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ARNOLD DIFFUSION IN HAMILTONIAN SYSTEMS: A PRIORI UNSTABLE CASE

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Abstract

By using variational method and under generic condition, we show that Arnold diffusion exists in *a priori* hyperbolic and time-periodic Hamiltonian systems with multiple degrees of freedom.

1. Introduction

In this paper we consider *a priori* hyperbolic and time-periodic Hamiltonian system with arbitrary $n+1$ degrees of freedom. The Hamiltonian has the form

$$H(u,v,t) = h_1(p) + h_2(x,y) + P(u,v,t) \tag{1.1}$$

where $u=(q,x)$, $v=(p,y)$, $(p,q)\in\mathbb{R}\times\mathbb{T}$, $(x,y)\in\mathbb{T}^n\times\mathbb{R}^n$, P is a time-1-periodic small perturbation. $H\in C^r$ $(r=3,4,\cdots,\infty)$ is assumed to satisfy the following hypothesis:

H1, h_1+h_2 is a convex function in v, i.e., Hessian matrix $\partial^2_{vv}(h_1+h_2)$ is positive definite. It is finite everywhere and has superlinear growth in v, i.e., $(h_1+h_2)/\|v\|\to\infty$ as $\|v\|\to\infty$.

H2, it is *a priori* hyperbolic in the sense that the Hamiltonian flow $\Phi^t_{h_2}$, determined by h_2, has a non-degenerate hyperbolic fixed point $(x,y)=(0,0)$ and the function $h_2(x,0):\mathbb{T}^n\to\mathbb{R}$ attains its strict maximum at $x=0$ mod 2π. We set $h_2(0,0)=0$.

Here, we do not assume the condition on the hyperbolic fixed point that its stable manifold coincides with its unstable manifold.

Let $\mathcal{B}_{\epsilon,K}$ denote a ball in the function space $C^r(\{(u,v,t)\in\mathbb{T}^{n+1}\times\mathbb{R}^{n+1}\times\mathbb{T}:\|v\|\le K\}\to\mathbb{R})$, centered at the origin with radius of ϵ. Now we can state the main result of this paper, it is a higher dimensional version of the theorem formulated by Arnold in [**Ar1**] where it was assumed that $n=1$.

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Theorem 1.1. *Let $A < B$ be two arbitrarily given numbers and assume H satisfies the hypotheses* **H1** *and* **H2**. *There exist a small number $\epsilon > 0$, a large number $K > 0$ and a residual set $\mathcal{S}_{\epsilon,K} \subset \mathcal{B}_{\epsilon,K}$ such that for each $P \in \mathcal{S}_{\epsilon,K}$ there exist orbits of the Hamiltonian flow which connect the region with $p < A$ to the region with $p > B$.*

Remark: From the proof we can see the following. For each $P \in \mathcal{S}_{\epsilon,K}$, there is an orbit which drifts from $\{p \leq A\}$ to $\{p \geq B\}$ in finite time. The smooth dependence of solutions of ODE's on parameters implies that the theorem holds if perturbation is in a small neighborhood of that residual set, i.e. the set $\mathcal{S}_{\epsilon,K}$ can be open and dense.

The main result of this paper can be extended to more general case:

$$H(u,v,t) = h(x,v) + P(u,v,t) \tag{1.2}$$

where $u = (q,x)$, $v = (p,y)$, $(p,q) \in \mathbb{R} \times \mathbb{T}$, $(x,y) \in \mathbb{T}^n \times \mathbb{R}^n$, P is a time-1-periodic small perturbation, $H \in C^r$ $(r = 3, 4, \cdots, \infty)$. We assume that h satisfies the following three hypothesis:

H1': h is a convex function in v, i.e. the Hessian matrix $\partial^2_{vv}h$ is positive definite. It is finite everywhere and has super-linear growth in v, i.e., $h/\|v\| \to \infty$ as $\|v\| \to \infty$.

Under this hypothesis, there exists a function $y = f(x,p)$ such that the equation $\frac{\partial h}{\partial y}(x,p,f(x,p)) = 0$ holds. Although h is not integrable in general, the Hamiltonian flow Φ^t_h, determined by h, has a family of invariant circles $\{q \in \mathbb{T}, p = \text{const.}, x = x(p), y = f(x(p),p)\}$ where $x(p)$ represents the critical point of $h(x,p,f(x,p))$, as the function of $x \in \mathbb{T}^n$.

H2': For each p, if $x(p)$ is a maximum point of $h(x,p,f(x,p))$, then it is also the maximum point of the function $h(x,p,f(x(p),p))$. There is a locally finite set $\mathbb{P} = \{p_j\} \subset \mathbb{R}$, for each $p \in \mathbb{R}\backslash\mathbb{P}$ there exists a unique $x(p) \in \mathbb{T}^n$ such that $h(x,p,f(x,p))$ attains its global maximum at $x(p)$; for each $p \in \mathbb{P}$, $h(x,p,f(x,p))$ has exactly two global maximum points $x(p)$ and $x'(p)$. For $p \in \mathbb{P}$, we also assume that $\partial_p h|_{x(p)} \neq \partial_p h|_{x'(p)}$.

Under the hypothesis **H2'**, if $h(x,p,f(x,p))$ has two maximum points at $x(p)$ and $x'(p)$, then we have $f(x'(p),p) = f(x(p),p)$. That follows from the positive definite condition of h in y and from the condition that $\partial_y h(x,p,f(x,p)) = 0$. So we can define $y(p) = f(x(p),p)$.

H3': h is *a priori* hyperbolic in the sense that the Hessian of $h(x,p,y)$ in x at its maximum point, $\frac{\partial^2 h}{\partial x^2}$ is negative definite.

Let $c = (p, y(p)) \in \mathbb{R}^{n+1}$. The hypothesis **H2'** is assumed so that the c-minimal measure has its support on the invariant circle $\{\mathbb{T} \times (x(p),p,y(p))\}$ for each $p \in \mathbb{R}\backslash\mathbb{P}$, the support is on two circles for each $p \in \mathbb{P}$. The hypothesis **H3'** guarantees that these invariant circles are normally hyperbolic.

The motivation for us to study such kind of *a priori* hyperbolic systems mainly comes from the problem of Arnold diffusion in nearly integrable Hamiltonian systems of KAM type:

$$H(u,v,t)=h(v)+\epsilon P(u,v,t,\epsilon), \qquad u\in\mathbb{T}^n, \quad v\in\mathbb{R}^n, \quad t\in\mathbb{T}.$$

Let Γ be a curve in the action variable space where the frequencies satisfy at least $n-1$ resonant conditions, i.e. there are $n-1$ linearly independent integer vectors $(k_i,l_i)\in\mathbb{Z}^{n+1}$, $(i=1,2,\cdots,n-1)$, such that

$$\left\langle \frac{\partial h(v)}{\partial v}, k_i \right\rangle + l_i = 0, \qquad \forall\ v\in\Gamma\subset\mathbb{R}^n.$$

For $v\in\Gamma$, the $n+1$-dimensional torus $v=$ const. admits a foliation of 2-dimensional invariant tori σ for Φ_h^t. Taking the average over these tori

$$h_1(\widehat{(u,t)},v,\epsilon)=\int_\sigma P(u,v,t,\epsilon)d\sigma,$$

we obtain an equivalent system

$$H(u,v,t)=h(v)+\epsilon h_1(\widehat{(u,t)},v,\epsilon)+\epsilon P_1(u,v,t,\epsilon);$$

here $\widehat{(u,t)}$ is clearly an $(n-1)$-dimensional variable. If the frequencies of the flow on these two dimensional tori are not strongly resonant, by one step of KAM iteration one can set

$$|\epsilon P_1(u,v,t,\epsilon)|=O(\epsilon^{1+\kappa}), \qquad \kappa>0.$$

By introducing a canonical coordinate transformation we can write the system in the form

$$H(q,x,p,y,t)=h_0(p,y)+\epsilon h_1(x,p,y,\epsilon)+O(\epsilon^{1+\kappa}),$$

where $q\in\mathbb{T}$, $x\in\mathbb{T}^{n-1}$, $p\in\mathbb{R}$ and $y\in\mathbb{R}^{n-1}$. In this case, the hypotheses **H1'∼3'** are satisfied if h_1 satisfies some generic conditions. To see that, we let $h=h_0+\epsilon h_1$, let $f(x,p)$ be the function such that $\partial_y h(x,p,f(x,p))=0$ and let $x(p)$ be the maximum point of the function $h(x,p,f(x,p))$. Since h_0 is independent of x, for all $\{x,p,f(x,p):x\in\mathbb{T}^{n-1}\}$ we have

$$\frac{d^2h}{dx^2}=\epsilon\frac{\partial^2 h_1}{\partial x^2}-\epsilon^2\frac{\partial^2 h_1}{\partial x\partial y}\left(\frac{\partial^2 h}{\partial y^2}\right)^{-1}\frac{\partial^2 h_1}{\partial y\partial x},$$

where we use $\frac{\partial}{\partial x}$ to denote the derivative of the function $h(x,p,y)$ in x and use $\frac{d}{dx}$ to denote the derivative of the function $h(x,p,f(x,p))$ in x. It follows that $x(p)$ is at least a local maximum point of the function $h(x,p,f(x(p),p))$ and the invariant circle $\{\mathbb{T}\times(x(p),p,f(x(p),p))\}$ is normally hyperbolic if h_1 is non-degenerate at its maximum point and if ϵ is sufficiently small. In fact, since the total variation of the function $\mathrm{Var}_{x\in\mathbb{T}^n} f(x,p) = O(\epsilon)$, we have

$$\max_x h(x,p,f(x(p),p)) - h(x(p),p,f(x(p),p)) \le O(\epsilon^2).$$

It follows that $x(p)$ is a maximum point of the function $h(x,p,f(x(p),p))$ if we perturb the function $h(x,p,y)$ by a small function $\epsilon^2\tilde{h}(x,p)$. We may think $\epsilon^2\tilde{h}(x,p)$ as a higher order perturbation. Thus, the invariant circle $\{\mathbb{T}\times(x(p),p,f(x(p),p))\}$ is the support of the c-minimal measure for the Lagrange action determined by $h_0+\epsilon h_1$ if h_1 has some non-degenerate property, where $c=(p,f(x(p),p))$. For the system (1.2) the theorem 1.1 also holds:

Theorem 1.2. *Let $A < B$ be two arbitrarily given numbers and assume $H = h + P$ satisfies the hypotheses* (**H1' ∼ 3'**)*. There exist a small number $\epsilon > 0$, a large number $K > 0$ and a residual set $\mathcal{S}_{\epsilon,K} \subset \mathcal{B}_{\epsilon,K}$ such that for each $P \in \mathcal{S}_{\epsilon,K}$ there exist orbits of the Hamiltonian flow which connect the region with $p < A$ to the region with $p > B$.*

In his celebrated paper [**Ar1**], Arnold constructed an example of nearly integrable Hamiltonian system with two and half degrees of freedom, in which there are some unstable orbits in the sense that the action undergoes substantial variation along these orbits. Such orbits are usually called diffusion orbits. Although this example does not have generic property, Arnold still asked whether there is such a phenomenon for a "typical" small perturbation (cf. [**Ar2**],[**Ar3**]).

Variational method has its advantage in the study of Arnold diffusion problem, it needs less geometrical structure information of the system. The pioneer work of Mather in [**Ma1**] and in [**Ma2**] provides a variational principle for time-periodically dependent positive definite Lagrangian systems. In our previous paper [**CY**], by using the variational method, we have shown that the diffusion orbits exist in generic *a priori* unstable time-periodic Hamiltonian systems with two degrees of freedom, this result was announced by Xia six years earlier ([**Xia**]). Using geometrical method, some substantial progresses has been made in [**DLS**] as well as in [**Tr**] that diffusion orbits exist in some types of *a priori* unstable and time-periodic Hamiltonian systems with two degrees of freedom. For *a priori* stable case, the only announcement was made by Mather in [**Ma3**] for systems with two degrees of freedom in time-periodic case, or with three degrees of freedom in autonomous case, under so-called cusp residual condition.

In this paper we still use variational arguments to construct diffusion orbits. In order to use variational method, we put the problem of consideration into Lagrangian formalism. Let M be a closed manifold, $H : T^*M\times\mathbb{T}\to\mathbb{R}$ be a smooth Hamiltonian which is positive definite on each cotangent fiber. Using Legendre transformation $\mathscr{L}^* : H \to L$

we obtain the Lagrangian

$$L(u,\dot u,t)=\max_v\{\langle v,\dot u\rangle - H(u,v,t)\}.$$

Here $\dot u=\dot u(u,v,t)$ is implicitly determined by $\dot u=\frac{\partial H}{\partial v}$. We denote by $\mathscr{L}:(u,v,t)\to(u,\dot u,t)$ the coordinate transformation determined by the Hamiltonian H.

In the Lagrangian formalism, the Hamiltonian equation (1.1) is equivalent to the Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot u}\right)-\frac{\partial L}{\partial u}=0. \tag{1.3}$$

This equation corresponds to the critical point of the functional

$$A_c(\gamma)=\int (L-\eta_c)(\gamma,\dot\gamma,t)dt,$$

where $\eta_c=\langle\eta_c(q),\dot q\rangle$, $\eta_c(q)$ denotes a closed 1-form $\langle\eta_c(q),dq\rangle$ evaluated at q, and its de-Rham cohomology $[\langle\eta_c(q),dq\rangle]=c\in H^1(M,\mathbb{R})$. For convenience and without danger of confusion, we call η_c closed 1-form also.

To apply the Mather theory directly, we introduce a modified Lagrangian

$$\tilde L=L_0(\dot u)\rho(\dot u)+(1-\rho(\dot u))L(u,\dot u,t), \tag{1.4}$$

in which $L_0(\dot u)$ is strictly convex in $\dot u$ and has super-linear growth in $\|\dot u\|$; $\rho(\dot u)=1$ when $\|\dot u\|\ge 2K$, $\rho(\dot u)=0$ when $\|\dot u\|\le K$. Clearly, we can choose some $\rho(\dot u)$ so that $\tilde L$ is convex in $\dot u$ also. This system is integrable near infinity, so each solution is defined for all $t\in\mathbb{R}$. We choose sufficiently large K so that the diffusion orbits we search for remain in the region $\{\|\dot u\|\le K\}$ where the Lagrangian $\tilde L$ and L generate the same phase flow. So, we can assume that the Lagrangian L satisfies the conditions:

Positive definiteness. For each $(u,t)\in M\times\mathbb{T}$, the Lagrangian function is strictly convex in velocity: the Hessian $L_{\dot u\dot u}$ is positive definite;

Super-linear growth. We suppose that L has fiber-wise super-linear growth: for each $(u,t)\in M\times\mathbb{T}$, we have $L/\|\dot u\|\to\infty$ as $\|\dot u\|\to\infty$.

Completeness. All solutions of the Lagrangian equations are well defined for all $t\in\mathbb{R}$.

Let $I=[a,b]$ be a compact interval of time. A curve $\gamma\in C^1(I,M)$ is called a c-minimizer or a c-minimal curve if it minimizes the action among all curves $\xi\in C^1(I,M)$ which satisfy the same boundary conditions:

$$A_c(\gamma)=\min_{\substack{\xi(a)=\gamma(a)\\ \xi(b)=\gamma(b)}}\int_a^b (L-\eta_c)(d\xi(t),t)dt.$$

If J is a non compact interval, the curve $\gamma \in C^1(J, M)$ is said a c-minimizer if $\gamma|_I$ is c-minimal for any compact interval $I \subset J$. Let ϕ_L^t be the flow determined by the Lagrangian L, an orbit $x(t)$ of ϕ_L^t is called c-minimizing if the curve $\pi \circ X$ is c-minimizing, where the operator π is the standard projection from tangent bundle to the underling manifold along the fibers, a point $(z, s) \in TM \times \mathbb{R}$ is c-minimizing if its orbit $\phi_L^t(z, s)$ is c-minimizing. We use $\tilde{\mathcal{G}}_L(c) \subset TM \times \mathbb{R}$ to denote the set of minimal orbits of $L - \eta_c$ (the c-minimal orbits of L). We shall drop the subscript L when it is clear which Lagrangian is under consideration. It is not necessary to assume the periodicity of L in t for the definition of $\tilde{\mathcal{G}}$. When it is periodic in t, $\tilde{\mathcal{G}}(c) \subset TM \times \mathbb{R}$ is a nonempty compact subset of $TM \times \mathbb{T}$, invariant for ϕ_L^t.

The definition of action along a C^1-curve can be extended to the action on a probability measure. Let $\mathfrak{M}$ be the set of Borel probability measures on $TM \times \mathbb{T}$, invariant for ϕ_L^t. For each $\nu \in \mathfrak{M}$, the action $A_c(\nu)$ is defined as the following:

$$A_c(\nu) = \int (L - \eta_c) d\nu.$$

Mather has proved in [**Ma1**] that for each first de Rham cohomology class c there is an invariant probability measure μ which minimizes the actions over $\mathfrak{M}$

$$A_c(\mu) = \inf_{\nu \in \mathfrak{M}} \int (L - \eta_c) d\nu.$$

We use $\tilde{\mathcal{M}}(c)$ to denote the support of the measure and call it Mather set. $\alpha(c) = -A_c(\mu) : H^1(M, \mathbb{R}) \to \mathbb{R}$ is called α-function. Its Legendre transformation β: $H_1(M, \mathbb{R}) \to \mathbb{R}$ is called β-function. Both functions are convex, finite everywhere and have super-linear growth.

To define Aubry set and Mañé set we let

$$h_c((m,t),(m',t')) = \min_{\substack{\gamma \in C^1([t,t'],M) \\ \gamma(t)=m, \gamma(t')=m'}} \int_t^{t'} (L - \eta_c)(d\gamma(s), s) ds + (t'-t)\alpha(c),$$

$$F_c((m,s),(m',s')) = \inf_{\substack{t=s \mod 1 \\ t'=s' \mod 1 \\ t'-t \geq 1}} h_c((m,t),(m',t'))$$

$$h_c^\infty((m,s),(m',s')) = \liminf_{\substack{s=t \mod 1 \\ t'=s' \mod 1 \\ t'-t \to \infty}} h_c((m,t),(m',t')),$$

$$h_c^k(m,m') = h_c((m,0),(m',k)),$$

$$h_c^\infty(m,m') = h_c^\infty((m,0),(m',0)),$$

$$F_c(m,m') = F_c((m,0),(m',0))$$

$$d_c(m,m') = h_c^\infty(m,m') + h_c^\infty(m',m).$$

It was showed in [**Ma2**] that d_c is a pseudo-metric on the set $\{x \in M : h_c^\infty(x,x) = 0\}$. A curve $\gamma \in C^1(\mathbb{R}, M)$ is called c-semi-static if

$$A_c(\gamma|_{[a,b]}) + \alpha(c)(b-a) = F_c(\gamma(a), \gamma(b), a \bmod 1, b \bmod 1)$$

for each $[a,b] \subset \mathbb{R}$. A curve $\gamma \in C^1(\mathbb{R}, M)$ is called c-static if, in addition

$$A_c(\gamma|_{[a,b]}) + \alpha(c)(b-a) = -F_c(\gamma(b), \gamma(a), b \bmod 1, a \bmod 1)$$

for each $[a,b] \subset \mathbb{R}$ with $b - a \geq 1$. An orbit $X(t) = (d\gamma(t), t \bmod 2\pi)$ is called c-static (semi-static) if γ is c-static (semi-static). We call the Mañé set $\tilde{\mathcal{N}}(c)$ the union of global c-semi-static orbits, and call the Aubry set $\tilde{\mathcal{A}}(c)$ the union of c-static orbits. We can also define corresponding Aubry sets and Mañé sets for some covering manifold $\tilde{M}$ respectively. Obviously, the c-static (semi-static) orbits for $\tilde{M}$ is not necessarily c-static (semi-static) for M.

