2 estimates (cf. [**Hor65**]) to solve the equation:

$$\bar\partial u = \bar\partial(\theta\, f) \quad \text{on } \Omega$$

with the strictly psh weight $m\varphi + (n+p)\log|z-x| + |z-x|^2$. There exists a C^∞ solution u satisfying the estimate:

$$\int\limits_\Omega \frac{|u|^2}{|z-x|^{2(n+p)}}\, e^{-2m\varphi}\, e^{-2|z-x|^2} \leq 2 \int\limits_\Omega \frac{|\bar\partial\theta|^2\, |f|^2}{|z-x|^{2(n+p)}}\, e^{-2m\varphi}\, e^{-2|z-x|^2}.$$

The non-integrability of $|z-x|^{-2(n+k)}$, $k = 0, 1, \ldots, p$, near x implies that $D^\alpha u(x) = 0$, $0 \leq |\alpha| \leq p$. If we set:

$$F := \theta\, f - u \in \mathcal{O}(\Omega),$$

we have $D^\alpha F(x) = D^\alpha f(x)$, $0 \leq |\alpha| \leq p$, and

$$\begin{aligned} \int\limits_\Omega |F|^2\, e^{-2m\varphi} &\leq 2\left(1 + C_n\, \frac{d^{2n}\, e^{2d^2}}{r^{2(n+1)}}\right)(d/r)^p \int\limits_B |f|^2\, e^{-2m\varphi} \\ &= C_{n,\,d,\,r}(d/r)^p, \end{aligned}$$

since $\int_B |f|^2\, e^{-2m\varphi} = 1$, with a constant $C_n > 0$ depending only on n and $C_{n,\,d,\,r}$ denoting the double of the parenthesis above. This means that $F/\sqrt{C_{n,\,d,\,r}(d/r)^p}$ belongs to the unit ball of $\mathcal{H}_\Omega(m\varphi)$. The second inequality follows from the following expressions holding at every $x \in B$:

$$B^{(p)}_{m\varphi,\,\Omega}(x) = \sum_{|\alpha|=0}^{p} \sup_{g \in \bar B_{m,\,\Omega}(1)} |D^\alpha g(x)|^2,$$

$$B^{(p)}_{m\varphi,\,B}(x) = \sum_{|\alpha|=0}^{p} \sup_{f \in \bar B_{m,\,B}(1)} |D^\alpha f(x)|^2,$$

where $\bar{B}_{m,\Omega}(1)$ and $\bar{B}_{m,B}(1)$ are the closed unit balls of $\mathcal{H}_\Omega(m\varphi)$ and respectively $\mathcal{H}_B(m\varphi)$. The calculation details are left to the reader.

q.e.d.

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