

POSITIVITY OF DIRECT IMAGE BUNDLES AND CONVEXITY ON THE SPACE OF KÄHLER METRICS

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Abstract

We develop some results from [4] on the positivity of direct image bundles in the particular case of a trivial fibration over a one-dimensional base. We also apply the results to study variations of Kähler metrics.

1. Introduction

In a previous paper, [4], we have studied curvature properties of vector bundles that arise as direct images of line bundles over a Kähler fibration. Here, we will continue this study in a very special case – trivial fibrations over a one-dimensional base – and elaborate on its connection with problems of variations of Kähler metrics on a compact manifold.

Let Z be a compact complex n -dimensional manifold, and let \hat{L} be a positive line bundle over Z . Put $X = U \times Z$, where U is a domain in \mathbb{C} , and denote by L the pullback of \hat{L} under the projection from X to Z . Let ϕ be a hermitian metric on L . Then ϕ can be written

$$\phi = \phi_0 + \psi,$$

where $\phi_0 = \phi_0(z)$ is some fixed metric on \hat{L} , pulled back to X , and $\psi = \psi(t, z)$ is a function of t in U and z in Z . We can thus think of ϕ as a family of metrics on \hat{L} , indexed by t in U , or as a (complex) curve in the affine space of all metrics on \hat{L} . Put

$$\hat{E} = H^0(Z, \hat{L} \otimes K_Z),$$

the space of global holomorphic \hat{L} -valued $(n, 0)$ -forms on Z . Denote by E the (trivial) vector bundle over U with fiber \hat{E} .

Even though E is a trivial bundle, it has a naturally defined metric H_ϕ which is not trivial. H_ϕ is defined by

$$\|u\|_t^2 = \int_{\pi^{-1}(t)} [u, u] e^{-\phi} = \int_Z [u, u] e^{-\phi_0 - \psi(t, z)}$$

if u is an element in $E_t = \hat{E}$, the fiber of E over t . Here $[u, u]e^{-\phi}$ is defined by

$$[u, u]e^{-\phi} = c_n u_j \wedge \bar{u}_j e^{-\phi_j}$$

if u_j and ϕ_j are local representatives of u and ϕ with respect to a local trivialization. The constant $c_n = i^{n^2}$ is chosen to make this expression nonnegative.

With this metric E becomes an Hermitian vector bundle, and it makes sense to consider its curvature Θ^E . It is proved in [4] that if ϕ is a (semi)positive metric on L , then H_ϕ is a (semi)positive metric on E . Moreover, assuming that Z has no nontrivial global holomorphic vector fields, Θ^E is strictly positive at a point t unless $\omega^t = i\partial\bar{\partial}_z\phi$ is stationary so that

$$\frac{\partial}{\partial t}\omega^t|_t = 0.$$

We are now ready to describe the content of this paper. In the next section, we will first reprove the formula for the curvature (Theorem 2.1) from [4], which is particularly simple in this situation. We then proceed to give exact conditions on ϕ that characterize when the curvature Θ^E has a null-vector, for the case when ϕ itself is semipositive. This result is implicit in [4], but not explicitly stated there. Then we go on to show that if the curvature is degenerate not only for one single t but for all t in an open neighbourhood of 0, then

$$\omega^t = T_t^*\omega^0,$$

where T_t is the flow of some holomorphic vector field on Z .

In Section 3, we then give a result corresponding to Theorem 2.1 for a trivial vector bundle F with fiber $H^0(Z, L)$ instead of $H^0(Z, L \otimes K_Z)$. In this case, we define our L^2 -norms on the fiber by

$$\|u\|^2 = \int_Z |u|^2 e^{-\phi} \omega_n$$

(see below). Surprisingly, we then get an estimate from *above* of the curvature, which implies among other things that if $\omega^{n+1} = 0$ on the total space $U \times Z$, then Θ^F is *seminegative*.

Next, we consider the Hermitian bundles $E(p)$ defined in the same way as E , but replacing L by L^p for some positive integer p . We then take $p\phi$ for our metric on L^p , and put $\Theta^p = \Theta^{E(p)}$ for ease of notation. It is an a priori non trivial fact, but an immediate consequence of the characterization mentioned above that if at some t and for some p_0 Θ^{p_0} is degenerate, then Θ^p actually vanishes at t for all p .

Motivated in part by this observation, we then study the asymptotic behaviour of Θ^p , as p tends to infinity. Let

$$\tau(p) = \mathbf{tr}\Theta^p/d_p,$$

where d_p is the rank of $E(p)$, i.e., the dimension of $H^0(Z, \hat{L}^p \otimes K_Z)$. Thus, $\tau(p)$ is the average of the eigenvalues of the curvature. In Section 4, we give an asymptotic formula for $\tau(p)$ containing one term of order p , one term of order zero, and one term that vanishes in the limit. It follows from this formula that if we assume $i\partial\bar{\partial}\phi$ to be semipositive, then

$$\liminf_{p \rightarrow \infty} \tau(p) = 0$$

implies that the condition characterizing degeneracy follows. Hence, in this case, Θ^p actually vanishes for all p . Ma and Zhang [12] have announced the existence of a complete asymptotic expansion for the (or ‘a’) kernel of the full curvature operator in a situation more general than ours and also computed the first two terms on the diagonal. I do not know if our result (which is much more elementary) follows from theirs. Our result does, however, follow from the relation between our vector bundles $E(p)$ and the *Mabuchi functional* from Kähler geometry that we discuss in Section 5. Hence, the reader who so wishes can skip directly to Section 5 after a brief look at our recollection of the Tian-Catlin-Zelditch asymptotic formula for the Bergman kernel in Section 4.

In Section 5, we relate our results to the study of variations of Kähler metrics on Z , cf. [14], [1], [7], [8], [15], [5]. In these works, one considers the space of all Kähler metrics ω on Z that lie in one fixed cohomology class, viz, $c(\hat{L})$, the Chern class of \hat{L} . Any such metric can be written

$$\omega = \omega^0 + i\partial\bar{\partial}\psi,$$

where ω^0 is the fixed reference metric, and ψ is a function on Z . The space of all such Kähler metrics can therefore be identified with the space \mathcal{K} of all - say smooth - metrics on \hat{L} , modulo constants, and it has a natural structure as a Riemannian manifold. A curve in \mathcal{K} can be represented by a function $\psi(t, z)$ on Z depending on an additional parameter t . Here t is naturally a real variable but following the usual convention we let t be complex, and assume ψ independent of the imaginary part of t . Our domain U is then a strip, and metrics ϕ satisfying $(i\partial\bar{\partial}\phi)^{n+1} = 0$ correspond to geodesics in \mathcal{K} .

The basic strategy in [7] and [8] is to approach the study of metrics on \hat{L} - and hence Kähler metrics on Z with Kähler form in $c(\hat{L})$ - through the induced metrics $\text{Hilb}(\phi)$ on $H^0(Z, \hat{L})$ defined by

$$\|u\|^2 = \int_Z |u|^2 e^{-p\phi} \omega_n^\phi,$$

and similar metrics on $H^0(Z, \hat{L}^p)$ for p large. Here $\omega_n = \omega^{\wedge n}/n!$ is the volume element on Z induced by ω . In our terminology, [7] and [8] thus deal with the bundle F .

In this paper, we will follow a different approach using the spaces $\hat{E} = H^0(Z, L \otimes K_Z)$ and the induced norms H_ϕ described above instead of $H^0(Z, L)$ and $\text{Hilb}(\phi)$.

In [8], Donaldson introduced certain, intimately related, functionals on \mathcal{K} , \mathcal{L}_p , and $\tilde{\mathcal{L}}_p$. Stationary points of $\tilde{\mathcal{L}}_p$ correspond to so called balanced metrics on \hat{L}^p . As p tends to infinity, $\tilde{\mathcal{L}}_p$ goes to a multiple of the so called Mabuchi functional, whose stationary points are precisely the metrics of constant scalar curvature on Z . Donaldson's \mathcal{L} functional can be defined in terms of the metric on $\det(F)$ induced by the metric on F . The negativity of F , and hence $\det(F)$, then give convexity of \mathcal{L} along geodesics (which is mentioned without proof in [8]).