We use $\mathcal{M}(c)$, $\mathcal{A}(c)$, $\mathcal{N}(c)$ and $\mathcal{G}(c)$ to denote the standard projection of $\tilde{\mathcal{M}}(c)$, $\tilde{\mathcal{A}}(c)$, $\tilde{\mathcal{N}}(c)$ and $\tilde{\mathcal{G}}(c)$ from $TM \times \mathbb{T}$ to $M \times \mathbb{T}$ respectively. We have the following inclusions ([**Be2**])

$$\tilde{\mathcal{M}}(c) \subseteq \tilde{\mathcal{A}}(c) \subseteq \tilde{\mathcal{N}}(c) \subseteq \tilde{\mathcal{G}}(c).$$

It was showed in [**Ma2**] that the inverse of the projection is Lipschitz when it is restricted to $\mathcal{A}(c)$ and $\mathcal{M}(c)$.

In the following we use the symbol $\tilde{\mathcal{N}}_s(c) = \tilde{\mathcal{N}}(c)|_{t=s}$ to denote the time s-section of a Mañé set, and so on. We use Φ_H to denote the time-1-map of $\Phi_H^t|_{t=1}$, and use ϕ_L to denote the time-1-map of $\phi_L^t|_{t=1}$ respectively.

This paper is organized as follows. In the section 2 we introduce so-called pseudo connecting orbit set and establish the upper semi-continuity of these sets. Such property is used to show the existence of local minimal orbits connecting some Mañé set to another Mañé set nearby. In the section 3, we investigate the topological structure of the Mañé sets and the pseudo connecting orbit sets, they correspond to those cohomology classes through which the diffusion orbits is constructed. The Mañé sets for a finite covering of the manifold $\tilde{M}$ may have very different structure from those for M. In the section 4, by making use of the upper semi-continuity of Mañé sets, the existence of local connecting orbits is established if the Mañé set has some kind of topological triviality. The section 5 is devoted to the construction of diffusion orbits if there is a so-called generalized transition chain along the corresponding path in the first de-Rham cohomology space. To show the generic condition we establish some Hölder continuity of the barrier functions in the section 6, with which the generic property is proved in the last section.

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After we finished the first version of this paper in 2004, we found the paper of P. Bernard ([**Be3**]).

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2. Upper semi-continuity

The variational construction of diffusion orbits depends on the upper semi-continuity of some set functions.

Lemma 2.1. *We assume $L \in C^r(TM \times \mathbb{R}, \mathbb{R})$ $(r \geq 2)$ satisfies the positive definite, superlinear-growth and completeness conditions, where M is a compact, connected Riemanian manifold. Considered as the function of t, L is assumed periodic for $t \in (-\infty, 0]$ and for $t \in [1, \infty)$. Then the map $L \to \tilde{\mathcal{G}}_L \subset TM \times \mathbb{R}$ is upper semi-continuous. As an immediate consequence, the map $c \to \tilde{\mathcal{G}}(c)$ is upper semi-continuous if L is periodic in t.*

The proof of this lemma was provided in [**Be2**] and [**CY**]. We can consider t is defined on $(\mathbb{T} \vee [0,1] \vee \mathbb{T})/\sim$, where $\sim$ is defined by identifying $\{0\} \in [0,1]$ with some point on one circle and identifying $\{1\} \in [0,1]$ with some point on another circle. Let $U_k = \{(\zeta, q, t) : (q,t) \in M \times (\mathbb{T} \vee [0,1] \vee \mathbb{T})/\sim, \|\zeta\| \leq k, \}$, $\cup_{k=1}^{\infty} U_k = TM \times \mathbb{R}$. Let $L_i \in C^r(TM \times \mathbb{T}, \mathbb{R})$. We say L_i converges to L if for each $\epsilon > 0$ and each U_k there exists i_0 such that $\|L - L_i\|_{C^r(U_k,\mathbb{R})} \leq \epsilon$ if $i \geq i_0$.

To establish some connection between two Mañé sets $\tilde{\mathcal{N}}(c)$ and $\tilde{\mathcal{N}}(c')$, we consider a modified Lagrangian

$$L_{\eta,\mu,\psi} = L - \eta - \mu - \psi$$

where η is a closed 1-form on M such that $[\eta] = c$, μ is a 1-form depending on t in the way that the restriction of μ on $\{t \leq 0\}$ is 0, the restriction on $\{t \geq 1\}$ is a closed 1-form $\bar{\mu}$ on M with $[\bar{\mu}] = c' - c$. Usually, this μ is called U-step 1-form ([**Be2**]). ψ is a function on $M \times \mathbb{R}$, $\psi(\cdot, t) = 0$ for all $t \in (-\infty, 0] \cup [1, \infty)$. Let $m, m' \in M$, we define

$$h_{\eta,\mu,\psi}^{T_0,T_1}(m,m') = \inf_{\substack{\gamma(-T_0)=m \\ \gamma(T_1)=m'}} \int_{-T_0}^{T_1} L_{\eta,\mu,\psi}(d\gamma(t),t)dt + T_0\alpha(c) + T_1\alpha(c').$$

Clearly $\exists\ m^* \in M$ and some constants $C_1,\ C_2$, independent of T_0, T_1, such that

$$\begin{aligned} h^{T_0,T_1}_{\eta,\mu,\psi}(m,m') &\le h^{T_0}_c(m,m^*) + h^{T_2}_{c'}(m^*,m') + C_1 \\ &\le C_2. \end{aligned}$$

Thus its limit infimum is bounded

$$h^{\infty}_{\eta,\mu,\psi}(m,m') = \liminf_{T_0,T_1\to\infty} h^{T_0,T_1}_{\eta,\mu,\psi}(m,m') \le C_2.$$

Let $\{T^i_0\}_{i\in\mathbb{Z}_+}$ and $\{T^i_1\}_{i\in\mathbb{Z}_+}$ be the sequence of positive integers such that $T^i_j \to \infty$ $(j = 0, 1)$ as $i \to \infty$ and the following limit exists

$$\lim_{i\to\infty} h^{T^i_0,T^i_1}_{\eta,\mu,\psi}(m,m') = h^{\infty}_{\eta,\mu,\psi}(m,m').$$

Let $\gamma_i(t,m,m')$: $[-T^i_0, T^i_1] \to M$ be a minimizer connecting m and m'

$$h^{T^i_0,T^i_1}_{\eta,\mu,\psi}(m,m') = \int_{-T^i_0}^{T^i_1} L_{\eta,\mu,\psi}(d\gamma_i(t),t)dt + T^i_0\alpha(c) + T^i_1\alpha(c').$$

It is not difficult to see that for any compact interval $[a,b]$, the set $\{\gamma_i\}$ is pre-compact in $C^1([a,b],M)$.

Lemma 2.2. *Let γ: $\mathbb{R} \to M$ be an accumulation point of $\{\gamma_i\}$. If $s \ge 1$ then*
(2.1)

$$A_{L_{\eta,\mu,\psi}}(\gamma|[s,\tau]) = \inf_{\substack{\tau_1-\tau\in\mathbb{Z},\tau_1>s\\ \gamma^*(s)=\gamma(s)\\ \gamma^*(\tau_1)=\gamma(\tau)}} \int_s^{\tau_1} L_{\eta,\mu,\psi}(d\gamma^*(t),t)dt + (\tau_1-\tau)\alpha(c');$$

if $\tau \le 0$ then

$$A_{L_{\eta,\mu,\psi}}(\gamma|[s,\tau]) = \inf_{\substack{s_1-s\in\mathbb{Z},s_1<\tau\\ \gamma^*(s_1)=\gamma(s)\\ \gamma^*(\tau)=\gamma(\tau)}} \int_{s_1}^{\tau} L_{\eta,\mu,\psi}(d\gamma^*(t),t)dt - (s_1-s)\alpha(c); \tag{2.2}$$

if $s \le 0$ and $\tau \ge 1$ then

$$\begin{aligned} A_{L_{\eta,\mu,\psi}}(\gamma|[s,\tau]) = \inf_{\substack{s_1-s\in\mathbb{Z},\ \tau_1-\tau\in\mathbb{Z}\\ s_1\le 0,\ \tau_1\ge 1\\ \gamma^*(s_1)=\gamma(s)\\ \gamma^*(\tau_1)=\gamma(\tau)}} &\int_{s_1}^{\tau_1} L_{\eta,\mu,\psi}(d\gamma^*(t),t)dt \\ &- (s_1-s)\alpha(c) + (\tau_1-\tau)\alpha(c'). \end{aligned} \tag{2.3}$$

Proof. : Let us suppose the contrary, for instance, (2.2) does not hold. Thus there would exist $\Delta > 0$, $s < \tau \le 0$, $s_1 < \tau \le 0$, $s_1 - s \in \mathbb{Z}$ and a curve γ^*: $[s_1,\tau] \to M$ with $\gamma^*(s_1) = \gamma(s)$, $\gamma^*(\tau) = \gamma(\tau)$ such that

$$A_{L_{\eta,\mu,\psi}}(\gamma|[s,\tau]) \ge \int_{s_1}^{\tau} L_{\eta,\mu,\psi}(d\gamma^*(t),t)dt - (s_1-s)\alpha(c) + \Delta.$$

Let $\epsilon = \frac{1}{4}\Delta$. By the definition of limit infimum there exist $T_0^{i_0} > 0$ and $T_1^{i_0} > 0$ such that

$$h^{T_0,T_1}_{\eta,\mu,\psi}(m_0,m_1) \geq h^{\infty}_{\eta,\mu,\psi}(m_0,m_1) - \epsilon, \qquad \forall\ T_0 \geq T_0^{i_0},\ T_1 \geq T_1^{i_0}, \tag{2.4}$$

and there exist subsequences $T_j^{i_k}$ ($j = 0, 1$, $k = 0, 1, 2, \cdots$) such that for all $k > 0$

$$T_0^{i_k} - T_0^{i_0} \geq s - s_1, \tag{2.5}$$

$$\left| h^{T_0^{i_k},T_1^{i_k}}_{\eta,\mu,\psi}(m_0,m_1) - h^{\infty}_{\eta,\mu,\psi}(m_0,m_1) \right| < \epsilon. \tag{2.6}$$

By taking a subsequence further we can assume $\gamma_{i_k} \to \gamma$. In this case, we can choose sufficiently large k such that $\gamma_{i_k}(s)$ and $\gamma_{i_k}(\tau)$ are so close to $\gamma(s)$ and $\gamma(\tau)$ respectively that we can construct a curve $\gamma^*_{i_k}$: $[s_1, \tau] \to M$ which has the same endpoints as γ_{i_k}: $\gamma_i^*(s_1) = \gamma_i(s)$, $\gamma_i^*(\tau) = \gamma_i(\tau)$ and satisfies the following

$$A_{L_{\eta,\mu,\psi}}(\gamma_{i_k}|[s,\tau]) \geq \int_{s_1}^{\tau} L_{\eta,\mu,\psi}(d\gamma^*_{i_k}(t), t)dt - (s_1 - s)\alpha(c) + \frac{3}{4}\Delta. \tag{2.7}$$

Let $T_0' = T_0^{i_k} + (s - s_1)$, if we extend $\gamma^*_{i_k}$ to $\mathbb{R} \to M$ such that

$$\gamma^*_{i_k} = \begin{cases} \gamma_{i_k}(t - s_1 + s), & t \leq s_1, \\ \gamma^*_{i_k}(t), & s_1 \leq t \leq \tau, \\ \gamma_{i_k}(t), & t \geq \tau, \end{cases}$$

then we obtain from (2.6) and (2.7) that

$$\begin{aligned} h^{T_0',T_1^{i_k}}_{\eta,\mu,\psi}(m_0,m_1) &\leq A_{L_{\eta,\mu,\psi}}(\gamma^*_{i_k}|[-T_0', T_1^{i_k}]) - T_1^{i_k}\alpha(c') - T_0'\alpha(c) \\ &\leq A_{L_{\eta,\mu,\psi}}(\gamma_{i_k}|[-T_0^{i_k}, T_1^{i_k}]) - T_1^{i_k}\alpha(c') - T_0^{i_k}\alpha(c) - \frac{3}{4}\Delta \\ &\leq h^{\infty}_{\eta,\mu,\psi}(m_0,m_1) - 2\epsilon. \end{aligned}$$

but this contradicts (2.4) since $T_0' \geq T_0^{i_0}$ and $T_1^{i_k} \geq T_1^{i_0}$, guaranteed by (2.5). (2.1) and (2.3) can be proved in the same way.

q.e.d.

With this lemma it is natural to define

$$\tilde{\mathcal{C}}_{\eta,\mu,\psi} = \{d\gamma \in \tilde{\mathcal{G}}_{L_{\eta,\mu,\psi}} :\ (2.1),\ (2.2) \text{ and } (2.3) \text{ hold }\}.$$

Although the elements in this set are not necessarily the orbits of the Lagrangian flow determined by L, the α-limit set of each element in this set is contained in $\tilde{\mathcal{N}}(c)$, the ω-limit set is contained in $\tilde{\mathcal{N}}(c')$. Due to this reason, we call it pseudo connecting orbit set. Obviously $\tilde{\mathcal{C}}_{\eta,0,0} = \tilde{\mathcal{N}}(c)$. For convenience we may drop the subscript ψ in the symbol when it is equal to zero, i.e. $\tilde{\mathcal{C}}_{\eta,\mu} := \tilde{\mathcal{C}}_{\eta,\mu,0}$.

Lemma 2.3. *The map $(\eta,\mu,\psi)\to\tilde{\mathcal{C}}_{\eta,\mu,\psi}$ is upper semi-continuous. $\tilde{\mathcal{C}}_{\eta,0,0}=\tilde{\mathcal{N}}([\eta])$. Consequently, the map $c\to\tilde{\mathcal{N}}(c)$ is upper semi-continuous too.*

Proof. : Let $\eta_i\to\eta$, $\mu_i\to\mu$ and $\psi_i\to\psi$, let $\gamma_i\in\tilde{\mathcal{C}}_{\eta_i,\mu_i,\psi_i}$ and let γ be an accumulation point of the set $\{\gamma_i\in\tilde{\mathcal{C}}_{\eta_i,\mu_i,\psi_i}\}_{i\in\mathbb{Z}^+}$. Clearly, $\gamma\in\tilde{\mathcal{C}}_{\eta,\mu,\psi}$. If $\gamma\notin\tilde{\mathcal{C}}_{\eta,\mu,\psi}$ there would be two point $\gamma(s)$,$\gamma(\tau)\in M$ such that one of the following three possible cases takes place. Either $\gamma(s)$ and $\gamma(\tau)\in M$ can be connected by another curve γ^*: $[s+n,\tau]\to M$ with smaller action

$$A_{L_{\eta,\mu,\psi}}(\gamma|[s,\tau])<A_{L_{\eta,\mu,\psi}}(\gamma^*|[s+n,\tau])-n\alpha(c)$$

in the case $\tau<0$; or there would a curve γ^*: $[s,\tau+n]\to M$ such that

$$A_{L_{\eta,\mu,\psi}}(\gamma|[s,\tau])<A_{L_{\eta,\mu,\psi}}(\gamma^*|[s,\tau+n])-n\alpha(c')$$

in the case $s\geq 1$, or when $s\leq 0$ and $\tau\geq 1$ there would be a curve γ^*: $[s+n_1,\tau+n_2]\to M$ such that

$$A_{L_{\eta,\mu,\psi}}(\gamma|[s,\tau])<A_{L_{\eta,\mu,\psi}}(\gamma^*|[s+n_1,\tau+n_2])-n_1\alpha(c)-n_2\alpha(c')$$

where $s+n_1\leq 0$, $\tau+n_2\geq 1$. Since γ is an accumulation point of γ_i, for any small $\epsilon>0$, there would be sufficiently large i such that $\|\gamma-\gamma_i\|_{C^1[s,t]}<\epsilon$, it follows that $\gamma_i\notin\tilde{\mathcal{C}}_{\eta_i,\mu_i,\psi_i}$ but that is absurd.

Let us consider the case that $\mu=0$ and $\psi=0$. In this case, $L-\eta$ is periodic in t. If some orbit $\gamma\in\tilde{\mathcal{C}}_{\eta,0,0}$: $\mathbb{R}\to M$ is not semi-static, then there exist $s<\tau\in\mathbb{R}$, $n\in\mathbb{Z}$, $\Delta>0$ and a curve γ^*: $[s,\tau+n]\to M$ such that $\gamma^*(s)=\gamma(s)$, $\gamma^*(\tau+n)=\gamma(\tau)$ and

$$A_{L_{\eta,0,0}}(\gamma|[s,\tau])\geq A_{L_{\eta,0,0}}(\gamma^*|[s,\tau+n])-n\alpha(c)+\Delta.$$

We can extend γ^* to $[s_1,\tau_1+n]\to M$ such that $s_1\leq\min\{s,0\}$, $\min\{\tau_1,\tau_1+n\}\geq 1$, $\tau_1\geq\tau$ and

$$\gamma^*=\begin{cases}\gamma(t), & s_1\leq t\leq s,\\ \gamma^*(t), & s\leq t\leq\tau+n,\\ \gamma(t-n), & \tau+n\leq t\leq\tau_1+n.\end{cases}$$

Since $L-\eta$ is periodic in t, we would have

$$A_{L_{\eta,0,0}}(\gamma|[s_1,\tau_1])\geq A_{L_{\eta,0,0}}(\tau^*\gamma|[s_1,\tau_1+n])-n\alpha(c)+\Delta.$$

but this contradicts to (2.3). q.e.d.

3. Structure of some $\tilde{\mathcal{N}}(c)$ and $\tilde{\mathcal{C}}_{\eta,\mu,\psi}$

It is natural to study the topological structure of the Mañé sets if we want to construct the connecting orbits between them.

