INSTANTON AND DONALDSON INVARIANTS

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Abstract

For a smooth projective toric surface we determine the Donaldson invariants and their wallcrossing in terms of the Nekrasov partition function. Using the solution of the Nekrasov conjecture [**33**, **38**, **3**] and its refinement [**34**], we apply this result to give a generating function for the wallcrossing of Donaldson invariants of good walls of simply connected projective surfaces with $b_+ = 1$ in terms of modular forms. This formula was proved earlier in [**19**] more generally for simply connected 4-manifolds with $b_+ = 1$, assuming the Kotschick-Morgan conjecture, and it was also derived by physical arguments in [**31**].

Introduction

Donaldson invariants have for a long time played an important role in the study and classification of differentiable 4-manifolds (see [**7**]). They are defined by moduli spaces of anti-self-dual connections on a principal $SO(3)$-bundle. The anti-self-duality equation depends on the choice of a Riemannian metric g. For generic g there are no reducible solutions to the equation and moduli spaces are smooth manifolds. In case $b_+ > 1$ two generic Riemannian metrics can be connected by a path. Then Donaldson invariants are independent of the choice of the metric, and they are invariants of a C^∞ compact oriented 4-manifold X.

On the other hand, in case $b_+ = 1$ nongeneric metrics form a real codimension 1 subset in the space of Riemannian metrics, i.e., a collection of *walls*, and two generic metrics cannot be connected by a path in general. As a consequence, Donaldson invariants are only piecewise constants as functions of the Riemannian metric g [**24**, **26**]. More precisely we have a chamber structure on the period domain, which is a connected component $\mathcal{C}$ of the positive cone in the second cohomology group $H^2(X, \mathbb{R})$, and the Donaldson invariants stay constant only when the period $\omega(g)$, which is the cohomology class of the self-dual

harmonic 2-form modulo scalars, stays in a chamber. The *wallcrossing terms* are the differences of Donaldson invariants when the metric moves to another chamber passing through a wall. In [**19**] the first author gave a formula for their generating function in terms of modular forms, assuming the Kotschick-Morgan conjecture[1], which states that the wallcrossing term is a polynomial in the intersection form and the multiplication by ξ, the cohomology class defining the wall (see §1.1 for more detail). The method of the proof was indirect and did not give a clear reason why modular forms appear.

A physical derivation of the wallcrossing formula was given by Moore-Witten [**31**]. We shall review their derivation and the physical background only very briefly here (see [**34**, Introduction] for a more detailed exposition for mathematicians). The work of Moore-Witten was based on Seiberg-Witten's ansatz [**40**] of the $\mathcal{N} = 2$ supersymmetric Yang-Mills theory on $\mathbb{R}^4$, which is a physical theory underlying Donaldson invariants [**41**]. The theory is controlled by a family of elliptic curves parametrized by a complex plane (called the u-plane). The modular forms that appear in the wallcrossing formula are related to this family. They expressed Donaldson invariants in terms of two contributions, the integral over the u-plane and the contribution from the points ± 2, where the corresponding elliptic curves are singular. The latter contribution corresponds to Seiberg-Witten invariants, which conjecturally contain the same information as Donaldson invariants [**42**]. Moore-Witten further studied the u-plane integral and its contribution to Donaldson invariants. They recovered the wallcrossing formula, as well as Fintushel-Stern's blowup formula [**15**], and also obtained new results, such as Seiberg-Witten contributions and calculation for $\mathbb{P}^2$ in terms of Hurwitz class numbers.

Seiberg-Witten and Moore-Witten's arguments clarified the reason why modular forms appear in Donaldson invariants. But they were physical and have no mathematically rigorous justification so far. A more rigorous approach was proposed much later by Nekrasov [**36**]. He introduced the partition function

$$Z^{\mathrm{inst}}(\varepsilon_1, \varepsilon_2, a; \Lambda) = \sum_{n \geq 0} \Lambda^{4n} \int_{M(n)} 1,$$

where $M(n)$ is the Gieseker's partial compactification of the framed moduli space of $SU(2)$-instantons on $\mathbb{A}^2$ and $\int_{M(n)}$ denotes the push-forward homomorphism to a point in the equivariant homology groups,

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[1]There are two preprints by Chen [**5**] and by Feehan-Leness [**14**], giving a proof and an announcement of a proof of the conjecture respectively. Frøyshov also gave a talk on a proof. Their approaches are differential geometric and quite different from ours, and the authors believe they are correct, but unfortunately do not have the ability to check their papers in full detail.

defined by a formal application of Bott's fixed point formula to the noncompact space $M(n)$. The variables ε_1, ε_2 are generators of the equivariant cohomology $H^*_{\mathbb{C}^* \times \mathbb{C}^*}(\mathrm{pt})$ of a point with respect to the two dimensional torus $\mathbb{C}^* \times \mathbb{C}^*$ acting on $\mathbb{A}^2$. The remaining variable a is also a generator of $H^*_{\mathbb{C}^*}(\mathrm{pt})$, where $\mathbb{C}^*$ acts on $M(n)$ by the change of the framing. This definition can be viewed as the generating function of the equivariant Donaldson invariants of $\mathbb{R}^4 = \mathbb{A}^2$. Although Nekrasov was motivated by a physical argument, the partition function is mathematically rigorously defined. He then conjectured

$$\varepsilon_1 \varepsilon_2 \log Z^{\mathrm{inst}}(\varepsilon_1, \varepsilon_2, a; \Lambda) = \mathcal{F}_0^{\mathrm{inst}}(a, \Lambda) + \text{higher terms} \quad \text{as } \varepsilon_1, \varepsilon_2 \to 0,$$

where $\mathcal{F}_0^{\mathrm{inst}}(a, \Lambda)$ is the instanton part of the Seiberg-Witten prepotential defined via periods of the elliptic curves mentioned above. The conjecture was proved by three groups, the second and third named authors [**33**], Nekrasov-Okounkov [**38**], and Braverman-Etingof [**3**] by completely different methods.

In this paper we express the wallcrossing terms of Donaldson invariants in terms of the Nekrasov partition function, under the assumption that the wall is good (see §2.1 for the definition). Thereby we give a partial mathematical justification of Moore-Witten's argument, where Seiberg-Witten's ansatz is replaced by the Nekrasov partition function. More precisely, we take a smooth toric surface X and consider equivariant Donaldson invariants. They also depend on the choice of a Riemannian metric as ordinary Donaldson invariants, and we have an equivariant wallcrossing term. The first main result (Theorem 3.3) expresses it as the residue at $a = \infty$ (corresponding to $u = \infty$ of the u-plane) of a product over contributions from fixed points in X, and the local contribution is essentially the Nekrasov partition function. This result comes from the following: In the wallcrossing the moduli space changes by replacing certain sheaves lying in extensions of ideal sheaves of zero-dimensional schemes twisted by line bundles by extensions the other way round. Using this fact one can express the change of Donaldson invariants under wallcrossing in terms of intersection numbers on the Hilbert schemes $X_2^{[l]}$ of points on two copies of X. For the wallcrossing of the Donaldson invariants without higher Chern characters this was already shown in [**8**, Th. 6.13] and [**17**, Th. 5.4, Th. 5.5]. These intersection numbers can be computed via equivariant localization on $X_2^{[l]}$. Every Γ-invariant scheme in $X_2^{[l]}$ is a union of Γ-invariant schemes with support one of the fixed points of X, and the contribution to the intersection number coming from invariant subschemes with support one of the fixed points of X is given by the Nekrasov partition function.

Then the second main result (Theorem 4.2) is about the nonequivariant limit $\varepsilon_1, \varepsilon_2 \to 0$, and we recover the formula in [**19**] via the

solution of Nekrasov's conjecture and its refinement [34], i.e., determination of several higher terms of $\varepsilon_1\varepsilon_2 \log Z(\varepsilon_1, \varepsilon_2, a; \Lambda)$. It is worthwhile remarking that the variable a appears in the wallcrossing term as an auxiliary variable, which is eventually integrated out. By contrast it plays a fundamental rôle in the Seiberg-Witten ansatz as a period of the Seiberg-Witten curve.

It is natural to expect that our equivariant wallcrossing formula is a special case of that for the Donaldson invariants for *families* whose definition was mentioned in [6]. Then we expect that higher coefficients of the Nekrasov partition function, which are higher genus Gromov-Witten invariants for a certain noncompact toric Calabi-Yau 3-fold, also play a role in 4-dimensional topology.

In §5 we show that the wallcrossing term for a good wall of an arbitrary projective surface X can be given by a universal polynomial depending on Chern classes $c_i(X)$, ξ and the intersection product on $H^*(X)$. The proof of this result does not yield an explicit form of the universal polynomial directly. But combining with the explicit form obtained for toric surfaces, we conclude that the same explicit formula holds for an arbitrary surface with $b_+ = 1$. In particular, it does not depend on $c_1(X)$ and satisfies the statements in the Kotschick-Morgan conjecture. (See Remark 5.8 for more explanation.) The 'goodness' of the wall means that the moduli space is smooth along sheaves replaced by the wallcrossing.

Results of Mochizuki show that the goodness assumption can be removed: [30, Thm 1.12] gives Proposition 2.8 for arbitrary walls if we replace vector bundles $\mathcal{A}_{\xi,+}$, $\mathcal{A}_{\xi,-}$ by the corresponding classes in K-theory. In the proof Mochizuki uses virtual fundamental classes and virtual localization. Therefore our main results (Theorem 4.2, Corollary 5.7) are true for any wall on a simply-connected projective surface.

In §6 we express the equivariant Donaldson invariants themselves for $\mathbb{P}^2$, instead of the wallcrossing terms, in terms of the Nekrasov partition function. The result here is independent of those in previous sections. However, we do not know how to deduce an explicit formula for ordinary Donaldson invariants via nonequivariant limit $\varepsilon_1, \varepsilon_2 \to 0$. Note also that we cannot extend this result to other toric surfaces, as fixed points are no longer isolated.

The Nekrasov partition function is defined for any rank. A higher rank generalization of Donaldson invariants was given recently by Kronheimer [27]. Though they are defined for $b_+ > 1$, many of his results are applicable to the $b_+ = 1$ case also. Therefore it is natural to hope that our results can be generalized to the higher rank cases. One of new difficulties appearing in higher rank cases is a recursive structure of the wallcrossing. We hope to come back this problem in the future.

Finally, let us mention that Nekrasov proposed that the equivariant Donaldson invariants for toric surfaces can be expressed as products of his partition functions over fixed points, integrated over a in any rank [**37**]. As equivariant Donaldson invariants vanish for a certain chamber for toric surfaces, our wallcrossing formula gives such an expression together with an explicit choice of contour for the a-integral, which was not specified in [loc. cit.]. It is an interesting problem to justify his argument more directly.

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1. Background Material

We will work over $\mathbb{C}$. We usually consider homology and cohomology with rational coefficients and for a variety Y we will write $H_i(Y)$, and $H^i(Y)$ for $H_i(Y, \mathbb{Q})$ and $H^i(Y, \mathbb{Q})$ respectively. If Y is projective and $\alpha \in H^*(Y)$, we denote $\int_Y \alpha$ its evaluation on the fundamental cycle of Y. If Y carries an action of a torus T, α is a T-equivariant class, and $p \colon X \to \mathrm{pt}$ is the projection to a point, we denote $\int_Y \alpha := p_*(\alpha) \in H^*(\mathrm{pt})$.

In this whole paper X will be a nonsingular projective surface over $\mathbb{C}$. Later we will specialize X to a smooth projective toric surface. For a class $\alpha \in H^*(X)$, we denote $\langle \alpha \rangle := \int_X \alpha$. If X is a toric surface, we use the same notation for the equivariant pushforward to a point.

1.1. Donaldson invariants. Let X be a smooth simply connected compact oriented 4-manifold with a Riemannian metric g. For $P \to X$ an $SO(3)$-bundle over X let $M(P)$ be the moduli space of irreducible anti-self-dual connections on P. For generic g this will be a manifold of dimension $d := -2p_1(P) - 3(1 + b^+(X))$. Let $\mathcal{P} \to X \times M(P)$ be the universal bundle. Then the Donaldson invariant of Y is a polynomial on $H_0(X) \oplus H_2(X)$, defined by

$$D^g_{c_1, d}(\alpha^n p^b) = \int_{M(P)} \mu(\alpha)^n \mu(p)^b.$$

Here c_1 is a lift of $w_2(P)$ to $H^2(X, \mathbb{Z})$, $p \in H_0(X)$ is the class of a point and $\alpha \in H_2(X)$, and for $\beta \in H_i(X)$ we define $\mu(\beta) := -\frac{1}{4}p_1(\mathcal{P})/\beta$. As $M(P)$ is not compact, this integral must be justified using the Uhlenbeck compactification of $M(P)$. Note that the orientation of $M(P)$ depends on the lift c_1 and a choice of a connected component $\mathcal{C}$ of the positive

cone in $H^2(X, \mathbb{R})$ which for algebraic surfaces we always take to be the component containing the ample cone. The generating function is

$$D_{c_1}^g(\exp(\alpha z + px)) := \sum_{d \geq 0} \sum_{n, m \geq 0} D_{c_1, d}^g \left(\frac{\alpha^n}{n!} \frac{p^m}{m!} \right) z^n x^m.$$

When $b_+(X) > 1$, then $D_{c_1, d}^g$ is independent of g as long as g is generic. If $b_+(X) = 1$, then $D_{c_1, d}^g$ depends on the period point $\omega(g) \in \mathcal{C}$.

In fact, the positive cone in $H^2(X, \mathbb{R})$ has a chamber structure (see [24],[26]): For a class $\xi \in H^2(X, \mathbb{Z}) \setminus \{0\}$, we put $W^\xi := \{ x \in \mathcal{C} \mid \langle x \cdot \xi \rangle = 0 \}$. Assume $W^\xi \neq \emptyset$. Then we call ξ a *class of type* (c_1, d) and call W^ξ a *wall of type* (c_1, d) if the following conditions hold

(1) $\xi + c_1$ is divisible by 2 in $H^2(X, \mathbb{Z})$,
(2) $d + 3 + \xi^2 \geq 0$.

We call ξ a *class of type* c_1 and call W^ξ a *wall of type* c_1, if $\xi + c_1$ is divisible by 2 in $H^2(X, \mathbb{Z})$. The chambers of type (c_1, d) are the connected components of the complement of the union of all walls of type (c_1, d) in $\mathcal{C}$. In [26] it is shown that $D_{c_1, d}^g$ depends only on the chamber of $\omega(g)$.

Let C_+, C_- be chambers of type (c_1, d) in $\mathcal{C}$ and g_+, g_- be Riemannian metrics with $\omega(g_\pm) \in C_\pm$. Then

$$D_{c_1, d}^{g_+}(\alpha^n p^b) - D_{c_1, d}^{g_-}(\alpha^n p^b) = \sum_\xi \Delta_{\xi, d}^X(\alpha^n p^b),$$

where the summation runs over the set of all classes ξ of type (c_1, d) with $\langle \xi \cdot C_+ \rangle > 0 > \langle \xi \cdot C_- \rangle$. The term $\Delta_{\xi, d}^X$ is called the *wallcrossing term*. The Kotschick-Morgan [26] conjecture says that $\Delta_{\xi, d}^X$ is a polynomial in the multiplication by ξ and the intersection form with coefficients depending only on ξ^2 and the homotopy type of X. Wallcrossing terms with small d had been calculated by various authors [24, 26, 8, 9, 17, 25, 29]. Then the first named author [19] gave a formula for the generating function of $\Delta_{\xi, d}^X$ in terms of modular forms, assuming the Kotschick-Morgan conjecture. See also [20].

Now we specialize to the case of a smooth projective surface X with $p_g(X) = 0$, in particular $b_+(X) = 1$. Let H be an ample divisor on X. Then the cohomology class H is a representative of the period point of the Fubini-Study metric of X associated to H. We write $D_{c_1, d}^H$ for the corresponding Donaldson invariants. By [28], [32], the $D_{c_1, d}^H$ can also be computed using moduli spaces of sheaves on X. We denote by $M_H^X(c_1, d)$ the moduli space of torsion-free H-semistable sheaves (in the sense of Gieseker and Maruyama) of rank 2 and with $c_1(E) = c_1$ and $4c_2(E) - c_1(E)^2 - 3 = d$. Let $M_H^X(c_1, d)_s$ be the open subset of stable sheaves. Assume that $M_H^X(c_1, d) = M_H^X(c_1, d)_s$ and that there exists a universal sheaf $\mathcal{E}$ on $X \times M_H^X(c_1, d)$. If there is no universal

sheaf, we can replace it by a quasiuniversal sheaf. When $p_g = 0$ (the case of our primary interest), then $\mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ is surjective, which means that $\chi(*, *)$ is unimodular on $K(X)$. Hence, there is a universal sheaf if $M_H^X(c_1, d) = M_H^X(c_1, d)_s$. For $\beta \in H_i(X, \mathbb{Q})$, we put $\mu(\beta) := \left(c_2(\mathcal{E}) - c_1(\mathcal{E})^2/4\right)/\beta \in H^{4-i}(M_H^X(c_1, d), \mathbb{Q})$, and define

$$\Phi_{c_1, d}^H(\alpha^n p^m) := \int_{M_H^X(c_1, d)} \mu(\alpha)^n \mu(p)^m$$

and

(1.1) $$\Phi_{c_1}^H(\exp(\alpha z + px)) := \sum_{d \geq 0} \Lambda^d \sum_{m, n} \Phi_{c_1, d}^H\left(\frac{\alpha^n}{n!} \frac{p^m}{m!}\right) z^n x^m$$

$$= \sum_{d \geq 0} \Lambda^d \int_{M_H^X(c_1, d)} \exp(\mu(\alpha z + px)).$$

Here if Y is a compact variety and $f = \sum_{i,j} a_{i,j} x^i z^j \in H^*(Y)[[x, z]]$, we write $\int_Y f = \sum_{i,j} x^i z^j \int_Y a_{i,j}$. Assume that $M_H^X(c_1, d)$ has the expected dimension d or is empty, and that H does not lie on a wall of type (c_1, d). Then by the results of [**32**],[**28**] one has

(1.2) $$\Phi_{c_1, d}^H(\alpha^n p^m) = (-1)^{(c_1^2 + \langle c_1 \cdot K_X \rangle)/2} D_{c_1, d}^H(\alpha^n p^m).$$

When $M_H^X(c_1, d)$ is not necessary of expected dimension, we define the invariants as follows (cf. [**16**, §3.8]): we consider blowup $P \colon \widehat{X} \to X$ at sufficiently many points $p_1, \ldots, p_N$ disjoint from cycles representing α, p. Let $C_1, \ldots, C_N$ denote the exceptional curves. We consider the moduli space $M_{P^*H}^{\widehat{X}}(P^*c_1, d+4N)$, where the polarization 'P^*H' means $P^*H - \varepsilon C_1 - \varepsilon C_2 - \cdots \varepsilon C_N$ for sufficiently small $\varepsilon > 0$. Then it has expected dimension for sufficiently large N by §A. We define

$$\int_{M_H^X(c_1, d)} \exp(\mu(\alpha z + px))$$

$$:= \left(-\frac{1}{2}\right)^N \int_{M_{P^*H}^{\widehat{X}}(P^*c_1, d+4N)} \mu(C_1)^4 \cdots \mu(C_N)^4 \exp(\mu(\alpha P^* z + p P^* x)).$$

By the blowup formula (see [**16**, Th. 8.1]), this definition is independent of N. From its definition, (1.2) remains to hold.

1.2. Nekrasov partition function. We briefly review the Nekrasov partition function in the case of rank 2. For more details see [**34**, Sections 3.1, 4]. Let ℓ_∞ be the line at infinity in $\mathbb{P}^2$. Let $M(n)$ be the moduli space of pairs (E, Φ), where E is a rank 2 torsion-free sheaf on $\mathbb{P}^2$ with $c_2(E) = n$, which is locally free in a neighbourhood of ℓ_∞ and $\Phi \colon E|_{\ell_\infty} \to \mathcal{O}_{\ell_\infty}^{\oplus 2}$ is an isomorphism. $M(n)$ is a nonsingular quasiprojective variety of dimension $4n$.

Let $\Gamma := \mathbb{C}^* \times \mathbb{C}^*$ and $\widetilde{T} := \Gamma \times \mathbb{C}^*$. $\widetilde{T}$ acts on $M(n)$ as follows: For $(t_1, t_2) \in \Gamma$, let F_{t_1,t_2} be the automorphism of $\mathbb{P}^2$ defined by $F_{t_1,t_2}([z_0, z_1, z_2]) \mapsto [z_0, t_1 z_1, t_2 z_2]$, and for $e \in \mathbb{C}^*$ let G_e be the automorphism of $\mathcal{O}_{\ell_\infty}^{\oplus 2}$ given by $(s_1, s_2) \mapsto (e^{-1} s_1, e s_2)$. Then for $(E, \Phi) \in M(n)$ we put $(t_1, t_2, e) \cdot (E, \Phi) := \left((F_{t_1,t_2}^{-1})^* E, \Phi' \right)$, where Φ' is the composition

$$(F_{t_1,t_2}^{-1})^*(E)|_{\ell_\infty} \xrightarrow{(F_{t_1,t_2}^{-1})^* \Phi} (F_{t_1,t_2}^{-1})^* \mathcal{O}_{\ell_\infty}^{\oplus 2} \longrightarrow \mathcal{O}_{\ell_\infty}^{\oplus 2} \longrightarrow \mathcal{O}_{\ell_\infty}^{\oplus 2}.$$

Let $\varepsilon_1, \varepsilon_2, a$ be the coordinates on the Lie algebra of $\widetilde{T}$, i.e., we can write $(t_1, t_2, e) = (e^{\varepsilon_1}, e^{\varepsilon_2}, e^a)$.

We briefly recall equivariant integration in the form we want to use it. Let T be a torus acting on a nonsingular variety Y with finitely many fixed points $q_1, \dots, q_s$. Let $e_1, \dots, e_n$ be the coordinates on the Lie algebra of T. The equivariant cohomology of a point is $H_T^*(\mathrm{pt}) = \mathbb{Q}[e_1, \dots, e_n]$. If $\alpha \in H_T^*(Y)$ is an equivariant cohomology class, then we put

$$\int_Y \alpha := \sum_{i=1}^s \frac{\iota_{q_i}^*(\alpha)}{e_T(T_{q_i} Y)} \in \mathbb{Q}(e_1, \dots, e_n).$$

Here $\iota_{q_i}^*$ is the equivariant pullback via the embedding $q_i \hookrightarrow Y$, and $e_T(T_{q_i} Y)$ is the equivariant Euler class of the tangent space of Y at q_i. If Y is also compact, then $\int_Y$ is the usual pushforward to a point in equivariant cohomology, in particular $\int_Y \alpha \in \mathbb{Q}[e_1, \dots, e_n]$.