Here we introduce the analogs of Donaldson's functionals in our setting and show that they have largely the same properties, with one important difference: In our setting, the \mathcal{L} functional is defined with the opposite sign compared to [7] and [8] reflecting the different estimates for the curvatures of the bundles F and E .

Nevertheless our \mathcal{L} -functional still converges to a multiple of the Mabuchi functional as p goes to infinity. This fact is actually somewhat easier to verify in our setting. We also show that the second derivative of the $\tilde{\mathcal{L}}_p$ -functional along a curve can be computed from the trace of the curvature of our associated vector bundles. From this, it follows that the second derivative of the Mabuchi functional along a geodesic in \mathcal{K} equals the limit

$$\lim_{p \rightarrow \infty} \tau(p).$$

From this, we see that our asymptotic formula for the trace $\tau(p)$ is, in fact, equivalent to a known formula for the second derivative of the Mabuchi functional, see [9].

As is well known, this formula gives a "formal proof" of the uniqueness properties of metrics of constant scalar curvature: Using the characterization of when the above limit vanishes, we see that two metrics of constant scalar curvature can be connected through the flow of a holomorphic vector field, provided that they can be connected with a smooth geodesic in the Riemannian space \mathcal{K} . This certainly does not give the results in [7], [1], [13], and [6], as the existence of smooth geodesics is an open and even dubious issue, (see [5] and [6] for the best results in this direction), but at least it indicates a seemingly interesting relation between our results on flatness of direct image bundles and the results of [1], [13], and [6].

It is proved in [5] that two metrics in \mathcal{K} can always be connected with a geodesic of class $C^{1,1}$. Phong and Sturm have showed that this geodesic can be obtained as an almost uniform limit of certain "finite dimensional geodesics" obtained from Bergman kernels in $H^0(Z, L^p)$. In the last section, we show that in our setting, we can improve on

this result somewhat, and prove that Bergman kernels for the spaces $H^0(Z, L^p \otimes K_Z)$ approximate the geodesic uniformly at the rate at least $p/\log p$. This section is independent of most of the rest of the paper, using only Theorem 2.1. Therefore, a reader mainly interested in the convergence of geodesics can go directly to the last section after reading Theorem 2.1.

Part of the results in this paper were announced in [3]. As is probably clear from the text, I am a novice in the study of extremal Kähler metrics, and I apologize for any omission in accrediting results properly. The motivation for this paper lies not so much in the particular results as in showing the link between $\bar{\partial}$ -theory and this beautiful area. It should also be stressed that the list of references in this paper is far from complete. Finally, I would like to thank Robert Berman, Sebastien Boucksom, and Yanir Rubinstein for helpful and stimulating discussions on these matters.

2. Positivity of direct images

We start by giving the precise form of the result from [4] in this particular setting, a trivial fibration with compact fibers over a one-dimensional base. We will assume all the time that the metric ϕ restricts to a positive metric on each fiber $\{t\} \times Z$. Most of the time, but not always, we will also assume that ϕ defines a semipositive metric on the total space X . With $\phi = \phi^0 + \psi$ given, we put

$$c(\phi) = \psi_{t\bar{t}} - |\bar{\partial}_z \psi_t|_{\omega^t}^2.$$

Here ψ_t means $\partial\psi/\partial t$ and the subscript ω^t indicates that we measure the $(0, 1)$ -form $\bar{\partial}_z \psi_t$ with the metric

$$\omega^t = i\partial\bar{\partial}_z \phi|_t.$$

This function plays a major role in our estimates and also in the theory of variations of Kähler metrics. A short calculation (see [14], [16], or [4]) shows that it is related to the complex Monge-Ampère operator through the formula

$$(2.1) \quad c(\phi)idt \wedge d\bar{t} \wedge \omega_n^t = (i\partial\bar{\partial}\phi)_{n+1}.$$

It is also, as proved in the references above, equal to the geodesic curvature of the path defined by ϕ in the Riemannian manifold \mathcal{K} .

To state our formula for the curvature of E , it is convenient to introduce yet another piece of notation. Let f be a $\bar{\partial}$ -closed form of bidegree $(n, 1)$ with values in \hat{L} on Z . Given a positive metric ϕ on \hat{L} , we can by the Hörmander $\bar{\partial}$ -estimate solve $\bar{\partial}v = f$ with the L^2 -estimate

$$\|v\|^2 \leq \|f\|^2,$$

where the norms are defined using the Kähler metric $i\partial\bar{\partial}\phi$ on Z . Let v be the L^2 -minimal solution and put

$$e(f) = \|f\|^2 - \|v\|^2.$$

Thus, e is a quadratic form in f , which by the Hörmander estimate is positively semidefinite. If we develop f

$$f = \sum f_j,$$

where the f_j are eigenforms of the $\bar{\partial}$ -Laplacian with eigenvalues λ_j , one can easily verify that

$$e(f) = \sum \left(1 - \frac{1}{\lambda_j}\right) \|f_j\|^2,$$

but we will not use this. We next restrict our quadratic form e to $\bar{\partial}$ -closed forms f of the form

$$f = \bar{\partial}\mu \wedge u$$

where μ is a smooth function, and u is a holomorphic L valued $(n, 0)$ -form. We then put

$$A(\mu, u) := e(\bar{\partial}\mu \wedge u).$$

A is a quadratic form in both u and μ separately, and it will play a main role in the sequel. We will give a conjectural geometric description of A as the Chern curvature form of a vector bundle over a certain infinite dimensional complex manifold in Section 5.

Theorem 2.1. *Let u be an element in E_t . Then*

$$\langle \Theta^E u, u \rangle = \int_{\pi^{-1}(t)} c(\phi)[u, u]e^{-\phi} + A(\psi_t, u).$$

This result is implicit in [4], but we shall indicate a simple proof in this special situation. Take $t = 0$. We extend u to a holomorphic section of E near the origin in such a way that $D'u = 0$ at 0, where D' is the Chern connection on E . Then

$$\langle \Theta^E u, u \rangle = -\frac{\partial^2}{\partial t \partial \bar{t}} \Big|_{t=0} \|u(t)\|_t^2.$$

It is easily verified, cf. e.g., [4], that

$$D'u = \Pi_{\text{holo}}(u_t - \psi_t u)$$

with Π_{holo} being the projection on the subspace of holomorphic sections. Hence, $D'u = 0$ means that $v = (u_t - \psi_t u)$ is orthogonal to the space of holomorphic forms. Therefore, for $t = 0$, v is the L^2 -minimal solution to the equation

$$\bar{\partial}_z v = -\bar{\partial}_z \psi_t \wedge u = -f.$$

Since

$$\frac{\partial}{\partial t} \|u(t)\|_t^2 = \langle u_t - \psi_t u, u \rangle,$$

we get

$$\frac{\partial^2}{\partial t \partial \bar{t}} \|u(t)\|_{t=0}^2 = \frac{\partial}{\partial \bar{t}} \langle (u_t - \psi_t u), u \rangle = - \int \psi_{t\bar{t}}[u, u] e^{-\phi} + \int [v, v] e^{-\phi}.$$

By definition

$$\int [v, v] e^{-\phi} = \|v\|^2 = \|f\|^2 - e(f),$$

so

$$\begin{aligned} \langle \Theta^E u, u \rangle &= \int_{\pi^{-1}(t)} (\psi_{t\bar{t}}[u, u] - |f|^2) e^{-\phi} + e(f) \\ &= \int_{\pi^{-1}(t)} c(\phi)[u, u] e^{-\phi} + e(f), \end{aligned}$$

and the proof is complete.

Since by Hörmander’s theorem $e(f)$ and hence A are always nonnegative, it follows immediately from Theorem 2.1 that

$$(2.2) \quad \langle \Theta^E u, u \rangle \geq \int_{\pi^{-1}(t)} c(\phi)[u, u] e^{-\phi},$$

with equality if and only if $A(\psi_t, u) = 0$. We also see that in case ϕ is assumed to be semipositive, then Θ^E is also semipositive and can have a null-vector u only if $c(\phi) = 0$, and $e(\bar{\partial}_z \psi_t \wedge u) = 0$. By (2.1), the first condition means that ϕ satisfies the homogenous Monge-Ampère equation

$$(i\bar{\partial}\bar{\partial}\phi)^{n+1} = 0.$$

To understand the meaning of the second condition, we need to analyze the quadratic form $e(f)$ a bit further.