Let L be the Lagrangian obtained from H in (1.2) by the Legendre transformation, it has the form as follows:

$$L(u,\dot{u},t)=\ell(\dot{q},x,\dot{x})+L_1(u,\dot{u},t), \tag{3.1}$$

here $\ell = \mathscr{L}^*(h)$, L_1 is a small perturbation. The perturbation term of the Lagrangian L_1 and the perturbation term of the Hamiltonian P is related by an operator $\Delta\mathscr{L}^*$ induced by the Legendre transformation $L_1 = \Delta\mathscr{L}^*(P) = \mathscr{L}^*(h+P) - \mathscr{L}^*(h)$. We also denote by $\mathcal{B}_{\epsilon,K}$ the ball in $C^r(\{(u,\dot{u},t) \in \mathbb{T}^{k+n} \times \mathbb{R}^{k+n} \times \mathbb{T} : \|\dot{u}\| \le K\} \to \mathbb{R})$, centered at the origin with radius of ϵ. Obviously, there exist $\varrho > 0$ and large $K' > 0$ such that

$$\Delta\mathscr{L}^*(\mathcal{B}_{\epsilon,K}) \subset \mathcal{B}_{\varrho\epsilon,K'}.$$

Let $c = (c_q, c_x)$ denote a cohomology class in $H^1(\mathbb{T}^{k+n}, \mathbb{R})$ where $c_q \in \mathbb{R}^k$ and $c_x \in \mathbb{R}^n$, let $\rho(\mu) = (\rho_q(\mu), \rho_x(\mu))$ denote the rotation vector of the minimal measure μ. Let

$$\ell_c(\dot{q}, x, \dot{x}) = \ell(\dot{q}, x, \dot{x}) - \langle \dot{q}, c_q \rangle - \langle \dot{x}, c_x \rangle.$$

When $c = (p, y(p))$, the function ℓ_c attains its global minimum at some circle $\{\mathscr{L}_h(x(p), q, p, y(p)) : q \in \mathbb{T}\}$ which is clearly invariant to the flow ϕ^t_ℓ. Here we use $\mathscr{L}_h : (u,v) \to (u,\dot{u})$ to denote the Legendre transformation determined by h. By the hypothesis **H2'** we find that there is only one action minimizing circle when $p \notin \mathbb{P}$ and there are two action minimizing circles, i.e. $\{\mathscr{L}_h(x(p), q, p, y(p)) : q \in \mathbb{T}\}$ and $\{\mathscr{L}_h(x'(p), q, p, y(p)) : q \in \mathbb{T}\}$. Obviously, there exists an n-dimensional convex disk $\mathcal{D}(c_q) \subset \mathbb{R}^n$ such that $c_x = y(c_q)$ is in its interior and for each $c \in \{c_q\} \times \mathcal{D}(c_q)$ the support of the c-minimal measure of ℓ is on these invariant circles.

To obtain the result of this paper we choose $k = 1$, but the demonstration in the following sections 3, 4 and 5 applies for arbitrary k. Recall k is the dimension of rotator.

Lemma 3.1. *Given large number $K > 0$ and a small number $\delta > 0$. There exists a small number $\epsilon = \epsilon(\delta) > 0$, if $c_q \in \{\max_{1 \le i \le k} |c_{q_i}| \le K\}$ and if $P \in \mathcal{B}_{\epsilon,K}$, then there exists an n-dimensional convex disk $\mathcal{D}(c_q)$ which contains the set $\{c = (c_q, c_x) : c_q = c^0_q, \|c_x - y(c^0_q)\| < C_q\}$ $(C_q > 0)$ such that*

1, for each $c \in \{\|c_x - y(c_q)\| < C_q\}$, if c_q is not close to $\mathbb{P}$ the Mañé set $\tilde{\mathcal{N}}(c) \subset \{\|x - x(c_q)\| < \delta\}$; if c_q is close to some point $p_i \in \mathbb{P}$, then the Aubry set $\tilde{\mathcal{A}}(c) \subset \{\|x - x(c_q)\| < \delta\} \cup \{\|x - x'(c_q)\| < \delta\}$. If $\tilde{\mathcal{A}}(c) \cap \{\|x - x(c_q)\| < \delta\} \neq \varnothing$ and $\tilde{\mathcal{A}}(c) \cap \{\|x - x'(c_q)\| < \delta\} \neq \varnothing$, then the Mañé set $\tilde{\mathcal{N}}(c)$ contains some orbit $d\gamma$: $\mathbb{R} \to TM$ such that $\alpha(d\gamma) \subset \{\|x - x(c_q)\| \le \delta\}$ and $\omega(d\gamma) \subset \{\|x - x'(c_q)\| \le \delta\}$, it also contain some orbit $d\gamma'$: $\mathbb{R} \to TM$ such that $\alpha(d\gamma') \subset \{\|x - x'(c_q)\| \le \delta\}$ and $\omega(d\gamma') \subset \{\|x - x(c_q)\| \le \delta\}$;

2, for each $c \in \mathrm{int}(\mathcal{D}(c_q))$, if c_q is not close to $\mathbb{P}$, the Mather set $\tilde{\mathcal{M}}(c) \subset \{\|x - x(c_q)\| < \delta\}$, if c_q is close to some point $p_i \in \mathbb{P}$, then the Mather set $\tilde{\mathcal{M}}(c) \subset \{\|x - x(c_q)\| < \delta\} \cup \{\|x - x'(c_q)\| < \delta\}$; for each $c \in \{c_q = \mathrm{constant}\} \backslash \overline{\mathcal{D}(c_q)}$ and each c-minimal measure μ, $\rho_x(\mu) \neq 0$;

3, if for each $c \in \mathrm{int}(\mathcal{D}(c_q))$, the c-minimal measure is uniquely ergodic then $\tilde{\mathcal{N}}(c) \subset \{\|x - x(c_q)\| < \delta\}$ for each $c \in \mathrm{int}(\mathcal{D}(c_q))$.

The interior of $\mathcal{D}(c_q)$ is in the sense that we think $\mathcal{D}(c_q)$ as a set in $\mathbb{R}^n$. We denote the rotation vector of μ by $\rho(\mu) = (\rho_q(\mu), \rho_x(\mu))$.

Proof. : Let $c^* = (c_q, y(c_q))$ and let us consider the case that c_q is not close to $\mathbb{P}$ first. The c^*-minimal measure of ℓ has its support at the invariant circle $\Gamma_{c_q} = \{\dot q = \omega_{c_q}, \dot x = 0, x = x(c_q)\}$. Let α_ℓ be the α-function of ℓ, then

$$\alpha_\ell(c^*) = -\ell_{c^*}|_{(u,\dot u)\in\Gamma}.$$

By the hypothesis (**H3'**) and the convex property of ℓ in $\dot u$, there exists $E_1 > 0$ such that

$$\ell_{c^*}(d\gamma(t)) + \alpha_\ell(c^*) \geq E_1 d^2$$

if $\gamma(t) \notin \{\|x - x(c_q)\| \leq d\}$. To each absolutely continuous curve $\gamma = (\gamma_q, \gamma_x) : [t_0, t_1] \to \mathbb{T}^k \times \mathbb{T}^n$ we associate a number

$$|[\gamma_x|_{[t_0,t_1]}]| = \frac{1}{2\pi}\sum_{i=1}^{n} |\bar\gamma_{x_i}(t_1) - \bar\gamma_{x_i}(t_0)|$$

where $\bar\gamma_x$ denotes the lift of γ_x to the universal covering $\mathbb{R}^n$. If $\gamma(t) \notin \{\|x - x(c_q)\| < \lambda\delta\}$ for all $t \in (t_0, t_1)$ and if there is some $t^* \in (t_0, t_1)$ such that $\gamma(t^*) \notin \{\|x - x(c_q)\| < \delta\}$ then there exists $E_2 > 0$ further such that

(3.2)
$$\int_{t_0}^{t_1} (\ell_{c^*}(d\gamma(t)) + \alpha_\ell(c^*))dt \geq E_1\lambda^2\delta^2(t_1 - t_0) + E_2(\delta^2 + |[\gamma_x|_{[T-0,t_1]}]|).$$

Here, we have made use of the super-linear growth in $\dot u$ also. Let $\xi : [t_0, t_1] \to \mathbb{T}^{k+n}$ ($t_1 - t_0 \geq 1$) be a minimal curve of ℓ joining two points in $\{\|x - x(c_q)\| \leq \lambda\delta\}$. Note that any vector $\omega \in \mathbb{R}^n$ can be approximated by some rational vector $\frac{m}{m_0}$, ($m \in \mathbb{Z}^n, m_0 \in \mathbb{Z}$) such that $\|\omega - \frac{m}{m_0}\| \leq \frac{1}{m_0^2}$, we can choose properly large $t_1 - t_0$ such that

$$\int_{t_0}^{t_1} (\ell_{c^*}(d\xi(t)) + \alpha_\ell(c^*))dt \leq E_3\Big(\lambda^2\delta^2 + \frac{1}{t_1 - t_0}\Big). \tag{3.3}$$

Let $\zeta_\ell, \zeta_L : [0,1] \to \mathbb{T}^{k+n}$ be the c-minimal curves of ℓ and L respectively. Clearly, there exist $E_4 > 0$ and $\Theta > 0$ such that

$$\int_{t_0}^{t_1} (\ell_c(d\zeta_\ell(t)) + \alpha_\ell(c))dt \leq E_4,$$

$$\int_{t_0}^{t_1} (L_c(d\zeta_L(t)) + \alpha_L(c))dt \leq E_4,$$

$$|\bar\zeta_{\ell x}(1) - \bar\zeta_{\ell x}(0)| \leq \Theta,$$

$$|\bar\zeta_{Lx}(1) - \bar\zeta_{Lx}(0)| \leq \Theta.$$

If there is a c-minimal curve γ: $\mathbb{R} \to \mathbb{T}^{k+n}$ of L such that $\gamma(t) \notin \{\|x - x(c_q)\| < \lambda\delta\}$ for all $t \in [t_0, t_1]$ and there is some $t^* \in (t_0, t_1)$ such that $\gamma(t^*) \notin \{\|x - x(c_q)\| < \delta\}$, then we construct a curve ξ^*: $\mathbb{R} \to \mathbb{T}^{k+n}$ such that

$$\xi^*(t) = \begin{cases} \gamma(t) & t \le t_0 - 1, \\ \xi_0(t) & t_0 - 1 \le t \le t_0, \\ \xi(t) & t_0 \le t \le t_1, \\ \xi_1(t) & t_1 \le t \le t_1 + 1, \\ \gamma(t) & t \ge t_1 + 1, \end{cases}$$

where ξ_0 is a c-minimal curve of ℓ connecting $\gamma(t_0 - 1)$ with some point $m_0 \in \{\|x - x(c_q)\| = \lambda\delta\}$, ξ_1 is a c-minimal curve of ℓ connecting $m_1 \in \{\|x - x(c_q)\| = \lambda\delta\}$ with $\gamma(t_1 + 1)$, ξ is a c^*-minimal curve of ℓ connecting m_0 with m_1. We compare the c-action of L along the curve γ_c with its c-action along ξ^*,

$$\begin{aligned} &\int_{t_0-1}^{t_1+1} \Big(L_c(d\gamma(t), t) - L_c(d\xi^*(t), t)\Big) dt \\ \ge\ &\int_{t_0}^{t_1} \Big(\ell_{c^*}(d\gamma_c(t)) - \ell_{c^*}(d\xi^*(t))\Big) dt \\ &- 2\pi C_q |[\gamma_x|_{[t_0,t_1]}]| - 2\varrho\epsilon(t_1 - t_0) - 4E_4 - 4\Theta \\ \ge\ &\frac{1}{2}\Big(E_1\lambda^2(t_1 - t_0) + E_2\Big)\delta^2 - \frac{E_3}{t_1 - t_0} - 4E_4 - 4\Theta > 0, \end{aligned}$$

if we set

$$\lambda \le \sqrt{\frac{E_2}{2E_3}}, \qquad C_q \le \frac{E_2}{2\pi} \qquad \epsilon \le \frac{E_1\lambda^2\delta^2}{4\varrho}, \tag{3.4}$$

and let $t_1 - t_0$ be sufficiently large. This contradiction verifies our claim. Therefore, each c-minimal orbit $d\gamma$ must enter the region $\{\|x - x(c_q)\| \le \lambda\delta\}$ for infinitely many times if (3.4) is satisfied.

Now we assume $\gamma = (\gamma_q, \gamma_x) : \mathbb{R} \to \mathbb{T}^k \times \mathbb{T}^n$ is a c-semi static curve for L such that $\gamma(t_0), \gamma(t_1) \in \{\|x - x(c_q)\| = \lambda\delta\}$, $\gamma(t) \notin \{\|x - x(c_q)\| < \lambda\delta\}$ for all $t \in [t_0, t_1]$, $\gamma(t^*) \notin \{\|x - x(c_q)\| < \delta\}$ for some $t^* \in (t_0, t_1)$. In this case, we construct a curve ξ: $\mathbb{R} \to \mathbb{T}^{k+n}$ such that

$$\xi^*(t) = \begin{cases} \gamma(t) & t \le t_0, \\ \xi(t) & t_0 \le t \le t_1 + T, \\ \gamma(t) & t \ge t_1 + T \end{cases}$$

where ξ is the c^*-minimal curve of ℓ: $[t_0, t_1 + T] \to \mathbb{T}^{k+n}$ joining $\gamma(t_0)$ with $\gamma(t_1)$. T is carefully chosen so that (3.3) holds. Clearly, T is uniformly bounded for any $\gamma(t_0), \gamma(t_1) \in \mathbb{T}^{k+n}$. Note we have $|\alpha_L(c) -$

$\alpha_\ell(c)| \le \varrho\epsilon$ and $|\alpha_\ell(c) - \alpha_\ell(c^*)| \le E_5|c - c^*|$ if $|c|, |c^*| \le K$,

$$\begin{aligned}
&\int_{t_0}^{t_1} \Big(L_c(d\gamma(t), t) + \alpha_L(c)\Big)dt - \int_{t_0}^{t_1+T} \Big(L_c(d\xi(t), t) + \alpha_L(c)\Big)dt \\
\ge\ & E_1\lambda^2\delta^2(t_1 - t_0) + E_2(\delta^2 + |[\gamma_x|_{[t_0,t_1]}]|) - 2\pi C_q|[\gamma_x|_{[t_0,t_1]}]| \\
& - E_3\Big(\lambda^2\delta^2 + \frac{1}{T + t_1 - t_0}\Big) - 2\varrho\epsilon|t_1 - t_0| - \varrho\epsilon T \\
& - |\alpha_L(c) - \alpha_\ell(c^*)|T \\
\ge\ & \frac{1}{2}\Big(E_1\lambda^2\delta^2(t_1 - t_0) + E_2\delta^2\Big) - \frac{E_3}{T + t_1 - t_0} - 2\varrho\epsilon T - E_5 C_q T - T\epsilon \\
\ge\ & \frac{1}{4}E_1\lambda^2\delta^2(t_1 - t_0) > 0,
\end{aligned}$$

if ϵ is suitably small so that (3.4) as well as the following holds

$$\frac{1}{T + t_1 - t_0} \le \frac{E_2\delta^2}{4E_3}, \qquad \epsilon \le \frac{E_2\delta^2}{8\varrho T}, \qquad C_q \le \frac{\lambda^2\delta^2(t_1 - t_0)}{4E_5 T}.$$

But this contradicts again the fact that γ is c-semi static.

Let us consider the case when c_q is close to $\mathbb{P}$. In this case, if $\|c_x - y(c_q)\| \le C_q$, the c-minimal measure may have some support in $\{\|x - x(c_q)\| \le \delta\}$, and have other support in $\{\|x - x'(c_q)\| \le \delta\}$ also. It is easy to see that the Aubry set $\tilde{\mathcal{A}}(c) \subseteq \{\|x - x(c_q)\| \le \delta\} \cup \{\|x - x'(c_q)\| \le \delta\}$. If the Aubry set is in a neighborhood of one sub-torus $x =$ const., then the Mañé set is also in this neighborhood. If both $\{\|x - x(c_q)\| \le \delta\}$ and $\{\|x - x'(c_q)\| \le \delta\}$ contains the support of the minimal measure, let us take $\xi \in \mathcal{M}_0(c) \cap \{\|x - x(c_q)\| \le \delta\}$, take $\zeta \in \mathcal{M}_0(c) \cap \{\|x - x'(c_q)\| \le \delta\}$ and calculate the quantity $h_c^\infty(\xi, \zeta)$. There must be some point $m \notin \{\|x - x(c_q)\| \le \delta\} \cup \{\|x - x'(c_q)\| \le \delta\}$ such that

$$h_c^\infty(\xi, \zeta) = h_c^\infty(\xi, m) + h_c^\infty(m, \zeta).$$

Similarly, there exists $m' \notin \{\|x - x(c_q)\| \le \delta\} \cup \{\|x - x'(c_q)\| \le \delta\}$ such that

$$h_c^\infty(\zeta, \xi) = h_c^\infty(\zeta, m') + h_c^\infty(m', \xi).$$

Recall the definition of the barrier function B_c^*:

$$B_c^*(m) = \min\{h_c^\infty(\xi, m) + h_c^\infty(m, \zeta) - h_c^\infty(\xi, \zeta) : \xi, \zeta \in \mathcal{M}_0(c)\}$$

we see that the Mañé set contains some orbits connecting $\{\|x - x(c_q)\| \le \delta\}$ to $\{\|x - x'(c_q)\| \le \delta\}$ or vice versa. This proves the first part of the lemma.

To continue the proof, we define

$$\begin{aligned}
\mathcal{D}(c_q) = \Big\{&c \in H^1(\mathbb{T}^k \times \mathbb{T}^n, \mathbb{R}) : c_q = \text{constant}, \\
&\exists\, c\text{-minimal measure } \mu \text{ such that } \rho_x(\mu) = 0\Big\}.
\end{aligned}$$

Obviously, it is an n-dimensional convex disk and contains the set $\{c_q =$ constant, $\|c_x - y(c_q)\| \le C_q\}$. In fact, if μ is a c-minimal measure for some $c \in \mathrm{int}(\mathcal{D}(c_q))$ then it is also a $(c_q, y(c_q))$-minimal measure. To see it, let us note a fact:

Proposition 3.1. *Let $c', c^* \in H^1(M,\mathbb{R})$, μ' and μ^* be the corresponding minimal measures respectively. If $\langle c' - c^*, \rho(\mu')\rangle = \langle c' - c^*, \rho(\mu^*)\rangle = 0$, then $\alpha(c') = \alpha(c^*)$.*

Proof. : By the definition of the α-function we find that

$$\begin{aligned} -\alpha(c') &= \int (L - \eta_{c'}) d\mu' \\ &= \int (L - \eta_{c^*}) d\mu' + \langle c^* - c', \rho(\mu')\rangle \\ &\ge -\alpha(c^*). \end{aligned}$$

In the same way, we have $-\alpha(c^*) \ge -\alpha(c')$. q.e.d.