Let x, y be the coordinates on $\mathbb{A}^2 = \mathbb{P}^2 \setminus \ell_\infty$. The fixed point set $M(n)^{\widetilde{T}}$ is a set of $(\mathcal{I}_{Z_1}, \Phi_1) \oplus (\mathcal{I}_{Z_2}, \Phi_2)$, where the $\mathcal{I}_{Z_i}$ are ideal sheaves of zero dimensional schemes Z_1, Z_2 with support in the origin of $\mathbb{A}^2$ with $\mathrm{len}(Z_1) + \mathrm{len}(Z_2) = n$ and Φ_α ($\alpha = 1, 2$) are isomorphisms of $\mathcal{I}_{Z_\alpha}|_{\ell_\infty}$ with the α-th factor of $\mathcal{O}_{\ell_\infty}^{\oplus 2}$. Write I_α for the ideal of Z_α in $\mathbb{C}[x, y]$. Then the above is a fixed point if and only if I_1 and I_2 are generated by monomials in x, y.

A Young diagram is a set

$$Y := \left\{ (i, j) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \mid j \le \lambda_i \right\},$$

where $(\lambda_i)_{i \in \mathbb{Z}_{>0}}$ is a partition, i.e., $\lambda_i \in \mathbb{Z}_{\ge 0}$, $\lambda_i \ge \lambda_{i+1}$ for all i and only finitely many λ_i are nonzero. Thus λ_i is the length of the i-th column of Y. Let $|Y|$ be the number of elements of Y, so that (λ_i) is a partition of $|Y|$. We denote by $(\lambda_j')_j$ the transpose of λ, thus λ_j' is the length of the j-th row of Y. For elements $s = (i, j) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ we put

$$a_Y(i, j) = \lambda_i - j, \quad l_Y(i, j) = \lambda_j' - i, \quad a'(i, j) = j - 1, \quad l'(i, j) = i - 1.$$

Let $I_Z \subset \mathbb{C}[x, y]$ be the ideal of a finite subscheme of $\mathbb{A}^2$ supported in the origin which is generated by monomials in x, y. To Z we associate the Young diagram

$$Y = Y_Z := \left\{ (i, j) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \mid x^{i-1} y^{j-1} \notin I_Z \right\},$$

with $|Y| = \text{len}(Z)$. To a fixed point $(\mathcal{I}_{Z_1}, \Phi_1) \oplus (\mathcal{I}_{Z_2}, \Phi_2)$ of the $\widetilde{T}$-action on $M(n)$ we associate $\vec{Y} = (Y_1, Y_2)$ with $Y_i = Y_{Z_i}$. This gives a bijection of the fixed point set $M(n)^{\widetilde{T}}$ with the set of pairs of Young diagrams $\vec{Y} = (Y_1, Y_2)$ with $|\vec{Y}| := |Y_1| + |Y_2| = n$.

Notation 1.3. We denote e the one-dimensional $\widetilde{T}$-module given by $(t_1, t_2, e) \mapsto e$, and similarly we write t_i $(i = 1, 2)$ for the 1-dimensional $\widetilde{T}$ modules given by $(t_1, t_2, e) \mapsto t_i$. We also write $e_1 := e^{-1}$, $e_2 := e$. And we write $a_1 := -a$, $a_2 := a$.

Following [**33**], [**34**], let, for $\alpha, \beta \in \{0, 1\}$, $N_{\alpha,\beta}^{\vec{Y}}(t_1, t_2, e)$ be the $\widetilde{T}$-equivariant character of $\text{Ext}^1(\mathcal{I}_{Z_\alpha}, \mathcal{I}_{Z_\beta}(-\ell_\infty))$ and $n_{\alpha,\beta}^{\vec{Y}}(\varepsilon_1, \varepsilon_2, a)$ the equivariant Euler class. It can be written by using a_Y, l_Y (see [**33**, Th. 2.11, Section 6] or [**34**, Sections 3, 4]), but an explicit form will not be used in this paper. Now the instanton part of the Nekrasov partition function is defined as

$$Z^{\text{inst}}(\varepsilon_1, \varepsilon_2, a, \Lambda) := \sum_{n \geq 0} \Lambda^{4n}\left(\int_{M(n)} 1\right) = \sum_{\vec{Y}} \frac{\Lambda^{4|\vec{Y}|}}{\prod_{\alpha,\beta=1}^2 n_{\alpha,\beta}^{\vec{Y}}(\varepsilon_1, \varepsilon_2, a)}.$$

More generally we will consider the following: For variables $\vec{\tau} := (\tau_\rho)_{\rho \geq 1}$ let

$$(1.4) \quad E^{\vec{Y}}(\varepsilon_1, \varepsilon_2, a, \vec{\tau}) := \exp\left(\sum_{\rho=1}^\infty \sum_{\alpha=1}^2 \tau_\rho \left[\frac{e^{a_\alpha}}{\varepsilon_1 \varepsilon_2}\right.\right.$$

$$\left.\left.\cdot \left(1 - (1 - e^{-\varepsilon_1})(1 - e^{-\varepsilon_2}) \sum_{s \in Y_\alpha} e^{-l'(s)\varepsilon_1 - a'(s)\varepsilon_2}\right)\right]_{\rho-1}\right).$$

(The sign in [**34**, (4.1)] is not correct. See the first claim in the proof of Lemma 3.9.) Here $[\cdot]_{\rho-1}$ means the part of degree $\rho - 1$, where $a, \varepsilon_1, \varepsilon_2$ have degree 1. Then the instanton part of the partition function is defined as
(1.5)

$$Z^{\text{inst}}(\varepsilon_1, \varepsilon_2, a; \Lambda, \vec{\tau}) := \sum_{\vec{Y}} \frac{\Lambda^{4|\vec{Y}|} E^{\vec{Y}}(\varepsilon_1, \varepsilon_2, a, \vec{\tau})}{\prod_{\alpha,\beta=1}^2 n_{\alpha,\beta}^{\vec{Y}}(\varepsilon_1, \varepsilon_2, a)} \in \mathbb{Q}(\varepsilon_1, \varepsilon_2, a)[[\Lambda]].$$

In particular, $Z^{\text{inst}}(\varepsilon_1, \varepsilon_2, a; \Lambda, \vec{0}) = Z^{\text{inst}}(\varepsilon_1, \varepsilon_2, a; \Lambda)$. As a power series in Λ, $Z^{\text{inst}}(\varepsilon_1, \varepsilon_2, a; \Lambda, \vec{\tau})$ starts with 1. Thus

$$F^{\text{inst}}(\varepsilon_1, \varepsilon_2, a; \Lambda, \vec{\tau}) := \log Z^{\text{inst}}(\varepsilon_1, \varepsilon_2, a; \Lambda, \vec{\tau}) \in \mathbb{Q}(\varepsilon_1, \varepsilon_2, a)[[\Lambda]]$$

is well-defined and we put $F^{\text{inst}}(\varepsilon_1, \varepsilon_2, a; \Lambda) := F^{\text{inst}}(\varepsilon_1, \varepsilon_2, a; \Lambda, \vec{0})$. Finally we define the perturbation part. We define c_n $(n \in \mathbb{Z}_{\geq 0})$ by

$$(1.6) \qquad \frac{1}{(e^{\varepsilon_1 t} - 1)(e^{\varepsilon_2 t} - 1)} = \sum_{n \geq 0} \frac{c_n}{n!} t^{n-2},$$

and define

(1.7) $\gamma_{\varepsilon_1,\varepsilon_2}(x;\Lambda)$

$$:= \frac{1}{\varepsilon_1\varepsilon_2}\Big\{ -\frac{1}{2}x^2\log\Big(\frac{x}{\Lambda}\Big) + \frac{3}{4}x^2\Big\} + \frac{\varepsilon_1+\varepsilon_2}{2\varepsilon_1\varepsilon_2}\Big\{ -x\log\Big(\frac{x}{\Lambda}\Big) + x\Big\}$$

$$- \frac{\varepsilon_1^2+\varepsilon_2^2+3\varepsilon_1\varepsilon_2}{12\varepsilon_1\varepsilon_2}\log\Big(\frac{x}{\Lambda}\Big) + \sum_{n=3}^{\infty}\frac{c_n x^{2-n}}{n(n-1)(n-2)}.$$

We put

$$F^{\mathrm{pert}}(\varepsilon_1,\varepsilon_2,a;\Lambda) := -\gamma_{\varepsilon_1,\varepsilon_2}(2a;\Lambda) - \gamma_{\varepsilon_1,\varepsilon_2}(-2a;\Lambda).$$

Then $F^{\mathrm{pert}}(\varepsilon_1,\varepsilon_2,a;\Lambda)$ is a Laurent series in $\varepsilon_1,\varepsilon_2$, whose coefficients are multiple-valued meromorphic functions in a,Λ. See [**34**, Appendix E] for the details. Finally we define

$$F(\varepsilon_1,\varepsilon_2,a;\Lambda,\vec{\tau}) := F^{\mathrm{pert}}(\varepsilon_1,\varepsilon_2,a;\Lambda) + F^{\mathrm{inst}}(\varepsilon_1,\varepsilon_2,a;\Lambda,\vec{\tau}),$$

$$F(\varepsilon_1,\varepsilon_2,a;\Lambda) := F(\varepsilon_1,\varepsilon_2,a;\Lambda,\vec{0}).$$

Formally one defines

$$Z(\varepsilon_1,\varepsilon_2,a;\Lambda,\vec{\tau})) := \exp(F^{\mathrm{pert}}(\varepsilon_1,\varepsilon_2,a;\Lambda))Z^{\mathrm{inst}}(\varepsilon_1,\varepsilon_2,a;\Lambda,\vec{\tau}).$$

2. Computation of the wallcrossing in terms of Hilbert schemes

Let X be a simply connected smooth projective surface with $p_g = 0$. In this section we will compute the wallcrossing of the Donaldson invariants of X in terms of intersection numbers of Hilbert schemes of points on X. Our result will be more generally about a refinement of the Donaldson invariants, also involving higher order μ-classes. In the next two sections we will specialize to the case that X is a smooth toric surface and relate this result to the Nekrasov partition function.

Notation 2.1. Let t be a variable. If Y is a variety and $b \in H^*(Y)[t]$, we denote by $[b]_d$ its part of degree d, where elements in $H^{2n}(Y)$ have degree n and t has degree 1.

If R is a ring, t a variable and $b \in R((t))$, we will denote for $i \in \mathbb{Z}$ by $[b]_{t^i}$ the coefficient of t^i of b.

If E is a torsion free sheaf of rank r on Y, then we put $\overline{\mathrm{ch}}(E) := \mathrm{ch}(E)\exp(-c_1(E)/r)$. We write $\overline{\mathrm{ch}}_i(E) := [\overline{\mathrm{ch}}(E)]_i$. We can view this as Chern character of E normalized by a twist with a rational line bundle, so that its first Chern class is zero. Note that in case $r = 2$, we have $-\overline{\mathrm{ch}}_2(E) = c_2(E) - c_1^2(E)/4$.

If E is a vector bundle of rank r on Y we write $c^t(E) := \sum_i c_i(E)t^{r-i}$ $= t^r c_{1/t}(E)$, with $c_t(E) := \sum_i c_i(E)t^i$.

Now we define a generalization of the μ-map and of the Donaldson invariants.

Definition 2.2. Fix an ample divisor H on X and fix c_1 and d. Assume that there is a universal sheaf $\mathcal{E}$ over $X \times M_H^X(c_1, d)$, and that $M_H^X(c_1, d)$ is of expected dimension d or empty for all $d \geq 0$. For a class $a \in H_i(X)$, and an integer $\rho \geq 1$, we put $\mu_\rho(a) := (-1)^\rho \overline{\mathrm{ch}}_{\rho+1}(\mathcal{E})/a \in H^{2\rho+2-i}(M_H^X(c_1, d))$. Note that the universal sheaf is well-defined up to a twist by the pullback of a line bundle from $M_H^X(c_1, d)$, thus μ_ρ is independent of the choice of the universal sheaf.

Let $b_1, \ldots, b_s$ be a homogeneous basis of $H_*(X)$. For all $\rho \geq 1$, let $\tau_1^\rho, \ldots, \tau_s^\rho$ be indeterminates, and put $\alpha_\rho := \sum_{k=1}^s a_k^\rho b_k \tau_k^\rho$, with $a_k^\rho \in \mathbb{Q}$. This means that $(\tau_k^\rho)_{k=1\ldots s}^{\rho \geq 1}$ is a coordinate system on the "large phase space" $\bigoplus_{\rho \geq 1} H_*(X)[\rho]$. We define

$$(2.3) \qquad \Phi_{c_1}^H\left(\exp\left(\sum_{\rho \geq 1} \alpha_\rho \right) \right) = \sum_{d \geq 0} \Lambda^d \int_{M_H^X(c_1,d)} \exp\left(\sum_{\rho \geq 1} \mu_\rho(\alpha_\rho) \right).$$

This is an element of $\mathbb{Q}[[\Lambda, (\tau_k^\rho)]]$. As by definition $\mu_1 = \mu$, our previous definition of $\Phi_{c_1}^H(\exp(\alpha z + px))$ is obtained by specializing $\alpha_\rho := 0$ for all $\rho > 1$.

We believe that we can define the invariants without the assumption that the moduli spaces are of expected dimensions as in the case of ordinary Donaldson invariants. This can be done once we generalize the blowup formula. This is a little delicate as higher Chern classes do not descend to Uhlenbeck compactifications.

2.1. The wallcrossing term. Let $\xi \in H^2(X, \mathbb{Z}) \setminus \{0\}$ be a class of type c_1. We say that ξ is *good* and W^ξ is a *good wall* if

(1) there is an ample divisor in W^ξ,
(2) $D + K_X$ is not effective for any divisor D with $W^{c_1(D)} = W^\xi$.

A sufficient condition for ξ to be good is that W^ξ contains an ample divisor H with $H \cdot K_X < 0$. Let ξ be a good class of type c_1.

Let $X^{[n]}$ be the Hilbert scheme of subschemes of length n on X. Let $Z_n(X) \subset X \times X^{[n]}$ be the universal subscheme. We write $X_2 := X \sqcup X$ and $X_2^{[l]} := \coprod_{n+m=l} X^{[n]} \times X^{[m]}$. Fix $l \in \mathbb{Z}_{\geq 0}$. Let $\mathcal{I}_1$ (resp. $\mathcal{I}_2$) be the sheaf on $X \times X_2^{[l]}$ whose restriction to $X \times X^{[n]} \times X^{[m]}$ is $p_{1,2}^*(\mathcal{I}_{Z_n(X)})$ (resp. $p_{1,3}^*(\mathcal{I}_{Z_m(X)})$), where $p_{1,2}: X \times X^{[n]} \times X^{[m]} \to X \times X^{[n]}$ (resp. $p_{1,3}: X \times X^{[n]} \times X^{[m]} \to X \times X^{[m]}$). Let $p : X \times X_2^{[l]} \to X_2^{[l]}$ be the projection. On $X_2^{[l]}$ we define

$$\mathcal{A}_{\xi,-} := \mathrm{Ext}_p^1(\mathcal{I}_2, \mathcal{I}_1(\xi)), \qquad \mathcal{A}_{\xi,+} := \mathrm{Ext}_p^1(\mathcal{I}_1, \mathcal{I}_2(-\xi)).$$

As ξ is good, $\mathcal{A}_{\xi,-}$, $\mathcal{A}_{\xi,+}$ are locally free on $X_2^{[l]}$. If ξ is understood, we also just write $\mathcal{A}_-$ and $\mathcal{A}_+$ instead of $\mathcal{A}_{\xi,-}$, $\mathcal{A}_{\xi,+}$. Let $\mathbb{P}_- := \mathbb{P}(\mathcal{A}_-^\vee)$ and $\mathbb{P}_+ := \mathbb{P}(\mathcal{A}_+^\vee)$ (we use the Grothendieck notation, i.e., this is the bundle

of 1-dimensional quotients). Let $\pi_\pm \colon \mathbb{P}_\pm \to X_2^{[l]}$ be the projection. Then $\mathbb{P}_\pm = \coprod_{n+m=l} \mathbb{P}_\pm^{n,m}$ with $\mathbb{P}_\pm^{n,m} = \pi_\pm^{-1}(X^{[n]} \times X^{[m]})$.

Now we define the wallcrossing term. We use the notations of the last section. For a coherent sheaf E of rank r on a variety Y, we view $1/c^t(E)$ as an element of $H^*(Y)[t^{-1}]$ via the formula

$$(2.4) \qquad \frac{1}{c^t(E)} = \frac{1}{t^r} \frac{1}{\sum_{i=0}^r c_i(E)\frac{1}{t^i}} = t^{-r} \sum_i s_i(E)t^{-i},$$

where r is the rank of E and the $s_i(E)$ are the Segre classes of E. If Y carries a Γ-action and E is equivariant, then $1/c^t(E) \in H_\Gamma^*(Y)[[t^{-1}]]$.

Definition 2.5. Let $\xi \in H^2(X, \mathbb{Z})$ be a good class of type c_1. For all $\rho \geq 1$ let α_ρ be as in (2.3). The *wallcrossing terms* are

$$\delta_{\xi,t}^X\Big(\exp\Big(\sum_{\rho\geq 1}\alpha_\rho)\Big)\Big) := \sum_{l\geq 0}\Lambda^{4l-\xi^2-3}$$

$$(2.6) \qquad \cdot \int_{X_2^{[l]}} \frac{\exp\Big(\sum_{\rho\geq 1}(-1)^\rho\big[\operatorname{ch}(\mathcal{I}_1)e^{\frac{\xi-t}{2}} + \operatorname{ch}(\mathcal{I}_2)e^{\frac{t-\xi}{2}}\big]_{\rho+1}/\alpha_\rho\Big)}{c^t(\mathcal{A}_{\xi,+})c^{-t}(\mathcal{A}_{\xi,-})},$$

$$\delta_\xi^X\Big(\exp\Big(\sum_{\rho\geq 1}\alpha_\rho\Big)\Big) := \Big[\delta_{\xi,t}^X\Big(\exp\Big(\sum_{\rho\geq 1}\alpha_\rho\Big)\Big)\Big]_{t^{-1}}.$$

$\delta_{\xi,t}^X\big(\exp(\alpha z + px)\big)$ and $\delta_\xi^X\big(\exp(\alpha z + px)\big)$ are defined by replacing α_1 by $\alpha z + px$ and α_ρ by 0 for $\rho \geq 2$ in $\delta_{\xi,t}^X\big(\exp\big(\sum_\rho \alpha_\rho\big)\big)$ and $\delta_\xi^X\big(\exp\big(\sum_\rho \alpha_\rho\big)\big)$. By (2.4) we see that

$$\delta_{\xi,t}^X\Big(\exp\Big(\sum_{\rho\geq 1}\alpha_\rho\Big)\Big) \in \Lambda^{-\xi^2-3}\mathbb{Q}[t, t^{-1}][[\Lambda, (\tau_k^\rho)]],$$

$$\delta_\xi^X\Big(\exp\Big(\sum_{\rho\geq 1}\alpha_\rho\Big)\Big) \in \mathbb{Q}[[\Lambda, (\tau_k^\rho)]]$$

(see Remark 2.7(1)).

Remark 2.7. (1) Fix $l \geq 0$. Write $d := 4l - \xi^2 - 3$ and let $E(t) := 1/(c^t(\mathcal{A}_{\xi,+})c^{-t}(\mathcal{A}_{\xi,-}))$ on $X_2^{[l]}$. Note that $\operatorname{rank}(\mathcal{A}_{\xi,+} \oplus \mathcal{A}_{\xi,-}) = d+1-2l$ (this follows from [8, Lemma 4.3]). If $d < 0$, then $d + 1 - 2l \leq 0$, thus $\mathcal{A}_{\xi,+} = 0 = \mathcal{A}_{\xi,-}$ and $E(t) = 1$, and thus the coefficient of Λ^d in $\delta_{\xi,t}^X\big(\exp\big(\sum \alpha_\rho\big)\big)$ is a polynomial in t. Let again d be arbitrary. We can write $E(t) = \sum_{i=0}^{2l} b_i t^{-(i+d+1-2l)}$, with $b_i \in H^{2i}(X_2^{[l]})$. Thus, if we give elements of $H^{2i}(X_2^{[l]})$ the degree i and t the degree 1, then $E(t)$ is homogeneous of degree $2l - d - 1$.

(2) Note that the factor Λ^d both in the definition of $\Phi_{c_1}^H$ and of δ_ξ^X is redundant. The coefficient of Λ^d in $\Phi_{c_1}^H(\exp(\sum \alpha_\rho))$ and $\delta_\xi^X(\exp(\sum \alpha_\rho))$ is a polynomial of weight d in the τ_k^ρ. Here the weight of τ_k^ρ is $\rho+1-i$ if

$b_k \in H_{2i}(X)$. For $\Phi_{c_1}^X$, this is clear because d is the complex dimension of $M_X^H(c_1, d)$. For δ_ξ^X, this follows easily from the last sentence of (1) and the fact that $X_2^{[l]}$ has dimension $2l$.

The aim of this section is to prove that the wallcrossing for the Donaldson invariants can be expressed as a sum over δ_ξ^X.

Proposition 2.8. *Let H_-, H_+ be ample divisors on X, which do not lie on a wall of type (c_1, d) for any $d \geq 0$. Let B_+ be the set of all classes ξ of type c_1 with $\langle \xi \cdot H_+ \rangle > 0 > \langle \xi \cdot H_- \rangle$. Assume that all classes in B_+ are good. Then*

$$\Phi_{c_1}^{H_+} \left(\exp \left(\sum_\rho \alpha_\rho \right) \right) - \Phi_{c_1}^{H_-} \left(\exp \left(\sum_\rho \alpha_\rho \right) \right)$$

$$= \sum_{\xi \in B_+} \delta_\xi^X \left(\exp \left(\sum_\rho \alpha_\rho \right) \right).$$

Remark 2.9.

From our final expression in Corollary 5.7, $\delta_\xi^X(\exp(\alpha z + px))$ is compatible with Fintushel-Stern's blowup formula [**15**]. (See [**20**, §4.2] and [**34**, §6].) Therefore it is enough to prove the proposition after we blowup X at sufficiently many times, as we did for the definition of $\Phi_{c_1}^H$. In particular, we may assume $M_{H_\pm}^X(c_1, d)$ is of expected dimension without loss of generality. However, the blowup does not make walls good in general, so we one needs different methods to prove the proposition for general wall. Let $p \colon X \times X_2^{[l]} \to X_2^{[l]}$ and $q \colon X \times X_2^{[l]} \to X$ be the projections. In [**30**, Thm 1.12] the proposition is proved for general walls with $\mathcal{A}_{\xi,+}$, $\mathcal{A}_{\xi,-}$ replaced by $-p_!(\mathcal{I}_2^\vee \otimes \mathcal{I}_1 \otimes q^! \xi)$, $-p_!(\mathcal{I}_1^\vee \otimes \mathcal{I}_2 \otimes q^! \xi^\vee))$. The proof uses virtual fundamental classes and virtual localization.