Recall that the minimal solution v to the equation $\bar{\partial}v = f$ can be written

$$v = \bar{\partial}^* \alpha$$

for some $(n, 1)$ -form α . This is simply because on a compact manifold the range of $\bar{\partial}$ and $\bar{\partial}^*$ are closed, so any form orthogonal to the null-space of $\bar{\partial}$ lies in the image of the adjoint operator. Moreover, by taking α orthogonal to the kernel of $\bar{\partial}^*$, we can assume that $\bar{\partial}\alpha = 0$, and α is then uniquely determined (as follows from (2.4) below). We will use the Kodaira-Nakano identity in the following form:

$$(2.3) \quad \|\alpha\|^2 + \|v\|^2 + \|\bar{\partial} * \alpha\|^2 = 2\langle f, \alpha \rangle,$$

where all norms are taken with respect to the Kähler metric given by the curvature form of the metric on \hat{L} . This slightly nonstandard formula follows from the more standard

$$(2.4) \quad \|\alpha\|^2 + \|\bar{\partial} * \alpha\|^2 = \|\bar{\partial}\alpha\|^2 + \|\bar{\partial}^* \alpha\|^2$$

if we add $\|\bar{\partial}^* \alpha\|^2 = \|v\|^2$ to both sides and use

$$\|\bar{\partial}^* \alpha\|^2 = \langle f, \alpha \rangle.$$

Now,

$$2\langle f, \alpha \rangle = 2\operatorname{Re} \langle f, \alpha \rangle = \|f\|^2 + \|\alpha\|^2 - \|f - \alpha\|^2.$$

Inserting this in (2.3) and simplifying, we find that

$$(2.5) \quad e(f) = \|f - \alpha\|^2 + \|\bar{\partial} * \alpha\|^2.$$

Thus, $e(f) = 0$ if and only if $f = \alpha$ and $\bar{\partial} * \alpha = 0$, so if $e(f)$ vanishes, we must have $\bar{\partial} * f = 0$. Conversely, if $\bar{\partial} * f = 0$, then $\bar{\partial}\bar{\partial}^* f = f$ since

$$\bar{\partial}\bar{\partial}^* f = \bar{\partial}(-e^\phi \partial e^{-\phi} * f) = \omega \wedge * f = f.$$

Hence $\alpha = f$, and we see that $e(f) = 0$. We have therefore proved the next proposition.

Proposition 2.2.

$$e(f) = 0,$$

i.e., equality holds in the Hörmander estimate for the L^2 -minimal solution to

$$\bar{\partial} v = f$$

*if and only if the $(n - 1, 0)$ -form $*f$ is holomorphic.*

For later use, we also give a variant of the argument above that leads to a more precise statement.

Proposition 2.3. *$e(f)$ is the minimum of the quadratic expression*

$$q(f, g) := \|f - g\|^2 + \|\bar{\partial} * g\|^2$$

where g ranges over all (L -valued) $\bar{\partial}$ -closed $(n, 1)$ -forms. Hence, if we take $g = f$, it follows that

$$e(f) \leq \|\bar{\partial} * f\|^2.$$

Proof. We already know that $q(f, \alpha) = e(f)$ if α is chosen so that $\bar{\partial}\bar{\partial}^* \alpha = f$ and $\bar{\partial} \alpha = 0$, so we need only prove that for any g as above, $q(f, g) \geq e(f)$. Write $g = \alpha + \gamma$. We claim that

$$\langle \alpha - f, \gamma \rangle + \langle \bar{\partial} * \alpha, \bar{\partial} * \gamma \rangle = 0.$$

This follows from polarizing (2.4); if we recall that $\bar{\partial} \alpha = \bar{\partial} \gamma = 0$, we get

$$\langle \alpha, \gamma \rangle + \langle \bar{\partial} * \alpha, \bar{\partial} * \gamma \rangle = \langle \bar{\partial}^* \alpha, \bar{\partial}^* \gamma \rangle = \langle f, \gamma \rangle.$$

Hence,

$$q(f, \alpha + \gamma) = q(f, \alpha) + \|\gamma\|^2 + \|\bar{\partial} * \gamma\|^2 \geq e(f).$$

q.e.d.

Recall that we are interested in when $e(f) = 0$ when $f = \bar{\partial}_z \psi_t \wedge u$. Let V be the complex gradient of ψ_t on a fiber $\{t\} \times Z$. This is a vector field of type $(1, 0)$ defined by

$$\delta_V \omega^t = \bar{\partial} \psi_t,$$

where δ_V means contraction of a form with a vector field. We claim that $\ast(\bar{\partial}_z\psi_t \wedge u) = -\delta_V u$. To see this, note that

$$\omega^t \wedge u = 0$$

for degree reasons, so

$$0 = \delta_V(\omega^t \wedge u) = \bar{\partial}_z\psi_t \wedge u + \omega^t \wedge \delta_V u.$$

Hence

$$\ast\delta_V u = -\bar{\partial}_z\psi_t \wedge u,$$

which proves the claim.

Since u is a holomorphic form, $\delta_V u$ is holomorphic if and only if V is a holomorphic vector field. This is clear outside of the zeros of u , and since V is automatically smooth, it must hold everywhere. We therefore see that for any choice of u in E_t and $f = \bar{\partial}_z\psi_t \wedge u$, we have $e(f) = A(\psi_t, u) = 0$ if and only if V , the complex gradient of ψ_t , is a holomorphic vector field. Note that, somewhat surprisingly, this condition is independent of u . We have therefore proved the next theorem.

Theorem 2.4. *Equality holds, for some u , in the inequality*

$$\langle \Theta^E u, u \rangle \geq \int_{\pi^{-1}(t)} c(\phi)[u, u] e^{-\phi}$$

if and only if V , the complex gradient of ψ_t on the fiber $Z_t = \{t\} \times Z$, is a holomorphic vector field. If ϕ is semipositive, then Θ^E has a null vector in E_t if and only if $c(\phi) = 0$ and V is holomorphic on Z_t . In this case, Θ^E vanishes on all of E_t .

We shall next see that if the conditions of the previous theorem - vanishing of $c(\phi)$ and holomorphicity of V on Z_t - are satisfied for t in an entire neighbourhood of the origin, then the variation of the metrics on Z comes from the flow of the holomorphic field V , the complex gradient of ψ_t .

Let us first recall a few basic facts about real and complex vector fields. If v is a real vector field on X , then v generates a flow, a one-parameter group F_τ , of diffeomorphisms of X satisfying

$$\frac{d}{d\tau} g(F_\tau(x)) = v(g)(F_\tau(x))$$

for any smooth function g . The Lie derivative of a form α on X with respect to v is defined by

$$\mathcal{L}_v \alpha = \left. \frac{d}{d\tau} \right|_{\tau=0} F_\tau^*(\alpha).$$

By a classical formula of Cartan,

$$\mathcal{L}_v \alpha = (d\delta_v + \delta_v d)\alpha.$$

If V is a complex vector field of type $(1, 0)$, we similarly let the *complex Lie derivative* be

$$\mathcal{L}_V^{\mathbb{C}}\alpha = (\partial\delta_V + \delta_V\partial)\alpha.$$

When V is a holomorphic field, and $v = \operatorname{Re} V$, one easily checks that

$$\mathcal{L}_v = \operatorname{Re} \mathcal{L}_V^{\mathbb{C}},$$

so information about the complex Lie derivative of V enables us to draw conclusions about the flow of $v = \operatorname{Re} V$.

Recall that the complex gradient of ψ_t is defined by

$$\bar{\partial}\psi_t = \delta_V\omega^t.$$

Applying $i\partial$ to both sides, we get (since $\partial\omega = 0$)

$$\frac{\partial}{\partial t}\omega = i\partial\delta_V\omega = i\mathcal{L}_V^{\mathbb{C}}\omega,$$

so we have a formula for the complex Lie derivative of ω . To use this formula, we need, however, to know that V is a holomorphic field, not just on each slice X_t , but also that it depends holomorphically on t . That is the object of the next lemma.

