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LOWER SEMICONTINUITY OF THE WILLMORE FUNCTIONAL FOR CURRENTS

Reiner Schätzle

Abstract

The weak mean curvature is lower semicontinuous under weak convergence of varifolds, that is, if $\mu_k \to \mu$ weakly as varifolds then $\| \vec{\mathbf{H}}_\mu \|_{L^p(\mu)} \leq \liminf_{k\to\infty} \| \vec{\mathbf{H}}_{\mu_k} \|_{L^p(\mu_k)}$. In contrast, if $T_k \to T$ weakly as integral currents, then μ_T may not have a locally bounded first variation even if $\| \vec{\mathbf{H}}_{\mu_{T_k}} \|_{L^\infty(\mu_k)}$ is bounded.

In 1999, Luigi Ambrosio asked the question whether lower semicontinuity of the weak mean curvature is true when T is assumed to be smooth. This was proved in [**AmMa03**] for $p > n = \dim T$ in $\mathbb{R}^{n+1}$ using results from [**Sch04**]. Here we prove this in any dimension and codimension down to the desired exponent $p = 2$. For $p = n = 2$, this corresponds to the Willmore functional.

In a forthcoming joint work [**RoSch06**], main steps of the present article are used to prove a modified conjecture of De Giorgi that the sum of the area and the Willmore functional is the Γ-limit of a diffuse Landau-Ginzburg approximation.

1. Introduction

The Willmore functional of a surface immersed into Euclidian space is up to a factor the integral of the square mean curvature. For an integral $2-$varifold μ in $\mathbb{R}^m$ this extends to

$$\mathcal{W}(\mu) := \frac{1}{4} \int |\vec{\mathbf{H}}_\mu|^2 \ \mathrm{d}\mu.$$

We recall that the mean curvature of a submanifold is given in classical differential geometry as the trace of second derivatives. Elementary calculations show that the mean curvature determines the change of the area of the submanifold under local variations. In presence of singularities, this variational property is used to define the weak mean curvature,

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more precisely for a rectifiable n-varifold μ the weak mean curvature $\vec{\mathbf{H}}_\mu$ is defined by

$$(\delta\mu)(\eta) := \int \operatorname{div}_{T\mu}(\eta)\ \mathrm{d}\mu = -\int \vec{\mathbf{H}}_\mu \eta\ \mathrm{d}\mu \quad \forall \eta \in C_0^1,$$

if such $\vec{\mathbf{H}}_\mu \in L^1_{\mathrm{loc}}(\mu)$ exists; see [**Sim**], §16. $\delta\mu$ is called the first variation of μ.

By this definition, lower semicontinuity of the weak mean curvature and in particular of the Willmore functional is immediate in the sense that if $\mu_k \to \mu$ weakly as varifolds then

$$\| \vec{\mathbf{H}}_\mu \|_{L^p(\mu)} \leq \liminf_{k\to\infty} \| \vec{\mathbf{H}}_{\mu_k} \|_{L^p(\mu_k)} . \tag{1.1}$$

If in contrast $T_k \to T$ weakly as integral currents and the mean curvature of the underlying integral varifolds μ_{T_k} is bounded in $L^p(\mu_{T_k})$, then the first variation of μ_T need not be locally bounded. Passing to the limits for a subsequence $\mu_{T_k} \to \mu_\infty$, we know $\mu_T \leq \mu_\infty$ and can estimate $\vec{\mathbf{H}}_{\mu_\infty}$ as above.

In 1999, Luigi Ambrosio pointed out that even assuming smoothness of T lower semicontinuity

$$\| \vec{\mathbf{H}}_{\mu_T} \|_{L^p(\mu_T)} \leq \liminf_{k\to\infty} \| \vec{\mathbf{H}}_{\mu_{T_k}} \|_{L^p(\mu_{T_k})} \quad \forall 2 \leq p \leq \infty \tag{1.2}$$

was not proved at that time, but would be a consequence of

$$\vec{\mathbf{H}}_{\mu_T} = \vec{\mathbf{H}}_{\mu_\infty} \quad \mu_T - \text{almost everywhere}, \tag{1.3}$$

as this implies by $\mu_T \leq \mu_\infty$ and (1.1) that

$$\| \vec{\mathbf{H}}_{\mu_T} \|_{L^p(\mu_T)} \leq \| \vec{\mathbf{H}}_{\mu_\infty} \|_{L^p(\mu_\infty)} \leq \liminf_{k\to\infty} \| \vec{\mathbf{H}}_{\mu_{T_k}} \|_{L^p(\mu_{T_k})},$$

which is (1.2).

Using the techniques of [**Sch01**] and [**Sch04**], Ambrosio and Masnou proved (1.2) for $p > n = \dim T, p \geq 2$ in $\mathbb{R}^{n+1}$ in [**AmMa03**]. In this article, we improve the integrability order of the mean curvature in (1.2) in any codimension down to the desired exponent of $p = 2$ which includes the Willmore functional.

Theorem 5.1 (Lower semicontinuity of the weak mean curvature for currents)**.** *Let $(T_k)_{k\in\mathbb{N}}$ be a sequence of integral n-currents with locally uniformly bounded total variation measures μ_{T_k} in an open set $\Omega \subseteq \mathbb{R}^m$ converging weakly as currents $T_k \to T$. If T is an integral current and μ_T is C^2-rectifiable with locally bounded first variation $\delta\mu_T = -\vec{\mathbf{H}}_{\mu_T}\mu_T + \delta\mu_{T,sing}$, then*

$$\| \vec{\mathbf{H}}_{\mu_T} \|_{L^p(\mu_T)} \leq \liminf_{k\to\infty} \| \vec{\mathbf{H}}_{\mu_{T_k}} \|_{L^p(\mu_{T_k})} \quad \forall 2 \leq p \leq \infty.$$

We know that the local geometries are contained in each other in the sense that $[\theta^{*n}(\mu_T) > 0] \subseteq [\theta^{*n}(\mu_\infty) > 0]$. But as the definition of the weak mean curvature is variational, it is unclear how the local geometries determine the weak mean curvature, and a proof of (1.3) is not obvious. Expressions of the weak mean curvature as derivatives of the local geometry were developed in [**Sch01**] and [**Sch04**]. There strong use was made of estimates for fully non-linear elliptic equations which restrict these results to $p > n$ so far. Even the case $p = n = 2$ in $\mathbb{R}^{n+1}$ is unclear.

Instead, here we observe in §3 via an adaptation of the blow up argument in [**Bra78**] in §2 that a quadratic approximation of a significant part of the support of μ by the tangent plane at a point $x \in [\theta^{*n}(\mu) > 0]$ outside a certain null set and $\vec{\mathbf{H}}_\mu \in L^2_{\mathrm{loc}}(\mu)$ imply quadratic decay of the height- and tilt-excess

$$\mathrm{heightex}_\mu(x, \varrho, T_x\mu) := \varrho^{-n-2} \int\limits_{B_\varrho(x)} \mathrm{dist}\,(\xi - x, T_x\mu)^2 \,\mathrm{d}\mu(\xi),$$

$$\mathrm{tiltex}_\mu(x, \varrho, T_x\mu) := \varrho^{-n} \int\limits_{B_\varrho(x)} \parallel T_\xi\mu - T_x\mu \parallel^2 \,\mathrm{d}\mu(\xi) = O_x(\varrho^2).$$

More precisely, let us recall the following definition adapted to [**AnSe94**].

Definition 1.1. A $\mathcal{H}^n$-measurable set $M \subseteq \mathbb{R}^m$ is called countably $C^2 - n$-rectifiable if

$$M \subseteq M_0 \cup \bigcup_{k=1}^{\infty} M_k$$

where $\mathcal{H}^n(M_0) = 0$ and $M_k, k \geq 1,$ are $C^2 - n$-submanifolds of $\mathbb{R}^m$.