In the rest of this section we will show Prop. 2.8. Let $d \geq 0$ be arbitrary. It is enough to show that the coefficients of Λ^d on both sides are equal. It is known that $M_H^X(c_1, d)$ and $\Phi_{c_1}^H$ are constant as long as H stays in the same chamber of type (c_1, d) and only changes when H crosses a wall of type (c_1, d). Following [**8**] and [**17**] we get the following description of the change of moduli spaces. Let B_d be the set of all $\xi \in B_+$ which define a wall of type (c_1, d). For the moment assume for simplicity that B_d consists of a single element ξ. Let $l := (d + 3 + \xi^2)/4 \in \mathbb{Z}_{\geq 0}$. Write $M_{0,l} := M_{H_-}^X(c_1, d)$. Then successively for all $n = 0, \ldots, l$ write $m := l - n$. Then one has the following: $M_{n,m}$ contains a closed subscheme $E_-^{n,m}$ isomorphic to $\mathbb{P}_-^{n,m}$ and $M_{n,m}$ is nonsingular in a neighbourhood of $E_-^{n,m}$. Let $\widehat{M}_{n,m}$ be the blow up of $M_{n,m}$ along $E_-^{n,m}$. The exceptional divisor is isomorphic to the fiber product $D^{n,m} := \mathbb{P}_-^{n,m} \times_{X^{[n]} \times X^{[m]}} \mathbb{P}_+^{n,m}$. We can blow down $\widehat{M}_{n,m}$ in $D^{n,m}$ in the other fiber direction to obtain a new variety $M_{n+1,m-1}$.

The image of $D^{n,m}$ is a closed subset $E_+^{n,m}$ isomorphic to $\mathbb{P}_+^{n,m}$ and $M_{n+1,m-1}$ is smooth in a neighbourhood of $E_+^{n,m}$.

The transformation from $M_{n,m}$ to $M_{n+1,m-1}$ does not have to be birational. It is possible that $E_+^{n,m} = \emptyset$, i.e., $\mathcal{A}_+ = 0$. As we know that $\mathrm{rank}(\mathcal{A}_-) + \mathrm{rank}(\mathcal{A}_+) + 2l = d + 1$, this happens if and only if $E_-^{n,m}$ has dimension d and thus by the smoothness of $M_{n,m}$ near $E_-^{n,m}$, we get that $E_-^{n,m}$ is a connected component of $M_{n,m}$. Then blowing up along $E_-^{n,m}$ just means deleting $E_-^{n,m}$. Thus in this case $M_{n+1,m-1} = M_{n,m} \setminus E_-^{n,m}$. Similarly we have $E_-^{n,m} = \emptyset$, i.e., $\mathcal{A}_- = 0$, if and only if $E_+^{n,m}$ is a connected component of $M_{n+1,m-1}$ and $M_{n+1,m-1} = M_{n,m} \sqcup E_+^{n,m}$. Below, if the transformation from $M_{n,m}$ to $M_{n+1,m-1}$ is birational, we say we are in case (1), otherwise in case (2).

Finally we have $M_{l+1,-1} = M_{H_+}^X(c_1, d)$. If B_d consists of more than one element, one obtains $M_{H_+}(c_1, d)$ from $M_{H_-}(c_1, d)$ by iterating this procedure in a suitable order over all $\xi \in B_+$.

Fix ξ in B_d. Fix $n, m \in \mathbb{Z}_{\geq 0}$ with $n + m = l := (d + 3 + \xi^2)/4$. We write $M_- := M_{n,m}$, $M_+ := M_{n+1,m-1}$. Let $\overline{\mathcal{E}}_\pm$ be universal sheaves on $X \times M_\pm$ respectively. Let $E_- := E_-^{n,m}$, $E_+ = E_+^{n,m}$. Let $\widetilde{M}$ be the blowup of M_- along E_-, and denote by D the exceptional divisor (which is also the exceptional divisor of the blowup of M_+ along E_+). Write $D' := X \times D$ and let $j : D \to \widetilde{M}$, $j' : X \times D \to X \times \widetilde{M}$ be the embeddings. Let $\mathcal{E}_-, \mathcal{E}_+$ be the pullbacks of $\overline{\mathcal{E}}_-, \overline{\mathcal{E}}_+$ to $X \times \widetilde{M}$.

Notation 2.10. Let $H \in \mathbb{Q}[(x_n)_{n>0}]$ be a polynomial. Let $a := (a_n)_{n>0}$ with $a_n \in H_*(X)$. For any variety Y and any class $A \in H^*(X \times Y)[[t]]$ we put $H(A/a) := H(([A]_n/a_n)_{n>0}) \in H^*(Y)[[t]]$. On $X \times X^{[n]} \times X^{[m]}$, denote $C(t) := \mathrm{ch}(\mathcal{I}_1)e^{\frac{\xi-t}{2}} + \mathrm{ch}(\mathcal{I}_2)e^{\frac{-\xi+t}{2}}$, and $C_i(t) := [C(t)]_i$.

We denote by τ_- (resp. τ_+) the universal quotient line bundle on $\mathbb{P}_- = \mathbb{P}(\mathcal{A}_-^\vee)$ (resp. $\mathbb{P}_+ = \mathbb{P}(\mathcal{A}_+^\vee)$). For a sheaf $\mathcal{F}$ and a divisor B, we write $\mathcal{F}(B)$ instead of $\mathcal{F} \otimes \mathcal{O}(B)$.

For a class $b \in H^*(X)$ we also denote by b its pullback to $X \times Y$ for a variety Y. We write $\mathcal{I}_1, \mathcal{I}_2$ for the pullback of $\mathcal{I}_1, \mathcal{I}_2$ to D' and we write τ_+, τ_- also for their pullbacks to D and D'.

We will show

$$(2.11) \quad \int_{M_+} H(\overline{\mathrm{ch}}(\overline{\mathcal{E}}_+)/a) - \int_{M_-} H(\overline{\mathrm{ch}}(\overline{\mathcal{E}}_-)/a)$$

$$= \int_{X^{[n]} \times X^{[m]}} \left[\frac{H(C(t)/a)}{c^t(\mathcal{A}_{\xi,+})c^{-t}(\mathcal{A}_{\xi,-})} \right]_{t^{-1}}.$$

Formula (2.11) implies Proposition 2.8 by summing over all $\xi \in B_+$, all $d \geq 0$ and over all n, m with $n + m = (d + \xi^2 + 3)/4$.

For the next three Lemmas assume that we are in case (1). Then by the projection formula $\int_{M_\pm} H(\overline{\mathrm{ch}}(\overline{\mathcal{E}}_\pm/a))) = \int_{\widetilde{M}} H(\overline{\mathrm{ch}}(\mathcal{E}_\pm/a))$, thus it is

enough to prove (2.11) with the left-hand side replaced by

$$\int_{\widetilde{M}} \big(H(\overline{\mathrm{ch}}(\mathcal{E}_+)/a) - H(\overline{\mathrm{ch}}(\mathcal{E}_-)/a)\big).$$

Lemma 2.12. $C(-\tau_-) = \overline{\mathrm{ch}}((j')^*(\mathcal{E}_-))$, $C(\tau_+) = \overline{\mathrm{ch}}((j')^*(\mathcal{E}_+))$ *and*

$$\overline{\mathrm{ch}}(\mathcal{E}_+) - \overline{\mathrm{ch}}(\mathcal{E}_-) = -j'_*\Big(\frac{C(t) - C(-s)}{s + t}\big|_{\substack{s=\tau_- \\ t=\tau_+}}\Big).$$

Proof. Write $\mathcal{F}_1 := \mathcal{I}_1(\frac{c_1+\xi}{2})$, $\mathcal{F}_2 := \mathcal{I}_2(\frac{c_1-\xi}{2})$. By [**8**, Section 5] we have the following facts:

(1) There exist a line bundle λ on D and an exact sequence $0 \to \mathcal{F}_1(\lambda) \to (j')^*(\mathcal{E}_-) \to \mathcal{F}_2(-\tau_- + \lambda) \to 0$,

(2) $\mathcal{E}_+$ can be defined by the exact sequence $0 \to \mathcal{E}_+ \to \mathcal{E}_- \to j'_*(\mathcal{F}_2(-\tau_- + \lambda)) \to 0$.

(3) We have the exact sequence $0 \to \mathcal{F}_2(\tau_+ + \lambda) \to (j')^*(\mathcal{E}_+) \to \mathcal{F}_1(\lambda) \to 0$.

In particular, $\overline{\mathrm{ch}}((j')^*\mathcal{E}_-) = C(-\tau_-)$, $\overline{\mathrm{ch}}((j')^*\mathcal{E}_+) = C(\tau_+)$.

Write $c_+ := c_1(\mathcal{E}_+), c_- := c_1(\mathcal{E}_-)$. As $c_1(j'_*(\mathcal{F}_2(-\tau_- + \lambda))) = D'$, we see that $c_+ = c_- - D'$. We also have $(j')^*(c_+) = c_1 + \tau_+ + 2\lambda$. Thus we get

$$\overline{\mathrm{ch}}(\mathcal{E}_+) = \big(\mathrm{ch}(\mathcal{E}_-) - \mathrm{ch}(j'_*\mathcal{F}_2(-\tau_- + \lambda))\big)e^{-c_+/2}$$
$$= \overline{\mathrm{ch}}(\mathcal{E}_-)e^{D'/2} - \mathrm{ch}(j'_*\mathcal{F}_2(-\tau_- + \lambda))e^{-c_+/2}.$$

Thus $\overline{\mathrm{ch}}(\mathcal{E}_+) - \overline{\mathrm{ch}}(\mathcal{E}_-) = (e^{D'/2} - 1)\overline{\mathrm{ch}}(\mathcal{E}_-) - \mathrm{ch}(j'_*\mathcal{F}_2(-\tau_- + \lambda))e^{-c_+/2}$. As $(j')^*D' = -\tau_+ - \tau_-$ by [**8**, Cor. 4.7], we get by the Grothendieck-Riemann-Roch Theorem and the projection formula

$$\mathrm{ch}\big(j'_*\mathcal{F}_2(-\tau_- + \lambda)\big)e^{-c_+/2}$$
$$= j'_*\Big(\frac{1 - e^t}{-t}\big|_{t=\tau_+ + \tau_-}\mathrm{ch}(\mathcal{F}_2(-\tau_- + \lambda))\Big)e^{-c_+/2}$$
$$= j'_*\Big(\frac{1 - e^t}{-t}\big|_{t=\tau_+ + \tau_-}\mathrm{ch}(\mathcal{I}_2)e^{\frac{-\xi - 2\tau_- - \tau_+}{2}}\Big)$$
$$= -j'_*\Big(\Big(\frac{1}{s + t}\big(\mathrm{ch}(\mathcal{I}_2)e^{\frac{-\xi - 2s - t}{2}} - \mathrm{ch}(\mathcal{I}_2)e^{\frac{-\xi + t}{2}}\big)\Big)\big|_{\substack{s=\tau_- \\ t=\tau_+}}\Big).$$

On the other hand, as $e^{D'/2} - 1$ is divisible by D', we get

$$(e^{D'/2} - 1)\overline{\mathrm{ch}}(\mathcal{E}_-) = j'_*\Big(\frac{1 - e^{-t/2}}{t}\big|_{t=\tau_+ + \tau_-}\overline{\mathrm{ch}}((j')^*\mathcal{E}_-)\Big)$$
$$= j'_*\Big(\frac{1}{s + t}\big(\mathrm{ch}(\mathcal{I}_1)e^{\frac{\xi + s}{2}} + \mathrm{ch}(\mathcal{I}_2)e^{\frac{-\xi - s}{2}}$$
$$- \mathrm{ch}(\mathcal{I}_1)e^{\frac{\xi - t}{2}} - \mathrm{ch}(\mathcal{I}_2)e^{\frac{-\xi - 2s - t}{2}}\big)\big|_{\substack{s=\tau_- \\ t=\tau_+}}\Big),$$

and the result follows.   q.e.d.

Lemma 2.13.

$$H(\overline{\mathrm{ch}}(\mathcal{E}_+)/a) - H(\overline{\mathrm{ch}}(\mathcal{E}_-)/a) = -j_* \Big(\frac{H(C(t)/a) - H(C(-s)/a)}{s+t} \Big|_{\substack{s=\tau_- \\ t=\tau_+}} \Big).$$

In particular,

$$\int_{\widetilde{M}} \big(H(\overline{\mathrm{ch}}(\mathcal{E}_+)/a) - H(\overline{\mathrm{ch}}(\mathcal{E}_-)/a) \big)$$

$$= - \int_D \frac{H(C(t)/a) - H(C(-s)/a)}{s+t} \Big|_{\substack{s=\tau_- \\ t=\tau_+}}.$$

Proof. We can assume that H is homogeneous of degree k. We make induction over k, the case $k = 0$ being trivial. In case $k = 1$, we have by the previous Lemma

$$\overline{\mathrm{ch}}_i(\mathcal{E}_+)/a_i - \overline{\mathrm{ch}}_i(\mathcal{E}_-)/a_i$$

$$= -j'_* \Big(\frac{C_i(t) - C_i(-s)}{s+t} \Big|_{\substack{s=\tau_- \\ t=\tau_+}} \Big)/a_i$$

$$= -j_* \Big(\frac{(C_i(t) - C_i(-s))/a_i}{s+t} \Big|_{\substack{s=\tau_- \\ t=\tau_+}} \Big).$$

Now let k be general. As the claim is linear in H, we can assume that $H = x_i H'$, with $\deg(H') = k - 1$. Thus we get by induction

$$H(\overline{\mathrm{ch}}(\mathcal{E}_+)/a) - H(\overline{\mathrm{ch}}(\mathcal{E}_-)/a)$$

$$= \big(\overline{\mathrm{ch}}_i(\mathcal{E}_+)/a_i - \overline{\mathrm{ch}}_i(\mathcal{E}_-)/a_i \big) H'(\overline{\mathrm{ch}}(\mathcal{E}_+)/a)$$

$$\quad + \overline{\mathrm{ch}}_i(\mathcal{E}_-)/a_i \cdot \big(H'(\overline{\mathrm{ch}}(\mathcal{E}_+)/a) - H'(\overline{\mathrm{ch}}(\mathcal{E}_-)/a) \big)$$

$$= -j_* \Big(\Big(\frac{(C_i(t) - C_i(-s))/a_i}{s+t} H'\big(\overline{\mathrm{ch}}((j')^*\mathcal{E}_+)/a \big)$$

$$\quad + \big(\overline{\mathrm{ch}}_i((j')^*\mathcal{E}_-)/a_i \big) \frac{H'(C(t)/a) - H'(C(-s)/a)}{s+t} \Big) \Big|_{\substack{s=\tau_- \\ t=\tau_+}} \Big)$$

$$= -j_* \Big(\frac{1}{s+t} \big((C_i(t) - C_i(-s))/a_i \cdot H'(C(t)/a)$$

$$\quad + C_i(-s)/a_i \cdot (H'(C(t)/a) - H'(C(-s)/a)) \big) \Big|_{\substack{s=\tau_- \\ t=\tau_+}} \Big)$$

$$= -j_* \Big(\frac{H((C(t)/a) - H(C(-s))/a)}{s+t} \Big|_{\substack{s=\tau_- \\ t=\tau_+}} \Big).$$

This shows the first statement; the second follows immediately by the projection formula.   q.e.d.

Recall that $D = \mathbb{P}(\mathcal{A}_-^\vee) \times_{X^{[n]} \times X^{[m]}} \mathbb{P}(\mathcal{A}_+^\vee)$. Let $\pi \colon D \to X^{[n]} \times X^{[m]}$ and $p_\pm \colon \mathbb{P}(\mathcal{A}_\pm^\vee) \to X^{[n]} \times X^{[m]}$ be the projections. We have reduced the computation of $\int_{\widetilde{M}} \big(H(\overline{\mathrm{ch}}(\mathcal{E}_+)/a) - H(\overline{\mathrm{ch}}(\mathcal{E}_+)/a) \big)$ to an integral over D, which we now push down to $X^{[n]} \times X^{[m]}$.

Lemma 2.14.
$$\pi_*\left(-\frac{H(C(t)/a) - H(C(-s)/a)}{s+t}\Big|_{\substack{s=\tau_- \\ t=\tau_+}}\right) = \left[\frac{H(C(t)/a)}{c^t(\mathcal{A}_+)c^{-t}(\mathcal{A}_-)}\right]_{t^{-1}}.$$

Proof. For a vector bundle E of rank e on a variety Y, let τ be the tautological quotient line bundle on $p : \mathbb{P}(E^\vee) \to Y$. Then

$$\sum_{n\geq 0} p_*(\tau^n)t^{-n-1} = t^{-e}\sum_n p_*(\tau^{n+e-1})t^{-n} = t^{-e}\frac{1}{\sum_n c_n(E)t^{-n}} = \frac{1}{c^t(E)},$$

and similarly $\sum_n p_*((-\tau)^n)t^{-n-1} = -\frac{1}{c^{-t}(E)}$. Thus we get

$$\pi_*\left(\frac{\tau_+^k - (-\tau_-)^k}{\tau_+ + \tau_-}\right)$$

$$= \sum_{i+j=k-1} \pi_{+*}(\tau_+^i)\pi_{-*}((-\tau_-)^j)$$

$$= \left[\left(\sum_n \pi_{+*}(\tau_+^n)t^{-n-1}\right)\left(\sum_n \pi_{-*}((-\tau_-)^n)t^{-n-1}\right)\right]_{t^{-k-1}}$$

$$= -\left[\frac{t^k}{c^t(\mathcal{A}_+)c^{-t}(\mathcal{A}_-)}\right]_{t^{-1}}.$$

We write $H(C(t)/a) = \sum_{k\geq 0} t^k Q_k$ with $Q_k \in H^*(X^{[n]} \times X^{[m]})$. Then

$$\pi_*\left(-\frac{H(C(t)/a) - H(C(-s)/a)}{t+s}\Big|_{\substack{s=\tau_- \\ t=\tau_+}}\right)$$

$$= -\sum_k \pi_*\left(\frac{\tau_+^k - (-\tau_-)^k}{\tau_+ + \tau_-}\right)Q_k$$

$$= \sum_k \left[\frac{t^k Q_k}{c^t(\mathcal{A}_+)c^{-t}(\mathcal{A}_-)}\right]_{t^{-1}} = \left[\frac{H(C(t)/a)}{c^t(\mathcal{A}_+)c^{-t}(\mathcal{A}_-)}\right]_{t^{-1}}.$$

q.e.d.

The projection formula and Lemmas 2.13, 2.14 imply formula (2.11). Thus we have shown (2.11) in case (1).

In case (2), we can assume by symmetry that $\mathbb{P}_+ = \emptyset$, thus $\mathcal{A}_+ = 0$ and $\mathcal{A}_-$ has rank $d+1-2l$. Then we have

$$\int_{M_+} H(\overline{\mathrm{ch}}(\overline{\mathcal{E}}_+)/a) - \int_{M_-} H(\overline{\mathrm{ch}}(\overline{\mathcal{E}}_-)/a)$$

$$= -\int_{\mathbb{P}_-} H(\overline{\mathrm{ch}}(\overline{\mathcal{E}}_-)/a) = -\int_{X^{[n]} \times X^{[m]}} \pi_{-*}(H(\overline{\mathrm{ch}}(\overline{\mathcal{E}}_-)/a)).$$

Denote by $j : \mathbb{P}_- \to M_-$ and $j' : X \times \mathbb{P}_- \to X \times M_-$ the embeddings. As before write $\mathcal{F}_1 := \mathcal{I}_1(\frac{c_1+\xi}{2})$, $\mathcal{F}_2 := \mathcal{I}_2(\frac{c_1-\xi}{2})$. By [**8**, Lemma 4.3] and the universal property of M_- there is line bundle λ on $\mathbb{P}_-$ and an exact sequence $0 \to \mathcal{F}_1(\lambda) \to (j')^*(\overline{\mathcal{E}}_-) \to \mathcal{F}_2(-\tau_- + \lambda) \to 0$. In particular,

as before, $\overline{\mathrm{ch}}((j')^*\overline{\mathcal{E}}_-) = C(-\tau_-)$. The arguments of Lemma 2.14 show that $-\pi_{-*}(-\tau_-)^k = \left[\frac{t^k}{c^{-t}(\mathcal{A}_-)}\right]_{t^{-1}}$, and in the same way as in the proof of Lemma 2.14 it follows that $-\pi_{-*}(H(C(-\tau_-)/a)) = \left[\frac{H(C(t)/a)}{c^{-t}(\mathcal{A}_-)}\right]_{t^{-1}}$. As $c^t(\mathcal{A}_+) = 1$, this shows (2.11) also in case (2) and thus finishes the proof of Proposition 2.8.

3. Comparison with the partition function

For the next two sections let X be a smooth projective toric surface over $\mathbb{C}$, in particular X is simply connected and $p_g(X) = 0$. X carries an action of $\Gamma := \mathbb{C}^* \times \mathbb{C}^*$ with finitely many fixed points, which we will denote by $p_1, \ldots, p_\chi$, where χ is the Euler number of X. Let $w(x_i), w(y_i)$ be the weights of the Γ-action on $T_{p_i}X$. Then there are local coordinates x_i, y_i at p_i, so that $(t_1, t_2)x_i = e^{-w(x_i)}x_i$, $(t_1, t_2)y_i = e^{-w(y_i)}y_i$. By definition $w(x_i)$ and $w(y_i)$ are linear forms in ε_1 and ε_2. For $\beta \in H^*_\Gamma(X)$ or $\beta \in H^\Gamma_*(X)$, we denote by $\iota^*_{p_i}\beta$ its pullback to the fixed point p_i. More generally, if Γ acts on a nonsingular variety Y and $W \subset Y$ is invariant under the Γ-action, we denote by $\iota^*_W : H^*_\Gamma(Y) \to H^*_\Gamma(W)$ the pullback homomorphism.

Note that T_X and the canonical bundle are canonically equivariant. Thus any polynomial in the Chern classes $c_i(X)$ and K_X is canonically an element of $H^*_\Gamma(X)$.