It follows from this proposition that $\alpha(c) =$ constant for all $c \in \mathcal{D}(c_q)$. For each $c \in \mathrm{int}(\mathcal{D}(c_q))$ if there was a c-minimal measure μ_1 such that $\rho_x(\mu_1) \neq 0$, then $\exists\, c' = (c_q, c'_x) \in \mathrm{int}(\mathcal{D}(c_q))$ such that $\langle c_x - c'_x, \rho_x(\mu_1)\rangle < 0$. Thus

$$\begin{aligned} -\alpha(c^*) &= A_{c^*}(\mu_1) \\ &= A_{c'}(\mu_1) + \langle c_x - c'_x, \rho(\mu_1)\rangle \\ &> -\alpha(c'). \end{aligned}$$

On the other hand, from the definition of $\mathcal{D}(c_q)$ and from the Proposition 3.1 we obtain that $\alpha(c') = \alpha(c^*)$. The contradiction implies that $\rho_x(\mu) = 0$ for each c-minimal measure when $c \in \mathrm{int}(\mathcal{D}(c_q))$. Consequently, for each c-minimal measure μ

$$\int (L - \eta_c) d\mu = \int (L - \eta_{c_q}) d\mu,$$

here, $\eta_c = (\eta_{c_q}, \eta_{c_x})$ is a closed 1-form with $[\eta_c] = (c_q, c_x) \in H^1(\mathbb{T}^{k+n}, \mathbb{R})$. Therefore, $\mathrm{supp}(\mu) \subset \{\|x - x(c_q)\| \le \delta\}$ if c_q is not close to $\mathbb{P}$, $\mathrm{supp}(\mu) \subset \{\|x - x(c_q)\| \le \delta\} \cup \{\|x - x'(c_q)\| \le \delta\}$ if c_q is close to $\mathbb{P}$. This proves the second part of the lemma.

Finally, let us consider the case that the c-minimal measure μ_c is always uniquely ergodic for each $c \in \mathrm{int}(\mathcal{D}(c_q))$. Obviously, there exists an invariant measure μ such that $\mu = \mu_c$ for all $c \in \mathrm{int}(\mathcal{D}(c_q))$. Note $\tilde{\mathcal{N}}(c) = \tilde{\mathcal{A}}(c)$ in this case. We claim that for each $d\gamma \in \tilde{\mathcal{N}}(c)$ and each $\xi \in \mathcal{M}_0(c)$, if $k_{ij} \to \infty$ $(i = 1, 2)$ as $j \to \infty$ are the two sequences such that $d\gamma(-k_{1j}), d\gamma(k_{2j}) \to \pi^{-1}(\xi)$, then

$$\lim_{j\to\infty} \int_{-k_{1j}}^{k_{2j}} \dot{\gamma}_{x_i}(t) dt = 0, \qquad \forall\, 1 \le i \le n. \tag{3.5}$$

In fact, for any $\xi \in \mathcal{M}_0(c)$ there exist two sequences $k_{ij} \to \infty$ as $j \to \infty$ $(i = 1, 2)$ such that $d\gamma(-k_{1j}) \to \pi^{-1}(\xi)$ and $d\gamma(k_{2j}) \to \pi^{-1}(\xi)$ as $j \to \infty$. Since γ is c-static, it follows that

$$h_c^{k_{1j}}(\gamma(-k_{1j}), \gamma(0)) + h_c^{k_{2j}}(\gamma(0), \gamma(k_{2j})) \to 0.$$

If (3.5) does not hold, by choosing a subsequence again (we use the same symbol) there would be some $1 \le i \le n$ such that

$$\left| \lim_{j\to\infty} \int_{-k_{1j}}^{k_{2j}} \dot{\gamma}_{x_i}(t)dt \right| \ge 2\pi > 0.$$

Under this assumption, let us consider the barrier function $B^*_{c'}$ where all other components of $c' \in \mathbb{R}^{k+n}$ are the same as those of c except for the component for x_i. Since $c - c' = (0, \cdots, 0, c_{x_i} - c'_{x_i}, 0, \cdots, 0)$, we obtain from the Proposition 3.1 that $\alpha(c') = \alpha(c)$, so

$$\begin{aligned}
B_{c'}(\gamma(0)) \ &\le \ \liminf_{j\to\infty} \int_{-k_{1j}}^{k_{2j}} \big(L(d\gamma(t), t) - \langle c', \dot{\gamma}(t)\rangle + \alpha(c')\big)\, dt \\
&\le \ \liminf_{j\to\infty} \int_{-k_{1j}}^{k_{2j}} (L(d\gamma(t), t) - \langle c, \dot{\gamma}(t)\rangle + \alpha(c))\, dt \\
&\quad + (c_{x_i} - c'_{x_i}) \lim_{j\to\infty} \int_{-k_{1j}}^{k_{2j}} \dot{\gamma}_{x_i}(t)dt \\
&\le \ -2|c_{x_i} - c'_{x_i}|\pi < 0
\end{aligned}$$

as we can choose $c_{x_i} > c'_{x_i}$ or $c_{x_i} < c'_{x_i}$ accordingly. But this is absurd since barrier function is non-negative.

Let us derive from (3.5) that each c-semi-static orbit $d\gamma$ is contained in $\{\|x - x(c_q)\| \le \delta\}$. In fact, we find that $d\gamma \in \tilde{\mathcal{N}}((c_q, y(c_q)))$. To see that, we find from (3.5) that the term $\langle c_x, \dot{\gamma}_x\rangle$ has no contribution to the action along the curve $\gamma|_{[-k_{1j}, k_{2j}]}$:

$$\int_{-k_{1j}}^{k_{2j}} (L - \langle c_q, \dot{\gamma}_q\rangle - \langle c_x, \dot{\gamma}_x\rangle)dt \to \int_{-k_{1j}}^{k_{2j}} (L - \langle c_q, \dot{\gamma}_q\rangle)dt, \tag{3.6}$$

as $j \to \infty$. If $d\gamma \notin \tilde{\mathcal{N}}((c_q, y(c_q)))$, there would exist $j' \in \mathbb{Z}^+$, $k' \in \mathbb{Z}$, $E > 0$ and a curve ζ: $[-k_{1j}, k_{2j} + k'] \to M$ such that $\zeta(-k_{1j'}) = \gamma(-k_{1j'})$, $\zeta(k_{2j'} + k') = \gamma(k_{2j'})$

$$\begin{aligned}
(3.7) \qquad & \int_{-k_{1j'}}^{k_{2j'}} (L(d\gamma(t), t) - \langle c_q, \dot{\gamma}_q\rangle - \langle y(c_q), \dot{\gamma}_x\rangle + \alpha(c))dt \\
\ge \ & \int_{-k_{1j'}}^{k_{2j'}+k'} (L(d\zeta(t), t) - \langle c_q, \dot{\zeta}_q\rangle - \langle y(c_q), \dot{\zeta}_x\rangle + \alpha(c))dt + E \\
\ge \ & F_{(c_q, y(c_q))}(\gamma(-k_{1j'}), \gamma(k_{2j'})) + E
\end{aligned}$$

and

$$\left|\int_{-k_{1j'}}^{k_{2j'}+k'} \dot{\zeta}_{x_i} dt\right| \to 0, \qquad \forall\ 1 \le i \le n. \tag{3.8}$$

(3.8) follows from the facts that $\tilde{\mathcal{N}}((c_q, y(c_q))) \subset \{\|x - x(c_q)\| \le \delta\}$ and $\gamma(-k_{ij}) \to \xi \in \mathcal{M}_0((c_q, y(c_q))) = \mathcal{M}_0(c)$. Let $j - j'$ be sufficiently large, we construct a curve ζ': $[-k_{1j}, k_{2j} + k'] \to M$ such that

$$\zeta'(t) = \begin{cases} \gamma(t), & t \in [-k_{1j}, -k_{1j'}]; \\ \zeta(t), & t \in [-k_{1j'}, k_{2j'} + k']; \\ \gamma(t - k'), & t \in [k_{2j'} + k', k_{2j} + k']. \end{cases}$$

It follows from (3.5–3.8) that

$$\begin{aligned} &\int_{-k_{1j}}^{k_{2j}+k'} (L(d\zeta'(t), t) - \langle c, \dot{\zeta}'\rangle)dt \\ &< \int_{-k_{1j}}^{k_{2j}} (L(d\gamma(t) - \langle c_q, \dot{\gamma}_q\rangle - \langle y(c_q), \dot{\gamma}_x\rangle)dt - E \\ &\le \int_{-k_{1j}}^{k_{2j}} (L(d\gamma(t), t) - \langle c, \dot{\gamma}\rangle)dt - \frac{E}{2}, \end{aligned}$$

but this contradicts to the property that $d\gamma \in \tilde{\mathcal{N}}(c)$. q.e.d.

Remark: The first part of the lemma can be proved by using the upper-semi continuity of Mañé set on Lagrangian functions. But the dependence on ϵ is not so clear as here (cf (3.4)) if we prove it in that way.

From the proof of the first part of the Lemma 3.1 we can see

Lemma 3.2. *Let $c \in \{\|c_x - y(c_q)\| < C_q\}$ and $b - a \ge 1$. For small number $d > 0$ there exits $\epsilon > 0$ and $\delta > 0$, such that if $\|P\| \le \epsilon$ and if γ: $[a, b] \to \mathbb{T}^{k+n}$ is a c-minimizer connecting points $\gamma(a), \gamma(b) \in \{\|x - x(c_q)\| \le \delta\}$, then $\|\gamma_x(t) - x(c_q)\| < d$ for all $t \in [a, b]$.*

The structure of Mañé set and pseudo connecting orbit set depends on what configuration manifold we choose for our consideration. In the following, when necessary, we use $\tilde{\mathcal{N}}(c, M)$, $\tilde{\mathcal{C}}_{\eta,\mu,\psi}(M)$ to specify the manifold on which these sets are defined. We shall omit M in that symbol when it is clearly defined. We do not intend to consider the most general case. Instead, let us consider some special case which is sufficient for the purpose of this paper.

According to the Lemma 3.1, the Mañé set $\tilde{\mathcal{N}}(c)$ is contained in $N_\delta := \{\|x - x(c_q)\| \le \delta\}$ for the cohomology class $c = (c_q, y(c_q))$ if c_q is not close to $\mathbb{P}$. To each curve γ: $(a, b) \to M$ such that $\gamma(a) \in N_\delta$ and $\gamma(b) \in N_\delta$ we can associate an element $[\gamma] = ([\gamma]_1, [\gamma]_2, \cdots, [\gamma]_n) \in H_1(M, N_\delta, \mathbb{Z})$. Here, a is a finite number or $-\infty$, b is a finite number or

∞. From the proof in [**Be1**] we can see that there exists a homoclinic orbit $d\gamma$ such that the first component of its relative homology is not zero: $[\gamma]_1 \neq 0$. The term "homoclinic" here means that both the α-limit set and the ω-limit set of the orbit are contained in the same Mañé set: $\alpha(d\gamma) \subseteq \tilde{\mathcal{N}}(c)$, $\omega(d\gamma) \subseteq \tilde{\mathcal{N}}(c)$. These homoclinic orbits are not in the Mañé set $\tilde{\mathcal{N}}(c)$ if $c \in \{\|c_x - y(c_q)\| < C_x\}$, but some of them are in the Mañé set $\tilde{\mathcal{N}}(c, \tilde{M})$ for a finite covering space $\tilde{M}$.

Let $\tilde{M} = \mathbb{T}^k \times (2\mathbb{T}) \times \mathbb{T}^{n-1}$ be the covering space of $M = \mathbb{T}^{k+n}$, let π_1 be the covering map $\tilde{M} \to M$:

$$\pi_1(q_1, \cdots, q_k, x_1, \cdots, x_n) = (q_1, \cdots, q_k, [x_1], x_2, \cdots, x_n)$$

where $[x_1] = x_1$ if $x_1 \leq 2\pi$, $[x_1] = x_1 - 2\pi$ if $2\pi \leq x_1 \leq 4\pi$.

Lemma 3.3. *Let $c = (c_q, y(c_q))$, $\tilde{M} = \mathbb{T}^k \times (2\mathbb{T}) \times \mathbb{T}^{n-1}$. If $d\gamma$: $\mathbb{R} \to TM$ is a homoclinic orbit to the set $\tilde{\mathcal{N}}(c) \subset N_\delta$ with the property*

$$\begin{aligned} &\liminf_{\substack{T_0\to\infty\\ T_1\to\infty}} \Big\{ \int_{-T_0}^{T_1} (L - \eta_c)(d\gamma(t), t)dt + (T_0 + T_1)\alpha(c) \Big\} \\ = &\liminf_{\substack{T_0\to\infty\\ T_1\to\infty}} \min_{\substack{\xi(-T_0)\in N_\delta\\ \xi(T_1)\in N_\delta\\ [\xi]_1\neq 0}} \Big\{ \int_{-T_0}^{T_1} (L - \eta_c)(d\xi(t), t)dt + (T_0 + T_1)\alpha(c) \Big\}, \end{aligned}$$

then $\{d\gamma(t), t\} \subset \pi_1 \tilde{\mathcal{N}}(c, \tilde{M})$.

Here we also use π_1: $T\tilde{M} \to TM$ to denote the standard projection, $\pi_1(u, \dot{u}) = (\pi_1 u, \dot{u})$.

Proof. : If we think $\tilde{M}$ as the configuration manifold, N_δ has two lifts denoted by N'_δ and N^*_δ. In this case, the minimal measure has at least two ergodic components, the support of one component is in N'_δ, another one is in N^*_δ. The lift of the homoclinic orbit founded in [**Be1**] is just an orbit joining the lift of the support of the minimal measure in N'_δ with another lift in N^*_δ. Recall the definition of the barrier function introduced by Mather in [**Ma2**]

$$B^*_c(m) = \min\{h^\infty_c(\xi, m) + h^\infty_c(m, \zeta) - h^\infty_c(\xi, \zeta) : \forall\ \xi, \zeta \in \mathcal{M}_0(c)\},$$

we obtain the result immediately. q.e.d.

Mañé announced in [**Me1**] that all c-static classes should be topologically transitive, it has been partially proved in [**CP**]. Mañé's announcement is true when there are finitely many Aubry classes, which is proved a generic property for all cohomology classes in [**BC**].

We can also define the Mañé set $\tilde{\mathcal{N}}(c,\tilde{M})$ from another point of view. Let $c=(c_q,y(c_q))$, $m_0\in N_\delta$, $m_1\in N_\delta$, we define

$$h^k_{c,e_1}(m_0,m_1)=\inf_{\substack{\gamma(0)=m_0\\ \gamma(k)=m_1\\ [\gamma]_1\neq 0}}\int_0^k (L-\eta_c)(d\gamma(t),t)dt+k\alpha(c),$$

$$h^{k_1,k_2}_{c,e_1}(m_0,\xi,m_1)=\inf_{\substack{\gamma(-k_1)=m_0\\ \gamma(0)=\xi\\ \gamma(k_2)=m_1\\ [\gamma]_1\neq 0}}\int_{-k_1}^{k_2} (L-\eta_c)(d\gamma(t),t)dt+(k_1+k_2)\alpha(c),$$

$$h^\infty_{c,e_1}(m_0,m_1)=\liminf_{k\to\infty} h^k_{c,e_1}(\xi,\zeta),$$

$$h^\infty_{c,e_1}(m_0,\xi,m_1)=\liminf_{\substack{k_1\to\infty\\ k_2\to\infty}} h^{k_1,k_2}_{c,e_1}(m_0,\xi,m_1),$$

$$B^*_{c,e_1}(\xi)=\inf\{h^\infty_{c,e_1}(m_0,\xi,m_1)-h^\infty_{c,e_1}(m_0,m_1):\ m_0,m_1\in\mathcal{M}_0(c)\}.$$

Recall we have introduced a modified Lagrangian $L_{\eta,\mu,\psi}=L-\eta-\mu-\psi$. Let $T_0\in\mathbb{Z}_+$, $T_1\in\mathbb{Z}_+$, we define

$$h^{T_0,T_1}_{\eta,\mu,\psi,e_1}(m_0,m_1)=\inf_{\substack{\xi(-T_0)=m_0\\ \xi(T_1)=m_1\\ [\xi]_1\neq 0}}\int_{-T_0}^{T_1} L_{\eta,\mu,\psi}(d\gamma(t),t)+T_0\alpha(c)+T_1\alpha(c'),$$

$$h^{T_0,T_1}_{\eta,\mu,\psi,e_1}(m_0,\xi,m_1)=\inf_{\substack{\xi(-T_0)=m_0\\ \xi(T_1)=m_1\\ \xi(0)=\xi\\ [\xi]_1\neq 0}}\int_{-T_0}^{T_1} L_{\eta,\mu,\psi}(d\gamma(t),t)+T_0\alpha(c)+T_1\alpha(c'),$$

$$h^\infty_{\eta,\mu,\psi,e_1}(m_0,m_1)=\liminf_{\substack{T_1\to\infty\\ T_2\to\infty}} h^{T_0,T_1}_{\eta,\mu,\psi,e_1}(m_0,m_1),$$

$$h^\infty_{\eta,\mu,\psi,e_1}(m_0,\xi,m_1)=\liminf_{\substack{T_1\to\infty\\ T_2\to\infty}} h^{T_0,T_1}_{\eta,\mu,\psi,e_1}(m_0,\xi,m_1).$$

Clearly, we have

Lemma 3.4. *Assume L has the form of* (3.1), *$c=(c_q,y(c_q))$, then*

$$\pi_1\mathcal{N}_0(c,\tilde{M})=\{B^*_{c,e_1}=0\}\cup\{B^*_c=0\},$$

$$\pi_1\mathcal{N}_0(c,\tilde{M})\backslash\mathcal{N}_0(c,M)\neq\varnothing.$$

For $\tilde{\mathcal{C}}_{\eta,\mu,\psi}(\tilde{M})$, we have the similar result:

Lemma 3.5. *Let $c=(c_q,y(c_q))$, $c'=(c'_q,y(c'_q))$, $[\eta]=c$ and μ is a U-step 1-form with $[\bar{\mu}]=c'-c$. If $\mathcal{N}(c)\subset N_\delta$, $\|\psi\|_{C^0}$ is suitably small and* $\operatorname{supp}(\psi)\cap N_\delta=\varnothing$, *then*

$$\pi_1\mathcal{C}_{\eta,\mu,\psi}(\tilde{M})\backslash\mathcal{C}_{\eta,\mu,\psi}(M)\neq\varnothing.$$

Proof. : For m_0, $m_1 \in N_\delta$, positive integers T_0^i, $T_1^i \in \mathbb{Z}_+$, let $\gamma_i(t, m_0, m_1, e_1)$: $[-T_0^i, T_1^i] \to M$ be a minimal curve joining m_0 and m_1 such that $[\gamma_i]_1 \neq 0$ and

$$h_{\eta,\mu,\psi,e_1}^{T_0^i,T_1^i}(m_0, m_1) = \int_{-T_0^i}^{T_1^i} L_{\eta,\mu,\psi}(d\gamma_i(t), t)dt + T_0^i\alpha(c) + T_1^i\alpha(c').$$

Let $\{T_0^i\}_{i\in\mathbb{Z}_+}$ and $\{T_1^i\}_{i\in\mathbb{Z}_+}$ be the sequence of positive integers such that $T_j^i \to \infty$ $(j = 0, 1)$ as $i \to \infty$ and the following limit exists

$$\lim_{i\to\infty} h_{\eta,\mu,\psi,e_1}^{T_0^i,T_1^i}(m_0, m_1) = \liminf_{T_0,T_1\to\infty} h_{\eta,\mu,\psi,e_1}^{T_0,T_1}(m_0, m_1) = h_{\eta,\mu,\psi,e_1}^{\infty}(m_0, m_1).$$

Let $\tilde{\gamma}_i$ be the lift of γ_i in the covering space $\tilde{M}$, it is a $\tilde{M}$-minimal curve. Clearly, the set of accumulation points of the set $\{\gamma_i\}$ contains a curve γ: $\mathbb{R} \to M$ with $[\gamma]_1 \neq 0$.