A rectifiable n-varifold $\mu = \theta\mathcal{H}^n \lfloor M, \theta > 0$ on M, is called C^2-rectifiable, if M or likewise $[\theta^{*n}(\mu) > 0]$ is countably $C^2 - n$-rectifiable.

We prove that the height- and tilt-excess decay quadratically almost everywhere on countably $C^2 - n$-rectifiable subsets of $[\theta^{*n}(\mu) > 0]$, if $\vec{\mathbf{H}}_\mu \in L^2_{loc}(\mu)$. As a remarkable consequence, we obtain for $\vec{\mathbf{H}}_\mu \in L^2_{loc}(\mu)$ that μ is C^2-rectifiable if and only if the height- and tilt-excess decay quadratically almost everywhere.

Theorem 3.1. *Let μ be an integral varifold in $\Omega \subseteq \mathbb{R}^m$ open with weak mean curvature $\vec{\mathbf{H}}_\mu \in L^2_{\mathrm{loc}}(\mu)$. Then μ is $C^2 - n$-rectifiable if and only if for μ-almost all $x \in \Omega$ the height-excess and the tilt-excess decay quadratically*

$$\mathrm{heightex}_\mu(x, \varrho, T_x\mu), \mathrm{tiltex}_\mu(x, \varrho, T_x\mu) = O_x(\varrho^2).$$

Combining with the calculation of the weak mean curvature in [**Sch04**] §6, we get the following very general expression of the weak mean curvature as derivatives of the local geometry in §4.

Theorem 4.1. *Let μ_0, μ be integral varifolds in $\Omega \subseteq \mathbb{R}^m$ open with locally bounded first variation $\delta\mu_0 = -\vec{\mathbf{H}}_{\mu_0}\mu_0 + \delta\mu_{0,sing}$, weak mean curvature $\vec{\mathbf{H}}_\mu \in L^2_{loc}(\mu)$ and*

$$\mu_0 \leq \mu.$$

Further, let $\Phi : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be $\mathcal{L}^n$-measurable with

$$\Phi(A) \subseteq [\theta^{*n}(\mu_0) > 0] \subseteq [\theta^{*n}(\mu) > 0]$$

and Φ be twice approximately differentiable with rank $D\Phi = n$ almost everywhere with respect to $\mathcal{L}^n$ on A. Then

$$\vec{\mathbf{H}}_{\mu_0}(\Phi) = \Delta_g \Phi = \frac{1}{\sqrt{g}} \partial_i(\sqrt{g} g^{ij} \partial_j \Phi) \quad \mathcal{L}^n - \textit{almost everywhere on } A,$$

where $g_{ij}(D\Phi) = \partial_i \Phi \partial_j \Phi, g = \det(g_{ij}), (g^{ij})_{ij} = (g_{ij})^{-1}_{ij}$.

Clearly, assuming smoothness of T, this implies (1.3), and hence proves (1.2).

A conjecture of De Giorgi

In [**DG91**], De Giorgi made the conjecture that the sum of the area and the Willmore functional is the Γ-limit of a diffuse Landau-Ginzburg approximation. In the form modified in [**LoMa00**] this reads putting (1.4)

$$\mathcal{F}_\varepsilon(u) := \int\limits_\Omega \Big(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\Big) \, \mathrm{d}\mathcal{L}^n + \int\limits_\Omega \frac{1}{\varepsilon}\Big(- \varepsilon \Delta u + \frac{1}{\varepsilon}W'(u)\Big)^2 \, \mathrm{d}\mathcal{L}^n$$

for $u \in W^{1,2}(\Omega)$ with $W(t) := (t^2-1)^2$ and

$$\mathcal{F}(E) := \sigma\Bigg(\mathcal{H}^{n-1}(\partial^* E \cap \Omega) + \int\limits_{\partial^* E \cap \Omega} |\vec{\mathbf{H}}_{\partial^* E}|^2 \, \mathrm{d}\mathcal{H}^{n-1}\Bigg)$$

for $E \subseteq \Omega$ with finite perimeter in $\Omega, \sigma := \int_{-1}^1 \sqrt{2W}$, that

$$\mathcal{F}(E) = \Gamma - L^1(\Omega) - \lim_{\varepsilon \to 0} \mathcal{F}_\varepsilon(E) \quad \text{for } \partial E \in C^2. \tag{1.5}$$

It is well known that the Γ-limit of the first integral in (1.4) is the perimeter of E times σ, see [**MoMor77**] and [**Mo87**]. The term $-\varepsilon\Delta u + \varepsilon^{-1}W'(u)$ appearing in the second integral is the L^2-gradient of the first integral, and is therefore related in the limit to the mean curvature which is the first variation of the area or perimeter functional. The $\Gamma - \limsup$-inequality of Γ-convergence of $\mathcal{F}_\varepsilon$ to $\mathcal{F}$ for $\partial E \in C^2$ was proved by Bellettini and Paolini in [**BePa93**], see also in [**BeMu05**] §5. For rotationally symmetric data in two dimensions, the full Γ-convergence $\mathcal{F}_\varepsilon$ to $\mathcal{F}$ was proved by Bellettini and Mugnai in [**BeMu05**].

The lower semicontinuity for $p = 2$ proved in this article implies that $\mathcal{F}$ is lower semicontinuous at E with $\partial E \in C^2$ which is a necessary condition for $\mathcal{F}$ being a Γ-limit in (1.5). In [**RoSch06**] we prove the above modified conjecture of De Giorgi (1.5) for surfaces ∂E in three dimensions which corresponds to the Willmore functional. There we define for $u_\varepsilon \to 2\chi_E - 1$ in $L^1(\Omega)$ the measures

$$\mu_\varepsilon := \Big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\Big)\mathcal{L}^n,$$
$$\alpha_\varepsilon := \frac{1}{\varepsilon}\Big(- \varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon)\Big)^2\mathcal{L}^n,$$

and see for subsequences $\mu_\varepsilon \to \mu, \alpha_\varepsilon \to \alpha$ that

$$\mathcal{H}^{n-1}\lfloor\partial^* E \leq \sigma^{-1}\mu \quad \text{and} \quad |\vec{\mathbf{H}}_\mu|^2\mu \leq \alpha. \tag{1.6}$$

The main difficulty in [**RoSch06**] is to prove that $\sigma^{-1}\mu$ is an integral varifold. After this is established, we conclude as in (1.3) with Corollary 4.3 that

$$\vec{\mathbf{H}}_{\partial E} = \vec{\mathbf{H}}_{\sigma^{-1}\mu} \quad \text{for } \partial E \in C^2,$$

and (1.5) follows with (1.6).