3.1. Equivariant Donaldson invariants and equivariant wall-crossing. We start by defining an equivariant version of the Donaldson invariants and the wallcrossing terms. For $t \in \Gamma$ denote by F_t the automorphism $X \to X; x \mapsto t \cdot x$. Then Γ acts on $X_2^{[l]}$ by $t \cdot (\mathcal{I}_{Y_1}, \mathcal{I}_{Y_2}) = ((F_t^{-1})^*\mathcal{I}_{Y_1}, (F_t^{-1})^*\mathcal{I}_{Y_2})$ and on $X \times X_2^{[l]}$ by $t \cdot (x, \mathcal{I}_{Y_1}, \mathcal{I}_{Y_2}) = (F_t(x), (F_t^{-1})^*\mathcal{I}_{Y_1}, (F_t^{-1})^*\mathcal{I}_{Y_2})$ and the sheaves $\mathcal{I}_1, \mathcal{I}_2$ are Γ-equivariant. Similarly Γ acts on $X \times M_X^H(c_1, d)$ by $t \cdot (x, E) = (F_t(x), (F_t^{-1})^*E)$. Let $\mathcal{E}$ be a universal sheaf over $X \times M_X^H(c_1, d)$; then one can show that $\mathcal{E}$ has a lifting to a Γ-equivariant sheaf, unique up to twist by a character. Thus an equivariant universal sheaf is unique up to twist by an equivariant line bundle.

Definition 3.1. We define the equivariant Donaldson invariants $\widetilde{\Phi}^H_{c_1}(\exp(\alpha z + px))$ by the right-hand side of (1.1), where now $\alpha \in H^\Gamma_2(X)$ and $p \in H^\Gamma_0(X)$ is a lift of the class of a point, μ is defined using the equivariant Chern classes of $\mathcal{E}$, and $\int_{M^X_H(c_1,d)}$ means pushforward to a point in equivariant cohomology. We assume that the moduli spaces $M^X_H(c_1, d)$ have dimension equal to the expected dimension d. If $\mathcal{E}$ is an equivariant torsion-free sheaf of rank r we define $\overline{\mathrm{ch}}(\mathcal{E}) := \mathrm{ch}(\mathcal{E})e^{-\frac{c_1(\mathcal{E})}{r}}$, where we now use equivariant Chern character and first Chern class and define $\mu_\rho(\beta) := (-1)^\rho \overline{\mathrm{ch}}_{\rho+1}(\mathcal{E})/\beta$. Let $b_1, \ldots, b_s$ be a homogeneous

basis of $H^*_\Gamma(X)$ as a free $\mathbb{Q}[\varepsilon_1, \varepsilon_2]$-module. For all $\rho \geq 1$ let $\tau^\rho_1, \ldots \tau^\rho_s$ be indeterminates, and put $\alpha_\rho := \sum_{k=1}^s a^\rho_k b_k \tau^\rho_k$, with $a^\rho_k \in \mathbb{Q}[\varepsilon_1, \varepsilon_2]$. Using this we define $\widetilde{\Phi}^H_{c_1}\big(\exp\big(\sum_\rho \alpha_\rho\big)\big) \in \mathbb{Q}[\varepsilon_1, \varepsilon_2][[\Lambda, (\tau^\rho_k)]]$ by the right-hand side of (2.3). As the equivariant universal sheaf is unique up to twist by an equivariant line bundle, $\widetilde{\Phi}^H_{c_1}(\exp(\alpha z + px))$ and $\widetilde{\Phi}^H_{c_1}\big(\exp\big(\sum_\rho \alpha_\rho\big)\big)$ are independent of the choice of equivariant universal bundle.

We cannot hope to extend this naive definition without the assumption that the moduli spaces are of expected dimensions. This is because we can blowup only at the fixed points of the torus action and cannot avoid the support of the cycles representing α_ρ. Here we probably need to use virtual fundamental classes as in [**30**]. Then to prove that its specialization coincides with the ordinary invariants, we need to prove the blowup formula in the context of virtual fundamental classes.

Let $\xi \in H^2(X, \mathbb{Z})$ be an equivariant lifting of a good class of type c_1. Then $\mathcal{I}_1, \mathcal{I}_2, \mathcal{A}_{\xi,+}$ and $\mathcal{A}_{\xi,-}$ are in a natural way equivariant sheaves on $X^{[l]}_2$, and the equivariant wallcrossing terms $\widetilde{\delta}^X_{\xi,t}(\exp(\sum_{\rho \geq 1} \alpha_\rho))$, $\widetilde{\delta}^X_{\xi,t}(\exp(\alpha z + px))$ are defined by the right-hand side of formulas (2.6), where now $\int_{X^{[l]}_2}$ stands for equivariant pushforward to a point, and

$$\widetilde{\delta}^X_\xi\Big(\exp\Big(\sum_{\rho \geq 1} \alpha_\rho\Big)\Big) := \Big[\widetilde{\delta}^X_{\xi,t}\Big(\exp\Big(\sum_{\rho \geq 1} \alpha_\rho\Big)\Big)\Big]_{t^{-1}},$$

$$\widetilde{\delta}^X_\xi(\exp(\alpha z + px)) := [\widetilde{\delta}^X_{\xi,t}(\exp(\alpha z + px))]_{t^{-1}}.$$

By (2.4) we see that $\widetilde{\delta}^X_{\xi,t}(\exp(\sum_{\rho \geq 1} \alpha_\rho)) \in \Lambda^{-\xi^2 - 3}\mathbb{Q}[\varepsilon_1, \varepsilon_2]((t^{-1}))[[\Lambda, (\tau^\rho_k)]]$. Thus $\widetilde{\delta}^X_\xi(\exp(\sum_{\rho \geq 1} \alpha_\rho)) \in \mathbb{Q}[\varepsilon_1, \varepsilon_2][[\Lambda, (\tau^\rho_k)]]$ and by definition $\widetilde{\delta}^X_{\xi,t}(\exp(\sum_{\rho \geq 1} \alpha_\rho))|_{\varepsilon_1 = \varepsilon_2 = 0} = \delta^X_{\xi,t}(\exp(\sum_{\rho \geq 1} \alpha_\rho))$.

Remark 3.2. Note that the coefficient of Λ^d in $\widetilde{\delta}^X_\xi(\exp(\sum_\rho \alpha_\rho))$ is not a polynomial of weight d in the τ^ρ_k (as in Remark 2.7) but has contributions of different weights. Arguing as in Rem. 2.7 one sees that the coefficient of Λ^d is a sum of terms of weight $\geq d$. Thus the variable Λ in the definition of $\widetilde{\delta}^X_\xi$ is not redundant.

Under the assumptions of Proposition 2.8 let $\widetilde{B}_+$ be a set consisting of one equivariant lift ξ for each class of type c_1 with $\langle \xi, H_+ \rangle > 0 > \langle \xi, H_- \rangle$. Then the same proof as before (with all sheaves and classes replaced by the equivariant versions) shows that the statement of the proposition holds with $\Phi^{H_+}_{c_1}$, $\Phi^{H_-}_{c_1}$, B_+, δ^X_ξ replaced by $\widetilde{\Phi}^{H_+}_{c_1}$, $\widetilde{\Phi}^{H_-}_{c_1}$, $\widetilde{B}_+$, $\widetilde{\delta}^X_\xi$ respectively, i.e., the wallcrossing of the equivariant Donaldson invariants is given by the equivariant wallcrossing terms.

In this section we want to give a formula expressing $\widetilde{\delta}^X_{\xi,t}$ in terms of the Nekrasov partition function Z.

Theorem 3.3.

$$\widetilde{\delta}^X_{\xi,t}\Big(\exp\Big(\sum_\rho \alpha_\rho\Big)\Big)$$

$$= \frac{1}{\Lambda}\exp\Big(\sum_{i=1}^{\chi} F\big(w(x_i), w(y_i), \tfrac{t-\iota^*_{p_i}\xi}{2}; \Lambda, ((-1)^\rho\iota^*_{p_i}\alpha_\rho)_\rho)\big)\Big).$$

Note that the left-hand side lies in $\Lambda^{-\xi^2-3}\mathbb{Q}[\varepsilon_1, \varepsilon_2]((t^{-1}))[[\Lambda, (\tau^\rho_k)]]$. In the course of the proof we will also have to show how one can interpret the right-hand side, so that both sides lie in the same ring.

It is tempting to write Theorem 3.3 as

$$\widetilde{\delta}^X_{\xi,t}\Big(\exp\Big(\sum_\rho \alpha_\rho\Big)\Big) = \frac{1}{\Lambda}\prod_{i=1}^{\chi} Z\big(w(x_i), w(y_i), \tfrac{t-\iota^*_{p_i}\xi}{2}; \Lambda, ((-1)^\rho\iota^*_{p_i}\alpha_\rho)_\rho\big),$$

but it appears difficult to give a meaning to the right-hand side of this equation (other than as an abbreviation for the right-hand side of Theorem 3.3).

As a first step we will show that, up to a correction term, there is an expression for $\widetilde{\delta}^X_{\xi,t}$ in terms of the instanton part of the partition function. In a second step we will see that this correction term is accounted for by the perturbation part.

3.2. The instanton part. We start by reviewing some results and definitions from [**9**]. The fixed points of the Γ-action on $X^{[l]}_2$ are the pairs (Z_1, Z_2) of zero-dimensional subschemes with support in $\{p_1, \ldots, p_\chi\}$ with $\mathrm{len}(Z_1) + \mathrm{len}(Z_2) = l$ and such that each I_{Z_α, p_i} is generated by monomials in x_i, y_i. We associate to (Z_1, Z_2) the χ-tuple $(\vec{Y}^1, \ldots, \vec{Y}^\chi)$ with $\vec{Y}^i = (Y^i_1, Y^i_2)$, where

$$Y^i_\alpha = \big\{(n,m) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \mid x^{n-1}_i y^{m-1}_i \notin I_{Z_\alpha, p_i}\big\}.$$

We write $|Y^i_\alpha|$ for the number of elements of Y^i_α and $|\vec{Y}^i| := |Y^i_1| + |Y^i_2|$. This gives a bijection from the fixed point set $(X^{[l]}_2)^\Gamma$ to the set of the χ-tuples of pairs of Young diagrams $(\vec{Y}^1, \ldots, \vec{Y}^\chi)$, with $\sum_i |Y^i| = l$.

We denote also by x_i, y_i the one dimensional Γ-modules given by $t \cdot x_i = e^{-w(x_i)}x_i$, $t \cdot y_i = e^{-w(y_i)}y_i$. If L is an equivariant line bundle on X, the fiber $L(p_i)$ at a fixed point and the cohomology groups $H^i(X, L)$ are in a natural way Γ-modules.

The following follows easily from of $N^{\vec{Y}^i}_{\alpha,\beta}(\varepsilon_1, \varepsilon_2, a)$ in [**33**] and [**9**, Lemma 3.2]. In fact it is basically a reformulation of [**9**, Lemma 3.2] and a straightforward generalization of [**33**, Thm. 3.4]. In order to get the correct result one has to take into account the following:

(1) The formulas in [**9**] are for $V = L \oplus (L \otimes K_X)$ instead of L. But the proof only uses that $H^0(V) = H^2(V) = 0$.

(2) Our convention for the Γ-action on $X_2^{[l]}$ differs from that in [**9**], which is $t \cdot (\mathcal{I}_{Y_1}, \mathcal{I}_{Y_2}) = (F_t^* \mathcal{I}_{Y_1}, F_t^* \mathcal{I}_{Y_2}))$. This changes the Γ-modules x_i, y_i to x_i^{-1}, y_i^{-1}.

(3) In [**33**, Thm. 3.4] the case of $\widehat{\mathbb{C}}^2$ was studied and the argument shows that $\mathrm{Ext}^1(\mathcal{I}_{Z_\alpha}, \mathcal{I}_{Z_\beta} \otimes L) = H^1(X, L) + \bigoplus_i \mathrm{Ext}^1(\mathcal{I}_{Z_\alpha, p_i}, \mathcal{I}_{Z_\beta, p_i}(-\ell_\infty)) \otimes L(p_i)$.

Lemma 3.4. *Let* $(Z_1, Z_2) \in (X_2^{[l]})^\Gamma$ *correspond to* $(\vec{Y}^1, \ldots, \vec{Y}^\chi)$ *under the above bijection. Let* L *be a* Γ-*equivariant line bundle on* X, *such that* $c_1(L)$ *is good. We have in the Grothendieck group of* Γ-*modules*

$$(3.5) \qquad T_{(Z_1, Z_2)} X_2^{[l]} = \sum_{i=1}^{\chi} \sum_{\gamma=1}^{2} N_{\gamma, \gamma}^{\vec{Y}^i}(x_i, y_i, L(p_i)^{-\frac{1}{2}}),$$

$$(3.6) \qquad \mathrm{Ext}^1(\mathcal{I}_{Z_2}, \mathcal{I}_{Z_1} \otimes L) = H^1(X, L) + \sum_{i=1}^{\chi} N_{2,1}^{\vec{Y}^i}(x_i, y_i, L(p_i)^{-\frac{1}{2}}),$$

$$(3.7) \quad \mathrm{Ext}^1(\mathcal{I}_{Z_1}, \mathcal{I}_{Z_2} \otimes L^{-1}) = H^1(X, L^{-1}) + \sum_{i=1}^{\chi} N_{1,2}^{\vec{Y}^i}(x_i, y_i, L(p_i)^{-\frac{1}{2}}).$$

Let $F = \sum_{i=1}^{r} F_i$ be a decomposition of a Γ-module into 1-dimensional modules in the Grothendieck group of Γ-modules, and let $w(F_i)$ be the weight of F_i. Then in the equivariant cohomology we get $c^t(F) = \prod_{i=1}^{r}(w(F_i) + t)$. Thus we have the following corollary.

Corollary 3.8. *Let* $(Z_1, Z_2) \in (X_2^{[l]})^\Gamma$ *correspond to* $(\vec{Y}^1, \ldots, \vec{Y}^\chi)$. *Write* L *for the equivariant line bundle on* X *whose first Chern class is (our chosen lifting of)* ξ. *Then in* $\mathbb{Q}[\varepsilon_1, \varepsilon_2, t]$, *we have the identity*

$$e(T_{(Z_1, Z_2)} X_2^{[l]}) \, c^{-t}(\mathrm{Ext}^1(\mathcal{I}_{Z_2}, \mathcal{I}_{Z_1} \otimes L)) \, c^t(\mathrm{Ext}^1(\mathcal{I}_{Z_1}, \mathcal{I}_{Z_2} \otimes L^\vee))$$

$$= c^{-t}(H^1(X, L)) \, c^t(H^1(X, L^\vee)) \prod_{i=1}^{\chi} \prod_{\alpha, \beta=1}^{2} n_{\alpha, \beta}^{\vec{Y}^i} \left(w(x_i), w(y_i), \frac{t - \iota_{p_i}^* \xi}{2} \right).$$

Lemma 3.9. *In* $\Lambda^{-\xi^2 - 3} \mathbb{Q}(\varepsilon_1, \varepsilon_2)((t^{-1}))[[\Lambda, (\tau_k^\rho)]]$ *we have*

$$\widetilde{\delta}_{\xi, t}^X \left(\exp \left(\sum_\rho \alpha_\rho \right) \right)$$

$$= \frac{\prod_{i=1}^{\chi} Z^{\mathrm{inst}} \left(w(x_i), w(y_i), \frac{t - \iota_{p_i}^* \xi}{2}; \Lambda, ((-1)^\rho \iota_{p_i}^* \alpha_\rho)_\rho \right)}{\Lambda^{\xi^2 + 3} c^{-t}(H^1(X, L)) c^t(H^1(X, L^\vee))}.$$

Remark 3.10. By Definition 3.1 the left-hand side is an element of $\Lambda^{-\xi^2 - 3} \mathbb{Q}[\varepsilon_1, \varepsilon_2]((t^{-1}))[[\Lambda, (\tau_k^\rho)]]$. Note that by (1.5) the right-hand side is an element of $\Lambda^{-\xi^2 - 3} \mathbb{Q}(\varepsilon_1, \varepsilon_2, t)[[\Lambda, (\tau_k^\rho)]]$. Using $\frac{1}{b+t} = t^{-1}(\sum b^i t^{-i})$, we can view $1/\prod_{\alpha, \beta=1}^{2} n_{\alpha, \beta}(w(x_i), w(y_i), (t - \iota_{p_i}^* \xi)/2)$ as an element of $\mathbb{Q}(\varepsilon_1, \varepsilon_2)[[t^{-1}]]$. Then by (2.4) the right-hand side of the Lemma is

interpreted as an element of $\Lambda^{-\xi^2-3}\mathbb{Q}(\varepsilon_1,\varepsilon_2)((t^{-1}))[[\Lambda,(\tau_k^\rho)]]$, and we will show that the equality holds here. The lemma shows that the right-hand side lies even in $\Lambda^{-\xi^2-3}\mathbb{Q}[\varepsilon_1,\varepsilon_2]((t^{-1}))[[\Lambda,(\tau_k^\rho)]]$.

Proof. Let $(Z_1,Z_2) \in (X_2^{[l]})^\Gamma$ correspond to $(\vec{Y}^1,\ldots,\vec{Y}^\chi)$. Let $\alpha \in \{1,2\}$, and let $p_i \in X^\Gamma$. We claim that

$$
\iota^*_{(p_i,(Z_1,Z_2))} \mathrm{ch}(\mathcal{I}_\alpha)
$$
$$
= 1 - (1 - e^{-w(x_i)})(1 - e^{-w(y_i)}) \cdot \sum_{s \in Y_\alpha^i} e^{-l'(s)w(x_i)-a'(s)w(y_i)}.
$$

Let $\mathcal{O}_1$ (resp. $\mathcal{O}_2$) be the sheaf on $X \times X_2^{[l]}$ whose restriction to $X \times X^{[n]} \times X^{[m]}$ is the pushforward of $\mathcal{O}_{Z_n(X)}$ (resp. $\mathcal{O}_{Z_m(X)}$) via the inclusion. For $\alpha = 1,2$ we have $\iota^*_{X\times(Z_1,Z_2)}(\mathcal{O}_\alpha) = \sum_{i=1}^\chi (\iota_{p_i})_*(\mathcal{O}_{Z_\alpha,p_i})$. By definition an equivariant basis of $\mathcal{O}_{Z_\alpha,p_i}$ is $\{x_i^{n-1}y_i^{m-1} \mid (n,m) \in Y_\alpha^i\}$, thus $\mathcal{O}_{Z_\alpha,p_i} = \sum_{s\in Y_\alpha^i} x_i^{l'(s)} y_i^{a'(s)}$ as Γ-modules. By localization we get

$$
i^*_{(p_i,(Z_1,Z_2))} \mathrm{ch}(\mathcal{O}_\alpha) = \iota^*_{p_i}\Big(\sum_{j=1}^\chi (\iota_{p_j})_* \mathrm{ch}(\mathcal{O}_{Z_\alpha,p_j})\Big) = \iota^*_{p_i}(\iota_{p_i})_* \mathrm{ch}(\mathcal{O}_{Z_\alpha,p_i})
$$
$$
= (1 - e^{-w(x_i)})(1 - e^{-w(y_i)}) \mathrm{ch}(\mathcal{O}_{Z_\alpha,p_i}).
$$

Using that $\mathrm{ch}(\mathcal{I}_\alpha) = 1 - \mathrm{ch}(\mathcal{O}_\alpha)$, we get the claim.

We put $f_1 := \frac{\xi-t}{2}$, $f_2 := \frac{t-\xi}{2}$. Then the claim implies

$$
\iota^*_{(p_i,(Z_1,Z_2))}\big(\mathrm{ch}(\mathcal{I}_1)e^{\frac{\xi-t}{2}} + \mathrm{ch}(\mathcal{I}_2)e^{\frac{t-\xi}{2}}\big)
$$
$$
= \sum_{\alpha=1}^2 e^{\iota^*_{p_i}(f_\alpha)}\Big(1-(1-e^{-w(x_i)})(1-e^{-w(y_i)})\sum_{s\in Y_\alpha^i} e^{-l'(s)w(x_i)-a'(s)w(y_i)}\Big).
$$

By the definition (1.4) of $E^{\vec{Y}}(\varepsilon_1,\varepsilon_2,a,\vec{\tau})$ this gives

$$
(3.11) \qquad \iota^*_{(Z_1,Z_2)} \exp\Big(\sum_\rho (-1)^\rho\big[\mathrm{ch}(\mathcal{I}_1)e^{\frac{\xi-t}{2}} \oplus \mathrm{ch}(\mathcal{I}_2)e^{\frac{t-\xi}{2}}\big]_{\rho+1}/\alpha_\rho\Big)
$$
$$
= \prod_{i=1}^\chi E^{\vec{Y}}\big(w(x_i),w(y_i),\frac{t-\iota^*_{p_i}\xi}{2},((-1)^\rho\iota^*_{p_i}\alpha_\rho)_\rho\big).
$$

Write $|Y| := |\vec{Y}_1| + \ldots + |\vec{Y}_\chi|$, and write (Z_1^Y, Z_2^Y) for the point of $X_2^{[|Y|]}$ determined by an χ-tuple $Y = (\vec{Y}_1,\ldots,\vec{Y}_\chi)$ of pairs of Young diagrams.

Then we get by localization

$$\widetilde{\delta}_{\xi,t}^{X}\Big(\exp\Big(\sum_{\rho}\alpha_{\rho}\Big)\Big)$$

$$= \sum_{Y=(\vec{Y}_1,\dots,\vec{Y}_\chi)} \Lambda^{4|Y|-\xi^2-3} \prod_{i=1}^{\chi} E^{\vec{Y}}\Big(w(x_i),w(y_i),\frac{t-i_{p_i}^*\xi}{2},((-1)^{\rho}\iota_{p_i}^*\alpha_{\rho})_{\rho}\Big)$$

$$\cdot \frac{1}{e(T_{(Z_1^Y,Z_2^Y)}X_2^{[|Y|]})c^{-t}(\mathrm{Ext}^1(\mathcal{I}_{Z_2^Y},\mathcal{I}_{Z_1^Y}\otimes L))c^t(\mathrm{Ext}^1(\mathcal{I}_{Z_1^Y},\mathcal{I}_{Z_2^Y}\otimes L^{\vee}))}$$

$$= \Lambda^{-\xi^2-3}\frac{\prod_{i=1}^{\chi}Z^{\mathrm{inst}}\big(w(x_i),w(y_i),\frac{t-\iota_{p_i}^*\xi}{2};\Lambda,((-1)^{\rho}\iota_{p_i}^*\alpha_{\rho})_{\rho}\big)}{c^{-t}(H^1(X,L))c^t(H^1(X,L^{\vee}))},$$

where the last step is by Cor. 3.8.   q.e.d.