On the other hand, if $\|\psi\|_{C^0}$ is suitably small and m_0, $m_1 \in N_\delta$, the hyperbolic structure of ℓ_2 guarantees that

$$h_{\eta,\mu,\psi}^{\infty}(m_0, m_1) < h_{\eta,\mu,\psi,e_1}^{\infty}(m_0, m_1).$$

In other words, these $\tilde{M}$-minimal curves $\{\gamma_i\}$ are not M-minimal curve. Consequently, γ is not a M-minimal curve. This completes the proof.
q.e.d.

Lemma 3.6. *Assume that* $\mathcal{A}_0(c) \subset (\{\|x-x_0\| < \delta\} \cup \{\|x-x_1\| < \delta\})$, $\mathcal{M}_0(c) \cap \{\|x-x_0\| < \delta\} \neq \varnothing$ *and* $\mathcal{M}_0(c) \cap \{\|x-x_1\| < \delta\} \neq \varnothing$ $(x_0 \neq x_1)$. *Then, for suitably small* δ *we have*

$$\mathcal{N}_0(c) \backslash (\{\|x - x_0\| < \delta\} \cup \{\|x - x_1\| < \delta\}) \neq \varnothing.$$

Proof. : Let $m_0 \in \mathcal{M}_0 \cap \{\|x - x_0\| < \delta\}$, $m_1 \in \mathcal{M}_0 \cap \{\|x - x_1\| < \delta\}$ and consider the function $h^\infty(m_0, m_1)$. q.e.d.

4. Existence of local connecting orbits

To begin with, let us consider the construction of diffusion orbits in Arnold's example from the variational point of view. There, each Mañé set under consideration properly contains the corresponding Mather set if we study the problem in a covering manifold $\tilde{M} = 2\mathbb{T} \times \mathbb{T}^n$. Under the small perturbation the stable manifold of each invariant circle transversally intersects the unstable manifold of the same invariant circle. It implies that the set $\pi_1\mathcal{N}_0(c, \tilde{M}) \backslash (\mathcal{M}_0(c, M) + \delta)$ is non-empty but its first homology is trivial for each c under consideration, here we use $A + \delta$ to denote the δ-neighborhood of A. The main goal of this section is to show that if the Mañé set $\mathcal{N}_0(c)$ has some kind of topological triviality, then for all c' sufficiently close to c, $\tilde{\mathcal{N}}_{c'}$ can be connected with $\tilde{\mathcal{N}}_c$ by some local minimal orbits.

Definition 4.1. Let $c = (c_q, y(c_q))$, $c' = (c'_q, y(c'_q))$, $N_\delta = \{\|x - x(c_q)\| \le \delta\}$. We assume that $\tilde{\mathcal{N}}(c) \subset \mathrm{int}(N_\delta)$ and $\tilde{\mathcal{N}}(c') \subset \mathrm{int}(N_\delta)$. Let γ: $\mathbb{R} \to M$ be an absolutely continuous curve such that $\gamma(t) \in N_\delta$ when $|t| \ge T$, and such that $[\gamma]_1 \ne 0$ where $[\gamma] = ([\gamma]_1, \cdots, [\gamma]_n) \in H_1(M, N_\delta, \mathbb{Z})$. We say $d\gamma$ is a local minimal orbit of L of the first type that connects $\tilde{\mathcal{N}}(c)$ to $\tilde{\mathcal{N}}(c')$ if

1, $d\gamma(t)$ is the solution of the Euler-Lagrange equation (1.3), the α- and ω-limit sets of $d\gamma$ are in $\tilde{\mathcal{N}}(c)$ and $\tilde{\mathcal{N}}(c')$ respectively;

2, There exist a closed 1-form η with $[\eta] = c$, a U-step 1-form μ with $[\bar{\mu}] = c' - c$ and a bump function ψ such that $d\gamma(t) \in \tilde{\mathcal{C}}_{\eta,\mu,\psi}(t)$ is a local minimal curve of the Lagrangian $L_{\eta,\mu,\psi}$ in the following sense: there exist two open balls V_0, V_1 and two positive integers T_0, T_1 such that $\bar{V}_0 \subset N_\delta \backslash \mathcal{M}_0(c)$, $\bar{V}_1 \subset N_\delta \backslash \mathcal{M}_0(c')$, $\gamma(-T_0) \in V_0$, $\gamma(T_1) \in V_1$ and

$$\begin{aligned}(4.1)\qquad &\min\Big\{h^{T_0,T_1}_{\eta,\mu,\psi,e_1}(m_0, m_1) + h^\infty_c(\xi, m_0) + h^\infty_{c'}(m_1, \zeta) :\\ &\qquad \xi \in \mathcal{M}_0(c) \cap \pi(\alpha(d\gamma)|_{t=0}), \zeta \in \mathcal{M}_0(c') \cap \pi(\omega(d\gamma)|_{t=0})\Big\}\\ &- \liminf_{\substack{T'_0 \to \infty \\ T'_1 \to \infty}} \int_{-T'_0}^{T'_1} L_{\eta,\mu,\psi}(d\gamma(t), t)dt - T'_0\alpha(c) - T'_1\alpha(c') \;> 0\end{aligned}$$

holds for any $(m_0, m_1) \in \partial(V_0 \times V_1)$.

Since $\pi(\omega(d\gamma)) \subset \mathcal{N}(c') \subset N_\delta$ and $\pi(\alpha(d\gamma)) \subset \mathcal{N}(c) \subset N_\delta$, $[\gamma|_{T_1 \le t < \infty}]$ and $[\gamma|_{-\infty < t \le -T_0}]$ are well defined. Indeed, recall the Lemma 3.2, we can see $[\gamma|_{T_1 \le t < \infty}] = 0$ and $[\gamma|_{-\infty < t \le -T_0}] = 0$. That is why we can use $h^\infty_c(\xi, m_0)$ and $h^\infty_{c'}(m_1, \zeta)$ in this definition. We do not intend to discuss local minimal curves in the most general case, the above definition is introduced for the special purpose of this paper.

Obviously, (4.1) is equivalent to that

$$\begin{aligned}(4.2)\qquad &h^{T_0,T_1}_{\eta,\mu,\psi,e_1}(m_0, m_1) + h^\infty_c(\xi, m_0) + h^\infty_{c'}(m_1, \zeta)\\ &- \int_{-T_0}^{T_1} L_{\eta,\mu,\psi}(d\gamma(t), t)dt - T'_0\alpha(c) - T'_1\alpha(c')\\ &- h^\infty_c(\xi, \gamma(-T_0)) - h^\infty_{c'}(\gamma(T_1), \zeta) \;> 0\end{aligned}$$

for each $\xi \in \mathcal{M}_0(c) \cap \pi(\alpha(d\gamma)|_{t=0})$ and each $\zeta \in \mathcal{M}_0(c') \cap \pi(\omega(d\gamma)|_{t=0})$.

Lemma 4.1. *If $\mathcal{N}_0(c, M) \subset N_\delta$ and $\pi_1\mathcal{A}_0(c, \tilde{M}) \backslash N_\delta$ is totally disconnected, then there exist $\epsilon_1 > 0$, a U-step form μ, a bump function ψ, a small number t_0 and an open disk O such that if $[\mu] = c'$, $\|c' - c\| < \epsilon_1$, then*

$$(4.3)\qquad \varnothing \ne \Big\{\pi_1\mathcal{C}_{\eta,\mu,\psi}(\tilde{M}) \backslash \mathcal{C}_{\eta,\mu,\psi}(M)\Big\}_{0 \le t \le t_0} \subset O$$

and each $d\gamma(t) \in \tilde{\mathcal{C}}_{\eta,\mu,\psi}(\tilde{M}) \backslash \tilde{\mathcal{C}}_{\eta,\mu,\psi}(M)|_t$ determines a minimal orbit of L of first type which connecting $\tilde{\mathcal{N}}_c$ with $\tilde{\mathcal{N}}_{c'}$.

Proof. : Since $\pi_1\mathcal{N}_0(c,\tilde{M})\backslash N_\delta$ is totally disconnected, there exist an open, connected set O which can shrink to one point by continuous deformation, and a small positive number $t_0>0$ such that

$$O\cap\pi_1\mathcal{N}(c,\tilde{M})|_{0\le t\le t_0}\backslash N_\delta\neq\varnothing,$$

$$O\cap N_\delta=\varnothing,\qquad \partial O\cap\pi_1\mathcal{N}(c,\tilde{M})|_{0\le t\le t_0}=\varnothing.$$

Clearly, we can find a small $\delta_1>0$ and define a non-negative function $f\in C^r(M,\mathbb{R})$ such that

$$f(q,x)\begin{cases}=0 & (q,x)\in N_\delta\cup\Big(\pi_1\mathcal{N}(c,\tilde{M})|_{0\le t\le t_0}\backslash(O+\delta_1)\Big),\\ =1 & (q,x)\in O,\\ <1 & \text{elsewhere.}\end{cases}$$

We choose a C^r-function $\rho:\mathbb{R}\to[0,1]$ such that $\rho=0$ on $t\in(-\infty,0]\cup[t_0,\infty)$, $0<\rho\le 1$ on $t\in(0,t_0)$. Let $\lambda\ge 0$ be a positive number,

$$\psi(q,x,t)=\lambda\rho(t)f(q,x),$$

By the upper semi-continuity of the set function $(\eta,\mu,\psi)\to\mathcal{C}_{\eta,\mu,\psi}(\tilde{M})$ we see that $\mathcal{C}_{\eta,0,\psi}(\tilde{M})|_{0\le t\le t_0}\cap\partial O=\varnothing$ if $\lambda>0$ is suitably small. By the choice of ψ, we have $\tilde{\mathcal{C}}_{\eta,0,\psi}(M)=\tilde{\mathcal{N}}(c,M)$. Consequently, by using the similar argument to prove the Lemma 3.5 we find

$$\varnothing\neq\Big\{\pi_1\mathcal{C}_{\eta,0,\psi}(\tilde{M})\backslash\mathcal{C}_{\eta,0,0}(M)\Big\}_{0\le t\le t_0}\subset O.$$

Since O is homotopically trivial, for any cohomology class c', there exists a closed 1-form $\bar{\mu}$ such that $[\bar{\mu}]=c'-c$ and $\mathrm{supp}(\bar{\mu})\cap O=\varnothing$. Let $\rho_1\in C^r(\mathbb{R},[0,1])$ such that $\rho_1=0$ on $(-\infty,0]$, $0<\rho_1<1$ on $(0,t_0)$ and $\rho_1=1$ on $[t_0,\infty)$, let $\mu=\rho_1(t)\bar{\mu}$ and set

$$L_{\eta,\mu,\psi}=L-\eta-\mu-\psi.$$

By using the upper semi-continuity and the similar argument to prove the Lemma 3.5 again we obtain (4.3) if $\|\mu\|$ is suitably small. Let $d\gamma\in\pi_1\tilde{\mathcal{C}}_{\eta,\mu,\psi}(\tilde{M})\backslash\tilde{\mathcal{C}}_{\eta,\mu,\psi}(M)$. Note that $f\equiv 1$ in O, $\mathrm{supp}(\bar{\mu})\cap O=\varnothing$, $d\gamma$: $TM\to\mathbb{R}$ is obviously a solution of the Euler-Lagrange equation, $\alpha(d\gamma)\subset\tilde{\mathcal{N}}(c)$ and $\omega(d\gamma)\subset\tilde{\mathcal{N}}(c')$.

Since we have assumed that $\pi_1\mathcal{C}_0(c,\tilde{M})\backslash N_\delta$ is totally disconnected in O, by the upper semi-continuity, there obviously are two open and connected sets V_0 and V_1 such that $\bar{V}_0\subset N_\delta\backslash\mathcal{M}_0(c)$, $\bar{V}_1\subset N_\delta\backslash\mathcal{M}_0(c')$ and (4.1) holds. q.e.d.

Let us compare $\pi_1\mathcal{C}_{\eta,0,\psi}(\tilde{M})\backslash\mathcal{C}_{\eta,0,\psi}(M)$ with $\pi_1\mathcal{N}(c,\tilde{M})\backslash\mathcal{N}(c,M)$. If $\gamma(t)$ is a minimal curve in $\pi_1\mathcal{N}(c,\tilde{M})\backslash\mathcal{N}(c,M)$, then its time k translation $\gamma(t+k)$ is also a minimal curve for each $k\in\mathbb{Z}$. By the choice of the open set O and the function ψ, we see that each orbit $d\gamma$ in $\pi_1\tilde{\mathcal{N}}(c,\tilde{M})\backslash\tilde{\mathcal{N}}(c,M)$ might be an orbit of the Euler-Lagrange equation determined by $L-\psi$ still, but only those curves remain to be minimal if they pass through O when $t\in[0,t_0]$.

We now consider some c-minimal measure which has more than one ergodic components.

Lemma 4.2. *Let $\Gamma:[-\varepsilon,\varepsilon]\to H^1(M,\mathbb{R})$ be a continuous curve. We assume that for each $-\varepsilon<s<0$, $\mathcal{N}_0(\Gamma(s))\subset\{\|x-x_0\|<\delta\}$, for each $0<s<\varepsilon$, $\mathcal{N}_0(\Gamma(s))\subset\{\|x-x_1\|<\delta\}$ while $\mathcal{A}_0(\Gamma(0))\subset(\{\|x-x_0\|<\delta\}\cup\{\|x-x_1\|<\delta\})$, $\mathcal{M}_0(\Gamma(0))\cap\{\|x-x_0\|<\delta\}\neq\varnothing$ and $\mathcal{M}_0(\Gamma(0))\cap\{\|x-x_1\|<\delta\}\neq\varnothing$. We also assume $\mathcal{N}_0(\Gamma(0))\backslash(\{\|x-x_0\|<\delta\}\cup\{\|x-x_1\|<\delta\})$ is totally disconnected. Then, there exists $0<\varepsilon'\leq\varepsilon$, for each $s_0\in(-\varepsilon',0)$ and each $s_1\in(0,\varepsilon')$, there are two closed 1-forms ν_0, ν_1 with $[\nu_0]=\Gamma(0)-\Gamma(s_0)$, $[\nu_1]=\Gamma(s_1)-\Gamma(0)$, a U-step 1-form μ with $[\bar\mu]=\Gamma(s_1)-\Gamma(s_0)$ and a bump function ψ such that each orbits $d\gamma(t)\in\tilde{\mathcal{C}}_{\eta-\nu_0,\mu,\psi}(t)$ is an orbit of the Lagrange flow determined by L and*

$$\varnothing\neq\mathcal{C}_{\eta-\nu_0,\mu,\psi}(M)|_{0\leq t\leq t_0}\subset O. \tag{4.4}$$

Proof. : According to the Lemma 3.6, $\mathcal{N}_0(\Gamma(0))\backslash(\{\|x-x_0\|<\delta\}\cup\{\|x-x_1\|<\delta\}$ is non-empty. In this case we do not need to lift M to its finite covering. Since $\mathcal{N}_0(\Gamma(0))\backslash(\{\|x-x_0\|<\delta\}\cup\{\|x-x_1\|<\delta\}$ is totally disconnected, there is a shrinkable open set O and a small positive number $t_0>0$ such that

$$O\cap\mathcal{N}(\Gamma(0))|_{0\leq t\leq t_0}\backslash(\{\|x-x_0\|<\delta\}\cup\{\|x-x_1\|<\delta\}\neq\varnothing,$$

$$\partial O\cap\mathcal{N}(\Gamma(0))|_{0\leq t\leq t_0}=\varnothing,$$

$$O\cap(\{\|x-x_0\|<\delta\}\cup\{\|x-x_1\|<\delta\})=\varnothing.$$

Remaining argument for the proof is similar to the proof of the Lemma 4.1. q.e.d.

The orbits in $\tilde{\mathcal{C}}_{\eta-\nu_0,\mu,\psi}$ has some local minimal property as the orbits of the first type have. For each $d\gamma\in\tilde{\mathcal{C}}_{\eta-\nu_0,\mu,\psi}$, there exist two open balls V_0 and V_1 such that $\bar V_0\subset\{\|x-x_0\|<\delta\}$, $\bar V_1\subset\{\|x-x_1\|<\delta\}$, $\gamma(-T_0)\in V_0$, $\gamma(T_1)\in V_1$ and

$$\begin{aligned} &h^{T_0,T_1}_{\eta-\nu_0,\mu,\psi}(m_0,m_1)+h^{\infty}_{\Gamma(s_0)}(\xi,m_0)+h^{\infty}_{\Gamma(s_1)}(m_1,\zeta)\\ &\quad-\int_{-T_0}^{T_1}L_{\eta-\nu_0,\mu,\psi}(d\gamma(t),t)dt-T_0'\alpha(c)-T_1'\alpha(c')\ >\ 0\end{aligned}\tag{4.5}$$

holds for each $(m_0,m_1)\in\partial(V_0\times V_1)$, each $\xi\in\mathcal{M}_0(c)\cap\pi(\alpha(d\gamma)|_{t=0})$ and each $\zeta\in\mathcal{M}_0(c')\cap\pi(\omega(d\gamma)|_{t=0})$. In this case, we call the element of $\tilde{\mathcal{C}}_{\eta-\nu_0,\mu,\psi}$ local minimal orbits of the second type.

We consider another type of local minimal orbits which connects $\tilde{\mathcal{N}}(c)$ with $\tilde{\mathcal{N}}(c')$.

Lemma 4.3. *If there is an open neighborhood V of $\mathcal{N}_0(c)$ such that $H_1(V,\mathbb{R})=0$, then there exists small $\varepsilon>0$, for each c' with $\|c'-c\|\le\varepsilon$ there exist a closed 1-form η and a U-step 1-form μ such that $[\eta]=c$, $\bar\mu=c'-c$ and each orbit in $\tilde{\mathcal{C}}_{\eta,\mu}$ is an orbit of the Lagrange flow of L.*

We call $d\gamma$ in such $\tilde{\mathcal{C}}_{\eta,\mu}$ local minimal orbit of the third type. Another version of the Lemma 4.3 was formulated by Mather in [**Ma2**].