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2. Blow-up

In this section, we reexamine the blow-up procedure used by Brakke in [**Bra78**] Theorem 5.6. We fix $n < m, \theta_0 \in \mathbb{N}$ and consider a sequence of integral n-varifolds μ_j in $B^m_8(0)$ with generalized mean curvature $\vec{\mathbf{H}}_{\mu_j} \in L^2(\mu_j)$ and $T \in G(m,n)$ with orthogonal projection $\pi : \mathbb{R}^m \to T$ satisfying

$$0 \in [\theta^n(\mu_j) > 0], \tag{2.1}$$

$$|(\omega_n\varrho^n)^{-1}\mu_j(B^m_\varrho(0)) - \theta_0| \leq \varepsilon_j \to 0 \quad \forall 0 < \varrho \leq 8, \tag{2.2}$$

$$\varrho^{-n}\mu_j(B^m_\varrho(0) \cap [\theta^n(\mu_j) \neq \theta_0]) \leq \varepsilon_j \quad \forall 0 < \varrho \leq 8. \tag{2.3}$$

We put

$$\alpha_{j\varrho} := \Big(\varrho^{2-n} \int\limits_{B^m_\varrho(0)} |\vec{\mathbf{H}}_{\mu_j}|^2 \,\mathrm{d}\mu_j\Big)^{1/2} \quad \forall 0 < \varrho \leq 8, \alpha_j := \alpha_{j8}, \tag{2.4}$$

$$\gamma_{j\varrho}^2 := \text{heightex}_{\mu_j}(0,\varrho,T) = \varrho^{-n-2} \int\limits_{B_\varrho^m(0)} |\pi_T^\perp(x)|^2 \, \mathrm{d}\mu_j(x) \quad \forall 0 < \varrho \le 8,\ \gamma_j := \gamma_{j8}, \tag{2.5}$$

$$\beta_{j\varrho}^2 := \text{tiltex}_{\mu_j}(0,\varrho,T) = \varrho^{-n} \int\limits_{B_\varrho^m(0)} \| \, T_x\mu_j - T \, \|^2 \, \mathrm{d}\mu_j(x) \quad \forall 0 < \varrho \le 8, \tag{2.6}$$

and assume

$$\max(\alpha_j,\gamma_j) \le \delta_j \to 0, \quad \delta_j \ne 0. \tag{2.7}$$

We get from [**Bra78**] Theorem 5.5 or [**Sim**] Lemma 22.2 that

$$\beta_{j,7}^2 \le C(\alpha_j\gamma_j + \gamma_j^2) \le C\delta_j^2. \tag{2.8}$$

For j large enough such that $\varepsilon_j < 1/2$, we get from [**Bra78**] Theorem 5.4 a lipschitz approximation of μ_j over T; that is there exists a θ_0-valued lipschitz map

$$f = (f_{j1},\dots,f_{j\theta_0}) : B_1^n(0) \subseteq T \to T_{\theta_0}^\perp,$$
$$F = (F_{j1},\dots,F_{j\theta_0}) : B_1^n(0) \subseteq T \to T \times T_{\theta_0}^\perp, \quad F_{ji}(y) = (y, f_{ji}(y)),$$

satisfying

$$\begin{gathered} \text{lip}\, f_j \le 1, \\ \| \, f_j \, \|_{L^\infty(B_1^n(0))} \le C(n)\gamma_j^{2/(n+2)}, \end{gathered} \tag{2.9}$$

and there exists a Borel set $Y_j \subseteq B_1^n(0)$ such that

$$\theta^n(\mu_j,(y,z)) = \#\{i \mid f_{ji}(y) = z\} \quad \text{for all } y \in Y_j \subseteq T,\ z \in B_1^{m-n}(0) \subseteq T^\perp. \tag{2.10}$$

And setting

$$X_j := [\theta^n(\mu_j) > 0] \cap (Y_j \times B_1^{m-n}(0)) = \cup_{i=1}^{\theta_0} F_i(Y_j), \tag{2.11}$$

then

$$\mu_j((B_1^n(0) \times B_1^{m-n}(0)) - X_j) + \mathcal{L}^n(B_1^n(0) - Y_j) \le C\delta_j^2, \tag{2.12}$$

where $C = C(n,m,\theta_0) < \infty$.

Selecting an appropriate subsequence (see the proof below), we obtain for $i = 1,\dots,\theta_0$

$$\delta_j^{-1} f_{ji} \to \bar f \quad \text{weakly in } W^{1,2}(B_1^n(0)) \text{ and strongly in } L^2(B_1^n(0)), \tag{2.13}$$

$$\| \, \bar f \, \|_{L^2(B_1^n(0))} \le C_n, \tag{2.14}$$

$$f_{ji} \to 0 \quad \text{strongly in } W^{1,2}(B_1^n(0)). \tag{2.15}$$

After these preliminaries, we estimate the height-excess on balls $B^m_\sigma(0)$ with $0 < \sigma \leq 1$.

Proposition 2.1. *There exists $C(n,\theta_0) < \infty$ such that for any $0 < \sigma \leq 1$*

$$\limsup_{j\to\infty} \delta_j^{-1}\gamma_{j\sigma} \leq C(n,\theta_0)\sigma^{-\frac{n}{2}-1} \parallel \bar f \parallel_{L^2(B^n_\sigma(0))} \tag{2.16}$$

for some $\bar f$ occurring as limit in (2.13).

Proof. First, we justify the limit procedure in (2.13). From (2.10), (2.11) and the Co-Area formula, we see for $\Phi \in (C^0 \cap L^\infty)(B \times B^{m-n}_{1/2}(0) \times G(m,n))$ that

$$\int_{X_j} \Phi(x, T_x\mu_j) J_{\mu_j}\pi(x)\ \mathrm{d}\mu_j(x) = \int_{Y_j} \sum_{i=1}^{\theta_0} \Phi(F_{ji}(y), im(DF_{ji}(y)))\ \mathrm{d}y$$

and

$$\begin{aligned}\int_{X_j} \Phi(x, T_x\mu_j)\ \mathrm{d}\mu_j(x) \\ = \int_{Y_j} \sum_{i=1}^{\theta_0} \Phi(F_{ji}(y), im(DF_{ji}(y)))\ \sqrt{Gr_n(DF_{ji}(y))}\ \mathrm{d}y\end{aligned} \tag{2.17}$$

where $Gr_n(DF_{ji}(y))$ denotes the Gram-Determinant of the columns of $DF_{ji}(y) \in \mathbb{R}^{n,m}$.

We establish a $W^{1,2}(B^n_1(0))$-bound on f_{ji}. By (2.17),

$$\begin{aligned}\int_{Y_j} \sum_{i=1}^{\theta_0} |f_{ji}(y)|^2\ \mathrm{d}y &\leq \int_{Y_j} \sum_{i=1}^{\theta_0} |f_{ji}(y)|^2\ \sqrt{Gr_n(DF_{ji}(y))}\ \mathrm{d}y \\ &\leq \int_{X_j} |\pi_T^\perp(x)|^2\ \mathrm{d}\mu_j(x) \leq 8^{n+2}\gamma_j^2 \leq C_n\delta_j^2.\end{aligned}$$

Next (2.9) and (2.12) yield

$$\int_{B^n_1(0)-Y_j} \sum_{i=1}^{\theta_0} |f_{ji}|^2 \leq C\delta_j^2\ \theta_0 C_n \gamma_j^{4/(n+2)} \leq C\delta_j^{2+4/(n+2)}.$$

Combining the two estimates, we obtain

$$\limsup_{j\to\infty} \delta_j^{-2} \int_B \sum_{i=1}^{\theta_0} |f_{ji}|^2 \leq C_n. \tag{2.18}$$

(2.9) yields

$$|\nabla f_{ji}(y)| = \| \pi_T^\perp DF_{ji}(y) \| \leq \| imDF_{ji}(y) - T \| \, \| DF_{ji}(y) \| \\ \leq C_{n,m} \| imDF_{ji}(y) - T \|,$$

hence by (2.17)

$$\int_{Y_j} \sum_{i=1}^{\theta_0} |\nabla f_{ji}(y)|^2 \, \mathrm{d}y \leq C_{n,m} \int_{Y_j} \sum_{i=1}^{\theta_0} \| imDF_{ji}(y) - T \|^2 \, \mathrm{d}y \\ \leq C_{n,m} \int_{X_j} \| T_x \mu_j - T \|^2 \, \mathrm{d}\mu_j(x) \\ \leq C_{n,m} \beta_{j,7}^2 \leq C_{n,m} \delta_j^2.$$

From (2.9) and (2.12), we see

$$\int_{B_1^n(0) - Y_j} \sum_{i=1}^{\theta_0} |\nabla f_{ji}|^2 \leq C \delta_j^2.$$

Combining the two estimates, we obtain

$$\limsup_{j \to \infty} \delta_j^{-2} \int_{B_1^n(0)} \sum_{i=1}^{\theta_0} |\nabla f_{ji}|^2 < \infty. \tag{2.19}$$

From (2.10), we see

$$Y_j - [f_{j1} = \cdots = f_{j\theta_0}] \subseteq \pi(B_1^m(0) - [\theta^n(\mu_j) = \theta_0]),$$

hence by (2.3)

$$\mathcal{L}^n(Y_j - [f_{j1} = \cdots = f_{j\theta_0}]) \leq \varepsilon_j. \tag{2.20}$$

Combining (2.18)–(2.20), we can select a subsequence converging according to (2.13)–(2.15).