3.3. The perturbation part.

Now we want to identify the contribution of the perturbation part. We first need to review the perturbation part of the K-theoretic Nekrasov partition function from [**35**, Section 4.2]. We set

(3.12)

$$\gamma_{\varepsilon_1,\varepsilon_2}(x|\beta;\Lambda) := \frac{1}{2\varepsilon_1\varepsilon_2}\left(-\frac{\beta}{6}\left(x+\frac{1}{2}(\varepsilon_1+\varepsilon_2)\right)^3 + x^2\log(\beta\Lambda)\right)$$

$$+ \sum_{n\geq 1}\frac{1}{n}\frac{e^{-\beta nx}}{(e^{\beta n\varepsilon_1}-1)(e^{\beta n\varepsilon_2}-1)},$$

$$\widetilde{\gamma}_{\varepsilon_1,\varepsilon_2}(x|\beta;\Lambda)$$

$$:= \gamma_{\varepsilon_1,\varepsilon_2}(x|\beta;\Lambda) + \frac{1}{\varepsilon_1\varepsilon_2}\left(\frac{\pi^2 x}{6\beta}-\frac{\zeta(3)}{\beta^2}\right)$$

$$+ \frac{\varepsilon_1+\varepsilon_2}{2\varepsilon_1\varepsilon_2}\left(x\log(\beta\Lambda)+\frac{\pi^2}{6\beta}\right) + \frac{\varepsilon_1^2+\varepsilon_2^2+3\varepsilon_1\varepsilon_2}{12\varepsilon_1\varepsilon_2}\log(\beta\Lambda)$$

for (x,β,Λ) in a neighbourhood of $\sqrt{-1}\mathbb{R}_{>0}\times\sqrt{-1}\mathbb{R}_{<0}\times\sqrt{-1}\mathbb{R}_{>0}$. We formally expand $\varepsilon_1\varepsilon_2\widetilde{\gamma}_{\varepsilon_1,\varepsilon_2}(x|\beta;\Lambda)$ as a power series of $\varepsilon_1,\varepsilon_2$ (around $\varepsilon_1=\varepsilon_2=0$). By the expansion (1.6) we obtain

$$\sum_{n\geq 1}\frac{1}{n}\frac{e^{-\beta nx}}{(e^{\beta n\varepsilon_1}-1)(e^{\beta n\varepsilon_2}-1)} = \sum_{m\geq 0}\frac{c_m}{m!}\beta^{m-2}\mathrm{Li}_{3-m}(e^{-\beta x}),$$

where Li_{3-m} is the polylogarithm (see [**35**, Appendix B] for details). Here we choose the branch of log by $\log(r\cdot e^{i\phi})=\log(r)+i\phi$ with $\log(r)\in\mathbb{R}$ for $\phi\in(-\pi/2,3\pi/2)$ and $r\in\mathbb{R}$. We define $\gamma_{\varepsilon_1,\varepsilon_2}(-x|\beta;\Lambda)$ by analytic continuation along circles in a counter-clockwise way. Finally we define

$$F_K^{\mathrm{pert}}(\varepsilon_1,\varepsilon_2,x|\beta;\Lambda) := -\gamma_{\varepsilon_1,\varepsilon_2}(2x|\beta;\Lambda)-\gamma_{\varepsilon_1,\varepsilon_2}(-2x|\beta;\Lambda).$$

Then $F_K^{\mathrm{pert}}(\varepsilon_1, \varepsilon_2, x|\beta; \Lambda)$ is a formal power series in $\varepsilon_1, \varepsilon_2$ whose coefficients are holomorphic functions in $\Lambda \in \mathbb{C} \setminus \sqrt{-1}\mathbb{R}_{\leq 0}$, $x \in \mathbb{C} \setminus \sqrt{-1}\mathbb{R}_{\leq 0}$, $\beta \in \mathbb{C}$ with $|\beta| < \pi/|x|$. In [**35**, Section 4.2] it is shown that $F_K^{\mathrm{pert}}(\varepsilon_1, \varepsilon_2, x|\beta; \Lambda)$ converges to $F^{\mathrm{pert}}(\varepsilon_1, \varepsilon_2, x; \Lambda)$ when β goes to 0.

We will use the following obvious consequence of the localization formula on X.

Remark 3.13. For any class $\gamma \in H_\Gamma^j(X)$ we have

$$\sum_{i=1}^\chi \frac{\iota_{p_i}^* \gamma}{w(x_i)w(y_i)} = \int_X \gamma \in H_\Gamma^{j-4}(pt).$$

In particular if $\gamma = 1$ or $\gamma \in H_\Gamma^2(X)$, then $\sum_{i=1}^\chi \frac{\iota_{p_i}^* \gamma}{w(x_i)w(y_i)} = 0$.

Lemma 3.14.

$$\sum_{i=1}^\chi F^{\mathrm{pert}}\left(w(x_i), w(y_i), \frac{t - \iota_{p_i}^* \xi}{2}; \Lambda\right)$$
$$= (-\xi^2 - 2)\log \Lambda - \log(c^{-t}(H^1(X,L))) - \log(c^t(H^1(X,L^\vee)))$$

holds in $\mathcal{O}[[\varepsilon_1, \varepsilon_2]]$, where $\mathcal{O}$ denotes the holomorphic functions in (t, Λ) on $(\mathbb{C} \setminus \sqrt{-1}\mathbb{R}_{\leq 0})^2$.

A priori, the left-hand side lives in $(\prod_{i=1}^\chi w(x_i)w(y_i))^{-1}\mathcal{O}[[\varepsilon_1, \varepsilon_2]]$, but in the course of the proof we show that it is, in fact, in $\mathcal{O}[[\varepsilon_1, \varepsilon_2]]$, and the equality holds in $\mathcal{O}[[\varepsilon_1, \varepsilon_2]]$. In $\mathcal{O}[[\varepsilon_1, \varepsilon_2]]$ we can take the exponential of both sides of the equation. Note that the exponential of the right-hand side also lives in $\Lambda^{-\xi^2-2}\mathbb{Q}[\varepsilon_1, \varepsilon_2]((t^{-1}))[[\Lambda, (\tau_k^\rho)]]$. With this remark Lemma 3.14 and Lemma 3.9 together imply Theorem 3.3.

Proof. Let L be an equivariant line bundle on X whose equivariant first Chern class is ξ. In particular $H^i(X,L) = 0$ and $H^i(X,L^\vee) = 0$ for $i \neq 1$. Let $\ell = h^1(X,L)$, $\ell' = h^1(X,L^\vee)$, and let $\alpha_1, \ldots, \alpha_\ell$ (resp. $\alpha'_1, \ldots, \alpha'_{\ell'}$) be the weights of Γ on $H^1(X,L)$ (resp. $H^1(X,L^\vee)$). Then in Γ-equivariant cohomology we get

$$c^{-t}(H^1(X,L)) = \prod_{j=1}^\ell (\alpha_j - t), \qquad c^t(H^1(X,L^\vee)) = \prod_{k=1}^{\ell'} (\alpha'_k + t).$$

Write $p : X \to pt$ for the map to a point. Then the Riemann-Roch theorem gives

$$\sum_{k=1}^{\ell'} e^{\alpha'_k + t} = -\mathrm{ch}(p_!(L^\vee))e^t = -\sum_{k=1}^\chi \frac{e^{-i_{p_i}^* \xi + t}}{(1 - e^{-w(x_i)})(1 - e^{-w(y_i)})}.$$

Thus, we get

$$(3.15) \qquad \sum_{i=1}^{\chi} \sum_{n>0} \frac{e^{(-i_{p_i}^* \xi + t)n(-\beta)}}{n(1 - e^{-w(x_i)n(-\beta)})(1 - e^{-w(y_i)n(-\beta)})}$$

$$= -\sum_{n>0} \sum_{k=1}^{\ell'} \frac{e^{(\alpha_k' + t)n(-\beta)}}{n} = \sum_{k=1}^{\ell'} \log(1 - e^{-(\alpha_k' + t)\beta})$$

in $\widetilde{\mathcal{O}}[[\varepsilon_1, \varepsilon_2]][\prod_i (w(x_i)w(y_i))^{-1}]$, where $\widetilde{\mathcal{O}}$ is the ring of holomorphic functions in (t, β, Λ) in a neighborhood of $\sqrt{-1}\mathbb{R}_{>0} \times \sqrt{-1}\mathbb{R}_{<0} \times \sqrt{-1}\mathbb{R}_{>0}$.

Now we apply the localization formula on X. Using (3.12) and Remark 3.13 we obtain

$$(3.16) \quad \sum_{i=1}^{\chi} \widetilde{\gamma}_{w(x_i), w(y_i)}(-i_{p_i}^* \xi + t | \beta; \Lambda)$$

$$= \sum_{k=1}^{\ell'} \log(1 - e^{-(\alpha_k' + t)\beta}) + \chi(L^\vee) \log(\beta\Lambda) - \frac{\beta}{12}\left(-\xi + t - \frac{K_X}{2}\right)^3,$$

in $\widetilde{\mathcal{O}}[[\varepsilon_1, \varepsilon_2]][\prod_i (w(x_i)w(y_i))^{-1}]$. Here we have used

$$\int_X \left(\frac{(-\xi + t - K_X) \cdot (-\xi + t)}{2} + \mathrm{Todd}_2(X)\right) = \chi(L^\vee),$$

which follows from Remark 3.13 and the Riemann-Roch theorem.

Since $\widetilde{\gamma}_{\varepsilon_1, \varepsilon_2}(-x|\beta; \Lambda)$, is defined by an analytic continuation, we derive from (3.16):

$$\sum_i \widetilde{\gamma}_{w(x_i), w(y_i)}(i_{p_i}^* \xi - t)|\beta; \Lambda)$$

$$= \sum_{k=1}^{\ell} \log(1 - e^{-(\alpha_k - t)\beta}) + \chi(L) \log(\beta\Lambda) - \frac{\beta}{12} \int_X \left(\xi - t - \frac{K_X}{2}\right)^3.$$

Thus we get in $\widetilde{\mathcal{O}}[[\varepsilon_1, \varepsilon_2]][\prod_i (w(x_i)w(y_i))^{-1}]$ that

$$(3.17)$$

$$\sum_{i=1}^{\chi} F_K^{\mathrm{pert}}(w(x_i), w(y_i), \tfrac{t - i_{p_i}^* \xi}{2} | \beta; \Lambda)$$

$$= -\sum_i \widetilde{\gamma}_{w(x_i), w(y_i)}(i_{p_i}^* \xi - t|\beta; \Lambda) - \sum_i \widetilde{\gamma}_{w(x_i), w(y_i)}(-i_{p_i}^* \xi + t)|\beta; \Lambda)$$

$$= -(\chi(L) + \chi(L^\vee))(\log(\Lambda) + \log(\beta))$$

$$- \beta\left(\frac{\langle K_X^3 \rangle}{48} + \frac{\langle K_X \xi^2 \rangle}{4} - \frac{t \langle K_X \xi \rangle}{2}\right)$$

$$+ \sum_{j=1}^{\ell} \log \left(\frac{1}{1 - e^{-(\alpha_j - t)\beta}} \right) + \sum_{k=1}^{\ell'} \log \left(\frac{1}{1 - e^{-(\alpha'_k + t)\beta}} \right).$$

As both sides are defined around $\beta = 0$, the equality holds there. Thus we can take $\beta = 0$. Using that $\lim_{\beta \to 0} \log \left(\frac{\beta}{1 - e^{-(\alpha_j - t)\beta}} \right) = \log(\alpha_j - t)$, $\lim_{\beta \to 0} \log \left(\frac{\beta}{1 - e^{-(\alpha'_k + t)\beta}} \right) = \log(\alpha'_k + t)$, and that $\ell = -\chi(L)$, $\ell' = -\chi(L^\vee)$, we obtain

$$F^{\mathrm{pert}}(w(x_i), w(y_i), \tfrac{t - i_{p_i}^* \xi}{2}; \Lambda)$$
$$= (-\xi^2 - 2) \log(\Lambda) - \sum_{j=1}^{\ell} \log(\alpha_j - t) - \sum_{k=1}^{\ell'} \log(\alpha'_k + t).$$

Note that the right-hand side of this equation is in $\mathcal{O}[[\varepsilon_1, \varepsilon_2]]$. Thus, while the individual summands of the left-hand side only lie in $\mathcal{O}[[\varepsilon_1, \varepsilon_2]][\prod_i (w(x_i) w(y_i))^{-1}]$, their sum lies in $\mathcal{O}[[\varepsilon_1, \varepsilon_2]]$. This shows Lemma 3.14.   q.e.d.

Now we want to express the wallcrossing for the Donaldson invariants in terms of the Nekrasov partition function $Z(\varepsilon_1, \varepsilon_2, a; \Lambda)$. This will be necessary because the Nekrasov conjecture determines the lowest order terms in $\varepsilon_1, \varepsilon_2$ of $F(\varepsilon_1, \varepsilon_2, a; \Lambda)$, but not of $F(\varepsilon_1, \varepsilon_2, a; \Lambda, \vec{\tau})$.

Corollary 3.18.

(1)

$$\widetilde{\delta}_{\xi,t}^X(\exp(\alpha z + px)) = \frac{1}{\Lambda} \exp \left(\frac{1}{2} \langle \mathrm{Todd}_2(X)(\alpha z + px) \rangle \right.$$
$$\left. \cdot \sum_{i=1}^{\chi} F\left(w(x_i), w(y_i), \tfrac{t - \iota_{p_i}^* \xi}{2}; \Lambda e^{\iota_{p_i}^*(\alpha z + px)/4} \right) \right),$$

(2) $\delta_{\xi,t}^X(\exp(\alpha z + px))$

$$= \frac{1}{\Lambda} \exp \left(\sum_{i=1}^{\chi} F\left(w(x_i), w(y_i), \tfrac{t - \iota_{p_i}^* \xi}{2}; \Lambda e^{\iota_{p_i}^*(\alpha z + px)/4} \right) \right) \Big|_{\varepsilon_1, \varepsilon_2 = 0}.$$

Proof. Let $\vec{\tau}_1 := (\tau_1, 0, 0, \ldots)$ be a vector with only the first entry nonzero. Then in [**34**, Section 4.5] it is shown that

$$(3.19) \quad F(\varepsilon_1, \varepsilon_2, a; \Lambda, \vec{\tau}_1) = -\frac{\tau_1(\varepsilon_1^2 + \varepsilon_2^2 + 3\varepsilon_1\varepsilon_2)}{24\varepsilon_1\varepsilon_2} + F(\varepsilon_1, \varepsilon_2, a; \Lambda e^{-\tau_1/4}).$$

Thus we get

$$\widetilde{\delta}^X_{\xi,t}(\exp(\alpha z + px))$$

$$= \frac{1}{\Lambda} \exp\left(\sum_{i=1}^{\chi} \frac{\iota^*_{p_i}(\alpha z + px)(w(x_i)^2 + w(y_i)^2 + 3w(x_i)w(y_i))}{24w(x_i)w(y_i)}\right)$$

$$\cdot \exp\left(\sum_{i=1}^{\chi} F\left(w(x_i), w(y_i), \frac{t - \iota^*_{p_i}\xi}{2}; \Lambda e^{\iota^*_{p_i}(\alpha z + px)/4}\right)\right).$$

By localization we get

$$\sum_{i=1}^{\chi} \frac{\iota^*_{p_i}(\alpha z + px)(w(x_i)^2 + w(y_i)^2 + 3w(x_i)w(y_i))}{24w(x_i)w(y_i)}$$

$$= \frac{1}{2}\big\langle (\alpha z + px)\,\mathrm{Todd}_2(X)\big\rangle.$$

This shows (1). (2) follows immediately, because $\langle(\alpha z + px)\,\mathrm{Todd}_2(X)\rangle = 0$ in nonequivariant cohomology. q.e.d.

4. Explicit formulas in terms of modular forms

We have expressed the wallcrossing δ^X_ξ in terms of the Nekrasov partition function. Now we want to use the Nekrasov conjecture to give an explicit formula in terms of the q-development of modular forms.

Let $q := e^{2\pi\sqrt{-1}\tau}$ for $\tau \in \mathcal{H} := \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$. Recall the theta functions

$$\theta_{00}(\tau) := \sum_{n\in\mathbb{Z}} q^{n^2/2}, \quad \theta_{01}(\tau) := \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2/2}, \quad \theta_{10}(\tau) := \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+1/2)^2}.$$

Write $E_2(\tau) := 1 - 24\sum_n \sigma_1(n)q^n$ for the normalized Eisenstein series of weight 2. Denote
(4.1)

$$u := -\frac{\theta_{00}^4 + \theta_{10}^4}{\theta_{00}^2\theta_{10}^2}\Lambda^2, \quad \frac{du}{da} := \frac{2\sqrt{-1}}{\theta_{00}\theta_{10}}\Lambda, \quad a := \sqrt{-1}\frac{2E_2 + \theta_{00}^4 + \theta_{10}^4}{3\theta_{00}\theta_{10}}\Lambda.$$

Finally put

$$T := \frac{1}{24}\left(\frac{du}{da}\right)^2 E_2 - \frac{u}{6}.$$

We can now state the formula for δ^X_ξ in terms of the q-development of these functions.

Theorem 4.2. *Let ξ be a good class. Then*

$$\delta^X_\xi(\exp(\alpha z + px)) = \sqrt{-1}^{\langle\xi K_X\rangle - 1}[\heartsuit]_{q^0}$$

with

$$\heartsuit = q^{-\frac{1}{2}(\frac{\xi}{2})^2} \exp\left(\frac{du}{da}\langle\alpha,\xi/2\rangle z + T\langle\alpha^2\rangle z^2 - ux\right)\left(\frac{\sqrt{-1}}{\Lambda}\frac{du}{da}\right)^3 \theta_{01}^{K_X^2}.$$

We briefly review the Nekrasov conjecture. For this we define u, a in a different way. Consider the family of elliptic curves $C_u : y^2 = (z^2 - u)^2 - 4\Lambda^4$, parametrized by $u \in \mathbb{C}$, which we call the u-plane. The Seiberg-Witten differential $dS := -\frac{1}{2\pi} \frac{z P'(z) dz}{y}$ is a meromorphic differential form on C_u. For suitable cycles A, B on C_u (for the definition see [**34**, Section 2.1], here they are called A_2, B_2) put $a := \int_A dS$, $a^D := 2\pi\sqrt{-1} \int_B dS$. These are functions on the u-plane ($|u| \gg |\Lambda|$). By definition a and a^D are functions of u, but conversely we will consider u and a^D as functions of a and Λ. The *period* of C_u is $\tau := \frac{1}{2\pi\sqrt{-1}} \frac{\partial a^D}{\partial a}$. The *Seiberg-Witten prepotential* $\mathcal{F}_0$ is the (suitably normalized) locally defined function on the u-plane with $a^D = -\frac{\partial \mathcal{F}_0}{\partial a}$. We choose the branch of the logarithm as $\log(re^{i\theta}) = \log(r) + i\theta$ for $r \in \mathbb{R}^+$ and $\theta \in (-\pi, \pi)$, with $\log(r) \in \mathbb{R}$. By [**34**, Sections 2.1, 2.3] for (a, Λ) in a neighborhood $U \subset \mathbb{C} \times \mathbb{C}$ of the set of $(a, \Lambda) \in \sqrt{-1}\mathbb{R}^+ \times \sqrt{-1}\mathbb{R}^+$, with $|a| \gg |\Lambda|$, $\mathcal{F}_0$ is a holomorphic function of a and Λ, which we write as $\mathcal{F}_0(a; \Lambda)$. By definition we have $\tau = -\frac{1}{2\pi\sqrt{-1}} \frac{\partial^2 \mathcal{F}_0}{(\partial a)^2}$ and $q = \exp(-\frac{\partial^2 \mathcal{F}_0}{(\partial a)^2})$. Then with this definition of τ the formulas (4.1) hold [**34**, equation (1.3)].

The Nekrasov conjecture [**36**] (proved in [**33**], [**34**], [**38**], [**3**]) says that

(1) $\varepsilon_1 \varepsilon_2 F(\varepsilon_1, \varepsilon_2, a; \Lambda)$ is regular at $\varepsilon_1, \varepsilon_2 = 0$,
(2) $(\varepsilon_1 \varepsilon_2 F(\varepsilon_1, \varepsilon_2, a; \Lambda))|_{\varepsilon_1 = \varepsilon_2 = 0} = \mathcal{F}_0(a; \Lambda)$.

Here we understand the equation (2) as follows: It is an abbreviation of two equations, one for the perturbation part and the other for the instanton part. The former is an equality for holomorphic functions in $(a, \Lambda) \in U$, and the latter is for formal power series in $\mathbb{C}[[1/a, \Lambda]]$. Equations appearing below should be understood in the same way, until the ambiguity of the branch of the logarithm in the perturbation part will disappear in the expression.

In [**34**] also the next higher order terms of $F(\varepsilon_1, \varepsilon_2, a, \Lambda)$ in $\varepsilon_1, \varepsilon_2$ are determined: We write

$$(4.3) \quad \varepsilon_1 \varepsilon_2 F(\varepsilon_1, \varepsilon_2, a; \Lambda) = \mathcal{F}_0(a; \Lambda) + (\varepsilon_1 + \varepsilon_2) H(a; \Lambda)$$
$$+ \varepsilon_1 \varepsilon_2 A(a; \Lambda) + \frac{\varepsilon_1^2 + \varepsilon_2^2}{3} B(a; \Lambda) + O$$

where O stands for terms of degree at least 3 in ε_1 and ε_2. It is also proved that H, A and B are holomorphic functions on U. In [**34**, Section 5.3] it is shown that $H(a, \Lambda) = \pi\sqrt{-1}a$. By [**34**, Section 7.1] we have

$$(4.4) \quad \exp(A) = \left(\frac{2}{\theta_{00}\theta_{10}}\right)^{1/2} = \left(\frac{\sqrt{-1}}{\Lambda} \frac{du}{da}\right)^{1/2}, \qquad \exp(B - A) = \theta_{01}.$$

Remark 4.5. The sign of H in [**34**, Section 5.3] was wrong and we have $H(\vec{a}, \mathfrak{q}, \vec{\tau}) = -\pi\sqrt{-1}\langle \vec{a}, \rho \rangle$ in the first displayed formula in [loc. cit., p. 66]. Therefore we have $H(a; \Lambda) = \pi\sqrt{-1}a$ in our case. The

mistake occurred when we take the sum of [loc. cit., (E.5)] over $\alpha < \beta$. Accordingly blowup formulas in [loc. cit., Section 6] must be corrected.