Proof. : Since V is topologically trivial, for any $c'\in H^1(M,\mathbb{R})$ there exists a closed 1-form $\bar\mu$ such that $\mathrm{supp}\bar\mu\cap V=\varnothing$. We take the U-step 1-form in the way such that $\mu=0$ when $t\le 0$ and $\mu=\bar\mu$ when $t\ge t_0$ where $t_0>0$ is suitably small. By the upper-semi continuity of the map $(\eta,\mu)\to\tilde{\mathcal{C}}_{\eta,\mu}$, we find that $d\gamma(t)$ $(0\le t\le t_0)$ is in V if $d\gamma\in\tilde{\mathcal{C}}_{\eta,\mu}$ and if $\|c'-c\|$ is sufficiently small. Therefore, $d\gamma$ is a solution of the Euler-Lagrangian equation determined by L. q.e.d.

5. Construction of global connecting orbits

The goal of this section is to construct some orbits which connect $\tilde{\mathcal{N}}(c)$ with $\tilde{\mathcal{N}}(c')$ if c and c' are connected by a generalized transition chain in $H^1(\mathbb{T}^k\times\mathbb{T}^n,\mathbb{R})$.

Definition 5.1. Let $\tilde M$ be a finite covering of a compact manifold M and let c, c' be two cohomolgy classes in $H^1(M,\mathbb{R})$. We say that c is joined with c' by a generalized transition chain if there is a continuous curve Γ: $[0,1]\to H^1(M,\mathbb{R})$ such that $\Gamma(0)=c$, $\Gamma(1)=c'$ and for each $\tau\in[0,1]$ at least one of the following cases takes place:

(I), there is small $\delta_\tau>0$ such that $\pi_1\mathcal{N}_0(\Gamma(\tau),\tilde M)\backslash(\mathcal{A}_0(\Gamma(\tau),M)+\delta_\tau)$ is non-empty and totally disconnected;

(II), $\mathcal{N}_0(\Gamma(\tau),M)$ is homologically trivial, i.e. it has a neighborhood U_τ such that the inclusion map $H_1(U_\tau,\mathbb{R})\to H_1(M,\mathbb{R})$ is the zero map.

In this paper, we do not intend to establish a theorem of the existence of connecting orbits between two cohomology classes in the most general case when they are joined by a generalized transition chain. Instead, we restrict ourselves to a special case:

Theorem 5.1. *Let $M=\mathbb{T}^k\times\mathbb{T}^n$, $\tilde M=\mathbb{T}^k\times 2\mathbb{T}\times\mathbb{T}^{n-1}$, the Lagrangian L be given by* (3.1)*. Let $x:[0,1]\to\mathbb{T}^n$ be a piecewise continuous curve, not continuous only at l points τ_j $(0<\tau_1<\cdots<\tau_l<1)$ where its left and right limit exist $x_j^-:=\lim_{\tau\to\tau_j^-}x(\tau)\ne\lim_{\tau\to\tau_j^+}x(\tau):=x_j^+$. We assume:*

i*, the two first cohomology classes $c=(c_q,y(c_q))$ and $c'=(c'_q,y(c'_q))$ are joined by a generalized transition chain Γ: $[0,1]\to H^1(M,\mathbb{R})\cap\{c_x=y(c_q)\}$;*

ii, *for each* $\tau \in (\tau_{j-1}, \tau_j)$, $\mathcal{N}_0(\Gamma(\tau)) \subset \{\|x - x_j(\tau)\| < \delta\}$, $\Gamma(\tau)$-*minimal measure is uniquely ergodic when* $\mathcal{N}_0(\Gamma(\tau), M)$ *is not homologically trivial;*

iii, *when* $\tau = \tau_j$ *for* $1 \le j \le l$, $\mathcal{A}_0(\Gamma(\tau_j)) \subset (\{\|x - x_j^-\| < \delta\} \cup \{\|x - x_j^+\| < \delta\})$, $\mathcal{M}_0(\Gamma(\tau_j)) \cap \{\|x - x_j^-\| < \delta\} \neq \varnothing$ *and* $\mathcal{M}_0(\Gamma(\tau_j)) \cap \{\|x - x_j^+\| < \delta\} \neq \varnothing$.

Then there exists an orbit of the Euler-Lagrange equation (1.3) $d\gamma$: $\mathbb{R} \to TM$ *that has the property:* $\alpha(d\gamma) \subset \tilde{\mathcal{N}}(c)$ *and* $\omega(d\gamma) \subset \tilde{\mathcal{N}}(c')$.

Proof. : According to the Lemma 4.2, there exist two numbers $\tau_j^- < \tau_j < \tau_j^+$ for each $0 < j < l$ such that $|\tau_j^\pm - \tau_j|$ is small and $\tilde{\mathcal{N}}(\Gamma(\tau_j^-))$ is connected to $\tilde{\mathcal{N}}(\Gamma(\tau_j^+))$ by some local minimal orbits of the second type.

Since the map $c \to \mathcal{N}(c, M)$ is upper semi-continuous, there are finitely many open intervals $\{J_i\}_{0 \le i \le m}$ such that

1, $\cup J_i \supset [0,1]$, $J_i \cap J_{i+1} \neq \varnothing$ and $J_i \cap J_{i\pm 2} = \varnothing$;

2, each J_i is defined in this way: if for all $\tau \in J_i$ the case (I) happens, then for all $\tau \in J_{i-1} \cup J_{i+1}$ the case (II) happens.

By the assumptions, there exists a finite sequence $\{s_i\}_{0 \le i \le i_m}$ such that $s_i \in J_j$ for each integer $i \in [i_{j-1}, i_j]$ $(0 < j < m)$, $\tilde{\mathcal{N}}(\Gamma(s_i))$ is connected to $\tilde{\mathcal{N}}(\Gamma(s_{i+1})$ by some local minimal orbit $d\gamma_i$ of L. $d\gamma_i$ is of the first or second type when $i \in [0, i_1) \cup [i_2, i_3) \cup \cdots \cup [i_{m-1}, i_m)$ and of the third type when $i \in [i_1, i_2) \cup \cdots \cup [i_{m-2}, i_{m-1})$. We choose these $\{s_i\}$ such that if $\mathcal{N}_0(\Gamma(\tau_j^\pm))$ is homologically trivial, then $\exists$ some $s_i = \tau_j^-$, $s_{i+1} = \tau_j^+$. Let $c_i = \Gamma(s_i)$, we define $I = \{i\}$ be the index set that $\tilde{\mathcal{N}}(c_i)$ is connected to $\tilde{\mathcal{N}}(c_{i+1})$ by some local minimal orbits of the second type if $i \in I$.

More precisely, for each integer $i \in [0, i_1) \cup [i_2, i_3) \cup \cdots \cup [i_{m-1}, i_m)$:

1, There exists a local minimal orbit of the first type or of the second type $d\gamma_i$: $\mathbb{R} \to TM$ such that it solves the Euler-Lagrange equation determined by L, $\alpha(d\gamma) \subset \tilde{\mathcal{N}}(c_i)$ and $\omega(d\gamma) \subset \tilde{\mathcal{N}}(c_{i+1})$;

2, Given a small number λ_i there is a non-negative function $\psi_i(q, x, t)$ such that $\psi \le \lambda_i$, $\psi_i = 0$ when $t \in (-\infty, 0] \cup [1, \infty)$. For each fixed t, the support of ψ_i is contained in a small neighborhood of the open disk O and $\psi_i =$ constant when it is restricted in O_i. If $i \notin I$

$$O_i \cap (\mathcal{N}(c_i, \tilde{M})|_{0 \le t \le t_0} \backslash N_{i\delta}) \neq \varnothing,$$

$$\partial O_i \cap \mathcal{N}(c_i, \tilde{M})|_{0 \le t \le t_0} = \varnothing,$$

$$O_i \cap N_{i\delta} = \varnothing$$

where $N_{i\delta} = \{\|x - x(s_i)\| < \delta\}$; If $i \in I$, O_i satisfies the following

$$O_i \cap (\mathcal{N}(c_i, M)|_{0 \le t \le t_0} \backslash (N_{i\delta} \cup N_{(i+1)\delta}) \neq \varnothing,$$

$$\partial O_i \cap \mathcal{N}(c_i, M)|_{0 \le t \le t_0} = \varnothing,$$

$$O_i \cap (N_{i\delta} \cup N_{(i+1)\delta}) = \varnothing.$$

3, There exist a closed 1-forms η_i with $[\eta_i] = c_i$ and a U step 1-form μ_i such that the restriction on $\{t \geq t_0\}$ is a closed 1-form $\bar{\mu}_i$ on M with $[\bar{\mu}_i] = c_{i+1} - c_i$. The support of μ_i is disjoint with O_i. For $i \notin I$, according to the Lemma 4.1, we can see that the set $\tilde{\mathcal{C}}_{\eta_i,\mu_i,\psi_i}(\tilde{M})$ has the property:

$$\varnothing \neq \pi_1 \mathcal{C}_{\eta_i,\mu_i,\psi_i}(\tilde{M}) \backslash \mathcal{C}_{\eta_i,\mu_i,\psi_i}(M)|_{0 \leq t \leq t_0} \subset O_i, \tag{5.1}$$

each orbit $d\gamma(t) \in \tilde{\mathcal{C}}_{\eta_i,\mu_i,\psi_i}(\tilde{M}) \backslash \tilde{\mathcal{C}}_{\eta_i,\mu_i,\psi_i}(M)|_t$ determines a local minimal orbit of L of the first type, which connects $\tilde{\mathcal{N}}(c_i)$ to $\tilde{\mathcal{N}}(c_{i+1})$. Consequently, there exist two open $(k+n)$-dimensional disks V_i^+ and V_{i+1}^- with $\bar{V}_i^+ \subset N_{i\delta} \backslash \mathcal{M}_0(c_i)$, $\bar{V}_{i+1}^- \subset N_{(i+1)\delta} \backslash \mathcal{M}_0(c_{i+1})$, two positive integers $\tilde{T}_i^0$, $\tilde{T}_i^1$ and a positive small number $\epsilon_i^* > 0$ such that

$$\begin{aligned} \min \Big\{ & h_{c_i}^{\infty}(\xi, m_0) + h_{\eta_i,\mu_i,\psi_i,e_1}^{\tilde{T}_i^0,\tilde{T}_i^1}(m_0, m_1) + h_{c_{i+1}}^{\infty}(m_1, \zeta) : \\ & (m_0, m_1) \in \partial(V_i^+ \times V_{i+1}^-) \Big\} \\ \geq \min \Big\{ & h_{c_i}^{\infty}(\xi, m_0) + h_{\eta_i,\mu_i,\psi_i,e_1}^{\tilde{T}_i^0,\tilde{T}_i^1}(m_0, m_1) + h_{c_{i+1}}^{\infty}(m_1, \zeta) : \\ & (m_0, m_1) \in V_i^+ \times V_{i+1}^- \Big\} + 5\epsilon_i^* \end{aligned} \tag{5.2}$$

where $\xi \in \mathcal{M}_0(c_i), \zeta \in \mathcal{M}_0(c_{i+1})$. For $i \in I$, according to the Lemma 4.2 we find that

$$\varnothing \neq \mathcal{C}_{\eta_i - \nu_i - \psi_i}(M)|_{0 \leq t \leq t_0} \subset O_i \tag{5.3}$$

and each orbit $d\gamma(t) \in \tilde{\mathcal{C}}_{\eta_i - \nu_i - \psi_i}(M)|_t$ determines a local minimal orbit of L of the second type which connects $\tilde{\mathcal{O}}(c_i)$ to $\tilde{\mathcal{N}}(c_{i+1})$. Consequently, there exist two open $(k+n)$-dimensional disks V_i^+ and V_{i+1}^- with $\bar{V}_i^+ \subset N_{i\delta} \backslash \mathcal{M}_0(c_i)$, $\bar{V}_{i+1}^- \subset N_{(i+1)\delta} \backslash \mathcal{M}_0(c_{i+1})$, two positive integers $\tilde{T}_i^0$, $\tilde{T}_i^1$ and a positive small number $\epsilon_i^* > 0$ such that

$$\begin{aligned} \min \Big\{ & h_{c_i}^{\infty}(\xi, m_0) + h_{\eta_i,\mu_i,\psi_i}^{\tilde{T}_i^0,\tilde{T}_i^1}(m_0, m_1) + h_{c_{i+1}}^{\infty}(m_1, \zeta) : \\ & (m_0, m_1) \in \partial(V_i^+ \times V_{i+1}^-) \Big\} \\ \geq \min \Big\{ & h_{c_i}^{\infty}(\xi, m_0) + h_{\eta_i,\mu_i,\psi_i}^{\tilde{T}_i^0,\tilde{T}_i^1}(m_0, m_1) + h_{c_{i+1}}^{\infty}(m_1, \zeta) : \\ & (m_0, m_1) \in V_i^+ \times V_{i+1}^- \Big\} + 5\epsilon_i^*, \end{aligned} \tag{5.4}$$

where $\xi \in \mathcal{M}_0(c_i), \zeta \in \mathcal{M}_0(c_{i+1})$. Note (5.2) and (5.4) are independent of the choice of ξ and ζ since the ergodicity of c_i-minimal measure is assumed for each i.

For each integer $i \in [i_1, i_2) \cup \cdots \cup [i_{m-2}, i_{m-1})$, there exist two closed 1-forms η_i, $\bar{\mu}_i$ defined on M, a U-step 1-form μ_i defined on $(u, t) \in M \times \mathbb{R}$ and an open set $U_i \subset M$ such that $[\eta_i] = c_i$, μ_i is closed on $U_i \times [0, t_0]$,

$\mu_i = 0$ when $t \le 0$, $\mu_i = \bar\mu_i$ when $t \ge t_0 > 0$, $[\bar\mu_i] = c_{i+1} - c_i$ and there is a small number $\delta_i > 0$ such that

$$\mathcal{C}_{\eta_i,\mu_i}(t) + \delta_i \subset U_i, \qquad \text{when } t \in [0, t_0]. \tag{5.5}$$

All orbits in $\tilde{\mathcal{C}}_{\eta_i,\mu_i}$ are the local minimal orbits of the third type of L, they connect $\tilde{\mathcal{N}}(c_i)$ to $\tilde{\mathcal{N}}(c_{i+1})$.

By the compactness of the manifold M, for a small $\epsilon_i^* > 0$ there exists $(\breve T_i^0, \breve T_i^1) = (\breve T_i^0, \breve T_i^1)(\epsilon_i^*) \in (\mathbb{Z}^+, \mathbb{Z}^+)$ such that

$$h_{\eta_i,\mu_i}^{T_0,T_1}(m_0, m_1) \ge h_{\eta_i,\mu_i}^{\infty}(m_0, m_1) - \epsilon_i^* \tag{5.6}$$

holds for all $T_0 \ge T_i^0$, $T_1 \ge T_i^1$ and for all $(m_0, m_1) \in M \times M$. Obviously, given (m_0, m_1) there are infinitely many $T_0 \ge T_i^0$ and $T_1 \ge T_i^1$ such that

$$\left| h_{\eta_i,\mu_i}^{T_0,T_1}(m_0, m_1) - h_{\eta_i,\mu_i}^{\infty}(m_0, m_1) \right| \le \epsilon_i^*. \tag{5.7}$$

Let $\gamma_i(t, m_0, m_1, T_0, T_1) : [-T_0, T_1] \to M$ be the minimizer of $h_{\eta_i,\mu_i}^{T_0,T_1}(m_0, m_1)$, it follows from the Lemma 2.2 that if $\epsilon_i^* > 0$ is sufficiently small, $T_0 > \breve T_i^0$ and $T_1 > \breve T_i^1$ are chosen sufficiently large so that (5.7) holds, then

$$d\gamma_i(t, m_0, m_1, T_0, T_1) \in \tilde{\mathcal{C}}_{\eta_i,\mu_i}(t) + \delta_i, \qquad \forall\ 0 \le t \le 1. \tag{5.8}$$

From the Lipschitz property of $h_{\eta_i,\mu_i}^{T_0,T_1}(m_0, m_1)$ in (m_0, m_1) there exist $\hat T_i^0(\epsilon_i^*) > \breve T_i^0(\epsilon_i^*)$ and $\hat T_i^1(\epsilon_i^*) > \breve T_i^1(\epsilon_i^*)$ so that for each (m_0, m_1) there are $T_j = T_j(m_0, m_1)$ with $\breve T_i^j(\epsilon_i^*) \le T_j \le \hat T_i^j(\epsilon_i^*)$ $(j = 0, 1)$ such that both (5.7) and (5.8) hold. Note that for different (m_0, m_1) we may need different $T_j \ge \breve T_i^j$ $(j = 0, 1)$.

Before we go back to consider those integers $i \in [0, i_1) \cup [i_2, i_3) \cup \cdots \cup [i_{m-1}, i_m)$, let us observe some facts. We can define the set of forward and backward semi-static curves:

$$\tilde{\mathcal{N}}^+(c) = \{(z, s) \in TM \times \mathbb{T} :\ \pi \circ \phi_L^t(z, s)|_{[0,+\infty)} \text{ is } c\text{-semi-static}\},$$

$$\tilde{\mathcal{N}}^-(c) = \{(z, s) \in TM \times \mathbb{T} :\ \pi \circ \phi_L^t(z, s)|_{(-\infty,0]} \text{ is } c\text{-semi-static}\}.$$

Proposition 5.1. *If the c-minimal measure is uniquely ergodic, $u \in \mathcal{A}_0(c)$, then there exists a unique $v \in T_uM$ such that $(u, v) \in \tilde{\mathcal{N}}_0^+(c)$ (or $\tilde{\mathcal{N}}_0^-(c)$). Moreover, $(u, v) \in \tilde{\mathcal{A}}_0(c)$.*

Proof. : Let us suppose the contrary. Then there would exist $(u, v) \in \tilde{\mathcal{A}}_0(c)$ and a forward c-semi-static curve $\gamma_+(t)$ with $\gamma_+(0) = u$ and $\dot\gamma_+(0) \ne v$. In this case, for any $u_1 \in \mathcal{M}_0(c)$ there exist two sequences $k_i, k_i' \to \infty$ such that $\pi \circ \phi_L^{k_i}(u, v) \to u_1$, $\gamma_+(k_i') \to u_1$ and

$$\begin{aligned} h_c^\infty(u, u_1) &= \lim_{k_i \to \infty} \int_0^{k_i} (L - \eta_c)(\phi_L^t(u, v), t)dt + k_i\alpha(c) \\ &= \lim_{k_i' \to \infty} \int_0^{k_i'} (L - \eta_c)(d\gamma_+(t), t)dt + k_i\alpha(c). \end{aligned}$$

Thus, we obtain that

$$\begin{aligned}
&h_c^\infty(\pi\circ\phi_L^{-1}(u,v),u_1)\\
&=F_c(\pi\circ\phi_L^{-1}(u,v),u)+h_c^\infty(u,u_1)\\
&=F_c(\pi\circ\phi_L^{-1}(u,v),u)+\lim_{k_i'\to\infty}\int_0^{k_i'}(L-\eta_c)(d\gamma_+(t),t)dt\\
&>h_c^\infty(\pi\circ\phi_L^{-1}(u,v),u_1)
\end{aligned}$$

where the last inequality follows from the facts that $\dot\gamma_+(0)\neq v$ and the minimizer must be a C^1-curve. But this is absurd. q.e.d.