Next, we get from (2.9), (2.13), (2.14), and (2.17) that

$$\limsup_{j \to \infty} \delta_j^{-2} \int_{B_\sigma^m(0) \cap X_j} |\pi_T^\perp(x)|^2 \, \mathrm{d}\mu_j(x) \\ \leq \limsup_{j \to \infty} \int_{B_\sigma^n(0) \cap Y_j} \sum_{i=1}^{\theta_0} |\delta_j^{-1} f_{ji}(y)|^2 \sqrt{Gr_n(DF_{ji}(y))} \, \mathrm{d}y \\ \leq C(n, \theta_0) \| \bar{f} \|_{L^2(B_\sigma^n(0))}^2 . \tag{2.21}$$

On the complement of X_j, we estimate for $0 < \tau \leq 1$ by (2.12)

(2.22)
$$\delta_j^{-2} \int\limits_{B_\sigma^m(0) - X_j} |\pi_T^\perp(x)|^2 \, \mathrm{d}\mu_j(x) \leq \delta_j^{-2} \mu_j\Big(B_\sigma^m(0) \cap \Big[|\pi_T^\perp| \geq \tau\Big]\Big) + C\tau^2.$$

Putting

$$B_{j,\tau} := B_\sigma^m(0) \cap \Big[|\pi_T^\perp| \geq \tau\Big], \qquad (2.23)$$
$$\delta_j^{-2} \mu_j(B_{j,\tau}) = M_{j,\tau},$$

we obtain with (2.22)

$$\delta_j^{-2} \int\limits_{B_\sigma(0) - X_j} |\pi_T^\perp(x)|^2 \, \mathrm{d}\mu_j(x) \leq M_{j,\tau} + C\tau^2 \quad \forall 0 < \tau \leq 1. \qquad (2.24)$$

If $M_{j,\tau} > 0$, there exists by Besicovitch's covering theorem $x \in B_{j,\tau}$, $\theta^n(\mu, x) \geq 1$ and

$$\int\limits_{B_r(x)} |\vec{\mathbf{H}}_{\mu_j}|^2 \, \mathrm{d}\mu_j \leq C_{n,m} \frac{\int_{B_2(0)} |\vec{\mathbf{H}}_{\mu_j}|^2 \, \mathrm{d}\mu_j}{\mu_j(B_{j,\tau})} \mu_j(B_r(x)) \quad \forall 0 < r \leq 1.$$

With the Hölder inequality, (2.4) and (2.7), we get

$$\int\limits_{B_r(x)} |\vec{\mathbf{H}}_{\mu_j}| \, \mathrm{d}\mu_j \leq C_{n,m} M_{j,\tau}^{-1/2} \mu_j(B_r(x)) \quad \forall 0 < r \leq 1.$$

The monotonicity formula, see [**Sim**] 17.6, yields

$$\mu_j(B_{\tau/2}(x)) \geq \exp(-C_{n,m} M_{j,\tau}^{-1/2} \tau) \omega_n (\tau/2)^n.$$

Thus, taking into account that $|\pi_T^\perp| \geq \tau/2$ on $B_{\tau/2}(x)$,

$$\begin{aligned} \gamma_j^2 = \operatorname{heightex}_{\mu_j}(0, 8, T) &\geq 8^{-n-2} \int\limits_{B_{\tau/2}(x)} |\pi_T^\perp(\xi)|^2 \, \mathrm{d}\mu_j(\xi) \\ &\geq \exp(-C_{n,m} M_{j,\tau}^{-1/2} \tau) c_0(n) \tau^{n+2}. \end{aligned}$$

Now we choose

$$\tau_j := \gamma_j^{1/(n+2)} \to 0$$

by (2.7), and see

$$\exp(-C_{n,m} M_{j,\tau_j}^{-1/2} \tau_j) \leq C_n \gamma_j \to 0;$$

hence

$$M_{j,\tau_j}^{-1/2} \tau_j \to \infty$$

or likewise

$$M_{j,\tau_j} \tau_j^{-2} \to 0.$$

Now (2.21) and (2.24) imply

$$\limsup_{j\to 0} \delta_j^{-2}\gamma_{\sigma j}^2 = \limsup_{j\to 0} \delta_j^{-2}\sigma^{-n-2} \int\limits_{B_\sigma^m(0)} |\pi_T^\perp(x)|^2 \ \mathrm{d}\mu_j(x)$$
$$\leq C(n,\theta_0)\sigma^{-n-2} \parallel \bar{f} \parallel^2_{L^2(B_\sigma^n(0))} .$$

q.e.d.

3. C^2-rectifiability and quadratic height-excess decay

In this section, we prove the equivalence of C^2-rectifiability and quadratic height-excess decay under the assumption of square integrable weak mean curvature.

Theorem 3.1. *Let μ be an integral varifold in $\Omega \subseteq \mathbb{R}^m$ open with weak mean curvature $\vec{\mathbf{H}}_\mu \in L^2_{loc}(\mu)$. Then μ is $C^2 - n$-rectifiable if and only if for $\mu -$ almost all $x \in \Omega$ the height-excess and the tilt-excess decay quadratically*

$$\text{(3.1)} \qquad \operatorname{heightex}_\mu(x,\varrho,T_x\mu), \operatorname{tiltex}_\mu(x,\varrho,T_x\mu) = O_x(\varrho^2).$$

Proof. Quadratic height-excess decay almost everywhere implies easily C^2-rectifiability using the C^2-extension lemma A.1, even without the assumption of square integrable weak mean curvature.

The converse is a consequence of the following more precise lemma recalling that quadratic height-excess decay almost everywhere implies quadratic tilt-excess decay almost everywhere under the assumption of square integrable weak mean curvature using the Cacciopoli-type estimate in [**Bra78**] Theorem 5.5 or [**Sim**] Lemma 22.2. q.e.d.

Lemma 3.1. *Let μ be an integral varifold in $\Omega \subseteq \mathbb{R}^m$ open with weak mean curvature $\vec{\mathbf{H}}_\mu \in L^2_{loc}(\mu)$. Further let $\Phi : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be $\mathcal{L}^n$-measurable with*

$$\Phi(A) \subseteq [\theta^{*n}(\mu) > 0]$$

and Φ be twice approximately differentiable with rank $D\Phi = n$ almost everywhere with respect to $\mathcal{L}^n$ on A. Then

$$\text{(3.2)} \qquad \limsup_{\varrho\to 0} \varrho^{-2}\operatorname{heightex}_\mu(x,\varrho,T_x\mu)$$
$$\leq C(n,\theta^n(\mu,x))\Big(|\vec{\mathbf{H}}_\mu(x)|^2 + |A_\Phi(x)|^2\Big) < \infty$$

for $\mathcal{L}^n$-almost all $y \in A$ and $x = \Phi(y)$, where A_Φ denotes the second fundamental form given by Φ.