Lemma 2.5. *Assume V , the complex gradient of ψ_t is holomorphic on each slice $X_t = Z \times \{t\}$ for all t in U . Let $V_{\bar{t}} = \partial V / \partial \bar{t}$. Then $V_{\bar{t}}$ is the complex gradient of $c(\phi)$. In particular, if $c(\phi)$ is constant on slices, then V is a holomorphic field on all of $X = U \times Z$.*

Proof. Recall that

$$(2.6) \quad \frac{\partial}{\partial t}\omega = i\partial\delta_V\omega = i\mathcal{L}_V^{\mathbb{C}}\omega,$$

where $\mathcal{L}_V^{\mathbb{C}}$ is the complex Lie derivative with respect to V . Taking conjugates, it follows that

$$\frac{\partial}{\partial \bar{t}}\omega = -i\bar{\partial}\delta_{\bar{V}}\omega.$$

Differentiating the defining relation

$$\bar{\partial}\psi_t = \delta_V\omega^t$$

with respect to \bar{t} , we get

$$\begin{aligned} \bar{\partial}_z\psi_{t\bar{t}} &= \delta_{V_{\bar{t}}}\omega + \delta_V\frac{\partial}{\partial \bar{t}}\omega = \delta_{V_{\bar{t}}}\omega - i\delta_V\bar{\partial}\delta_{\bar{V}}\omega \\ &= \delta_{V_{\bar{t}}}\omega + i\bar{\partial}\delta_V\delta_{\bar{V}}\omega = \delta_{V_{\bar{t}}}\omega + \bar{\partial}|V|^2. \end{aligned}$$

Thus,

$$\delta_{V_{\bar{t}}}\omega = \bar{\partial}(\psi_{t\bar{t}} - |V|^2),$$

and the lemma follows since the norm of V is equal to the norm of $\bar{\partial}\psi_t$.

q.e.d.

Theorem 2.6. *Assume the curvature Θ^E is degenerate on E_t for all t in U . Then*

$$\omega^t = S_t^*(\omega^0),$$

where S_t is the flow of some holomorphic vector field on Z . It follows that

$$\phi(t, z) = \phi(0, S_t(z)) + \psi(t).$$

Proof. By Theorem 2.3 and Lemma 2.4, the hypothesis implies that V , the complex gradient of ψ_t , is holomorphic with respect to both t and z . Let

$$T_s(t, z)$$

be the flow of iV ; note that it acts fiberwise on X . Let

$$U_s(t, z)$$

be the flow of $W := iV - \partial/\partial t$. Then

$$U_s(t, z) = (t - s, T_s(t, z)).$$

By (2.7), the (complex) Lie derivative of ω with respect to W vanishes, so $U_s^*(\omega) = \omega$. Taking $s = t$, it follows that $\omega^0 = T_t^*(\omega^t)$, and the theorem follows with S equal to the inverse of T . (The last part follows since $\phi(t, z) - \phi(0, S_t(z))$ is pluriharmonic, hence constant, on fibers.) q.e.d.

3. Negativity of direct image bundles.

In this section, we give for comparison a theorem that very roughly corresponds to Theorem 2.1 for the (trivial) vector bundle F with fiber

$$\hat{F} = F_t = H^0(Z, L),$$

instead of $\hat{E} = H^0(Z, K_Z \otimes L)$. The difference is thus that we do not take tensor products with the canonical bundle here. We then use the metric

$$\|u\|_t^2 := \int_{\pi^{-1}(t)} |u|^2 e^{-\phi} \omega_n,$$

where $\omega = (i\partial\bar{\partial})_z \phi$, on F . We shall see that we then instead get an estimate of the curvature from above.

Theorem 3.1. *Let ϕ be a smooth metric on L over $Z \times U$ satisfying*

$$\omega^\phi := i\partial\bar{\partial}\phi \geq 0.$$

Let u be an element in F_t . Then the curvature Θ^F of F equipped with the metric described above satisfies

$$\langle \Theta^F u, u \rangle \leq (n + 1) \int_{\pi^{-1}(t)} |u|^2 c(\phi) e^{-\phi} \omega_n.$$

Proof. Let u be a holomorphic section of F . We start by computing

$$i\partial\bar{\partial}\|u\|_t^2.$$

For this, we note that $\|u\|^2$ is the push-forward of the form

$$R = |u|^2 e^{-\phi} \omega_n^\phi$$

under π . Hence, if we denote by ∂^ϕ the twisted derivative $e^\phi \partial e^{-\phi}$

$$(3.1) \quad \begin{aligned} i\partial\bar{\partial}\|u\|_t^2 &= \pi_*(i\partial\bar{\partial}R) \\ &= \pi_*(i\partial^\phi u \wedge \overline{\partial^\phi u} \wedge \omega_n^\phi e^{-\phi}) - \pi_* (|u|^2 \omega^\phi \wedge \omega_n^\phi e^{-\phi}). \end{aligned}$$

Here we have used the commutator rule

$$\bar{\partial}\partial^\phi + \partial^\phi\bar{\partial} = \partial\bar{\partial}\phi.$$

The first term in the right hand side of (3.1) is nonnegative, so using

$$\omega^\phi \wedge \omega_n^\phi = (n+1)c(\phi)idt \wedge d\bar{t} \wedge \omega_n^\phi,$$

we get that

$$(3.2) \quad i\partial\bar{\partial}\|u\|_t^2 \geq -(n+1) \int_{\pi^{-1}(t)} |u|^2 c(\phi) e^{-\phi} \omega_n idt \wedge d\bar{t}.$$

On the other hand,

$$i\partial\bar{\partial}\|u\|_t^2 = \langle D'u, D'u \rangle - \langle \Theta^F u, u \rangle$$

if D' is the $(1, 0)$ -part of the connection on F . If we combine this formula with (3.2), we see that

$$\langle \Theta^F u, u \rangle \leq \langle D'u, D'u \rangle + (n+1) \int_{\pi^{-1}(t)} |u|^2 c(\phi) e^{-\phi} \omega_n idt \wedge d\bar{t}.$$

Since we can make the first term in the right hand side vanishes at any given point by choosing the section u , so that $D'u = 0$ at that given point, it follows that

$$\langle \Theta^F u, u \rangle \leq (n+1) \int_{\pi^{-1}(t)} |u|^2 c(\phi) e^{-\phi} \omega_n.$$

q.e.d.

Corollary 3.2. *Assume in addition to the hypotheses in the previous theorem that $\omega_{n+1}^\phi = 0$, so that $c(\phi) = 0$. Then $\Theta^F \leq 0$.*

Remark. It might seem surprising (and even confusing) that we get different signs for the curvature of the very closely related bundles E and F . This can be clarified somewhat if we consider the real variable analogs of Theorems 2.1 and 3.1. In this analogy, a metric on L over $Z \times U$ with nonnegative curvature corresponds to a convex function on

$\mathbb{R}^n \times U$ with U now an interval in \mathbb{R} . As explained in [4], Theorem 2.1 corresponds to *Prekopa's Theorem*, which states that the function

$$-\log \int_{\mathbb{R}^n} e^{-\phi(x,t)} dx$$

is convex. On the other hand, Theorem 3.1 corresponds to the fact that

$$\log \left(\int e^{-\phi(x,t)} \det(\phi_{x_j x_k}) dx \right)$$

is convex if the determinant of the full hessian of ϕ vanishes.

4. Asymptotics

As in the introduction, we next let for p a positive integer $E(p)$ be the vector bundle defined in the same way as E , but replacing L by L^p . It follows immediately from Theorem 2.3 that if the curvature Θ^p of $E(p)$ is degenerate at t for p equal to some p_0 , then actually Θ^p vanishes completely at t for any $p = 1, 2, \dots$. We shall now see that we can draw a similar conclusion if Θ^p vanishes in the limit, as p tends to infinity. This requires an asymptotic study of the two terms in our curvature formula from Theorem 2.1.

The main point in our asymptotic study of Θ^p is the next asymptotic formula for the trace of the quadratic form $A(\mu, \cdot)$ introduced in Section 2.