Proof of Theorem 4.2. We apply the localization formula to X. Note that $w(x_i), w(y_i)$ are homogeneous of degree 1 in $\varepsilon_1, \varepsilon_2$. Furthermore, if $\beta \in H_\Gamma^{2i}(X)$, then $\iota_{p_i}^*(\beta)$ is homogeneous of degree i in $\varepsilon_1, \varepsilon_2$. Therefore we get the following expansion (where on the right-hand side we take the values of $\mathcal{F}_0$ and its derivatives at $(\frac{t}{2}, \Lambda)$):

$$\mathcal{F}_0\big(\tfrac{t-\iota_{p_i}^*(\xi)}{2}; \Lambda e^{\iota_{p_i}^*(\alpha z + px)/4}\big)$$

$$= \mathcal{F}_0 - \frac{1}{2}\frac{\partial \mathcal{F}_0}{\partial a}\iota_{p_i}^*(\xi) + \frac{\partial F_0}{\partial \log \Lambda}\iota_{p_i}^*(\alpha z + px)/4 - \frac{1}{8}\frac{\partial^2 F_0}{\partial a \partial \log \Lambda}\iota_{p_i}^*(\xi)\iota_{p_i}^*(\alpha)z$$

$$+ \frac{1}{2}\frac{\partial^2 F_0}{(\partial \log \Lambda)^2}\iota_{p_i}^*(\alpha/4)^2 z^2 + \frac{1}{8}\frac{\partial^2 F_0}{\partial a^2}\iota_{p_i}^*(\xi)^2 + O,$$

where O stands for terms of degree larger than 2 in $\varepsilon_1, \varepsilon_2$. By Remark 3.13 we see that

$$(4.6) \qquad \sum_{i=1}^{\chi} \frac{\mathcal{F}_0\big(\tfrac{t-\iota_{p_i}^*(\xi)}{2}; \Lambda e^{\iota_{p_i}^*(\alpha z + px)/4}\big)}{w(x_i)w(y_i)}$$

$$= \frac{1}{4}\frac{\partial \mathcal{F}_0}{\partial \log \Lambda}x - \frac{1}{8}\frac{\partial^2 \mathcal{F}_0}{\partial a \partial \log \Lambda}\langle \xi, \alpha\rangle z$$

$$+ \frac{1}{32}\frac{\partial^2 \mathcal{F}_0}{(\partial \log \Lambda)^2}\langle \alpha^2\rangle z^2 + \frac{1}{8}\frac{\partial^2 \mathcal{F}_0}{\partial a^2}\xi^2 + O,$$

where again O stands for terms of degree larger than 2 in $\varepsilon_1, \varepsilon_2$. Similarly, we get

$$(4.7) \qquad \sum_{i=1}^{\chi} \frac{w(x_i) + w(y_i)}{w(x_i)w(y_i)}H\big(\tfrac{t-\iota_{p_i}^*\xi}{2}; \Lambda e^{\iota_{p_i}^*(\alpha z + px)/4}\big)$$

$$= \sum_{i=1}^{\chi} \pi\sqrt{-1}\frac{w(x_i) + w(y_i)}{w(x_i)w(y_i)}\frac{t - \iota_{p_i}^*\xi}{2} = \pi\sqrt{-1}\langle \xi/2, K_X\rangle.$$

Finally,

$$(4.8) \quad \sum_{i=1}^{\chi} \Big(A\big(\tfrac{t-\iota_{p_i}^*\xi}{2}; \Lambda e^{\iota_{p_i}^*(\alpha z + px)}\big)$$

$$+ \frac{w(x_i)^2 + w(y_i)^2}{3w(x_i)w(y_i)}B\big(\tfrac{t-\iota_{p_i}^*\xi}{2}; \Lambda e^{\iota_{p_i}^*(\alpha z + px)/4}\big)\Big) = \chi A + \sigma B + O,$$

where $\sigma = \frac{1}{3}(c_1(X)^2 - 2\chi)$ is the signature of X, and the argument of A, B is $(t/2, \Lambda)$. By the formulas (4.3)–(4.8) we get that $\sum_{i=1}^{\chi} \frac{1}{w(x_i)w(y_i)}$ $F\big(w(x_i), w(y_i), \tfrac{t-\iota_{p_i}^*\xi}{2}, \Lambda e^{\iota_{p_i}^*(\alpha z + px)/4}\big)$ is regular in $\varepsilon_1, \varepsilon_2$. Therefore we can take its exponential, which is still regular in $\varepsilon_1, \varepsilon_2$.

Put $a := t/2$. Then we get by [**34**, Prop. 2.3] and [**34**, formula (2.12)] that

$$(4.9) \qquad \frac{\partial \mathcal{F}_0}{\partial \log \Lambda} = -4u, \quad \frac{1}{32}\frac{\partial^2 \mathcal{F}_0}{(\partial \log \Lambda)^2} = \frac{1}{24}E_2\Big(\frac{du}{da}\Big)^2 - \frac{u}{6} = T.$$

As X is a rational surface, we have $\chi = -\sigma + 4$, thus we get by (4.4)

$$(4.10) \qquad \exp(\chi A + \sigma B) = \Big(\frac{\sqrt{-1}}{\Lambda}\frac{du}{da}\Big)^2 \theta_{01}^{\sigma}.$$

These relations and (4.1) hold in a neighborhood of $(a, \Lambda) = (\infty, 0)$. Thus they are equalities in $\mathbb{C}[[\Lambda/a]][\Lambda, a/\Lambda]$. Note that the q-development of

$$(4.11) \qquad \frac{\Lambda}{a} = \Big(\sqrt{-1}\frac{2E_2 + \theta_{00}^4 + \theta_{10}^4}{3\theta_{00}\theta_{10}}\Big)^{-1}$$

starts with $\frac{2}{\sqrt{-1}}q^{\frac{1}{8}}$. Thus $\mathbb{C}[[q^{1/8}]] \cong \mathbb{C}[[\Lambda/a]]$. Putting (4.3)–(4.10) into Corollary 3.18 we get

(4.12)

$\delta_{\xi,t}^X(\exp(\alpha z + px))$

$$= \frac{1}{\Lambda}\lim_{\varepsilon_1,\varepsilon_2\to 0}\exp\Big(\sum_{i=1}^{\chi}\frac{1}{w(x_i)w(y_i)}F\big(w(x_i), w(y_i), \tfrac{t-\iota_{p_i}^*\xi}{2}, \Lambda e^{\iota_{p_i}^*(\alpha z+px)/4}\big)\Big)$$

$$= \lim_{\varepsilon_1,\varepsilon_2\to 0}\exp\Big(\sum_{i=1}^{\chi}\Big(\frac{\mathcal{F}_0(\tfrac{t-\iota_{p_i}^*\xi}{2}; \Lambda e^{\iota_{p_i}^*(\alpha z+px)/4})}{w(x_i)w(y_i)}$$

$$+ \frac{w(x_i) + w(y_i)}{w(x_i)w(y_i)}H\big(\tfrac{t-\iota_{p_i}^*\xi}{2}; \Lambda e^{\iota_{p_i}^*(\alpha z+px)/4}\big)$$

$$+ A\big(\tfrac{t-\iota_{p_i}^*\xi}{2}; \Lambda e^{\iota_{p_i}^*(\alpha z+px)/4}\big)$$

$$+ \frac{w(x_i)^2 + w(y_i)^2}{3w(x_i)w(y_i)}B\big(\tfrac{t-\iota_{p_i}^*\xi}{2}; \Lambda e^{\iota_{p_i}^*(\alpha z+px)/4}\big)\Big)\Big)$$

$$= \sqrt{-1}^{\langle\xi,K_X\rangle}\Big(\frac{1}{\Lambda}q^{-\frac{1}{2}(\frac{\xi}{2})^2}\exp\Big(\frac{du}{da}\langle\alpha, \xi/2\rangle z + T\langle\alpha^2\rangle z^2 - ux\Big)$$

$$\cdot \Big(\frac{\sqrt{-1}}{\Lambda}\frac{du}{da}\Big)^2 \theta_{01}^{\sigma}\Big).$$

The final equality, i.e., the first term = the last term, holds in $\mathbb{C}((\Lambda))((\Lambda/a))[[z, x]] = \mathbb{C}((\Lambda))((1/t))[[z, x]]$. Indeed, for the left-hand side the coefficient of $z^n x^m$ is in $\Lambda^{-\xi^2-3} \times t^{\xi^2+2+n+2m}\mathbb{C}[[\Lambda, 1/t]]$. Thus we get

$$\delta_\xi^X(\exp(\alpha z + px)) = -\sqrt{-1}^{\langle\xi,K_X\rangle}\operatorname*{res}_{t=\infty}\spadesuit$$

with

$$\spadesuit = \frac{1}{\Lambda} q^{-\frac{1}{2}(\frac{\xi}{2})^2} \exp\left(\frac{du}{da}\langle\alpha, \xi/2\rangle z + T\langle\alpha^2\rangle z^2 - ux\right)\left(\frac{\sqrt{-1}}{\Lambda}\frac{du}{da}\right)^2 \theta_{01}^\sigma dt.$$

Finally we want to express this result in terms of the q development of the modular forms involved. That is, we change the variable from t to q. First we determine $\frac{da}{d\tau}$. Combining formulas (V.4.1), (V.5.2) and (V.5.6) of [4] (note that in the notation of [4] $\theta_{00}(\tau) = \theta_3(0, \tau)$, $\theta_{01}(\tau) = \theta_2(0, \tau)$, $\theta_{10}(\tau) = \theta_1(0, \tau)$), we get

$$\frac{d\log(\theta_{00})}{d\tau} - \frac{d\log(\theta_{10})}{d\tau} = \frac{\pi}{4\sqrt{-1}}\theta_{01}^4.$$

By [4, (VII.3.10)] we have $\theta_{00}^4 - \theta_{10}^4 = \theta_{01}^4$, and thus

$$\frac{du}{d\tau} = -\frac{d}{d\tau}\left(\frac{\theta_{00}^2}{\theta_{10}^2} + \frac{\theta_{10}^2}{\theta_{00}^2}\right)\Lambda^2 = -2\Lambda^2\frac{\theta_{00}^4 - \theta_{10}^4}{\theta_{10}^2\theta_{00}^2}\left(\frac{d\log(\theta_{00})}{d\tau} - \frac{d\log(\theta_{10})}{d\tau}\right)$$

$$= -\frac{\Lambda^2\pi}{2\sqrt{-1}}\frac{\theta_{01}^8}{\theta_{10}^2\theta_{00}^2} = -\frac{\pi\sqrt{-1}}{8}\left(\frac{du}{da}\right)^2\theta_{01}^8.$$

Thus we get

$$\frac{da}{d\tau} = \frac{da}{du}\frac{du}{d\tau} = -\frac{\pi\sqrt{-1}}{8}\frac{du}{da}\theta_{01}^8.$$

By $a = t/2$ and $q = e^{2\pi\sqrt{-1}\tau}$, we have

$$dt = 2da = 2\frac{da}{d\tau}d\tau = \frac{1}{\pi\sqrt{-1}}\frac{da}{d\tau}\frac{dq}{q} = -\frac{1}{8}\frac{du}{da}\theta_{01}^8\frac{dq}{q}.$$

By (4.11) the residue at $a = \infty$ is 8 times the residue at $q = 0$. Therefore we get

$$\delta_\xi^X(\exp(\alpha z + px)) = \sqrt{-1}^{\langle\xi, K_X\rangle - 1}\mathop{\mathrm{res}}_{q=0}\heartsuit$$

with

$$\heartsuit = \left[q^{-\frac{1}{2}(\frac{\xi}{2})^2}\exp\left(\frac{du}{da}\langle\alpha, \xi/2\rangle z + T\langle\alpha^2\rangle z^2 - ux\right)\left(\frac{\sqrt{-1}}{\Lambda}\frac{du}{da}\right)^3\theta_{01}^{\sigma+8}\frac{dq}{q}\right],$$

and Theorem 4.2 follows by $\sigma + 8 = 3\sigma + 2\chi = K_X^2$.   q.e.d.

Remark 4.13.

(1) Denote by u_{MW} and h the functions denoted by u, h in [31]. Note that in the notation of [31], $\lambda_0 = c_1/2$ and $\lambda = \xi/2$. We are computing the wallcrossing for $\Phi_{c_1}^X$, whereas in [31] the wallcrossing for $D_{c_1}^X$ is computed. Thus we have to multiply their formula (4.6) by $(-1)^{-(c_1^2 + c_1 K_X)/2}$ to compare it with ours. Write $\delta_{\xi,MW}^X$ for the wall-crossing formula obtained this way. Using the fact that $u = -2u_{MW}$, $\frac{du}{da} = \frac{\sqrt{-1}\Lambda}{h}$, we see that $\delta_{\xi,MW}^X = -\frac{1}{2}\delta_{-\xi}^X$. By definition $\delta_{-\xi}^X = -\delta_\xi^X$. Thus $\delta_{\xi,MW}^X = \frac{1}{2}\delta_\xi^X$. It was observed in [31] that the formula in [19]

gives $2\delta^X_{\xi,MW}$ for the wallcrossing of $D^X_{c_1}$. Thus our formula agrees with the results of [**19**].

(2) Denote by U, f, R the functions denoted by the same letters in [**20**]. Then it is easy to check that

$$U(\tau) = -\frac{1}{\Lambda^2}u(\tau+1), \quad \frac{1}{f(\tau)} = \frac{1}{\Lambda}\frac{du}{da}(\tau+1),$$

$$R(\tau) = -\left(\frac{1}{\Lambda}\frac{du}{da}(\tau+1)\right)^4\theta_{01}(\tau+1)^8.$$

Using these formulas it is also easy to see directly that Theorem 4.2 gives the same wallcrossing formula as [**19**],[**20**], after correcting for the different sign conventions.

5. Generalization to non-toric surfaces

In this section we show that the wallcrossing term is given by the same formula as in Theorem 4.2 for a good wall of an *arbitrary* simply connected projective surface X. The proof is based on [**10**] for Chern numbers of Hilbert schemes of points.

We consider the Grothendieck group $K(Y)$ of locally free sheaves on a smooth projective variety Y. It is isomorphic to that of coherent sheaves. It has a ring structure from the tensor product. We denote it by $\otimes$. For a morphism $f\colon Y_1 \to Y_2$ we have a pushforward homomorphism $f_!\colon K(Y_1) \to K(Y_2)$, and the pullback homomorphism $f^!\colon K(Y_2) \to K(Y_1)$. We also have the involution $\vee$ on $K(Y)$ given by the dual vector bundle for a vector bundle.

Let X be a projective surface and $X^{[n]}$ denote the Hilbert scheme of n points on X. As before let $X_2 = X \sqcup X$ be the disjoint union of two copies of X. Let $X_2^{[l]}$ be the Hilbert scheme of l points on X_2, i.e., $X_2^{[l]} = \bigsqcup_{m+n=l} X^{[m]} \times X^{[n]}$, and let $\mathcal{I}_1$ (resp. $\mathcal{I}_2$) be the sheaf on $X \times X_2^{[l]}$ whose restriction to $X \times X^{[m]} \times X^{[n]}$ is $p^*_{12}(\mathcal{I}_{Z_m(X)})$ (resp. $p^*_{13}(\mathcal{I}_{Z_n(X)})$). Let us define p, q by

$$p\colon X_2^{[l]} \times X \to X_2^{[l]}, \qquad q\colon X_2^{[l]} \times X \to X.$$

These maps depend on l, but we suppress the dependence from the notation hoping that they do not lead to confusion, though we will vary l later.

In this section we prove the following:

Theorem 5.1. *There exist universal power series* $A_i \in \mathbb{Q}((t^{-1}))[[\Lambda]]$, $i = 1, \ldots, 8$, *such that for all projective surfaces X and all $\xi \in \mathrm{Pic}(X)$*

$$(-1)^{\chi(\mathcal{O}_X)+\xi(\xi-K_X)/2}t^{-\xi^2-2\chi(\mathcal{O}_X)}\Lambda^{\xi^2+3\chi(\mathcal{O}_X)}\delta^X_{\xi,t}\left(\exp\left(\alpha z + px\right)\right)$$

$$= \exp(\xi^2 A_1 + \xi \cdot c_1(X)A_2 + c_1(X)^2 A_3 + c_2(X)A_4 + \alpha \cdot \xi A_5 z$$

$$+ \alpha \cdot c_1(X)A_6 z + \alpha^2 A_7 z^2 + xA_8).$$

Here $\delta_{\xi,t}^X$ is defined for arbitrary projective surface by the same formula (2.6) except that we change $\Lambda^{4l-\xi^2-3}$ into $\Lambda^{4l-\xi^2-3\chi(\mathcal{O}_X)}$ and also $\mathcal{A}_{\xi,-}$, $\mathcal{A}_{\xi,+}$ into

$$-p_!(\mathcal{I}_2^\vee \otimes \mathcal{I}_1 \otimes q^!\xi), \qquad -p_!(\mathcal{I}_1^\vee \otimes \mathcal{I}_2 \otimes q^!\xi^\vee) \in K(X_2^{[l]}),$$

respectively.

When ξ is good, both $\mathrm{Ext}_p^0(\mathcal{I}_2, \mathcal{I}_1(\xi))$, $\mathrm{Ext}_p^2(\mathcal{I}_2, \mathcal{I}_1(\xi))$ vanish [**8**, Lemma 4.3]. Therefore we have

$$\mathrm{Ext}_p^1(\mathcal{I}_2, \mathcal{I}_1(\xi)) = -p_!(\mathcal{I}_2^\vee \otimes \mathcal{I}_1 \otimes q^!\xi)$$

and the same for $\mathrm{Ext}_p^1(\mathcal{I}_1, \mathcal{I}_2(-\xi))$.

The proof is a straightforward modification of that of [**10**, Th. 4.2], so we only give a sketch of the proof. The essential point is to use the incidence variety to compute the intersection products on Hilbert schemes recursively. A slight difference is that we need to introduce *two* incidence varieties because we study Hilbert schemes of a nonconnected surface X_2.

For $\alpha = 1, 2$ let $X_{2,\alpha}^{[l,l+1]}$ be the variety of pairs Z, Z' in $X_2^{[l]} \times X_2^{[l+1]}$ satisfying $Z \subset Z'$ and $Z' \setminus Z$ is a point in the αth-factor of X_2. This is an obvious generalization of the incidence variety $X^{[l,l+1]}$, studied by various people and used in [**10**]. Let ϕ_α and ψ_α be the projections from $X_{2,\alpha}^{[l,l+1]}$ to $X_2^{[l]}$ and $X_2^{[l+1]}$ respectively. Let ρ_α be the map $X_{2,\alpha}^{[l,l+1]} \to X$ defined by letting $\rho(Z, Z')$ be the unique point in $Z' \setminus Z$. Let $\mathcal{L}$ be the line bundle whose fiber at (Z, Z') is the kernel of the homomorphism $H^0(\mathcal{O}_{Z'}) \to H^0(\mathcal{O}_Z)$. We have

$$X \xleftarrow{\ \rho_\alpha\ } X_{2,\alpha}^{[l,l+1]} \xrightarrow{\ \psi_\alpha\ } X_2^{[l+1]}$$
$$\downarrow{\phi_\alpha}$$
$$X_2^{[l]}$$

We also define $j_\alpha = \rho_\alpha \times \mathrm{id} \colon X_{2,\alpha}^{[l,l+1]} \to X \times X_{2,\alpha}^{[l,l+1]}$ and $\sigma_\alpha = \rho_\alpha \times \phi_\alpha \colon X_{2,\alpha}^{[l,l+1]} \to X \times X_2^{[l]}$.

We first have the following analog of [loc. cit., (5)]

(5.2)
$$\psi_{\alpha X}^! \mathcal{I}_\beta = \phi_{\alpha X}^! \mathcal{I}_\beta - \delta_{\alpha\beta} j_{\alpha!}\mathcal{L} = \phi_{\alpha X}^! \mathcal{I}_\beta - \delta_{\alpha\beta} p^!\mathcal{L} \otimes \rho_{\alpha X}^! \mathcal{O}_\Delta, \quad \text{for } \alpha, \beta = 1, 2,$$

where $p \colon X \times X_{2,\alpha}^{[l,l+1]} \to X_{2,\alpha}^{[l,l+1]}$ and $f_X = f \times \mathrm{id}_X$ for $f = \phi_\alpha, \psi_\alpha$.

Next we have an analog of [loc. cit., (8)]

(5.3) $$\psi_\alpha^* \mathrm{ch}(\mathcal{I}_\beta)/c = \phi_\alpha^* \mathrm{ch}(\mathcal{I}_\beta)/c - \delta_{\alpha\beta} \mathrm{ch}(\mathcal{L}) \cdot \rho_\alpha^* c$$

for $c \in H_*(X)$.

We also get an analog of [**10**, Prop. 2.3] using (5.2)

(5.4)
$$\psi^!_\alpha p_!(\mathcal{I}_2^\vee \otimes \mathcal{I}_1 \otimes q^!\xi) = \phi^!_\alpha p_!(\mathcal{I}_2^\vee \otimes \mathcal{I}_1 \otimes q^!\xi) - \delta_{\alpha 1}\sigma^!_\alpha \mathcal{I}_2^\vee \otimes \rho^!_\alpha \xi \otimes \mathcal{L}$$
$$-\delta_{\alpha 2}\sigma^!_\alpha \mathcal{I}_1 \otimes \rho^!_\alpha(\xi \otimes \omega_X^\vee) \otimes \mathcal{L}^\vee.$$

More precisely, we do not get a term corresponding to the third term in [loc. cit., (10)] coming from the product of two copies of the diagonal, because $\delta_{\alpha 1}\delta_{\alpha 2}$ is always 0.

Using these results, the same argument as in [**10**, Prop. 3.1, Thm 4.1] shows the following.