Proof. As the conclusion of the lemma is almost everywhere, we may assume by the C^2-extension Lemma A.1 that $\Phi \in C^2(U,\mathbb{R}^m)$ for some open $A \subseteq U \subseteq \mathbb{R}^n$. As the conclusion is local and $D\Phi$ has full rank on

A, and by continuity on all of U. For μ-almost all $x \in \Omega$, and for x being a Lebesgue point of $\vec{\mathbf{H}}_\mu \in L^2_{loc}(\mu)$,

$$\theta := \theta^n(\mu, x) \in \mathbb{N}, \tag{3.3}$$
$$T_x\mu = \theta T \text{ exists,}$$
$$\theta^n(\mu) \text{ is approximately continuous with respect to } \mu \text{ in } x,$$
$$D_\mu(|\vec{\mathbf{H}}_\mu|^2\mu)(x) := \lim_{\varrho \to 0} \mu(B_\varrho(x))^{-1} \int\limits_{B_\varrho(x)} |\vec{\mathbf{H}}_\mu|^2 \,\mathrm{d}\mu = |\vec{\mathbf{H}}_\mu(x)|^2 < \infty.$$

By Area formula and since $D\Phi$ has full rank on A, we see that for $\mathcal{L}^n$-almost all $y \in A$ the image $x = \Phi(y)$ satisfies (3.3). Moreover, $\mathcal{L}^n$-almost all $y \in A$ satisfy

$$\theta^n(\mathcal{L}^n \lfloor A, y) = 1. \tag{3.4}$$

We prove (3.2) for such y. After a translation, we may assume $y = 0, \Phi(y) = 0$, and after a rotation $T = \mathbb{R}^n \times \{0\}$. Moreover, after a C^1-parameter change, we may write Φ in the form $\Phi(y) = (y, \varphi(y))$ with $\varphi \in C^2(U, \mathbb{R}^{m-n}, \varphi(0) = 0, \nabla\varphi(0) = 0$.

If (3.2) does not hold, then there is a sequence $\varrho_j \to 0$ and

$$\Lambda^2 > \theta\omega_n 8^n |\vec{\mathbf{H}}_\mu(0)|^2 + |A_\Phi(0)|^2/4 \tag{3.5}$$

such that for $\mu_j := (x \mapsto \varrho_j^{-1}x)_\#\mu$ in the notation of §2 for

$$\gamma_j = \gamma_{8\varrho_j} \geq \Lambda\varrho_j. \tag{3.6}$$

Clearly by (3.3), we have $\alpha_j < \Lambda\varrho_j$ for j large, hence putting $\delta_j := \max(\alpha_j, \gamma_j)$ in (2.7), we obtain

$$\delta_j = \gamma_j \quad \text{for } j \text{ large, and} \quad \tau := \limsup_{j \to \infty} \gamma_j^{-1}\varrho_j \leq 1/\Lambda. \tag{3.7}$$

Putting $A_j := \varrho_j^{-1}A, U_j := \varrho_j^{-1}U, \varphi_j : U_j \to \mathbb{R}^{m-n}, \varphi_j(y) := \varrho_j^{-1}\varphi(\varrho_j y)$, we see

$$\operatorname{graph}(\varphi_j|A_j) \subseteq [\theta^{*n}(\mu_j) > 0];$$

hence by (2.10)

$$\varphi_j(y) \subseteq \{f_{j1}(y), \ldots, f_{j\theta_0}(y)\} \quad \forall y \in Y_j \cap A_j.$$

Since $\mathcal{L}^n(B_1^n(0) - A_j) \to 0$ by (3.4), we see from (2.20)

$$\mathcal{L}^n(Y_j - [\varphi_j = f_{j1} = \cdots = f_{j\theta_0}]) \to 0$$

and from (2.13) for appropriate subsequences

$$\delta_j^{-1}\varphi_j \to \bar{f} \quad \text{locally in measure.} \tag{3.8}$$

As φ is twice differentiable and $\varphi(0) = 0, \nabla\varphi(0) = 0$, we get putting

$$Q(y) := \frac{1}{2} y^T D^2\varphi(0) y \quad \text{for } y \in \mathbb{R}^n,$$

that

$$\varphi_j \to 0, \varrho_j^{-1}\varphi_j \to Q \quad \text{uniformly on compact subsets of } \mathbb{R}^n,$$

in particular by (3.7)

$$\limsup_{j\to\infty} |\delta_j^{-1}\varphi_j| = \limsup_{j\to\infty} |(\gamma_j^{-1}\varrho_j)\varrho_j^{-1}\varphi_j| \le \tau|Q|.$$

With (3.8), we get

$$|\bar f| \le \tau|Q|.$$

$2Q = A_\Phi(0)$ represents the second fundamental form of Φ, as $\nabla\varphi(0) = 0$, in particular

$$|\bar f(y)| \le \tau|A_\Phi(0)|\ |y|^2/2.$$

Then Proposition 2.1 yields by (3.7) for $0 < \sigma \le 1$

$$\begin{aligned}\limsup_{j\to\infty} \gamma_{8\varrho_j}^{-1}\gamma_{\sigma\varrho_j} &= \limsup_{j\to\infty} \gamma_j^{-1}\gamma_{j\sigma} \\ &\le C(n,\theta)\sigma^{-n/2-1} \parallel \bar f \parallel_{L^2(B_\sigma^n(0))} \\ &\le C(n,\theta)\Lambda\sigma\tau.\end{aligned} \tag{3.9}$$

Further assuming $\gamma_{8\varrho_j} \ge \Gamma\Lambda 8\varrho_j$ for some $\Gamma > 2C(n,\theta)$, we see $\tau \le 1/(8\Gamma\Lambda)$, hence $\limsup_{j\to\infty} \gamma_{8\varrho_j}^{-1}\gamma_{\sigma\varrho_j} < \sigma/16 =: \eta/2$, and conclude

$$(\eta\varrho)^{-1}\gamma_{\eta\varrho} \le \frac{1}{2}\varrho^{-1}\gamma_\varrho \quad \text{if } \varrho^{-1}\gamma_\varrho \ge \Gamma\Lambda \text{ and } \varrho \le \varrho_0 \tag{3.10}$$

for some $\varrho_0 > 0$ small enough.

Now if $(\eta\varrho)^{-1}\gamma_{\eta\varrho} \ge \eta^{-\nu}\Gamma\Lambda$ for $\nu = 1+(n+2)/2$ and some $0 < \varrho \le \varrho_0$, then $\Gamma\Lambda \le \eta^\nu(\eta\varrho)^{-1}\gamma_{\eta\varrho} \le \varrho^{-1}\gamma_\varrho$, hence by (3.10)

$$(\eta\varrho)^{-1}\gamma_{\eta\varrho} - \eta^{-\nu}\Gamma\Lambda \le \frac{1}{2}\varrho^{-1}\gamma_\varrho - \eta^{-\nu}\Gamma\Lambda \le \frac{1}{2}\Big(\varrho^{-1}\gamma_\varrho - \eta^{-\nu}\Gamma\Lambda\Big)$$

and in any case

$$\max\Big((\eta\varrho)^{-1}\gamma_{\eta\varrho} - \eta^{-\nu}\Gamma\Lambda, 0\Big) \le \frac{1}{2}\max\Big(\varrho^{-1}\gamma_\varrho - \eta^{-\nu}\Gamma\Lambda, 0\Big) \qquad \text{for all } 0 < \varrho \le \varrho_0.$$

This yields $\max(\varrho^{-1}\gamma_\varrho - \eta^{-\nu}\Gamma\Lambda, 0) \to 0$ for $\varrho \to 0$ and

$$\limsup_{\varrho\to 0} \varrho^{-1}\gamma_\varrho \le \eta^{-\nu}\Gamma\Lambda,$$

which proves (3.2). q.e.d.