Proposition 5.2. *Assume the c-minimal measure is uniquely ergodic, then for all $\zeta\in\mathcal{M}(c)$ and all $m_0,m_1\in M$, we have*

$$h_c^\infty(m_0,\zeta)+h_c^\infty(\zeta,m_1)=h_c^\infty(m_0,m_1).$$

Proof. : By definition,

$$h_c^\infty(m_0,\zeta)+h_c^\infty(\zeta,m_1)\ge h_c^\infty(m_0,m_1)$$

for all $m_0,m_1,\zeta\in M$. Let γ_T: $[0,T]\to M$ be a c-minimal curve connecting m_0 with m_1. As the c-minimal measure is uniquely ergodic, for any $\epsilon>0$, there exists a positive integer $T(\epsilon)$ such that for each integer $T\ge T(\epsilon)$ there is $T_1<T$ with the property $\gamma_T(T_1)\in\mathcal{M}(c)+\epsilon$. Let $T_2=T-T_1$. In this case, we have

$$h_c^T(m_0,m_1)=h_c^{T_1}(m_0,\gamma_T(T_1))+h_c^{T_2}(\gamma_T(T_1),m_1)$$

We claim that $T_1\to\infty$ and $T-T_1\to\infty$ as $\epsilon\to0$. Indeed, if T_1 is bounded by some finite number, then there would be a point $u\in\mathcal{M}_0(c)$ and a vector $v\in T_uM$ such that $\phi^t(u,v)$ is a forward c-semi-static orbit as $t\to\infty$ with $(u,v)\notin\tilde{\mathcal{A}}(c)$. But this contradicts to the Proposition 5.1. Clearly, there exist $\zeta\in\mathcal{M}(c)$ and a subsequence $\{T_i\}$ such that $\gamma_{T_i}(T_1)\to\zeta$ as $T_i\to\infty$. It implies that

$$h_c^\infty(m_0,\zeta)+h_c^\infty(\zeta,m_1)\le h_c^\infty(m_0,m_1).$$

As the c-minimal measure is uniquely ergodic, for any $\xi\in\mathcal{A}_0(c)$

$$\begin{aligned}
&h_c^\infty(m_0,\xi)+h_c^\infty(\xi,m_1)\\
&=h_c^\infty(m_0,\zeta)+h_c^\infty(\zeta,\xi)+h_c^\infty(\xi,\zeta)+h_c^\infty(\zeta,m_1)\\
&=h_c^\infty(m_0,\zeta)+h_c^\infty(\zeta,m_1).
\end{aligned}$$

This completes the proof. q.e.d.

Let $m_0,m_1\in M$, let γ_T: $[0,T]\to M$ be a c-minimizer connecting m_0 with m_1. For each integer $i\in[0,i_1]\cup[i_2,i_3]\cup\cdots\cup[i_{m-1}-i_m)$, in view of the Proposition 5.2, there exists $\breve{T}_i(\epsilon_i^*)>0$, independent of m_0 and m_1, such that

(5.9)
$$h_{c_i}^T(m_0,m_1)\ge h_{c_i}^\infty(m_0,\zeta)+h_{c_i}^\infty(\zeta,m_1)-\epsilon_i^*,\quad\forall\, T\ge\breve{T}_i(\epsilon_i^*),\ \zeta\in\mathcal{M}(c)$$

and there exists $\hat{T}_i(\epsilon_i^*) > \breve{T}_i(\epsilon_i^*)$ such that for each $(m_0, m_1) \in M \times M$ we have some integers T between $\breve{T}_i(\epsilon_i^*)$ and $\hat{T}_i(\epsilon_i^*)$ so that

$$\left|h_{c_i}^T(m_0, m_1) - h_{c_i}^\infty(m_0, \zeta) - h_{c_i}^\infty(\zeta, m_1)\right| \le \epsilon_i^*, \quad \forall\, \zeta \in \mathcal{M}(c). \tag{5.10}$$

We define τ_i inductively for $0 \le i \le i_m$. We let $\tau_0 = 0$, for $i \in [0, i_1) \cup [i_2, i_3) \cup \cdots \cup [i_{m-1}, i_m)$ we choose τ_i such that

$$\breve{T}_i + \tilde{T}_{i-1}^1 + \tilde{T}_i^0 \le \tau_i - \tau_{i-1} \le \hat{T}_i + \tilde{T}_{i-1}^1 + \tilde{T}_i^0 \tag{5.11}$$

(cf. (5.2) for the definition of $\tilde{T}_{i-1}^1$ and $\tilde{T}_i^0$). For $i \in [i_1, i_2) \cup \cdots \cup [i_{m-2}, i_{m-1})$ we choose those τ_i such that

$$\max\{\breve{T}_i^0, \breve{T}_{i-1}^1 + 1\} \le \tau_i - \tau_{i-1} \le \max\{\hat{T}_i^0, \hat{T}_{i-1}^1 + 1\}. \tag{5.12}$$

To consider the case that $i = i_1$ we note that both $\hat{T}_{i_1}$ and $\hat{T}_{i_1}^0$ can be taken large enough such that for any $m_0, m_1 \in M$ there exist $T(m_0, m_1)$, $T_0(m_0, m_1)$ with

$$\max\{\breve{T}_{i_1}, \breve{T}_{i_1}^0\} \le T(m_0, m_1), \qquad T_0(m_0, m_1) \le \max\{\hat{T}_{i_1}, \hat{T}_{i_1}^0\}$$

such that (5.7) holds provided we set $T_0 = T_0(m_0, m_1)$; (5.10) holds provided we set $T = T(m_0, m_1)$; and (5.6) and (5.9) hold for each $T_0, T \ge \max\{\breve{T}_{i_1}, \breve{T}_{i_1}^0\}$. Thus, we choose

$$\tilde{T}_{i_1-1}^1 + \max\{\breve{T}_{i_1}, \breve{T}_{i_1}^0\} \le \tau_{i_1} - \tau_{i_1-1} \le \tilde{T}_{i_1-1}^1 + \max\{\hat{T}_{i_1}, \hat{T}_{i_1}^0\}. \tag{5.13}$$

The case for $i = i_3, i_5, \cdots, i_m$ can be treated in the same way.

Similarly, we can choose suitably large $\hat{T}_{i_2}$ and $\hat{T}_{i_2}^1$ and set the range for τ_{i_2}:

$$\max\{\breve{T}_{i_2-1}^1, \breve{T}_{i_2}\} + \tilde{T}_{i_2}^0 \le \tau_{i_2} - \tau_{i_2-1} \le \max\{\hat{T}_{i_2-1}^1, \hat{T}_{i_2}\} + \tilde{T}_{i_2}^0. \tag{5.14}$$

The case for $i = i_4, i_6, \cdots, i_{m-1}$ can also be treated in the same way.

We define an index set for $\vec{\tau} = (\tau_1, \tau_2, \cdots, \tau_{i_3-2}, \tau_{i_3-1})$:

$$\Lambda = \Big\{\vec{\tau} \in \mathbb{Z}^{i_3-1} : \ (5.11 \sim 5.14) \text{ hold}\Big\}.$$

Consider τ as the time translation $\tau^*\phi(q, x, t) = \phi(q, x, t + \tau)$ on $M \times \mathbb{R}$, let $\psi_i \equiv 0$ for $i \in [i_1, i_2) \cup \cdots \cup [i_{m-2}, i_{m-1})$, we define a modified Lagrangian

$$\tilde{L} = L - \eta_0 - \sum_{i=0}^{i_m-1} (-\tau_i)^*(\mu_i + \psi_i). \tag{5.15}$$

Let $\mathbb{V} = V_0^+ \times V_1^- \times \cdots \times V_{i_1}^- \times V_{i_2}^+ \times \cdots \times V_{i_m-1}^+ \times V_{i_m}^-$. For $(m, m') \in M \times M$, $Z = (z_0^+, z_1^-, z_1^+, \cdots, z_{i_1-1}^+, z_{i_1}^-, z_{i_2}^+, z_{i_2+1}^-, \cdots, z_{i_m-1}^+, z_{i_m}^-) \in \mathbb{V}$

we define

$$h_{\tilde{L}}^{K,K'}(m_0,m_1,Z,\vec{\tau}) = \inf_{\substack{\gamma(-K)=m_0\\ \gamma(\bar{K}'+\tau_{i_m-1})=m_1\\ \gamma(\tau_i-\tilde{T}_i^0)=z_i^+\\ \gamma(\tau_i+\tilde{T}_i^1)=z_{i+1}^-\\ [\gamma|_{t\in J_i}]_1\neq 0\\ i\in\mathbb{I}}} \int_{-K}^{\bar{K}'+\tau_{i_m-1}} \tilde{L}(d\gamma(t),t)dt + \sum_{i=1}^{i_m-1}(\tau_i-\tau_{i-1})\alpha(c_i) + K\alpha(c_0) + K'\alpha(c_{i_m})$$

where $\bar{K}' = K' + \tilde{T}_{i_m-1}^1 + \hat{T}_{i_m}$, $J_i = [\tau_i - \tilde{T}_i^0, \tau_i + \tilde{T}_i^1]$ and $\mathbb{I} = \mathbb{N} \cap \Big([0,i_1) \cup [i_2,i_3) \cup \cdots \cup [i_{m-1},i_m) \backslash I\Big)$.

Let $h_{\tilde{L}}^{K,K'}(m_0,m_1)$ be the minimizer of $h_{\tilde{L}}^{K,K'}(m_0,m_1,Z,\vec{\tau})$ over $\mathbb{V}$ in z and over Λ in $\vec{\tau}$ respectively:

$$h_{\tilde{L}}^{K,K'}(m_0,m_1) = \min_{\vec{\tau}\in\Lambda, z\in\mathbb{V}} h_{\tilde{L}}^{K,K'}(m_0,m_1,Z,\vec{\tau}), \tag{5.16}$$

let $K_j, K_j' \to \infty$ be the subsequence such that

$$\lim_{K_j,K_j'\to\infty} h_{\tilde{L}}^{K_j,K_j'}(m_0,m_1) = \liminf_{\substack{K\to\infty\\ K'\to\infty}} h_{\tilde{L}}^{K,K'}(m_0,m_1), \tag{5.17}$$

and let $\gamma(t;K_j,K_j',m_0,m_1)$ be the minimal curve, we claim that

$$d\gamma(t;K_j,K_j',m_0,m_1)$$

is a solution of the Euler-Lagrange equation determined by L if K_j and K_j' are sufficiently large. Indeed,

1, for each $i \in [i_1,i_2) \cup \cdots \cup [i_{m-2},i_{m-1})$, we have

$$(-\tau_i)^*\gamma(t;K_j,K_j') \in \mathcal{C}_{\eta_i,\mu_i}(t) + \delta_i \subset U_i, \qquad \text{when } \tau_i \le t \le \tau_i + 1. \tag{5.18}$$

To see that, let us choose $m_i = \gamma(\tau_{i-1}+1)$, $m_i' = \gamma(\tau_{i+1})$. Since $\gamma(t;K_j,K_j',m_0,m_1)$ is the minimizer of $h_{\tilde{L}}^{K,K'}(m_0,m_1,Z,\vec{\tau})$ over Λ, thus

$$\begin{aligned} &A_{\tilde{L}}((-\tau_i)^*\gamma|_{\tau_{i-1}+1}^{\tau_{i+1}}) + (\tau_i - \tau_{i-1}+1)\alpha(c_i) + (\tau_{i+1}-\tau_i)\alpha(c_{i+1}) \\ = &\inf_{\substack{\xi(-T_0)=m_i\\ \xi(T_1)=m_i'\\ \check{T}_i^0\le T_0\le \hat{T}_i^0\\ \check{T}_i^1\le T_1\le\hat{T}_i^1}} \int_{-T_0}^{T_1} (L-\eta_i-\mu_i)(d\xi(t),t)dt + T_0\alpha(c_i) + T_1\alpha(c_{i+1}). \end{aligned}$$

Thus we obtain (5.18) from this formula, (5.5), (5.8) and (5.12). Consequently, $\gamma(t;K_j,K_j')|_{\tau_i\le t\le\tau_i+1}$ falls into the region where $(-\tau_i)^*\mu_i$ is closed. So, $d\gamma(t;K_j,K_j')$ is the solution of the Euler-Lagrange equation determined by L when $\tau_i \le t \le \tau_i + 1$;

2, for $i \in [0, i_1) \cup [i_2, i_3) \cup \cdots \cup [i_{m-1}, i_m)$, we claim that

$$(-\tau_i)^*\gamma(t)|_{0\le t\le t_0} \in \mathrm{int}(O_i). \tag{5.19}$$

It is actually the consequence of (5.1). In fact, if

$$d\gamma \in \tilde{\mathcal{C}}_{\eta_i,\mu_i,\psi_i}(\tilde{M})\backslash\tilde{\mathcal{C}}_{\eta_i,\mu_i,\psi_i}(M)$$

then γ must pass through O_i during the time interval $[0, t_0]$. We note that the function L_{η_i,μ_i,ψ_i} is not time-periodic, the translate of $d\gamma_i$ is no longer a minimizer of the same kind if $d\gamma_i \in \pi_1\tilde{\mathcal{C}}_{\eta_i,\mu_i,\psi_i}(\tilde{M})\backslash\tilde{\mathcal{C}}_{\eta_i,\mu_i,\psi_i}(M)$, $d\gamma_i(k) \notin \pi_1\tilde{\mathcal{C}}_{\eta_i,\mu_i,\psi_i}(\tilde{M})|_{t=0}$ for each $k \in \mathbb{Z}\backslash\{0\}$.

By the condition of the Theorem 5.1, both c_i- and c_{i+1}-minimal measure are uniquely ergodic. For any $m_i \in \mathcal{M}_0(c_i)$, $m_{i+1} \in \mathcal{M}_0(c_{i+1})$ and any smooth curve $d\gamma \in \tilde{\mathcal{C}}_{\eta_i,\mu_i,\psi_i}(\tilde{M})\backslash\tilde{\mathcal{C}}_{\eta_i,\mu_i,\psi_i}(M)$, if $\mathbb{Z}^+ \ni T_0^k \to \infty$ and $\mathbb{Z}^+ \ni T_1^k \to \infty$ (as $k \to \infty$) are two sequences such that $\gamma(-T_0^k) \to m_i$ and $\gamma(T_1^k) \to m_{i+1}$, then

$$\begin{aligned}\lim_{k\to\infty}\int_{-T_0^k}^{T_1^k} L_{\eta_i,\mu_i,\psi_i}(d\gamma(t),t)dt + T_0^k\alpha(c_i) + T_1^k\alpha(c_{i+1})&\\ = h^\infty_{\eta_i,\mu_i,\psi_i,e_1}(m_i, m_{i+1}).&\end{aligned}$$

Let ζ: $\mathbb{R} \to M$ be an absolutely continuous curve such that $[\zeta]_1 \neq 0$, $\zeta(t') \notin int(O_i)$ for some $t' \in [0, t_0]$, $\zeta(-T_0^k) \to m_i$, $\zeta(T_1^k) \to m_{i+1}$ as $k \to \infty$. Since $\tilde{\mathcal{C}}_{\eta_i,\mu_i,\psi_i}|_{t=\text{constant}}$ is closed, there exists a positive number $d > 0$ such that

$$\begin{aligned}\liminf_{\substack{T_0^k\to\infty\\ T_1^k\to\infty}}\int_{-T_0^k}^{T_1^k} L_{\eta_i,\mu_i,\psi_i}(d\zeta(t),t)dt + T_0^k\alpha(c_i) + T_1^k\alpha(c_{i+1})&\\ \ge h^\infty_{\eta_i,\mu_i,\psi_i,e_1}(m_i, m_{i+1}) + d.&\end{aligned}$$

Recall the construction of the modified Lagrangian $\tilde{L}$ (see (5.15)) and γ is the minimizer of $h^{K,K'}_{\tilde{L}}(m_0, m_1, Z, \vec{\tau})$ over $\mathbb{V}$ in z and over Λ in $\vec{\tau}$ respectively. Given any small number $\varepsilon > 0$, by choosing sufficiently large $\hat{T}_i - \breve{T}_i$, we can see that there are sufficiently large $K_i^-, K_i^+ \in \mathbb{Z}$ with the properties that $\tau_{i-1} + \tilde{T}^1_{i-1} + K_i^- + K_i^+ = \tau_i - \tilde{T}_i^0$, $\breve{T}_i \le K_i^- + K_i^+ \le \hat{T}_i$ and

$$\left\|\gamma(\tau_i - \tilde{T}_i^0 - K_i^+) - m_i\right\| < \varepsilon, \qquad \left\|\gamma_i(\tau_i + \tilde{T}_i^1 + K_{i+1}^-) - m_{i+1}\right\| \le \varepsilon.$$

If there was $t' \in [0, t_0]$ such that

$$(-\tau_i)^*\gamma(t') \notin int(O_i),$$

from the Lipschitz continuity of $h^{\infty}_{\eta_i,\mu_i,\psi_i,e_1}(m,m')$ in (m,m') we would obtain

$$\begin{aligned}&\int_{\tau_i-\tilde T_i^0-K_i^+}^{\tau_i+\tilde T_i^1+K_{i+1}^-} L_{\eta_i,\mu_i,\psi_i}(d\gamma(t),t)dt\\&+(\tilde T_i^0+K_{i-1}^+)\alpha(c_i)+(\tilde T_i^1+K_{i+1}^-)\alpha(c_{i+1})\\&\ge h^{\infty}_{\eta_i,\mu_i,\psi_i,e_1}(m_i,m_{i+1})+\frac{3}{4}d.\end{aligned}$$

On the other hand, there is suitably large $\bar K_i^-,\bar K_i^+\in\mathbb{Z}$ with the properties that $\breve T_i\le \bar K_i^-+\bar K_i^+\le \hat T_i$ and

$$\left\|\gamma_i(\tau_i-\tilde T_i^0-\bar K_i^+)-m_i\right\|\le\varepsilon,\qquad \left\|\gamma_i(\tau_i+\tilde T_i^1+\bar K_{i+1}^-)-m_{i+1}\right\|\le\varepsilon.$$

Because $d\gamma_i(t')\in\tilde{\mathcal{C}}_{\eta_i,\mu_i,\psi_i}|_{t=t'}$, we have

$$\begin{aligned}&\int_{\tau_i-\tilde T_i^0-\bar K_i^+}^{\tau_i+\tilde T_i^1+\bar K_{i+1}^-} L_{\eta_i,\mu_i,\psi_i}(d\gamma_i(t),t)dt\\&+(\tilde T_i^0+\bar K_{i-1}^+)\alpha(c_i)+(\tilde T_i^1+\bar K_{i+1}^-)\alpha(c_{i+1})\\&\le h^{\infty}_{\eta_i,\mu_i,\psi_i,e_1}(m_i,m_{i+1})+\frac{1}{4}d\end{aligned}$$

if ε is sufficiently small. It implies that γ is not a minimizer. This contradiction verifies our claim.