Theorem 4.1. *Let A_p be the quadratic form A with L replaced by L^p . Denote by d_p the dimension of $H^0(Z, K_Z \otimes L^p)$ and by $\text{Vol}(Z)$ the volume of Z with respect to the metric ω . Then*

$$\lim_{p \rightarrow \infty} \frac{1}{d_p} \text{tr} A_p(p\mu, \cdot) = \frac{1}{\text{Vol}(Z)} \int_Z |\bar{\partial} V_\mu|^2 \omega_n.$$

In the proof, we will use the asymptotic expansion for Bergman kernels of Tian-Catlin-Zelditch that we recall in the next subsection.

4.1. The Tian-Catlin-Zelditch asymptotic formula for Bergman kernels. Fix p for the moment, and let u_j be an orthonormal basis for $E(p)$. Then

$$\sum [u_j, u_j] e^{-p\phi} =: B_p e^{-p\phi}$$

is the Bergman form for $E(p) = H^0(Z, K_Z \otimes L^p)$. By e.g., [18], there are constants a_j so that

$$(4.1) \quad B_{p\phi} e^{-p\phi} = p^n \left(a_0 + \frac{a_1}{p} S + O\left(\frac{1}{p^2}\right) \right) \omega_n^\phi.$$

Here a_0 and a_1 are constants, and by the computation of the second term due to Lu, [11], $S = S_{\omega^\phi}$ is the scalar curvature of the metric with Kähler form ω^ϕ . Normally the expansion is given for the Bergman

kernel for the space of global sections of L^p , but the analogous formula for our Bergman kernel follows from this. More precisely, it follows from the known expansions for $L^p \otimes F$, where F is another line bundle with fixed metric – our case corresponding to $F = K_z$ with metric induced by ω^ϕ , see e.g., [2].

Since

$$\frac{d_p}{p^n} = b + c/p + O\left(\frac{1}{p^2}\right),$$

where b is again a positive constant, it follows from 2.7 that

$$(4.2) \quad \frac{B_{p\phi} e^{-p\phi}}{d_p} = \left(a'_0 + \frac{a'_1}{p}(S - \hat{S}) + O\left(\frac{1}{p^2}\right) \right) \omega_n^\phi$$

for certain constants a'_0 , a'_1 and \hat{S} . The left hand side here integrates to 1, so it follows that $a'_0 = 1/\text{Vol}(Z)$. Moreover, the second term in the right hand side must integrate to 0, so \hat{S} is the average of S over Z .

4.2. Proof of Theorem 2.7. We shall first prove that

$$\limsup_{p \rightarrow \infty} \frac{1}{d_p} \text{tr} A_p(p\mu, \cdot) \leq \frac{1}{\text{Vol}(Z)} \int_Z |\bar{\partial} V_\mu|^2 \omega_n.$$

This is a relatively simple consequence of Proposition 2.3, together with the formula for the asymptotic expansion of Bergman forms. Recall that by Proposition 2.3,

$$A_p(p\mu, u) \leq \|p\bar{\partial} *_p \bar{\partial}\mu \wedge u\|_p^2 = \|\bar{\partial} * \bar{\partial}\mu \wedge u\|^2.$$

Here the p in the subscript indicate norms and $*$ -operators with respect to the metric $p\omega$, and norms and $*$ -operators without subscripts are taken with respect to ω . (Notice that $p*_p = *$ and that the norm of a form of total degree n does not change when we multiply the metric with p .) But it is easily verified that

$$\|\bar{\partial} * \bar{\partial}\mu \wedge u\|^2 = \int |\bar{\partial} V_\mu|^2 [u, u].$$

This means that the trace of A_p is dominated by

$$\int |\bar{\partial} V_\mu|^2 B_{p\phi},$$

which after division with d_p tends to

$$\frac{1}{\text{Vol}(Z)} \int |\bar{\partial} V_\mu|^2 \omega_n$$

by the asymptotic formula for $B_{p\phi}$. This completes the proof of the upper estimate. We next turn to the estimate from below of the trace, i.e., we estimate $\bar{\partial} V_\mu$ from above.

For this, we shall estimate

$$I = \int (\bar{\partial}V_\mu, \xi)\omega_n/\text{Vol}(Z)$$

where ξ is a $(0, 1)$ with vector field coefficients. By the asymptotic expansion for Bergman forms, this is the limit of

$$I_p = \int (\bar{\partial}V_\mu, \xi) \sum [u_j, u_j]/d_p,$$

as p tends to infinity, if (u_j) is an orthonormal basis for $H^0(Z, K_Z \otimes L^p)$. But

$$I_p = \sum \langle \bar{\partial}\delta_{V_\mu} u_j, \delta_\xi u_j \rangle / d_p = \sum \langle \delta_{V_\mu} u_j, \bar{\partial}^* \delta_\xi u_j \rangle / d_p.$$

Recall that

$$A_p(p\mu, u_j) = \|p\bar{\partial}\mu \wedge u_j - p\alpha_j\|_p^2 + \|\bar{\partial} * \alpha_j\|_p^2$$

where α_j are $(n, 1)$ -forms. Translating to norms with respect to ω , this equals

$$p\|\bar{\partial}\mu \wedge u_j - \alpha_j\|^2 + \|\bar{\partial} * \alpha_j\|^2.$$

In particular,

$$\sum \|\bar{\partial}\mu \wedge u_j - \alpha_j\|^2 / d_p = \sum \|\delta_{V_\mu} u_j - * \alpha_j\|^2 / d_p \leq \frac{1}{p} \text{tr} A_p(p\mu, \cdot) / d_p.$$

Hence, up to an error of size $1/p$, I_p equals

$$\sum \langle * \alpha_j, \bar{\partial}^* \delta_\xi u_j \rangle / d_p = \sum \langle \bar{\partial} * \alpha_j, \delta_\xi u_j \rangle / d_p,$$

which by Cauchy's inequality is dominated by

$$\begin{aligned} & \left(\sum \|\bar{\partial} * \alpha_j\|^2 / d_p \right)^{1/2} \left(\sum \|\delta_\xi u_j\|^2 / d_p \right)^{1/2} \\ & \leq (\text{tr} A_p(p\mu, \cdot) / d_p)^{1/2} \left(\int |\xi|^2 B_\phi / d_p \right)^{1/2}. \end{aligned}$$

Therefore,

$$\frac{1}{\text{Vol}(Z)} \int (\bar{\partial}V_\mu, \xi)\omega_n \leq \liminf (\text{tr} A_p(p\mu, \cdot) / d_p)^{1/2} (\|\xi\|^2 / \text{Vol}(Z))^{1/2},$$

so

$$\frac{1}{\text{Vol}(Z)} \int |\bar{\partial}V_\mu|^2 \omega_n \leq \liminf \frac{1}{d_p} \text{tr} A_p(p\mu, \cdot).$$

This completes the proof of Theorem 2.7.

q.e.d.

4.3. Asymptotic behaviour of the curvature. In the previous subsection, we have used the asymptotics of Bergman forms to describe the asymptotic behaviour of the trace of A_p . The behaviour of the first term in the formula for the curvature Θ^p as p goes to infinity also follows from the Tian-Catlin-Zelditch formula. This term is

$$\int c(p\phi)[u, u],$$

so its trace is

$$\int pc(\phi)B_{p\phi}.$$

The results recalled in Subsection 2.1, therefore, lead to the next theorem.

Theorem 4.2. *As p goes to infinity,*

$$\begin{aligned} & \frac{1}{d_p} \operatorname{tr} \Theta^p \\ &= (p \int c(\phi) \omega_n^\phi + \int c(\phi)(S - \hat{S}) \omega_n^\phi + \int |\bar{\partial} V_{\psi_t}|^2 \omega_n^\phi) / \operatorname{Vol}(Z) + o(1). \end{aligned}$$

In particular,

$$\liminf \frac{1}{d_p} \operatorname{tr} \Theta^p = 0$$

(if and) only if $c(\phi) = 0$, and V_{ψ_t} is a holomorphic vector field.