4. Locality of the mean curvature

Theorem 4.1. *Let μ_0, μ be integral varifolds in $\Omega \subseteq \mathbb{R}^m$ open with locally bounded first variation $\delta\mu_0 = -\vec{\mathbf{H}}_{\mu_0}\mu_0 + \delta\mu_{0,\mathrm{sing}}$, weak mean curvature $\vec{\mathbf{H}}_\mu \in L^2_{\mathrm{loc}}(\mu)$ and*

$$\mu_0 \leq \mu.$$

Further, let $\Phi : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be $\mathcal{L}^n$-measurable with

$$\Phi(A) \subseteq [\theta^{*n}(\mu_0) > 0] \subseteq [\theta^{*n}(\mu) > 0]$$

and Φ be twice approximately differentiable with rank $D\Phi = n$ almost everywhere with respect to $\mathcal{L}^n$ on A. Then

(4.1)

$$\vec{\mathbf{H}}_{\mu_0}(\Phi) = \Delta_g \Phi = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j\Phi) \quad \mathcal{L}^n - \textit{almost everywhere on } A,$$

where $g_{ij}(D\Phi) = \partial_i\Phi\partial_j\Phi, g = \det(g_{ij}), (g^{ij})_{ij} = (g_{ij})^{-1}_{ij}$.

Proof. As in the proof of Lemma 3.1, we may assume that $\Phi(y) = (y, \varphi(y))$ for all $y \in U$ and some $\varphi \in C^2(U, \mathbb{R}^{m-n})$. Then $M := \operatorname{graph}\varphi$ is a C^2-submanifold of $\mathbb{R}^m$, and we have to prove

$$\vec{\mathbf{H}}_{\mu_0}(., \varphi) = \vec{\mathbf{H}}_M(., \varphi) \quad \mathcal{L}^n - \text{almost everywhere on } A. \tag{4.2}$$

Let $\pi : \mathbb{R}^m \to \mathbb{R}^n$ be the orthogonal projection and $\Sigma \subseteq [\theta^{*n}(\mu) > 0]$ be the set of points x such that

$$\begin{gathered} \theta := \theta^n(\mu, x), \theta_0 := \theta^n(\mu_0, x) \in \mathbb{N}, \\ T_x\mu = T_x\mu_0 \text{ exist}, \ J_{T_x\mu}\pi > 0, \end{gathered} \tag{4.3}$$

$\theta^n(\mu), \theta^n(\mu_0)$ are approximately continuous with respect to μ, μ_0 in x,

$$D_\mu(|H_\mu|^2\mu)(x) < \infty,$$

$$D_{\mu_0}(\delta\mu_0)(x) = -\vec{\mathbf{H}}_{\mu_0}(x),$$

$$\vec{\mathbf{H}}_{\mu_0}(x) \perp T_x\mu_0,$$

and

$$\limsup_{\varrho \to 0} \varrho^{-2}\mathrm{heightex}_\mu(x, \varrho, T_x\mu) < \infty. \tag{4.4}$$

By Co-Area formula, the differentiation theorem for measures, see [**Sim**] Theorem 4.7, [**Bra78**] Theorem 5.8 and Lemma 3.1

$$\mathcal{L}^n(\pi([\theta^{*n}(\mu) > 0] - \Sigma)) = 0. \tag{4.5}$$

We consider $y \in A$ satisfying

$$\begin{gathered} y \in A - \pi([\theta^{*n}(\mu) > 0] - \Sigma)), \\ \theta^n(\mathcal{L}^n\lfloor A, y) = 1. \end{gathered} \tag{4.6}$$

By (4.5), this includes almost all $y \in A$.

[**Bra78**] Theorem 5.5 or [**Sim**] 22.2 imply for $x = (y, \varphi(y)), y \in A$ by (4.3) and (4.4)

$$\limsup_{\varrho \to 0} \varrho^{-2} \mathrm{tiltex}_\mu(x, \varrho, T_x\mu) < \infty.$$

As $\mu_0 \leq \mu$ and $T_\xi \mu_0 = T_\xi \mu$ for $\mathcal{H}^n$ − almost all $\xi \in [\theta^{*n}(\mu_0) > 0] \subseteq [\theta^{*n}(\mu) > 0]$, we get for $x = (y, \varphi(y)), y \in A$

$$\limsup_{\varrho \to 0} \varrho^{-2} \mathrm{tiltex}_{\mu_0}(x, \varrho, T_x\mu_0) < \infty. \tag{4.7}$$

Now we follow the proof of [**Sch04**] Proposition 6.1. (4.2) will be proved when we establish

$$\vec{\mathbf{H}}_{\mu_0}(x)\nu = \vec{\mathbf{H}}_M(x)\nu \quad \forall \nu \perp T_x\mu_0 = T_xM =: T. \tag{4.8}$$

We put

$$\Sigma_0 := [\theta^n(\mu_0) = \theta_0] \cap M \subseteq [\theta^{*n}(\mu_0) > 0] \cap M$$

and see for $\mu_M := \theta_0 \mathcal{H}^n \lfloor M$

$$\mu \lfloor \Sigma_0 = \mu_M \lfloor \Sigma_0. \tag{4.9}$$

As $x = (y, \varphi(y)) \in \Sigma_0$, we may add to our assumptions (4.3) that

$$\varrho^{-n} \Big(\mu_0(B^m_\varrho(x) - \Sigma_0) + \mu_M(B^m_\varrho(x) - \Sigma_0) \Big) \leq \omega(\varrho), \tag{4.10}$$

where $\omega(\varrho) \to 0$ for $\varrho \to 0$. We choose $\chi \in C^\infty_0(B^m_1(0))$ rotationally symmetric with

$$0 \leq \chi \leq 1 \quad \text{and} \quad \chi \equiv 1 \text{ on } B^m_{1/2}(0)$$

and put $\chi_\varrho(\xi) := \chi(\varrho^{-1}(\xi - x))$.

We calculate by (4.3)

$$\begin{aligned} &\lim_{\varrho \to 0} (\omega_n \varrho^n)^{-1} \delta\mu_0(\chi_\varrho) \\ &= \omega_n^{-1} \theta_0 D_{\mu_0}(\delta\mu_0)(x) \int\limits_{T \cap B^m_1(0)} \chi \, \mathrm{d}\mathcal{L}^n \\ &= -\omega_n^{-1} \theta_0 \vec{\mathbf{H}}_{\mu_0}(x) \int\limits_{T \cap B^m_1(0)} \chi \, \mathrm{d}\mathcal{L}^n, \end{aligned}$$

and, as $\varphi \in C^2(U)$,

$$\begin{aligned} &\lim_{\varrho \to 0} (\omega_n \varrho^n)^{-1} \delta\mu_M(\chi_\varrho) \\ &= -\lim_{\varrho \to 0} (\omega_n \varrho^n)^{-1} \int\limits_{B^m_\varrho(x)} \chi_\varrho \, \vec{\mathbf{H}}_M \, \mathrm{d}\mu_M \\ &= -\omega_n^{-1} \theta_0 \vec{\mathbf{H}}_M(x) \int\limits_{T \cap B^m_1(0)} \chi \, \mathrm{d}\mathcal{L}^n. \end{aligned}$$

(4.8) will follow when we prove

$$I_\varrho := \varrho^{-n}(\delta\mu_0(\chi_\varrho) - \delta\mu_M(\chi_\varrho))\nu \to 0 \quad \text{for } \varrho \to 0. \tag{4.11}$$

We recall for $\tilde{\mu} = \mu_0, \mu_M$ that

$$\delta\tilde{\mu}(\chi_\varrho)\nu = \int_{B_\varrho^m(x)} D\chi_\varrho(\xi)\ T_\xi\tilde{\mu}\ \nu\ \mathrm{d}\tilde{\mu}(\xi)$$

and abbreviate

$$R_{\varrho,\tilde{\mu}} := \varrho^{-n} \int_{B_\varrho^m(x)-\Sigma_0} D\chi_\varrho(\xi)\ (T_\xi\tilde{\mu} - T_x\tilde{\mu})\ \nu\ \mathrm{d}\tilde{\mu}(\xi).$$