Lemma 5.5. *Fix $l \geq 0$. Let P be any polynomial in the $c_{i_1}(\mathcal{A}_+)$, $c_{i_2}(\mathcal{A}_-)$, $\mathrm{ch}_{i_3}(\mathcal{I}_1)\xi^{i_4}/(\alpha z + px)$, $\mathrm{ch}_{i_5}(\mathcal{I}_2)\xi^{i_6}/(\alpha z + px)$ for $i_1,\ldots,i_6 \in \mathbb{Z}_{\geq 0}$, then there exists a universal polynomial Q (depending only on P) in ξ^2, $\xi c_1(X)$, $c_1(X)^2$, $c_2(X)$, $\xi\alpha z$, $\alpha c_1(X)z$, $\alpha^2 z^2$, x, such that $\int_{X_2^{[l]}} P = Q$.*

We denote the left-hand-side of Theorem 5.1 by $\overline{\delta}^X_{\xi,t}$. By definition we have

$$(-1)^{rk(\mathcal{A}_-)}t^{rk(\mathcal{A}_+)+rk(\mathcal{A}_-)}\frac{1}{c^t(\mathcal{A}_+)c^{-t}(\mathcal{A}_-)}$$
$$= \sum_{i,j} s_i(\mathcal{A}_-)s_j(\mathcal{A}_+)(-1)^i t^{-i-j},$$

where $rk(\mathcal{A}_-) = l - \chi(\mathcal{O}_X) - \frac{\xi(\xi - K_X)}{2}$, $rk(\mathcal{A}_+) = l - \chi(\mathcal{O}_X) - \frac{\xi(\xi + K_X)}{2}$. Therefore by Lemma 5.5 we can write

$$\overline{\delta}^X_{\xi,t}(\alpha z + px) = \sum_{l \geq 0}\sum_{i \in \mathbb{Z}} \Lambda^{4l} P_{l,i}t^i,$$

where $P_{l,i}$ is a universal polynomial in ξ^2, $\xi c_1(X)$, $c_1(X)^2$, $c_2(X)$, $\xi\alpha z$, $\alpha c_1(X)z$, $\alpha^2 z^2$, x, depending only on l and i. It is easy to see from the definition that the coefficient of Λ^0 of $\overline{\delta}^X_{\xi,t}$ as a power series in Λ is 1. Thus there is a universal power series $G_{t,\Lambda} \in \mathbb{Q}((t^{-1}))[x_1,\ldots,x_8][[\Lambda]]$, such that

$$\overline{\delta}^X_{\xi,t}(\alpha z + px)$$
$$= \exp(G_{t,\Lambda}(\xi^2, \xi c_1(X), c_1(X)^2, c_2(X), \xi\alpha z, \alpha c_1(X)z, \alpha^2 z^2, x)).$$

Now assume that $X = Y \sqcup Z$ for Y, Z not necessarily connected projective surfaces, and $\xi \in \mathrm{Pic}(X)$, $\beta \in H_2(X)z \oplus H_0(X)x$ satisfy $\xi|_Y = \xi_1$, $\xi|_Z = \xi_2$, $\beta|_Y = \beta_1$, $\beta|_Z = \beta_2$. Then $X_2^{[l]} = \coprod_{n+m=l} Y_2^{[n]} \times Z_2^{[m]}$, and denoting $\mathcal{A}_{-,X}$, $\mathcal{A}_{-,Y}$, $\mathcal{A}_{-,Z}$ respectively, the bundles $\mathcal{A}_-$ on $X_2^{[l]}$, $Y_2^{[n]}$ and $Z_2^{[m]}$, it is obvious that $\mathcal{A}_{-,X}|_{Y_2^{[n]} \times Z_2^{[m]}} = \mathcal{A}_{-,Y} \boxplus \mathcal{A}_{-,Z}$ and similarly

for $\mathcal{A}_+, \mathcal{I}_1, \mathcal{I}_2$. Thus it follows from the definitions that

$$(5.6) \qquad \overline{\delta}^X_{\xi,t}(\beta) = \overline{\delta}^Y_{\xi_1,t}(\beta_1)\overline{\delta}^Z_{\xi_2,t}(\beta_2).$$

To a triple (X,ξ,β) of a projective surface X, a class $\xi \in \mathrm{Pic}(X)$ and $\beta \in H_2(X)z \oplus H_0(X)x$ we associate the vector $v(X,\xi,\beta) := (\xi^2, \xi c_1(X), c_1(X)^2, c_2(X), \xi\beta, \beta c_1(X), \beta^2, \beta) \in \mathbb{Z}^8$, where we suppress $\int_X$ in the notation. Then we know $\overline{\delta}^X_{\xi,t}(\beta) = \exp(G_{t,\Lambda}(v(X,\xi,\beta)))$. Choose triples (X_i,ξ_i,β_i), $i = 1,\ldots,8$ as above such that the $w_i := v(X_i,\xi_i,\beta_i)$ form a basis of $\mathbb{Q}^8$. Let $(a_{i,j})^8_{i,j=1}$ be the matrix such that $\sum_j a_{i,j}w_j = e_i$ for all i, where e_i is the vector with i-th entry 1 and all others zero. For all i put $A_i := \sum_j a_{i,j}G_{t,\Lambda}(w_j)$. Let (X,ξ,β) be a triple, such that $(v^1,\ldots,v^8) := v(X,\xi,\beta) = \sum^8_{i=1} n_i w_i$, with $n_i \in \mathbb{Z}_{\geq 0}$.

Then by (5.6) we get $\overline{\delta}^X_{\xi,t}(\beta) = \exp(\sum_j n_j G_{t,\Lambda}(w_j))$. Thus by $\sum_i v^i a_{i,j} = n_j$ for all j, we get $\overline{\delta}^X_{\xi,t}(\beta) = \exp\left(\sum_i v^i A_i\right)$. Note that the v^i are just the intersection numbers $\xi^2,\ldots,\beta$. As the set of all vectors $\sum^8_{i=1} n_i w_i$ with all $n_i \in \mathbb{Z}_{\geq 0}$ is Zariski dense in $\mathbb{Q}^8$, the last equality holds for all triples (X,ξ,β) of a projective surface X, a class $\xi \in \mathrm{Pic}(X)$ and $\beta \in H_2(X)z \oplus H_0(X)x$. This proves the Theorem.

Corollary 5.7.
(1) *Theorem 4.2 holds for any simply connected smooth projective surface with $p_g = 0$ and any good class ξ.*

(2) *More generally, for any smooth projective surface X and any $\xi \in \mathrm{Pic}(X)$, we have*

$$\delta^X_{\xi,t}(\exp(\alpha z + px))$$

$$= \sqrt{-1}^{\langle\xi,K_X\rangle}\left(\frac{q^{-\frac{1}{2}(\frac{\xi}{2})^2}}{\Lambda^{\chi(\mathcal{O}_X)}}\exp\left(\frac{du}{da}\langle\alpha,\xi/2\rangle z + T\langle\alpha^2\rangle z^2 - ux\right)\right.$$

$$\left. \cdot \left(\frac{\sqrt{-1}}{\Lambda}\frac{du}{da}\right)^{2\chi(\mathcal{O}_X)}\theta^\sigma_{01}\right).$$

Proof. In the notations of Section 4, putting $t = 2a$, we can rewrite Theorem 5.1 in terms of q. For $f,g \in \mathbb{C}((q^{1/8},\Lambda))$, we write $f \equiv g$ if $f/g = \exp(h)$ with $h \in q^{1/8}\mathbb{C}[[q^{1/2}]]$. Note that $\frac{du}{da} \equiv \sqrt{-1}\Lambda q^{-1/8}$, $t \equiv \sqrt{-1}\Lambda q^{-1/8}$. Thus

$$(-1)^{\chi(\mathcal{O}_X)+\xi(\xi-K_X)/2}t^{-\xi^2-2\chi(\mathcal{O}_X)}\Lambda^{\xi^2+3\chi(\mathcal{O}_X)}$$

$$\equiv \sqrt{-1}^{-\langle\xi,K_X\rangle}\Lambda^{\chi(\mathcal{O}_X)}q^{\frac{\xi^2}{8}}\left(\frac{\sqrt{-1}}{\Lambda}\frac{du}{da}\right)^{-2\chi(\mathcal{O}_X)}.$$

Thus for any triple (X,ξ,β) with $v(X,\xi,\beta) = (v^1,\ldots,v^8)$ we get

$$\delta^X_{\xi,t}(\beta) = \frac{\sqrt{-1}^{\langle\xi,K_X\rangle}}{\Lambda^{\chi(\mathcal{O}_X)}}q^{-\frac{\xi^2}{8}}\left(\frac{\sqrt{-1}}{\Lambda}\frac{du}{da}\right)^{2\chi(\mathcal{O}_X)}\exp\left(\sum^8_{i=1}v^i B_i\right),$$

for some universal power series $B_i \in \mathbb{C}((q^{-\frac{1}{8}}))[[\Lambda]]$. As the $v(X, \xi, \beta)$ with X a toric surface and ξ a good class generate $\mathbb{Q}^8$ as a vector space, the B_i are determined by their values for toric surfaces and good classes, i.e., they are given by (4.12). Note that the proof of (4.12) still works without any changes also if ξ is not good (replacing $\mathcal{A}_{\xi,-}$, $\mathcal{A}_{\xi,+}$ by $-p_!(\mathcal{I}_2^\vee \otimes \mathcal{I}_1 \otimes q^! \xi)$, $-p_!(\mathcal{I}_1^\vee \otimes \mathcal{I}_2 \otimes q^! \xi^\vee)$.)   q.e.d.

Remark 5.8.

(1) Using [**30**, Thm1.12], we get that Thm 3.3 and part (1) of Cor. 5.7 hold also if ξ is not good.

(2) As we mentioned in the introduction, the assertion that $\xi c_1(X)$ appears only as a sign in $\delta_{\xi,t}^X$ is one of statements of the Kotschick-Morgan conjecture. This comes from $H(a, \Lambda) = \pi\sqrt{-1}a$, as $\varepsilon_1 + \varepsilon_2$ is the equivariant first Chern class of $\mathbb{A}^2$. The latter statement, proved in [**34**, Section 5.3], is a consequence of the blowup equation [**33**, (6.14)]. This is by no means simple to check directly from the definition of Nekrasov partition function.

6. Equivariant Donaldson invariants for $\mathbb{P}^2$

Let us consider the complex projective plane $X = \mathbb{P}^2$ and let H be the hyperplane bundle. Let $M_H(n)$ be the moduli space of H-semistable sheaves E on X with $\operatorname{rank} E = 2$, $c_1(E) = H$, $c_2(E) - c_1(E)^2/4 \equiv \Delta(E) = n$. As $\operatorname{GCD}(2, c_1(E)) = 1$, $M_H(n)$ is nonsingular of dimension $4n - 3$. Let $\mathcal{E}$ be the universal bundle. Our method works also for $\operatorname{rank} E = 2$, $c_1(E) = 0$, $c_2(E) = n \equiv 1 \mod 2$. But the moduli space becomes singular when $c_2(E)$ is even, so our localization technique fails.

Let us consider the Donaldson invariants

$$\Phi_H^H(\alpha z + px) = \sum_{n \geq 0} \Lambda^{4n-3} \int_{M_H(n)} \exp\left(-\overline{\operatorname{ch}}_2(\mathcal{E})/(\alpha z + px)\right).$$

Hereafter we denote this just by $\Phi(\alpha z + px)$ for brevity as we will not vary H in this section.

Let Γ be the 2-dimensional torus acting on X by $[x : y : z] \mapsto [t_1 x : t_2 y : z]$. We have three fixed points $[1 : 0 : 0]$, $[0 : 1 : 0]$, $[0 : 0 : 1]$, and their characters of tangent spaces are $1/t_1 + t_2/t_1$, $t_1/t_2 + 1/t_2$ and $t_1 + t_2$ respectively. We set $p_x = [1 : 0 : 0]$, $p_y = [0 : 1 : 0]$, $p_z = [0 : 0 : 1]$. We take the coordinates around each p_i and define their weights as

$$(w(x_i), w(y_i)) = (-\varepsilon_1, \varepsilon_2 - \varepsilon_1), \qquad (\varepsilon_1 - \varepsilon_2, -\varepsilon_2), \qquad (\varepsilon_1, \varepsilon_2)$$

for $i = x, y, z$ respectively. We consider the induced Γ-action on $M_H(n)$. It also lifts to the universal bundle $\mathcal{E}$, so we can define the equivariant Donaldson invariants $\widetilde{\Phi}(\alpha z + px)$, where α, p are equivariant cohomology classes. In the nonequivariant limit $\varepsilon_1, \varepsilon_2 \to 0$, they go to the ordinary Donaldson invariants $\Phi(\alpha z + px)$, where α, p are replaced by their

nonequivariant limit. For example, there are three equivariant lifts $[p_x]$, $[p_y]$, $[p_z]$ of the point class p given by the three fixed points. Then $\widetilde{\Phi}(p_i x)$ depends on $i = x, y, z$, but its nonequivariant limit $\lim_{\varepsilon_1, \varepsilon_2 \to 0} \widetilde{\Phi}(p_i x)$ is equal to $\Phi(px)$.

Proposition 6.1 ([23]).
(1) *A sheaf $E \in M_H(n)$ is fixed by the Γ-action if and only if there exists a Γ-equivariant structure on E.*

(2) *A sheaf $E \in M_H(n)$ is fixed by the Γ-action if and only if both its reflexive hull $E^{\vee\vee}$ and the quotient $E^{\vee\vee}/E$ have Γ-equivariant structures.*

For a stable sheaf E, its Γ-equivariant structure is unique up to a twist by a character. We normalized it so that $\det E^{\vee\vee}$ is trivial. This may not be possible in general, but it is possible if we formally tensor by a square root of a line bundle. In particular, the actions on the fibers over fixed points are well-defined if we lift the action to a double covering $\widetilde{\Gamma} \to \Gamma$. We consider the $\widetilde{\Gamma}$-structure as if it is a Γ-structure hereafter.

Let $\mathcal{O}(x)$, $\mathcal{O}(y)$, $\mathcal{O}(z)$ be the Γ-equivariant line bundles, where the Γ-structures are given so that the homomorphism $x \colon \mathcal{O} \to \mathcal{O}(x)$ is equivariant, etc. The characters of the fiber of $\mathcal{O}(x)$ at the fixed points $[1 : 0 : 0]$, $[0 : 1 : 0]$, $[0 : 0 : 1]$ are given by $1, t_2/t_1, 1/t_1$ respectively.

By a result of [**2**, **23**], we have

Proposition 6.2.
(1) *A Γ-equivariant rank 2 vector bundle E with $c_1(E) = 1$ is classified by a triple $(p, q, r) \in \mathbb{Z}_{>0}^3$ with $p + q + r \equiv 1 \mod 2$.*

(2) *The above E is stable if and only if p, q, r satisfy the strict triangle inequality, i.e., $p + q < r$, $q + r < p$, $r + p < q$.*

In fact, the vector bundle E is given as a cokernel of

$$\mathcal{O} \to \mathcal{O}(px) \oplus \mathcal{O}(qy) \oplus \mathcal{O}(rz)$$

for some $(p, q, r) \in \mathbb{Z}_{>0}^3$ after a twist by a line bundle. Let $E^{(p,q,r)}$ be the corresponding Γ-equivariant vector bundle. We have

$$\Delta(E^{(p,q,r)}) = (pq + qr + rp)/2 - p^2/4 - q^2/4 - r^2/4.$$

Let us denote this by $\Delta(p, q, r)$. Note that $E^{(p,q,r)}$ is an isolated fixed point in $M(H, \Delta(p, q, r))$. This assertion fails for higher ranks or toric surfaces other than $\mathbb{P}^2$.

We have the decomposition of the fixed point set:

$$M_H(n)^\Gamma = \bigsqcup M_{p,q,r}(n - \Delta(p, q, r)),$$

where $M_{p,q,r}(m)$ denote the set of Γ-equivariant sheaves E with $E^{\vee\vee} = E^{(p,q,r)}$ and $\text{length}(E^{\vee\vee}/E) = m$. The quotient sheaf $E^{\vee\vee}/E$ is supported at $\{p_x, p_y, p_z\}$. Accordingly we have a factorization

$$M_{p,q,r}(m) = \bigsqcup_{m_x+m_y+m_z=m} M_{p,q,r}^x(m_x) \times M_{p,q,r}^y(m_y) \times M_{p,q,r}^z(m_z),$$

where $M_{p,q,r}^x(m_x)$ denotes the set of Γ-equivariant sheaves supported at p_x, etc.

The character of the fiber of $E^{(p,q,r)}$ at the fixed point $p_z = [0:0:1]$ is given by

$$(6.3) \qquad \text{ch}\, E_{p_z}^{(p,q,r)} = t_1^{p/2} t_2^{q/2} \left[t_1^{-p} + t_2^{-q} \right] = t_1^{-p/2} t_2^{q/2} + t_1^{p/2} t_2^{-q/2}$$

Similarly, the characters of the fibers at $p_y = [0:1:0]$, $p_x = [1:0:0]$ are given by

$$(6.4) \qquad \text{ch}\, E_{p_y}^{(p,q,r)} = (t_1/t_2)^{p/2}(1/t_2)^{r/2} \left[(t_1/t_2)^{-p} + (1/t_2)^{-r} \right]$$
$$= t_1^{-p/2} t_2^{(p-r)/2} + t_1^{p/2} t_2^{(r-p)/2},$$

$$(6.5) \qquad \text{ch}\, E_{p_x}^{(p,q,r)} = (1/t_1)^{r/2}(t_2/t_1)^{q/2} \left[(1/t_1)^{-r} + (t_2/t_1)^{-q} \right]$$
$$= t_1^{(r-q)/2} t_2^{q/2} + t_1^{(q-r)/2} t_2^{-q/2},$$

respectively.

Let us study a Γ-equivariant sheaf $E \in M_{p,q,r}^z(m_z)$, i.e., $E^{\vee\vee} = E^{(p,q,r)}$ and $\text{Supp}(E^{\vee\vee}/E) = \{p_z\}$. Using the coordinate system $(x/z, y/z)$ around p_z, we can identify $E^{\vee\vee}/E$ with a Γ-equivariant quotient sheaf $Q = \mathcal{O}^{\oplus 2}/F$, where Γ acts on the trivial bundle $\mathcal{O}^{\oplus 2}$ so that the character of the fiber at the origin is (6.3).

Let $M(n)$ be the framed moduli space of rank 2 torsion-free sheaves on $\mathbb{P}^2$ as in §1.2. This is the Gieseker partial compactification of framed moduli spaces of instantons on $\mathbb{R}^4$. Let $M_0(n)$ be the Uhlenbeck partial compactification of framed moduli spaces of instantons on $\mathbb{R}^4$, and let $\pi\colon M(n) \to M_0(n)$ be the natural projective morphism. (See [**33**, §2], [**34**, §3].) We have an action of $\widetilde{T} = (\mathbb{C}^*)^2 \times \mathbb{C}^*$ on $M(n)$, $M_0(n)$ such that π is equivariant. According to (6.3), we define $\rho_{p,q,r}^z\colon \Gamma \to \widetilde{T}$ by

$$\rho_{p,q,r}^z(t_1, t_2) = (t_1, t_2, t_1^{p/2} t_2^{-q/2}).$$

Note that there is no reason to prefer $t_1^{p/2} t_2^{-q/2}$ instead of $t_1^{-p/2} t_2^{q/2}$. Either choice will work in the following argument.

The following result follows from [**18**], but we give a direct proof:

Lemma 6.6.

(1) *The origin $(\mathcal{O}^{\oplus 2}, m[0])$ is the only Γ-fixed point in $M_0(m)$.*

(2) *A point $(F, \varphi) \in M(m)$ is fixed by the Γ-action if and only if $F^{\vee\vee} = \mathcal{O}^{\oplus 2}$ and $F^{\vee\vee}/F$ is a Γ-equivariant sheaf.*

Proof. (2) follows from (1). Let us prove (1). Let us use the ADHM description (B_1, B_2, i, j) for $M_0(m)$. The coordinate ring of $M_0(m)$ is generated by the following two types of functions

  a) $\mathrm{tr}(B_{\alpha_1} B_{\alpha_2} \cdots B_{\alpha_N})$,
  b) $\langle \chi, j B_{\alpha_1} B_{\alpha_2} \cdots B_{\alpha_N} i \rangle$

where $\alpha_i = 1$ or 2 and χ is a linear form on $\mathrm{End}(W)$. Let us take a Γ-equivariant form χ. Then its weight is either 1, $t_1^{-p} t_2^q$ or $t_1^p t_2^{-q}$. Under the Γ-action, $B_{\alpha_1} B_{\alpha_2} \cdots B_{\alpha_N}$ is multiplied by $t_{\alpha_1} t_{\alpha_2} \cdots t_{\alpha_N}$. Therefore the first type of functions are never preserved by the Γ-action. Similarly the second type of functions are multiplied by $t_{\alpha_1} t_{\alpha_2} \cdots t_{\alpha_N} t_1 t_2$, $t_{\alpha_1} t_{\alpha_2} \cdots t_{\alpha_N} t_1^{1-p} t_2^{q+1}$ or $t_{\alpha_1} t_{\alpha_2} \cdots t_{\alpha_N} t_1^{p+1} t_2^{1-q}$. These are never 1 as p, $q > 0$.   q.e.d.

Thanks to this lemma we have

Corollary 6.7.

$$M_{p,q,r}^z(m) \cong M(m)^{\rho_{p,q,r}^z(\Gamma)}.$$

We define $\rho_{p,q,r}^x$, $\rho_{p,q,r}^y : \Gamma \to \widetilde{T}$ by

$$\rho_{p,q,r}^x(t_1, t_2) = (1/t_1, t_2/t_1, t_1^{(q-r)/2} t_2^{-q/2}),$$

$$\rho_{p,q,r}^y(t_1, t_2) = (t_1/t_2, 1/t_2, t_1^{p/2} t_2^{(r-p)/2}).$$

(See (6.5) (resp. (6.4)).) The above lemma and corollary hold also for these homomorphisms.

Let $N_{p,q,r;m}$ be the normal bundle of $M_{p,q,r}(m)$ in $M(H, m + \Delta(p, q, r))$. Its fiber at E is the sum of nonzero weight spaces in $\mathrm{Ext}^1(E, E)$. We decompose $E^{\vee\vee}/E$ to Q^x, Q^y, Q^z according to the support p_x, p_y, p_z. By the above corollary, we identify them with (F^x, φ), (F^y, φ), (F^z, φ) as elements of $M(m_x)$, $M(m_y)$, $M(m_z)$. We have

$$\mathrm{Ext}^1(E, E) = \mathrm{Ext}^1(E^{(p,q,r)}, E^{(p,q,r)}) + \sum_{i=x,y,z} \mathrm{Ext}^1(F^i, F^i(-\ell_\infty))$$

in the Grothendieck group of Γ-equivariant vector bundles on $M_{p,q,r}(m)$ $= \bigsqcup_{m=m_x+m_y+m_z} \prod_{i=x,y,z} M(m_i)^{\rho_{p,q,r}^i(\Gamma)}$. The first factor of the right hand side is the tangent space of $M_H(n)$ at $E^{(p,q,r)}$. Let us denote it by $T_{p,q,r}$. Then the equivariant Euler class of $N_{p,q,r;m}$ is given by

$$e(T_{(p,q,r)}) \prod_{i=x,y,z} e(N_{(F^i, \varphi)}),$$

where $N_{(F^i, \varphi)}$ denotes the fiber of the normal bundle of the fixed point component containing (F^i, φ) in $M(m_i)$. We also have

$$-\overline{\mathrm{ch}}_2(\mathcal{E}) = -\overline{\mathrm{ch}}_2(E^{p,q,r}) + \mathrm{ch}_2(Q^x) + \mathrm{ch}_2(Q^y) + \mathrm{ch}_2(Q^z)$$

$$= -\overline{\mathrm{ch}}_2(E^{p,q,r}) + m_x[p_x] + m_y[p_y] + m_z[p_z],$$

where we have identified homology classes $[p_x]$, $[p_y]$, $[p_z]$ with their Poincaré dual. We get

(6.8)
$$\widetilde{\Phi}(\alpha z + px)$$

$$= \sum_{p,q,r} \Lambda^{4\Delta(p,q,r)-3} \sum_{m} \Lambda^{4m} \int_{M_{p,q,r}(m)} \frac{\exp\left(-\overline{\mathrm{ch}}_2(\mathcal{E})/(\alpha z + px)\right)}{e(N_{p,q,r}(m))}$$

$$= \sum_{p,q,r} \Lambda^{4\Delta(p,q,r)-3} \frac{\exp\left(-\overline{\mathrm{ch}}_2(E^{(p,q,r)})/(\alpha z + px)\right)}{e(T_{p,q,r})}$$

$$\cdot \prod_{i=x,y,z} \sum_{m_i} \Lambda^{4m_i} \exp\left(m_i \iota_{p_i}^*(\alpha z + px)\right) \int_{M(m_i)^{\rho_{p,q,r}^i(\Gamma)}} \frac{1}{e(N_{(F^i,\varphi)})},$$

where ι_{p_i} denotes the inclusion map $\{p_i\} \to X$.