5. Convexity on the space of Kähler metrics

We now return to the space \mathcal{K} of Kähler potentials on Z . Fixing a base metric ϕ_0 , any other metric ϕ can be written

$$\phi = \phi_0 + \psi$$

where ψ is a function on Z . The space \mathcal{K} is therefore affine, and its tangent space is the space of (say smooth) real valued functions on Z . Following [14] and [16], one defines a metric on the tangent space at ϕ by

$$\|\psi\|^2 = \int |\psi|^2 \omega_n^\phi,$$

where $\omega^\phi = i\partial\bar{\partial}\phi$. This gives \mathcal{K} the structure of a Riemannian manifold, and it is proved in the references above that the geodesic curvature of a path $\phi(t, z) = \phi_0(z) + \psi(t, z)$ is given by

$$c(\phi) = \psi_{t\bar{t}} - |\bar{\partial}_z \psi_t|_{\omega^\phi}^2.$$

Here we follow the convention that we let t be complex and let ψ be independent of the argument of t . By formula (2.1), ϕ defines a geodesic in \mathcal{K} precisely when ϕ satisfies the homogeneous Monge-Ampère equation with respect to both variables t and z .

Let us now consider a (smooth) function, \mathcal{F} on \mathcal{K} . Then

$$\frac{\partial}{\partial t} \mathcal{F}(\phi_t) = \mathcal{F}' \cdot \phi_t.$$

To write second order derivatives, we have two possibilities. The simplest alternative would be to write

$$\frac{\partial^2}{\partial t \partial \bar{t}} \mathcal{F}(\phi_t) = \mathcal{F}' \cdot \phi_{t\bar{t}} + D^2 \mathcal{F}(\phi_t, \phi_{\bar{t}}).$$

Then $D^2 \mathcal{F}$ is the Hessian of \mathcal{F} with respect to the affine structure on \mathcal{K} . The second possibility is to write

$$(5.1) \quad \frac{\partial^2}{\partial t \partial \bar{t}} \mathcal{F}(\phi_t) = \mathcal{F}' \cdot c(\phi) + \mathcal{F}''(\phi_t, \phi_{\bar{t}})$$

and let this formula define the Hessian of \mathcal{F} as a quadratic form on the tangent space to \mathcal{K} . This is the Hessian of \mathcal{F} determined by the Riemannian structure on \mathcal{K} , and it is this form of the Hessian that we will use. Then \mathcal{F} is said to be convex on \mathcal{K} if \mathcal{F}'' is positively semidefinite, so that, in particular, the restriction of \mathcal{F} to any smooth geodesic is convex. Notice, however, that the convexity is defined independently of the existence of smooth geodesics.

One classical example of such a function is I defined by

$$I' \cdot \phi_t = \int \phi_t \omega_n^\phi.$$

It is well known and not hard to compute that

$$\frac{\partial}{\partial \bar{t}} I' \cdot \phi_t = \int c(\phi) \omega_n^\phi = I' \cdot c(\phi),$$

so $I'' = 0$ and I is linear along geodesics.

In [8], Donaldson introduced another functional \mathcal{L} in the following manner. Let H be the space $H^0(Z, \hat{L})$ of global holomorphic sections to \hat{L} over Z . Let (h_j) be some fixed basis for H , and let for any ϕ in \mathcal{K} , $\mathcal{L}(\phi)$ be the logarithm of the determinant of the matrix of the metric $\text{Hilb}(\phi)$ expressed in the given basis. Here the metric $\text{Hilb}(\phi)$ is defined by

$$\|h\|^2 = \int |h|^2 e^{-\phi} \omega_n^\phi.$$

Note that \mathcal{L} depends on the basis chosen, but only up to an additive constant. Equivalently, Donaldson's \mathcal{L} -functional can be defined as the logarithm of the square of the norm of some fixed constant section of the line bundle $\det(F)$, where F is the vector bundle discussed in Section 3 (the notion of a constant section makes sense since F and hence $\det(F)$ are trivial bundles). Corollary 3.2 therefore implies that $\mathcal{L}(\phi(\cdot, t))$ is a subharmonic function of t if $\phi(\cdot, t)$ is any curve in \mathcal{K} satisfying $c(\phi) = 0$. Hence, in particular, \mathcal{L} is convex along geodesics. Notice, however, that

\mathcal{L} has no apparent convexity property along curves satisfying $i\partial\bar{\partial}\phi \geq 0$ or $i\partial\bar{\partial}\phi \leq 0$.

In this paper, we will change the setup and notation, and let \mathcal{L} be the similarly defined functional but *with opposite sign*, and using the space $\hat{E} = H^0(Z, L \otimes K_Z)$ instead of H , and our metrics H_ϕ instead of $\text{Hilb}(\phi)$. Hence, in our setup \mathcal{L} is the *negative* of the logarithm of the squared norm of some fixed constant section of the line bundle $\det(E)$ discussed in Section 2. Here the norm on $\det(E)$ is, of course, the norm induced by our norms H_ϕ on E .

Thus, \mathcal{L}' is the negative of the connection form on $\det(E)$ (with respect to the constant section chosen). Let (u_j) be an orthonormal basis of E for one fixed t . Then the connection form on $\det(E)$ is equal to the trace of the connection, θ on E .

$$\text{tr}\theta = \sum \langle \theta u_j, u_j \rangle = - \sum \int \phi_t [u_j, u_j] e^{-\phi} = - \int \phi_t B_\phi e^{-\phi},$$

where B_ϕ is the Bergman kernel. We have thus proved the first part of the following lemma.

Lemma 5.1.

$$\mathcal{L}' . \phi_t = \int \phi_t B_\phi e^{-\phi}$$

and

$$\frac{\partial^2}{\partial t \partial \bar{t}} \mathcal{L}(\phi_t) = \text{tr}\Theta^E.$$

Hence,

$$\mathcal{L}''(\mu, \mu) = \text{tr}A_\mu(\cdot, \cdot).$$

The second part of the lemma follows since the Laplacian of the logarithm of the square norm of a holomorphic (in this case constant) section to $\det E$ is the negative of the curvature of $\det E$, which is equal to the trace of the curvature of E . The last part then follows from the definition of the Hessian, formula (3.1), and Theorem 2.1.

Note that by Lemma 3.1, \mathcal{L}'' is nonnegative along geodesics and vanishes only if V , the complex gradient of $\psi_t = \phi_t$ is a holomorphic vector field on Z . Hence \mathcal{L} is convex along geodesics and also convex along any curve such that $c(\phi) \geq 0$.

Following [8], we next let

$$\tilde{\mathcal{L}} = \frac{1}{d}\mathcal{L} - \frac{1}{\text{Vol}}I,$$

where d is the dimension of \hat{E} , and Vol is the volume of Z . Then $\tilde{\mathcal{L}}' . \phi_t = 0$ if ϕ_t is constant on Z for some t , and $\tilde{\mathcal{L}}$ is also convex along geodesics (since I is linear). The stationary points of $\tilde{\mathcal{L}}$ are in this setting the points ϕ in \mathcal{K} such that

$$B_\phi e^{-\phi} = \frac{d}{\text{Vol}}\omega_n^\phi.$$

This follows immediately from

$$\tilde{\mathcal{L}}'.\mu = \int \mu \left(\frac{B_\phi e^{-\phi}}{d} - \frac{\omega_n^\phi}{\text{Vol}} \right).$$

We will refer to these points as balanced, but note that this term now has a slightly different meaning from what it has in [8].

Finally, we give also the definition of the *Mabuchi functional* \mathcal{M} . It is determined (up to a constant) by the formula for its derivative

$$\mathcal{M}'.\mu = \int \mu(S_{\omega^\phi} - \hat{S}_{\omega^\phi})\omega_n^\phi,$$

where S_{ω^ϕ} is the scalar curvature of the metric ω^ϕ , and \hat{S}_{ω^ϕ} is its average. Its critical points are precisely the metrics ϕ in \mathcal{K} that induce metrics of constant scalar curvature on Z .

Proposition 5.2. *Let ϕ_0 and ϕ_1 be balanced points. Assume they can be joined by a smooth geodesic. Then*

$$\omega^{\phi_0} = S^*(\omega^{\phi_1})$$

where S is the time 1 map of some holomorphic vector field on Z .

To see this, let ϕ be the geodesic and consider the restriction of $\tilde{\mathcal{L}}$ to the geodesic. Since the end points are balanced, the derivative of $\tilde{\mathcal{L}}$ vanishes at the end points, and since $\tilde{\mathcal{L}}$ is convex, it must be constant. Since I is linear, \mathcal{L}'' also vanishes so Θ^E is zero by Lemma 5.1. The proposition then follows from Theorem 2.4.