Since $T_\xi\mu_0 = T_\xi M$ for almost all $\xi \in \Sigma_0 \subseteq [\theta^{*n}(\mu_0) > 0] \cap M$, (4.9) and $T\nu = 0$, as ν is normal to T, we obtain that

$$I_\varrho = R_{\varrho,\mu_0} - R_{\varrho,\mu_M}.$$

We estimate

$$\begin{aligned}
&|R_{\varrho,\tilde{\mu}}| \\
&\leq C\varrho^{-n-1} \int_{B_\varrho^m(x)-\Sigma_0} \|\ T_\xi\tilde{\mu} - T_x\tilde{\mu}\ \|\ \ \mathrm{d}\tilde{\mu}(\xi) \\
&\leq C\varrho^{-1}\Big(\varrho^{-n}\tilde{\mu}(B_\varrho^m(x) - \Sigma_0)\Big)^{1/2}\Big(\varrho^{-n}\int_{B_\varrho^m(x)} \|\ T_\xi\tilde{\mu} - T_x\tilde{\mu}\ \|^2\ \ \mathrm{d}\tilde{\mu}(\xi)\Big)^{1/2} \\
&\leq C\varrho^{-1}\omega(\varrho)^{1/2}\,\mathrm{tiltex}_{\tilde{\mu}}(x,\varrho,T_x\tilde{\mu})^{1/2},
\end{aligned}$$

where we have used (4.10).

Now for $\tilde{\mu} = \mu_0$, we have quadratic decay of the tilt-excess at 0 for $\varrho \to 0$ by (4.7), whereas such decay is immediate for $\tilde{\mu} = \mu_M$, since $D^2\varphi \in C^0(U)$. Therefore,

$$|R_{\varrho,\tilde{\mu}}| \leq C\omega(\varrho)^{1/2},$$

which proves (4.11); hence (4.8) and (4.2). q.e.d.

We get two immediate corollaries.

Corollary 4.2. *Let μ_i be integral varifolds in $\Omega \subseteq \mathbb{R}^m$ with $H_{\mu_i} \in L^2_{\mathrm{loc}}(\mu_i), i = 1, 2$. If*

$$[\theta^{*n}(\mu_1) > 0] \cap [\theta^{*n}(\mu_2) > 0] \textit{ is countably } C^2 - \textit{rectifiable}, \tag{4.12}$$

in particular if μ_1 or μ_2 is C^2-rectifiable, then

(4.13)

$$\vec{\mathbf{H}}_{\mu_1} = \vec{\mathbf{H}}_{\mu_2} \quad \mathcal{H}^n - \textit{almost everywhere on } [\theta^{*n}(\mu_1) > 0] \cap [\theta^{*n}(\mu_2) > 0].$$

Corollary 4.3. *Let μ_1, μ_2 be integral varifolds in $\Omega \subseteq \mathbb{R}^m$ open with locally bounded first variation $\delta\mu_1 = -\vec{\mathbf{H}}_{\mu_1}\mu_1 + \delta\mu_{1,\mathrm{sing}}$, weak mean curvature $\vec{\mathbf{H}}_{\mu_2} \in L^2_{\mathrm{loc}}(\mu_2)$ and*

$$\mu_1 \leq \mu_2.$$

If

$$\mu_1 \text{ is } C^2 - \text{rectifiable,} \tag{4.14}$$

then

$$\vec{\mathbf{H}}_{\mu_1} = \vec{\mathbf{H}}_{\mu_2} \quad \mu_1 - \text{almost everywhere.} \tag{4.15}$$

5. Lower semicontinuity of the weak mean curvature for currents

Corollary 4.3 implies the lower semicontinuity in (1.2) down to the desired exponent of $p = 2$ for the integrability order of the mean curvature which corresponds in two dimensions to the Willmore functional. Still we have to assume that the limit current is smooth or at least C^2-rectifiable and has locally bounded first variation.

Theorem 5.1 (Lower semicontinuity of the weak mean curvature for currents)**.** *Let $(T_k)_{k\in\mathbb{N}}$ be a sequence of integral n-currents with locally uniformly bounded total variation measures μ_{T_k} in an open set $\Omega \subseteq \mathbb{R}^m$ converging weakly as currents $T_k \to T$. If T is an integral current and μ_T is C^2-rectifiable with locally bounded first variation $\delta\mu_T = -\vec{\mathbf{H}}_{\mu_T}\mu_T + \delta\mu_{T,sing}$, then*

$$\parallel \vec{\mathbf{H}}_{\mu_T} \parallel_{L^p(\mu_T)} \leq \liminf_{k\to\infty} \parallel \vec{\mathbf{H}}_{\mu_{T_k}} \parallel_{L^p(\mu_{T_k})} \quad \forall 2 \leq p \leq \infty.$$

Remark. The compactness theorem for integral currents, see [**Sim**] Theorem 27.3, implies that T is an integral current, if the boundary masses of T_k are locally uniformly bounded.

Proof. We may assume that $\mu_{T_k} \to \mu_\infty$ weakly as Radon-measures after passing to an appropriate subsequence. By lower semicontinuity of the masses and the weak mean curvature, we know

$$\mu_T \leq \mu_\infty,$$

$$\parallel \vec{\mathbf{H}}_{\mu_\infty} \parallel_{L^p(\mu_\infty)} \leq \liminf_{k\to\infty} \parallel \vec{\mathbf{H}}_{\mu_{T_k}} \parallel_{L^p(\mu_{T_k})} \quad \forall 2 \leq p \leq \infty.$$

We may assume that the liminf is finite for some $2 \leq p \leq \infty$, which implies that μ_∞ is an integral varifold by Allard's integral compactness theorem, see [**All72**] Theorem 6.4 or [**Sim**] Remark 42.8, and $\vec{\mathbf{H}}_{\mu_\infty} \in L^2_{\mathrm{loc}}(\mu_\infty)$. As the integral varifold μ_T is C^2-rectifiable and has locally bounded first variation $\delta\mu_T = -\vec{\mathbf{H}}_{\mu_T}\mu_T + \delta\mu_{T,\mathrm{sing}}$ by assumption, we conclude by Corollary 4.3

$$\vec{\mathbf{H}}_{\mu_T} = \vec{\mathbf{H}}_{\mu_\infty} \quad \mu_T - \text{almost everywhere},$$

and hence

$$\| \vec{\mathbf{H}}_{\mu_T} \|_{L^p(\mu_T)} \leq \| \vec{\mathbf{H}}_{\mu_\infty} \|_{L^p(\mu_\infty)} \quad \forall 2 \leq p \leq \infty,$$

and the theorem is proved. q.e.d.

Appendix A. C^2-extension lemma

The following C^2-extension lemma is an easy consequence of the technique used in [**F**] 3.1.8 and Whitney's extension theorem. Unfortunately, we were not able to find it in literature and include therefore its proof for the reader's convenience.