We study the first term and the second term separately.

6.1. Quotient sheaf part.

Recall that the instanton part of Nekrasov's partition function is written by $(\iota_{0*})^{-1}\pi_*[M(m)] = (\iota_{0*})^{-1}[M_0(m)]$, where ι_0 is the inclusion of the $\widetilde{T}$-fixed point in $M_0(m)$. Here the equivariant homology groups are taken with respect to the $\widetilde{T}$-action. By Lemma 6.6(1) we replace them by those with respect to the Γ-action and get an element $(\iota_{0*})^{-1}[M_0(m)]$ in the quotient field of $H_\Gamma^*(\mathrm{pt})$. In order to distinguish this from the above element, we denote them by $(\iota_{0*})^{-1}[M_0(m)]_{\widetilde{T}}$ and $(\iota_{0*})^{-1}[M_0(m)]_\Gamma$.

We set $S(\Gamma) = H_\Gamma^*(\mathrm{pt})$, $S(\widetilde{T}) = H_{\widetilde{T}}^*(\mathrm{pt})$. We denote their quotient fields by $\mathcal{S}(\Gamma)$ and $\mathcal{S}(\widetilde{T})$ respectively. Let $d\rho_{p,q,r}^i \colon \mathrm{Lie}(\Gamma) \to \mathrm{Lie}(\widetilde{T})$ be the differential of the homomorphism $\rho_{p,q,r}^i$. It induces the restriction homomorphism $\left(d\rho_{p,q,r}^i\right)^* \colon S(\widetilde{T}) \to S(\Gamma)$.

Lemma 6.9. *The rational function* $(\iota_{0*})^{-1}[M_0(m)]_{\widetilde{T}} \in \mathcal{S}(\widetilde{T})$ *can be restricted under the homomorphism* $(d\rho_{p,q,r}^i)^*$, *and is mapped to* $(\iota_{0*})^{-1}[M_0(m)]_\Gamma \in \mathcal{S}(\Gamma)$.

Proof. From the proof of the localization theorem (see e.g., [**1**]), $(\iota_{0*})^{-1}[M_0(m)]_{\widetilde{T}}$ can be defined in a localized module $S(\widetilde{T})_f$ with a polynomial f which vanishes on all Lie subalgebras of stabilizer subgroups $\neq \widetilde{T}$. Under the homomorphism $\rho_{p,q,r}^i \colon \Gamma \to \widetilde{T}$, stabilizer subgroups in $\widetilde{T}$ are mapped to stabilizer subgroups in Γ. By Lemma 6.6(1), if a stabilizer subgroup is not $\widetilde{T}$, then it is mapped to a subgroup $\neq \Gamma$. Therefore f is restricted to a nonzero polynomial under $d\rho_{p,q,r}^i \colon \mathrm{Lie}(\Gamma) \to \mathrm{Lie}(\widetilde{T})$ and we have an induced homomorphism

$$\left(d\rho_{p,q,r}^i\right)^* \colon S(\widetilde{T})_f \to S(\Gamma)_f.$$

From the definition we clearly have the assertion.   q.e.d.

By the localization theorem, $(\iota_{0*})^{-1}[M_0(m_i)]_\Gamma = (\iota_{0*})^{-1}\pi_*[M(m_i)]_\Gamma$ is equal to

$$\int_{M(m_i)^{\rho^i_{p,q,r}(\Gamma)}} \frac{1}{e(N_{(F^\alpha,\varphi)})}.$$

Therefore we get

(6.10) $$\sum_{m_i} \mathsf{q}^{m_i} \exp\left(m_i \iota^*_{p_i}(\alpha + pX)\right) \int_{M(m_i)^{\rho^i_{p,q,r}(\Gamma)}} \frac{1}{e(N_{(F^i,\varphi)})}$$

$$= Z^{\text{inst}}(w(x_i), w(y_i), -\tfrac{\xi^i_{p,q,r}}{2}; \mathsf{q}e^{\iota^*_{p_i}(\alpha+pX)}),$$

where
(6.11)
$$\xi^z_{p,q,r} = -p\varepsilon_1 + q\varepsilon_2, \quad \xi^y_{p,q,r} = -p\varepsilon_1 + (p-r)\varepsilon_2, \quad \xi^x_{p,q,r} = (r-q)\varepsilon_1 + q\varepsilon_2.$$

6.2. Vector bundle part. Let us calculate

$$\operatorname{ch} T^{(p,q,r)} = \operatorname{ch} \operatorname{Ext}^1(E^{(p,q,r)}, E^{(p,q,r)})$$

$$= \sum_{p=0}^{2} (-1)^{p+1} \operatorname{ch} H^p(\mathbb{P}^2, \mathcal{E}nd_0(E^{(p,q,r)}))$$

where $\mathcal{E}nd_0$ means the trace-free part. We calculate this by the localization theorem, i.e.,

$$\operatorname{ch} \operatorname{Ext}^1(E^{(p,q,r)}, E^{(p,q,r)})$$

$$= -\frac{\operatorname{ch} \mathcal{E}nd_0(E^{(p,q,r)})_{[0:0:1]}}{(1-t_1)(1-t_2)}$$

$$- \frac{\operatorname{ch} \mathcal{E}nd_0(E^{(p,q,r)})_{[0:1:0]}}{(1-t_1/t_2)(1-1/t_2)} - \frac{\operatorname{ch} \mathcal{E}nd_0(E^{(p,q,r)})_{[1:0:0]}}{(1-1/t_1)(1-t_2/t_1)}.$$

We have

$$\operatorname{ch} \mathcal{E}nd_0(E^{(p,q,r)})_{[0:0:1]} = 1 + t_1^{-p}t_2^q + t_1^p t_2^{-q},$$

$$\operatorname{ch} \mathcal{E}nd_0(E^{(p,q,r)})_{[0:1:0]} = 1 + t_1^{-p}t_2^{p-r} + t_1^p t_2^{r-p},$$

$$\operatorname{ch} \mathcal{E}nd_0(E^{(p,q,r)})_{[1:0:0]} = 1 + t_1^{r-q}t_2^q + t_1^{q-r}t_2^{-q}.$$

A calculation shows the following:

Lemma 6.12. *Let us define the convex region $D^{(p,q,r)}$ as follows:*

(1) Case $p = q = r = 1$: $D^{(p,q,r)} = \{(0,0)\}$.

(2) Case $p = 1$, $q = r \neq 1$: $D^{(p,q,r)} = \operatorname{Conv}((0, q-1), (-1, q-1), (-1, -q+2), (0, -q+1))$.

(3) Case $q = 1$, $p = r \neq 1$: $D^{(p,q,r)} = \operatorname{Conv}((p-1, 0), (-p+1, 0), (-p+2, -1), (p-1, -1))$.

(4) *Case $r = 1$, $p = q \neq 1$:* $D^{(p,q,r)} = \mathrm{Conv}((p-1, 1-p), (p-1, 2-p), (2-p, p-1), (1-p, p-1))$.

(5) *Case $p+q = r+1$, not above:* $D^{(p,q,r)} = \mathrm{Conv}((p-1, q-1), (-p, q-1), (-p, -q+2), (-p+2, -q), (p-1, -q))$.

(6) *Case $r+p = q+1$, not above:* $D^{(p,q,r)} = \mathrm{Conv}((p-1, 1-q), (p-1, r-p+1), (2-p, q-1), (-p, q-1), (-p, p-r+1))$.

(7) *Case $q+r = p+1$, not above:* $D^{(p,q,r)} = \mathrm{Conv}((p-1, -q), (p-1, r-p+1), (r-q+1, q-1), (1-p, q-1), (q-r+1, -q))$.

(8) *Otherwise:* $D^{(p,q,r)} = \mathrm{Conv}((p-1, -q), (p-1, r-p+1), (r-q+1, q-1), (-p, q-1), (-p, p-r+1), (q-r+1, -q))$

Here Conv *denotes the convex hull. Then* $\mathrm{ch}\,\mathrm{Ext}^1(E^{(p,q,r)}, E^{(p,q,r)})$ *is the sum of monomials* $t_1^m t_2^n$ *where* $(m,n) \in \mathbb{Z}^2$ *runs over* $D^{(p,q,r)} \setminus \{(0,0)\}$.

Note that the origin $(0,0)$ is in $D^{(p,q,r)}$ in all cases. We thus have

$$(6.13) \qquad e(T_{p,q,r}) = \prod_{(m,n) \in D^{(p,q,r)} \cap \mathbb{Z}^2 \setminus \{(0,0)\}} (m\varepsilon_1 + n\varepsilon_2).$$

We can also express $\mathrm{ch}_2(\mathcal{E})/(\alpha z + px)$ by the localization formula:

$$(6.14) \qquad -\mathrm{ch}_2(\mathcal{E})/(\alpha z + px) = -\frac{1}{4} \sum_{i=x,y,z} \frac{(\xi_{p,q,r}^i)^2 \iota_{p_i}^*(\alpha z + px)}{w(x_i) w(y_i)},$$

where $\xi_{p,q,r}^i$ is as in (6.11) and $w(x_i)w(y_i)$ appears as the Euler class $e(T_{p_i}\mathbb{P}^2)$ of the tangent space at p_i.

Substituting (6.10, 6.13, 6.14) into (6.8), we get the following:

Theorem 6.15. *The equivariant Donaldson invariants of $\mathbb{P}^2$ are given by*

$$\widetilde{\Phi}(\alpha z + px) = \sum_{p,q,r} \Lambda^{4\Delta(p,q,r)-3} \exp\left(-\frac{1}{4} \sum_{i=x,y,z} \frac{(\xi_{p,q,r}^i)^2 \iota_{p_i}^*(\alpha z + px)}{w(x_i)w(y_i)} \right)$$

$$\cdot \prod_{\substack{(m,n) \in D^{(p,q,r)} \cap \mathbb{Z}^2 \\ \setminus \{(0,0)\}}} \frac{1}{m\varepsilon_1 + n\varepsilon_2}$$

$$\cdot \prod_{i=x,y,z} Z^{\mathrm{inst}}\left(w(x_i), w(y_i), -\frac{\xi_{p,q,r}^i}{2}; \mathfrak{q} e^{\iota_{p_i}^*(\alpha z + px)} \right),$$

where p, q, r runs over $\mathbb{Z}_{>0}^3$ satisfying $p+q+r \equiv 1 \mod 2$ and the strict triangle inequality.

Ordinary Donaldson invariants $\Phi(\alpha z + px)$ are given by $\lim_{\varepsilon_1, \varepsilon_2 \to 0} \widetilde{\Phi}(\alpha z + px)$. But the solution of Nekrasov's conjecture does not say anything about this limit, so we do not know how to get an explicit formula from the above. Note that the summation over p,q,r is related

to Hurwitz class numbers (according to [**22**]), which appeared in [**31**, §9] in the formula for the Donaldson invariants of $\mathbb{P}^2$.

On the other hand we have:

Theorem 6.16. *Let* $P\colon Y \to \mathbb{P}^2$ *be the blowup of the fixed point* p_z *(different from* $p = p_x$*). Then*

$$\widetilde{\Phi}(Hz + px)$$

$$= \sum_{\substack{\xi = (2n-1)P^*H - 2aE \\ a \geq n \in \mathbb{Z}_{>0}}} \left[\frac{1}{\Lambda} \exp\left(\frac{1}{2} \langle \operatorname{Todd}_2(Y)(P^*Hz + P^*px) \rangle \right.\right.$$

$$\left.\left. \cdot \sum_{i = x, y, z_1, z_2} F\left(w(x_i), w(y_i), \frac{t - \iota_{p_i}^* \xi}{2}; \Lambda e^{\iota_{p_i}^*(P^*Hz + P^*px)/4}\right) \right)\right]_{t^{-1}},$$

where p_{z_1}, p_{z_2} *are the fixed points in the exceptional set of* Y.

The formula, when compared with the one in Theorem 6.15, probably gives us a nontrivial identity on the partition function.

The idea of the proof is the same as in [**19**, Th. 3.5], but we put a little more care as we consider the equivariant Donaldson invariants.

Proof. Let $P\colon Y \to \mathbb{P}^2$ be the blow up of the fixed point p_z. We first assume that the line H is H_{xy}. In particular, H does not pass through the point p_z which we blowup. Let $\widehat{M}_H(n)$ be the moduli space of $(P^*H - \varepsilon E)$-stable rank 2 sheaves on Y with $c_1 = P^*H$, $\Delta = n$. By [**34**, App. F] there exists a projective morphism $\widehat{\pi}\colon \widehat{M}_H(n) \to N_H(n)$, where $N_H^{\mathbb{P}^2}(n)$ is the Uhlenbeck compactification of the moduli space of locally free sheaves on $\mathbb{P}^2$ with $c_1 = H$, $\Delta = n$.

By the definition of $\widehat{\pi}$ the class $\mu(P^*H)$ on $\widehat{M}_H(n)$ is the pullback of the class $\mu(H)$ on $N_H(n)$ by $\widehat{\pi}$. In fact, by

$$H_{xz} = H_{xy} - \varepsilon_2, \qquad H_{yz} = H_{xy} - \varepsilon_1,$$

and

$$c_2(\mathcal{E}) - \frac{1}{4}c_1(\mathcal{E})^2/[\mathbb{P}^2] \in H_\Gamma^0(N_H(n)) \cong \mathbb{C},$$

$\mu(H_{xz})$, $\mu(H_{yz})$ are equal to $\mu(H) = \mu(H_{xy})$ modulo classes from $H_\Gamma^*(\mathrm{pt})$. Therefore this assertion is true for any H.

By [**16**, Th. 6.9], $\mu(p)$ extends to a class on the Donaldson compactification $N_H(n)$. The extension can be made so that the class is equivariant with respect to the compact form of Γ, and it is enough for our purpose. Then we have $\mu(P^*p) = \widehat{\pi}^*\mu(p)$ as we blow up at a point

different from p. Therefore

$$\exp(\mu(P^*H)z + \mu(P^*p)x) \cap [\widehat{M}_H(n)]$$

$$= \widehat{\pi}^* \left(\exp(\mu(H)z + \mu(p)x) \right) \cap [\widehat{M}_H(n)]$$

$$= \exp(\mu(H)z + \mu(p)x) \cap \widehat{\pi}_*[\widehat{M}_H(n)]$$

$$= \exp(\mu(H)z + \mu(p)x) \cap [N_H(n)].$$

There is an alternative way to prove this formula. Restrict the maps $\pi, \widehat{\pi}$ to the fixed point set: $\pi \colon M_H(n)^\Gamma \to N_H(n)^\Gamma$, $\widehat{\pi} \colon \widehat{M}_H(n)^\Gamma \to N_H(n)^\Gamma$. We have

$$N_H(n)^\Gamma = \bigsqcup_{\substack{p,q,r,m_x,m_y,m_z \\ m_x+m_y+m_z+\Delta(p,q,r)=n}} \{(E^{(p,q,r)}, m_x[p_x] + m_y[p_y] + m_z[p_z])\}$$

by the same argument as above. In particular, $N_H(n)^\Gamma$ consists of finitely many points. We have direct sum decompositions of $H_\Gamma^4(M_H(n))$ and $H_\Gamma^4(\widehat{M}_H(n))$ correspondingly. From the expression (1.4) we see that $\mu(p) \in H_\Gamma^4(M_H(n)^\Gamma)$ and $\mu(P^*p) \in H_\Gamma^4(\widehat{M}_H(n)^\Gamma)$ are pullbacks of the same class in $H_\Gamma^4(N_H(n)^\Gamma) = \bigoplus_{p,q,r,m_x,m_y,m_z} H_\Gamma^4(\mathrm{pt})$. This assertion is enough for the above calculation.

Therefore

$$\widetilde{\Phi}_H^{\mathbb{P}^2,H}(\exp(Hz + px)) = \widetilde{\Phi}_H^{Y,H-\varepsilon E}(\exp(P^*Hz + P^*px)).$$

On the other hand, we have

$$\widetilde{\Phi}_H^{Y,F+\varepsilon E}(\exp(P^*Hz + P^*px)) = 0$$

for $F = P^*H - E$ is the fiber class and ε is a sufficiently small number by [**39**]. Therefore, by the proof of [**19**, Th. 3.5] we have

$$\widetilde{\Phi}_H^{Y,H-\varepsilon E}(\exp(P^*Hz + P^*px))$$

$$= \sum_{\substack{\xi=(2n-1)P^*H-2aE \\ a \geq n \in \mathbb{Z}_{>0}}} \widetilde{\delta}_\xi^Y(\exp(P^*Hz + P^*px)).$$

Let p_x, p_y denote the inverse image of p_x, p_y under P. Let p_{z_1}, p_{z_2} be the two fixed points in the exceptional set E. By Corollary 3.18 we have

$$\widetilde{\delta}_{\xi,t}^Y(\exp(P^*Hz + P^*px))$$

$$= \frac{1}{\Lambda} \exp\left(\frac{1}{2} \langle \mathrm{Todd}_2(Y)(P^*Hz + P^*px) \rangle \right.$$

$$\left. \cdot \sum_{i=x,y,z_1,z_2} F\left(w(x_i), w(y_i), \tfrac{t-\iota_{p_i}^*\xi}{2}; \Lambda e^{\iota_{p_i}^*(P^*Hz+P^*px)/4}\right)\right).$$

Now the assertion follows.   q.e.d.

Appendix A. Generic smoothness after blowup

For a μ-semistable rank 2 sheaf F on X, there exists a constant β_∞ depending only on X (and the rank of F) (see [**21**, 4.5.7]) such that

$$\dim \mathrm{Ext}^2(F, F)_0 \leq \beta_\infty.$$

Therefore

(A.1) $$\dim M_H^X(c_1, n) \leq \exp \dim M_H^X(c_1, n) + \beta_\infty,$$

where $\exp \dim M_H^X(c_1, n)$ is the expected dimension of $M_H^X(c_1, n)$.

By the result of Donaldson, Zuo, Gieseker-Li, O'Grady (see [**21**, §9]) there exists a constant m_0 depending only on X, H (and rank) such that $M_H^X(c_1, m)$ is irreducible and of expected dimension for $m \geq m_0$.

Let $P \colon \widehat{X} \to X$ be blowup at points $p_1, \ldots, p_N$ as before. We take a polarization H on X and consider the polarization P^*H on $\widehat{X}$ as above. For simplicity we assume (c_1, H) is odd. By [**34**, App. F] we have a projective morphism $\widehat{\pi} \colon M_{P^*H}^{\widehat{X}}(P^*c_1, m) \to N_H^X(c_1, m)$, where $N_H^X(c_1, m)$ is the Uhlenbeck compactification, which is set-theoretically equal to $N_H^X(c_1, m) = \bigsqcup_k M_H^X(c_1, m-k)_{\mathrm{lf}} \times S^k X$, where $M_H^X(c_1, m-k)_{\mathrm{lf}}$ is the open subscheme of $M_H^X(c_1, m-k)$ consisting of stable vector bundles.

A point in $S^k X$ can be written as $[Z] = \sum m_i[p_i] + \sum \lambda_p[x_p]$ where p_i, x_p are disjoint and $\lambda_p \geq 1$. Then we have a stratification of $S^k X$ parametrized by $(m_i)_i \in \mathbb{Z}_{\geq 0}^N$ and the partition $\lambda = \{\lambda_p\}_p$ of $k - \sum m_i$. By [**34**, App. F] the fiber of $\widehat{\pi}$ over $(E, [Z]) \in M_H^X(c_1, m-k)_{\mathrm{lf}} \times S^k X$ depends only on $m - k$ and the stratum containing $[Z]$. And it is also equal to the fiber of the morphism defined for the framed moduli spaces on $\widehat{\mathbb{P}}^2$ and $\mathbb{P}^2$. The homology of central fibers (i.e., $\lambda = \emptyset$, $m_1 = n$, $m_i = 0$ $(i \geq 2)$) was calculated in [**34**, Th. 3.8~10]. We find its dimension is given by

$$2n + \max_{l \in \mathbb{Z} \colon l^2 \leq n} l \leq 3n.$$

Therefore

$$\dim \widehat{\pi}^{-1}(E, [Z]) \leq 3 \sum_{i=1}^N m_i + \sum_p (2\lambda_p - 1).$$

Therefore we have

$$\dim \widehat{\pi}^{-1}(M_H^X(c_1, m-k)_{\mathrm{lf}} \times S^k X)$$

$$\leq \dim M_H^X(c_1, m-k)_{\mathrm{lf}} + \max_{\sum m_i + |\lambda_p| = k} \left\{ \sum_i 3m_i + \sum_p (2\lambda_p + 1) \right\}$$

$$\leq \dim M_H^X(c_1, m-k)_{\mathrm{lf}} + 3k.$$

Let us take $m \geq m_0 + \beta_\infty$. For $k > \beta_\infty$ we have

$$\dim M_H^X(c_1, m-k)_{\mathrm{lf}} + 3k \leq \exp \dim M_H^X(c_1, m) - k + \beta_\infty$$
$$< \exp \dim M_H^X(c_1, m)$$

by (A.1). For $k \leq \beta_\infty$, we have $m - k \geq m_0$. Therefore $M_H^X(c_1, m-k)$ is of expected dimension. Therefore

$$\dim M_H^X(c_1, m-k)_{\mathrm{lf}} + 3k = \exp \dim M_H^X(c_1, m) - k < \dim M_H^X(c_1, m)$$

unless $k = 0$. The open locus $\widehat{\pi}^{-1}(M_H^X(c_1, m)_{\mathrm{lf}})$ consists of pullbacks P^*E of $E \in M_H^X(c_1, m)$ and $\widehat{\pi}$ is an isomorphism there. Therefore $M_{\widehat{X}}^{P^*H}(P^*c_1, m)$ is of expected dimension (and irreducible).

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