We will now apply the same reasoning to L^p , where p tends to infinity. Define \mathcal{L}_p the same way as \mathcal{L} , but replacing L by L^p . Put

$$(5.2) \quad \tilde{\mathcal{L}}_p = \frac{1}{d(p)}\mathcal{L} - \frac{1}{\text{Vol}}I.$$

Then

$$\tilde{\mathcal{L}}_p'.\mu = \int \mu\sigma_p$$

where

$$\sigma_p = \frac{B_{p\phi}e^{-p\phi}}{d(p)} - \frac{\omega_n^\phi}{\text{Vol}}.$$

It follows (cf. formula 4.2) that, as p tends to infinity, $p\sigma_p$ tends to (a constant times)

$$(S_\phi - \hat{S}_\phi)\omega_n^\phi,$$

where S_ϕ is the scalar curvature of the metric ω^ϕ , and \hat{S}_ϕ is the average of S_ϕ . By definition of the Mabuchi functional \mathcal{M} , this means that $\tilde{\mathcal{L}}_p'$ tends to \mathcal{M}' . The next result says the second derivatives also converge. The formula for the second derivative of the Mabuchi functional in this proposition can be found in [9].

Proposition 5.3.

$$\lim_{p \rightarrow \infty} \tilde{\mathcal{L}}_p''(\mu, \mu) = \frac{1}{\mathbf{Vol}(Z)} \int |V_\mu|^2 \omega_n = a_1' \mathcal{M}''(\mu, \mu).$$

Proof. The first equality follows directly from Lemma 5.1 and Theorem 4.1, and the remaining equality follows, of course, since we know that the first derivative of $\tilde{\mathcal{L}}_p$ converges to $a_1' \mathcal{M}'$. q.e.d.

Remark. Conversely, Theorem 4.1 follows from the formula for second derivative of the Mabuchi functional if we use that A is the Hessian of $\tilde{\mathcal{L}}$ and that $\tilde{\mathcal{L}}_p$ converges to a multiple of the Mabuchi functional. We have included the proof in Section 4 since it gives an approach to the Hessian of the Mabuchi functional using only data that can a priori be of very low regularity.

In particular, the Mabuchi functional is convex along geodesics, and even strictly convex if Z has no nontrivial holomorphic vector fields. Recall that the critical points of \mathcal{M} are precisely the Kähler potentials such that ω^ϕ has constant scalar curvature. The same argument that led to Proposition 3.2 now gives the next theorem.

Theorem 5.4. *Let ϕ_0 and ϕ_1 be points in \mathcal{K} that define metrics of constant scalar curvature on Z . Assume they can be joined by a smooth geodesic. Then*

$$\omega^{\phi_0} = S^*(\omega^{\phi_1})$$

where S is the time 1 map of some holomorphic vector field on Z .

5.1. An interpretation of the form A (?). Our form

$$A(\mu, u) = e(\bar{\partial}\mu \wedge u)$$

is defined on the tangent bundle of \mathcal{K} times the complex vector space

$$\hat{E} = H^0(Z, K_Z \otimes L),$$

and it is quadratic in both arguments so it resembles a curvature form on a vector bundle over \mathcal{K} with fiber \hat{E} . If we try to make this idea more precise, we must first find a connection. Clearly, the trivial bundle over \mathcal{K} with fiber \hat{E} has a natural metric if we define the norm over a point ϕ in \mathcal{K} to be just H_ϕ . A metric does not in itself induce a connection, however. It is therefore natural to consider complexifications of \mathcal{K} since holomorphic bundles over complex manifolds have canonical connections.

Let us, therefore, *assume* that we have a complex manifold $\tilde{\mathcal{K}}$ containing \mathcal{K} as a totally real submanifold together with a projection map

$$\pi : \tilde{\mathcal{K}} \rightarrow \mathcal{K},$$

having the property that any geodesic in \mathcal{K} lifts to a holomorphic curve in $\tilde{\mathcal{K}}$. Define a trivial vector bundle \mathcal{E} over $\tilde{\mathcal{K}}$ with fiber \hat{E} and the

'tautological' Hermitian metric $H_{\pi(z)}$ on \mathcal{E}_z . We claim that $A(\mu, u)$ must then be the Chern curvature form

$$\langle \Theta_{(\mu, i\mu)}^{\mathcal{E}} u, u \rangle.$$

To see this, take a piece of a geodesic curve through a point ϕ_0 in \mathcal{K} with tangent vector μ at ϕ_0 and lift it to a complex curve through a point z_0 in $\tilde{\mathcal{K}}$. Then \mathcal{E} restricted to this complex curve is a vector bundle E of the type considered in Section 2. By Theorem 2.1, its curvature is given by

$$\langle \Theta^E u, u \rangle = A(\mu, u)$$

(the first term in the curvature formula disappears for a geodesic). But the Chern curvature of the bundle restricted to a complex curve is the restriction of the Chern curvature of the full bundle, proving our claim.

This interpretation hinges, of course, on the existence of a complexification of \mathcal{K} with the properties above. As pointed out to me by Yanir Rubinstein, the at least 'moral' existence of such a complexification is an important motivation for the constructions in [16] and [9]. One can also compare to the work of Lempert and Szoke, [10], who show the existence of a complex structure with the properties we require on a neighbourhood of the zero section in the tangent bundle of any *finite dimensional* Riemannian manifold. At any rate, if $\tilde{\mathcal{K}}$ exists, we also see that Hörmander's L^2 -estimates imply (and are almost equivalent to) that \mathcal{E} has nonnegative curvature, and that the curvature is strictly positive if Z has no nontrivial holomorphic vector fields.

5.2. 'Finite dimensional geodesics'. A Hermitian metric H on the trivial vector bundle $E = U \times \hat{E}$ over U is a complex curve in the space of Hilbert norms on \hat{E} . The latter is a symmetric space (cf. [16] and [7]), and it turns out that geodesics in this space correspond exactly to metrics with zero curvature (at least if the metric depends only on the real part of t). Let us call a metric with semipositive curvature a *subgeodesic*, and in the same way call a complex curve in \mathcal{K} a subgeodesic if $i\partial\bar{\partial}\phi \geq 0$. With this terminology, Theorem 2.1 implies that if ϕ is a subgeodesic in \mathcal{K} , then the corresponding metric H_ϕ on E is also a subgeodesic. Moreover, H_ϕ is a geodesic, i.e., the curvature vanishes if and only if ϕ arises from one fixed metric on \hat{L} via the flow of a holomorphic vector field.

Conversely, any metric on a vector bundle $F = U \times H^0(Z, L)$ induces a corresponding curve in the space of metrics on L by taking the Bergman kernels of the norms on any fiber (this map from metrics on $U \times H^0(Z, L)$ to \mathcal{K} is called the Fubini-Study map, see [8]). Assuming that \hat{L} is sufficiently positive so that the Bergman metric is nondegenerate on fibers, we have here a completely symmetric situation: Subgeodesics on F map to subgeodesics in \mathcal{K} , and the image is a geodesic if and only

if the metric on F arises from one fix Hilbert norm on \hat{F} via the flow of a holomorphic vector field. The first part of this claim is clear from the explicit form of the Bergman kernels (see beginning of next section or [15]). The second part follows from an analysis of the foliation by complex curves that the homogenous Monge-Ampère equation

$$(i\partial\bar{\partial}\log B)^{n+1} = 0$$

induces, but we omit the proof.

6. Approximation of geodesics

In the previous section, we did not use the full curvature estimate in Theorem 2.1, but only the estimate of the trace of the curvature that follows from it. In this section, we shall use the full curvature estimate to show that a recent result of Phong and Sturm, [15], on approximation of geodesics can be sharpened a bit in our setting.

Recall that any ϕ in \mathcal{K} induces a Hilbert norm H_ϕ on \hat{E} . Let M be the space of all Hilbert norms on \hat{E} and let H_t be a curve in M , where t in U is a complex parameter. Then H_t defines an Hermitian structure on our vector bundle E over U . We say that H_t is flat if this Hermitian structure is flat, i.e., if it has vanishing curvature. In case H_t is independent of the argument of t , this means precisely that the corresponding real curve is a geodesic in the symmetric space of all Hermitian norms on \hat{E} , see [7] and [15].