Lemma A.1. *Let $\varphi : A \to \mathbb{R}^m$ be approximately differentiable on the $\mathcal{L}^n$-measurable set $A \subseteq \mathbb{R}^n$ which has full density in all its points and*

$$\text{(A.1)} \qquad \operatorname*{aplim\,sup}_{z\to y, z\in A} \frac{|\varphi(z) - \varphi(y) - \nabla\varphi(y)(z-y)|}{|z-y|^2} < \infty \quad \forall y \in A$$

or

(A.2)

$$\limsup_{\varrho\to 0} \varrho^{-n-2} \int_{B_\varrho(y)\cap A} |\varphi(z) - \varphi(y) - \nabla\varphi(y)(z-y)| \,\mathrm{d}z < \infty \quad \forall y \in A.$$

Then there exist countably many $\varphi_k \in C^2_{\mathrm{loc}}(U_k), U_k \subseteq \mathbb{R}^n$ open, satisfying

$$\text{(A.3)} \qquad \mathcal{L}^n\Big(A - \bigcup_{k=1}^\infty \Big([\varphi = \varphi_k] \cap [\nabla\varphi = \nabla\varphi_k]\Big)\Big) = 0,$$

in particular graph φ is countably $C^2 - n$-rectifiable in the sense of Definition 1.1.

Proof. For the proof of (A.3) it suffices to consider $m = 1$. Clearly, the approximate differentials $\nabla\varphi : A \to \mathbb{R}^n$ are $\mathcal{L}^n$-measurable. We put

$$l_y(w) := \varphi(y) + \nabla\varphi(y)(w-y) \quad \text{for } y \in A, w \in \mathbb{R}^n,$$

$$Q(y, \varrho, k) := \{w \in B_\varrho(y) \mid w \notin A \text{ or } |\varphi(w) - l_y(w)| \geq k|w-y|^2\}$$

and for $\varepsilon > 0$ small enough

$$A_k := \{y \in A \mid \forall 0 < \varrho \leq 1/k : \mathcal{L}^n(Q(y,\varrho,k)) \leq \varepsilon\varrho^n\}.$$

A_k is $\mathcal{L}^n$-measurable by Fubini's theorem; more precisely, see [**F**] 3.1.3, and

$$\text{(A.4)} \qquad A = \bigcup_{k=1}^\infty A_k$$

by (A.1). The same is true for (A.2) when we observe that (A.2) implies

$$\limsup_{\varrho \to 0} \varrho^{-n} \int\limits_{B_\varrho(y) \cap A} \frac{|\varphi(z) - \varphi(y) - \nabla\varphi(y)(z-y)|}{|z-y|^2} \, \mathrm{d}z < \infty \quad \forall y \in A.$$

For $y, z \in A_k, 0 < 2\varrho := |y-z| < 1/k, \bar{z} = (y+z)/2$,

$$w, w' \in B^n_\varrho(\bar{z}) - \Big(Q(y, 2\varrho, k) \cup Q(z, 2\varrho, k)\Big) =: W \subseteq B^n_{2\varrho}(y) \cap B^n_{2\varrho}(z),$$

we calculate

$$\begin{aligned}
&\Big|(\nabla\varphi(z) - \nabla\varphi(y))(w' - w)\Big| \\
&\leq |\varphi(w') - l_y(w')| + |\varphi(w) - l_y(w)| + |\varphi(w') - l_z(w')| + |\varphi(w) - l_z(w)| \\
&\leq k(|w'-y|^2 + |w-y|^2 + |w'-z|^2 + |w-z|^2) \leq 16k\varrho^2.
\end{aligned}$$

Integrating yields

$$\begin{aligned}
&c_n|\nabla\varphi(z) - \nabla\varphi(y)|\varrho \\
&\leq \fint\limits_{B^n_\varrho(\bar{z})} \fint\limits_{B^n_\varrho(\bar{z})} |(\nabla\varphi(z) - \nabla\varphi(y))(w' - w)| \, \mathrm{d}w \, \mathrm{d}w' \\
&\leq \fint\limits_{B^n_\varrho(\bar{z})} \fint\limits_{B^n_\varrho(\bar{z})} |(\nabla\varphi(z) - \nabla\varphi(y))(w' - w)| \; \chi_W(w) \; \chi_W(w') \, \mathrm{d}w \, \mathrm{d}w' \\
&\quad + 4|\nabla\varphi(z) - \nabla\varphi(y)|\varrho\Big((\omega_n\varrho^n)^{-1}\mathcal{L}^n(Q(y, 2\varrho, k) \cup Q(z, 2\varrho, k))\Big) \\
&\leq 16k\varrho^2 + 2^{n+2}\omega_n^{-1}\varepsilon|\nabla\varphi(z) - \nabla\varphi(y)|\varrho.
\end{aligned}$$

For $\varepsilon < \varepsilon_0(n)$ small enough, we get

$$\text{(A.5)} \qquad |\nabla\varphi(z) - \nabla\varphi(y)| \leq C_n k|z-y| \quad \forall y, z \in A_k, |y-z| < 1/k.$$

Observing

$$\mathcal{L}^n(W) \geq \omega_n\varrho^n - 2^{n+1}\varepsilon\varrho^n > 0$$

for $\varepsilon < \varepsilon_0(n)$ small enough, there exists $w \in W \neq \emptyset$, and we get using (A.5)

(A.6)

$$\begin{aligned}
&|\varphi(z) - l_y(z)| \\
&\leq |\varphi(w) - l_z(w)| + |\varphi(w) - l_y(w))| + |(\nabla\varphi(z) - \nabla\varphi(y))(w - z)| \\
&\leq k(|w-z|^2 + |w-y|^2) + |\nabla\varphi(z) - \nabla\varphi(y)| \; |w - z| \\
&\leq C_n k|z-y|^2 \quad \forall y, z \in A_k, |y-z| < 1/k.
\end{aligned}$$

Next we put $A_k(y_0) := A_k \cap B^n_{1/(2k)}(y_0)$ for $y_0 \in A_k$, and see from (A.5) and (A.6) for all $y \in A_k(y_0)$

$$\text{(A.7)} \qquad \begin{aligned}
|\nabla\varphi(y)| &\leq |\nabla\varphi(y_0)| + C_n, \\
|\varphi(y)| &\leq |\varphi(y_0)| + (|\nabla\varphi(y_0)| + C_n)/(2k).
\end{aligned}$$

Therefore

$$\varphi|A_k(y_0) \in Lip(2, A_k(y_0))$$

in the sense of [**St**] VI.2.3. Then by [**St**] VI.2.3 Theorem 4 and [**F**] Theorem 3.1.15, there exist $\varphi_{kj} \in C^2(U_{kj})$ satisfying

$$\mathcal{L}^n\Big(A_k - \bigcup_{j=1}^{\infty}[\varphi = \varphi_{kj}] \cap [\nabla\varphi = \nabla\varphi_{kj}]\Big) = 0.$$

Recalling (A.4), we see that $(\varphi_{kj})_{k,j\in\mathbb{N}}$ satisfy (A.3).

Putting $Q_k := A_k - \bigcup_{j=1}^{\infty}\Big([\varphi = \varphi_{kj}] \cap [\nabla\varphi = \nabla\varphi_{kj}]\Big)$, we see recalling (A.4)

$$\operatorname{graph}\varphi \subseteq \bigcup_{k=1}^{\infty} \operatorname{graph}(\varphi|Q_k) \cup \bigcup_{k,j=1}^{\infty} \operatorname{graph}\varphi_{kj}.$$

Since φ is approximately differentiable on all of A, we can decompose A into countably many $\mathcal{L}^n$-measurable sets on which φ is lipschitz, see [**F**] Theorem 3.1.8; hence

$$\mathcal{H}^n\Big(\operatorname{graph}(\varphi|Q_k)\Big) = 0,$$

as $\mathcal{L}^n(Q_k) = 0$ by (A.3). Since $\operatorname{graph}\varphi_{kj}$ are $C^2 - n$-submanifolds, we see that $\operatorname{graph}\varphi$ is countably $C^2 - n$-rectifiable. q.e.d.

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Mathematisches Institut der Eberhard-Karls-Universität Tübingen
Auf der Morgenstelle 10
D-72076 Tübingen, Germany

E-mail address: schaetz@everest.mathematik.uni-tuebingen.de