As in [15], we note that any two points H_0 and H_1 in M can be joined by a flat curve: Choose an orthonormal basis of H_0 , (u_j) that diagonalizes H_1 , so that

$$\langle u_j, u_k \rangle_{H_1} = \delta_{jk} e^{2\lambda_j}.$$

Then H_t defined by

$$\langle u_j, u_k \rangle_{H_t} = \delta_{jk} |t|^{2\lambda_j},$$

where $\log |t|$ ranges from 0 to 1, is a flat curve joining H_0 and H_1 .

For any curve in M , we get a curve of metrics on $L \otimes K_z$ by taking the logarithm of B_t , the Bergman kernel for \hat{E} with the metric H_t . Here B_t is defined by

$$B_t = \sum [u_{jt}(z), u_{jt}(z)],$$

where (u_{jt}) is an orthonormal basis for the scalar product H_t . Explicitly,

$$(6.1) \quad B_t = \sum [u_j(z), u_j(z)] |t|^{-2\lambda_j}$$

if (u_j) is the diagonalizing basis chosen above.

For p a positive integer, we can do the same construction for the space $\hat{E}(p)$ consisting of sections to $L^p \otimes K_Z$ and get metrics

$$p\phi_{(p)}(t, \cdot) = \log B_t(p)$$

on $L^p \otimes K_Z$.

Let ϕ_0 and ϕ_1 be two smooth positive metrics on \hat{L} , i.e., points in \mathcal{K} . Let U be the annulus $\{0 < \log |t| < 1\}$ and consider the space \mathcal{A} of all smooth semipositive metrics ϕ on L , the pull back of \hat{L} to $U \times Z$ such that

$$\phi \leq \phi_0$$

for $\log |t| = 0$ and

$$\phi \leq \phi_1$$

for $\log |t| = 1$.

Then $\phi^* := \sup_{\mathcal{A}} \phi$ is a moral candidate for a geodesic in \mathcal{K} , but its eventual smoothness properties is a very hard issue, see [5], [6].

Here we shall prove a variant of the result of Phong and Sturm, [15]. For p large, we consider the Hilbert norms $H_{p\phi_0}$ and $H_{p\phi_1}$ on $\hat{E}(p)$, connect them with a flat curve of Hilbert norms and define $B_t(p)$ and $\phi_{(p)}$ as above.

Theorem 6.1.

$$(6.2) \quad \sup |\phi_{(p)} - \phi^*| \leq C \frac{\log p}{p}.$$

The meaning of this statement is perhaps a bit obscure since $\phi_{(p)}$ and ϕ^* are metrics on different bundles. If we choose a fixed smooth metric χ on K_Z , the precise meaning of (6.2) is

$$\sup |(\phi_{(p)} - \chi/p) - \phi^*| \leq C \frac{\log p}{p}.$$

Note that in the case of principal interest when K_Z is negative, we can choose χ to have negative curvature so that our approximants are positively curved. In case \hat{L} is a power of the canonical bundle on Z , one can also avoid the introduction of χ by normalizing $\phi_{(p)}$ differently.

To prove Theorem 6.1, we first note that ϕ^* is bounded from above by $\max(\phi_0, \phi_1)$. This follows if we apply the maximum principle with respect to the t -variable for a general element in \mathcal{A} .

The direction of Theorem 6.1 that estimates ϕ^* from below is relatively straightforward. First note that when $\log |t| = 0$ by the Tian-Zelditch-Catlin formula

$$(6.3) \quad |\phi_{(p)} - \chi/p - \phi_0| \leq C \log p/p,$$

and that a similar estimate holds on the outer boundary of the annulus. We will use ϕ_0 as a fixed strictly positive auxiliary metric on \hat{L} . Notice that by (6.1), $\phi_{(p)}$ defines a semipositive metric on L over $U \times Z$ so that if a is a sufficiently large constant,

$$\phi_{(p)}(1 - a/p) + a\phi_0/p - \chi/p$$

is also semipositive for p large enough (the positivity of ϕ_0 compensates for the possible negativity of χ if a is large enough). Combining with (6.3), we see that

$$\xi_p = \phi_{(p)}(1 - a/p) + a\phi_0/p - \chi/p - C \log p/p$$

belongs to \mathcal{A} . Hence $\phi^* \geq \xi_p$ proving one direction of (6.2). Notice that this shows, in particular, that ϕ^* is uniformly bounded from below by the smooth metric ξ_{p_0} for some fixed large p_0 .

The proof of the other direction is divided into two steps. First we estimate $B_t(p)$ from below by $B_{p\phi^*}$ - the Bergman kernel associated to $p\phi^*$, and then we estimate $B_{p\phi^*}$ by $e^{p\phi^*}$. For the first step, we need a well known lemma, cf. [17].

Lemma 6.2. *Let E be a holomorphic vector bundle over (the closure of) a one-dimensional domain U , and let A and B be two Hermitian metrics on E that extend continuously to \bar{U} . Assume that the curvature of A is seminegative and that the curvature of B is semipositive. Then if $A \leq B$ on the boundary of U , it follows that $A \leq B$ in U .*

Let $E = E(p)$ be our trivial bundle with fiber $\hat{E}(p)$, and let A be the metric defined by the flat curve between $H_{p\phi_0}$ and $H_{p\phi_1}$. B is the metric defined by H_{ϕ^*} . By Theorem 2.1, B is semipositive and by definition A is flat. Thus by the lemma,

$$A \leq B.$$

This implies the opposite inequality for the Bergman kernels, so

$$(6.4) \quad B_t(p) \geq B_{\phi^*}.$$

One might object here that ϕ^* is not necessarily smooth, so Theorem 2.1 can not be applied directly. This can be circumvented by proving instead (6.4) with ϕ^* replaced by an arbitrary element in \mathcal{A} , which will suffice for our purposes.

The remaining part of the proof now follows from a variant of the Ohsawa-Takegoshi extension theorem.

Theorem 6.3. *Let L be a line bundle over a compact manifold Z with a positive metric ϕ_0 , and let ϕ be a semipositive metric on L such that $\phi_0 - \phi$ is uniformly bounded. Let χ be a fixed smooth metric on K_Z .*

Then for any point x in Z and any sufficiently large integer p , there is a holomorphic section h to $L^p \otimes K_Z$ such that

$$(6.5) \quad |h(x)|^2 \geq e^{p\phi + \chi}(x)$$

and

$$(6.6) \quad \int [h, h] e^{-p\phi} \leq C,$$

where C does not depend on p .

Accepting this for a moment, we first see how Theorem 6.1 follows. We will apply Theorem 6.3 to ϕ^* (or to an arbitrary element in \mathcal{A}) for one fixed value of t . We already know that ϕ^* is bounded from below by a smooth metric and from above by $\max(\phi_0, \phi_1)$, so Theorem 6.3 does apply to the couple ϕ^* and ϕ_0 . By the extremal characterization of Bergman kernels, it follows from Theorem 6.3 that

$$B_{p\phi^*} \geq Ce^{p\phi^* + \chi}.$$

Combining with (6.4), we find

$$(6.7) \quad B_t(p) \geq Ce^{p\phi^* + \chi},$$

from where it follows that

$$\phi_{(p)} \geq \phi^* + \chi/p + C/p;$$

so Theorem 6.1 follows.

To prove Theorem 6.3, first choose a trivializing neighbourhood W and local coordinates centered at x . By the Ohsawa-Takegoshi extension theorem for bounded domains in \mathbb{C}^n , we can find a section over W satisfying (6.5) over W with an integral estimate over W . A standard argument involving Hörmander L^2 -estimates over Z with respect to a singular weight with a logarithmic pole at x then extends h to a global section such that (6.5) still holds and

$$\int [h, h] e^{-(p-p_0)\phi + p_0\phi_0} \leq C.$$

Since $\phi - \phi_0$ is assumed to be uniformly bounded, this gives (6.6), and we are done.

As a final remark, we note that if we assume known that ϕ^* has a certain degree of smoothness, and that $\omega^t > 0$, then we can replace the crude lower bound in (6.7) from the Ohsawa-Takegoshi theorem by a few terms from the Tian-Zelditch-Catlin expansion. One then gets a very precise estimate from below of $B_t(p)$.

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