The formula (5.19) implies that $d\gamma(t;K_j,K_j')$ is the solution of the Euler-Lagrange equation determined by L when $\tau_i\le t\le\tau_i+1$ for $i\in[0,i_1)\cup[i_2,i_3)\cup\cdots\cup[i_{m-1},i_m)$;

3, We claim that the curve γ does not touch the boundary of V_i^+ at the time $t=\tau_i-\tilde T_i^0$ and does not touch the boundary of V_{i+1}^- at the time $t=\tau_i+\tilde T_i^1$ for each integer $i\in[0,i_1)\cup[i_2,i_3)\cup\cdots\cup[i_{m-1},i_m)$. If $(\gamma(\tau_i-\tilde T_i^0),\gamma(\tau_i+\tilde T_i^1))=(m_i,m_i')\in\partial(V_i^+\times V_{i+1}^-)$ for some $i\notin I$, let $m_{i-1}'=\gamma(\tau_{i-1}+\tilde T_{i-1}^1)$ and $m_{i+1}=\gamma(\tau_{i+1}-\tilde T_{i+1}^1)$, from (5.1) we can see that there exist $(\bar m_i,\bar m_i')\in V_i^+\times V_{i+1}^-$ such that for $\xi\in\mathcal{M}_0(c_i)$, $\zeta\in\mathcal{M}_0(c_{i+1})$:

$$\begin{aligned}&h_{c_i}^{T_i}(m_{i-1}',m_i)+h^{\tilde T_i^0,\tilde T_i^1}_{\eta_i,\mu_i,\psi_i,e_1}(m_i,m_i')+h_{c_{i+1}}^{T_{i+1}}(m_i',m_{i+1})\\\ge\ &h_{c_i}^{\infty}(\xi,m_i)+h^{\tilde T_i^0,\tilde T_i^1}_{\eta_i,\mu_i,\psi_i,e_1}(m_i,m_i')+h_{c_{i+1}}^{\infty}(m_i',\zeta)\\&+h_{c_i}^{\infty}(m_{i-1}',\xi)+h_{c_{i+1}}^{\infty}(\zeta,m_{i+1})-2\epsilon_i^*\\\ge\ &h_{c_i}^{\infty}(\xi,\bar m_i)+h^{\tilde T_i^0,\tilde T_i^1}_{\eta_i,\mu_i,\psi_i,e_1}(\bar m_i,\bar m_i')+h_{c_{i+1}}^{\infty}(\bar m_i',\zeta)\\&+h_{c_i}^{\infty}(m_{i-1}',\xi)+h_{c_{i+1}}^{\infty}(\zeta,m_{i+1})+3\epsilon_i^*\\\ge\ &h_{c_i}^{T_i'}(m_{i-1}',\bar m_i)+h^{\tilde T_i^0,\tilde T_i^1}_{\eta_i,\mu_i,\psi_i,e_1}(\bar m_i,\bar m_i')+h_{c_{i+1}}^{T_{i+1}'}(\bar m_i',m_{i+1})+\epsilon_i^*\end{aligned}$$

where $T_i, T_{i+1}, T'_i, T'_{i+1}$ satisfy the condition $\breve{T}_j \le T_j, T'_j \le \hat{T}_j$ ($j = i-1, i$). In above arguments, (5.9) and (5.10) are used to obtain the first and the third inequality, (5.2) is used to obtain the second inequality. But this contradicts to the fact that γ is a minimal curve of $\tilde{L}$ on $\mathbb{V}$ and Λ. The case for $i \in I$ can be treated in the same way. Therefore, the minimizer γ is differentiable at the time $t = \tau_i - \tilde{T}_i^0$ and $t = \tau_i + \tilde{T}_i^1$ for each i.

Let $K_j, K'_j \to \infty$, denote by γ_∞: $\mathbb{R} \to M$ an accumulation point of $\{\gamma(t, K_j, K'_j)\}$. Obviously, $\alpha(d\gamma_\infty) \subset \tilde{\mathcal{N}}(c)$ and $\omega(d\gamma_\infty) \subset \tilde{\mathcal{N}}(c')$. This completes the proof. q.e.d.

6. Hölder continuity

The task in this section is to build up some Hölder continuous dependence of h_c^∞ on some parameters if we set $k = 1$. These properties will be used to show that there is a generic set for perturbation where the conditions for the Theorem 5.1 are satisfied.

Let Φ_H^t be the Hamiltonian flow determined by H, it is a small perturbation of Φ_h^t. Let Φ_H and Φ_h be their time-1-maps. According to the hypothesis **H2'** and **H3'**, there are several cylinders $\Sigma_j^0 = \mathbb{T} \times \{p_{j-1} - 2\delta < p < p_j + 2\delta\} \times \{(x, y) = (x_j(p), y(p))\}$ $(1 \le j \le l)$ which are normally hyperbolic and invariant to the map Φ_h, here $\delta > 0$ is small. When p increases from $p_j - \delta$ to $p_j + \delta$, the global maximum point of $h(x, p, y(p))$ jumps from $x_j(p)$ to $x_{j+1}(p)$. It follows from the fundamental theorem of normally hyperbolic invariant manifold (cf. [**HPS**]) that there is $\epsilon = \epsilon(A, B, \delta) > 0$ such that if $\|P\|_{C^r} \le \epsilon$ on the region $\{\|(p, y)\| \le \max(|A|, |B|) + 1\}$ the map Φ_H^{s+k} $(k \in \mathbb{Z}, 0 < s < 1)$ also has several C^{r-1} invariant manifold $\Sigma_j(s) = \mathbb{T} \times \{p_{j-1} - \delta < p < p_j + \delta\} \times \{(x, y) = (x_j(p, q), y_j(p, q))\}$, provided that $r \ge 2$. These manifolds are the small deformation of the manifolds $\Sigma_j^0|_{p_{j-1}-\delta<p<p_j+\delta}$. Thus, they are also normally hyperbolic and time-1-periodic. Let $\Sigma_j = \Sigma_j(0)$, it can be considered as the image of a map ψ from the standard cylinder $\Sigma = \mathbb{T} \times \mathbb{R} \times \{(x, y) = (0, 0)\}$ to $\Sigma_j = \{q \in \mathbb{T}, p_{j-1} - \delta < p < p_j + \delta, (x, y) = (x_j(p, q), y_j(p, q))\}$. This map induces a 2-form $\Psi_j^*\omega$ on Σ

$$\Psi_j^*\omega = \left(1 + \sum_{i=1}^{n} \frac{\partial(x_{ji}, y_{ji})}{\partial(p, q)}\right) dp \wedge dq$$

and $\Psi_j^*\omega = dp \wedge dq$ when $P = 0$. Since the second de Rham co-homology group of Σ_0 is trivial, by using Moser's argument on the isotopy of symplectic forms [**Mo**], we find that, on $\Sigma_{\{|p| \le \max(|A|,|B|)+1\}}$, there exists a diffeomorphism Ψ such that

$$(\Psi_j \circ \Psi)^*\omega = dp \wedge dq.$$

Since Σ is invariant for Φ_H and $\Phi_H^*\omega=\omega$, we have

$$\Big((\Psi_j\circ\Psi)^{-1}\circ\Phi_H\circ(\Psi_j\circ\Psi)\Big)^*dp\wedge dq=dp\wedge dq$$

i.e. $(\Psi_j\circ\Psi)^{-1}\circ\Phi_H\circ(\Psi_j\circ\Psi)$ preserves the standard area. Clearly, it is exact and twist since it is a small perturbation of Φ_h. In this sense, we say that the restriction of Φ_H on Σ_j is obviously area-preserving and twist. If $r>4$ there are many invariant homotopically non-trivial curves, including many KAM curves. All these curves are Lipschitz. Given $\rho\in\mathbb{R}$ there is an Aubry-Mather set with rotation number ρ, which is either an invariant circle, or a Denjoy set if $\rho\in\mathbb{R}\backslash\mathbb{Q}$, or periodic orbits if $\rho\in\mathbb{Q}$. Under the generic condition we can assume there is no homotopically non-trivial invariant curves with rational rotation number for $\Phi_H|_{\Sigma_j}$, instead, there is only one minimal periodic orbit on Σ for each rational rotation number.

Let us consider the Legendre transformation $\mathscr{L}$. By abuse of terminology we continue to denote $\Sigma_j(s)$ and its image under the Legendre transformation by the same symbol. Let

$$\tilde\Sigma_j=\bigcup_{s\in\mathbb{T}}(\Sigma_j(s),s),$$

which has the normal hyperbolicity as well. Under the Legendre transformation those Aubry-Mather sets (invariant curves, Denjoy sets or minimal periodic orbits) on Σ correspond to the support of some c-minimal measures.

To continue the study, let us first consider the case when there is only one cylinder $\tilde\Sigma$. In this case, $\mathbb{P}=\varnothing$. Consequently, $\mathcal{N}(c)\subset N_\delta$ and the lemma 3.1 has the following form (k=1)

Proposition 6.1. *Given some large number $K>0$ and a small number $\delta>0$ there exists a small number $\epsilon=\epsilon(\delta)$, if $L_1\in\mathcal{B}_{\epsilon,K}$ and if $|c_q|\le K$ then there exists an n-dimensional convex set $\mathcal{D}(c_q)$ which contains a neighborhood of $(c_q,y(c_q))\cap\mathbb{R}^n$ such that for each $c\in\mathrm{int}(\mathcal{D}(c_q))$ the Mañé set $\tilde{\mathcal{N}}(c)\subset\tilde\Sigma$, the Mather set $\mathcal{M}_0(c)$ is the Aubry-Mather set for the twist map. If the rotation number is irrational, then $\mathcal{M}(c)$ is uniquely ergodic.*

Proof. : The normal hyperbolicity guarantees that the invariant set in N_δ must be in the invariant cylinder. The time-1-map restricted on the cylinder is then an area-preserving twist map. q.e.d.

Consider a cohomology class $c=(c_q,y(c_q))\in H^1(M,\mathbb{R})$ such that it corresponds to an invariant circle Γ in Σ with irrational rotation number. In the Hamiltonian formalism, $\Gamma=\{(p,q,x,y)\in\mathbb{R}^{n+1}\times\mathbb{T}^{n+1}:(p,x,y)=(p,x,y)(q),q\in\mathbb{T}\}$. Based on each point on this circle, there is a C^{r-1}-stable fiber as well as a C^{r-1}-unstable fiber. These stable (unstable) fibers C^{r-2}-depends on the base point and make up the local stable (unstable) manifold of that circle which are the graph of a Lipschitz function in a small neighborhood of the circle, i.e.

$$W^{u,s}_{loc}(\Gamma) = \Big\{(q,x,(p,y)^{u,s}(q,x)) : (q,x) \in N_\delta \subset T^{n+1}\Big\}$$

where $(p,y)(q,x)$ is a Lipschitz function of (q,x).

We use $C^{k,\alpha}$ to denote those functions whose k-th derivative is of α-Hölder.

Lemma 6.1. *There exists a $C^{1,1}$ function $S^{s,u}$: $N_\delta \to \mathbb{R}$ and a constant vector $c \in \mathbb{R}^{n+1}$ such that $W^{s,u}_{loc}(\Gamma) = \{(q,x), dS^{s,u}(q,x) + c : (q,x) \in N_\delta\}$.*

Proof. : Let us consider the stable manifold. By the condition there is a Lipschitz function (p,y): $N_\delta \to \mathbb{R}$ such that

$$W^s_{loc}(\Gamma) = \{(q,x,(p,y)^s(q,x)) : (q,x) \in N_\delta\}.$$

Let σ be an 2-dimensional disk in W^s_{loc}. Since σ is in the stable manifold, $\Phi^k_H(\partial\sigma)$ approaches uniformly to Γ, i.e. $\Phi^k_H(\partial\sigma)$ is such a closed curve going from a point to another point and returning back along almost the same path when k is sufficiently large. As Φ_H is a symplectic diffeomorphism we have

$$\begin{aligned}\iint_\sigma \Big(dp\wedge dq + \sum_{i=1}^n dy_i \wedge dx_i\Big) &= \oint_{\partial\sigma} \Big(pdq + \sum_{i=1}^n y_i dx_i\Big)\\ &= \oint_{\Phi^k_H(\partial\sigma)} \Big(pdq + \sum_{i=1}^n y_i dx_i\Big)\\ &= 0.\end{aligned}$$

Note the function $(p,y)^s(q,x)$ is Lipschitz, it is differentiable almost everywhere in N_δ. As σ is arbitrarily chosen, for almost $(q,x) \in N_\delta$ the following holds:

$$\frac{\partial p}{\partial x_i} = \frac{\partial y_i}{\partial q}, \qquad \frac{\partial y_i}{\partial x_j} = \frac{\partial y_j}{\partial x_i}, \qquad \forall\ 1 \le i,j \le n. \tag{6.1}$$

Consequently, there exists a $C^{1,1}$-function S^s_c and $c = (c_q, y(c_q)) \in \mathbb{R}^{n+1}$ such that $(p,y)^s = dS^s_c + c$. In the same way, we obtain a $C^{1,1}$-function S^u_c and $c' = (c'_q, y(c'_q)) \in \mathbb{R}^{n+1}$ such that $(p,y)^u = dS^u_c + c'$. Since W^s_{loc} intersects W^u_{loc} on the whole Γ, $c' = c$.

q.e.d.

Indeed, for almost all initial points $(q,x,(p,y)^s(q,x)) \in W^s$, $(p,y)^s$ is differentiable at all $\Phi^k_H(q,x,(p,y)^s(q,x))$ for all $k \in \mathbb{Z}^+$. To see that, let $O \subset N_\delta$ be an open set, for each k there is a full Lebesgue measure set $O_k \subset \pi(\Phi_H^k\{O,(p,y)^s(O)\})$ where $(p,y)^s$ is differentiable. Since Φ_H is a diffeomorphism, the set

$$O^* = \bigcap_{k=0}^{\infty} \pi\Big(\Phi_H^{-k}\{O_k,(p,y)^s(O_k)\}\Big)$$

is a full Lebesgue measure subset of O. For any point $(q,x) \in O^*$, $(p,y)^s$ is differentiable at the points $\pi(\Phi_H^k\{(q,x),(p,y)^s(q,x)\}$ for all $k \in \mathbb{Z}^+$.

Let us consider the Hamiltonian flow. The local stable (unstable) manifold is a graph of some function

$$\tilde{W}_{loc}^{s,u} = \{(q,x,t),(p,y)^{s,u}(q,x,t) : (q,x,t) \in N_\delta \times \mathbb{T}\}.$$

Obviously, we have $((p,y)^{s,u},t)^*\Omega = 0$ in $M \times \mathbb{T}$, where $\Omega = \sum dx_i \wedge dy_i + dq \wedge dp - dH \wedge dt$. Thus, in the covering space $\mathbb{R}^{n+2}$ there exists a $C^{1,1}$-function $\bar{S}_c^{s,u}(q,x,t)$ such that $d\bar{S}_c^{s,u} = (p,y)^{s,u}(q,x,t) - H((p,y)^{s,u}(q,x,t),q,x,t)dt$. Consequently, there exist a cohomology class $c = (c_q, y(c_q))$ and a function $S_c^{s,u}(q,x,t) \in C^{1,1}(N_\delta \times \mathbb{T},\mathbb{R})$ such that

$$L^{s,u} = L - c_q(\dot{q} + \partial_q S_c^{s,u}) - \langle \partial_x S_c^{s,u}, \dot{x}\rangle - \partial_t S_c^{s,u}$$

attains its minimum at $\mathscr{L}W^{s,u}$ as the function of $(\dot{q},\dot{x})$. Note $L^{s,u}_{(\dot{q},\dot{x})} - \partial_{(q,x)} S_c^{s,u}$ is Lipschitz, $dL^{s,u}_{(\dot{q},\dot{x})}/dt$ and $L^{s,u}_{(q,x)}$ exist almost everywhere. Since $\mathscr{L}W^{s,u}$ is a manifold made up by the trajectories of the Euler-Lagrange flow, it follows from Euler-Lagrange equation $dL_{\dot{q},\dot{x}}/dt = L_{q,x}$ and (6.1) that $L_{q,x}|_{\mathscr{L}W_{loc}^{s,u}} = 0$ almost everywhere. The absolute continuity of L implies that $L^{s,u}|_{\mathscr{L}W_{loc}^{s,u}}$ is a function of t alone. So, by adding some function of t to $S_c^{s,u}$, $L^{s,u}|_{\mathscr{L}W_{loc}^{s,u}} = -\alpha(c)$.

Lemma 6.2. *Let $c = (c_q, y(c_q))$. If Γ is an invariant circle in the cylinder, the Aubry-Mather set is uniquely ergodic, then $\exists\ S_c^{s,u} \in C^{1,1}(N_\delta,\mathbb{R})$ such that*

$$h_c^\infty(\xi,m) = S_c^u(m) - S_c^u(\xi), \qquad h_c^\infty(m,\xi) = S_c^s(\xi) - S_c^s(m) \tag{6.2}$$

hold for each $\xi \in \pi(\Gamma)$ and each $m \in N_\delta$.

Proof. : Since there are the local stable manifold $W^s(\Gamma)$ and the unstable manifold $W^u(\Gamma)$ to the invariant circle Γ, for each point $m \in N_\delta$ there is a unique c-minimal orbit $\gamma^{s,u}(t)$ such that $\gamma^{s,u}(0) = m$ and $\gamma^{s,u}(k) \to \pi(\Gamma)$ as $\mathbb{Z} \ni k \to \pm\infty$. Let $\xi \in \mathcal{M}_0(c)$, there is an integer subsequence $k_i^{s,u} \to \pm\infty$ as $i \to \infty$ such that $\gamma^{s,u}(k_i^{s,u}) \to \xi$ as $i \to \infty$. It means that

$$\lim_{i\to\infty} h_c^{k_i^s}(m,\gamma^s(k_i^s)) = h_c^\infty(m,\xi), \qquad \lim_{i\to\infty} h_c^{-k_i^u}(\gamma^u(k_i^u),m) = h_c^\infty(\xi,m).$$

Since $L^{s,u}+\alpha(c)=0$ on $W^{s,u}$, we have

$$\begin{aligned}\int_{k_i^u}^{0}\Big(L(d\gamma_c^u(k_i^u)-\langle c,\dot\gamma_c^u(k_i^u)\rangle+\alpha(c)\Big)dt =&S^u(\gamma_c^u(0))-S^u(\gamma_c^u(k_i^u)),\\ \int_0^{k_i^s}\Big(L(d\gamma_c^s(k_i^u)-\langle c,\dot\gamma_c^s(k_i^u)\rangle+\alpha(c)\Big)dt =&S^s(\gamma_c^s(k_i^u))-S^s(\gamma_c^s(0)).\end{aligned}$$

That implies that (6.2) holds for each $m\in N_\delta$ and each $\xi\in\mathcal{M}_0(c)$. To see that (6.2) holds for each $\xi\in\pi(\Gamma)$, let us recall that, for a twist map, the sufficient and necessary condition for the existence of an invariant circle is that the Peierl's barrier function is identically equal to zero. Consequently, passing each $\zeta\in\pi(\Gamma)$ there is a regular c-minimal configuration $(\cdots,m_i,\cdots)$ such that $\zeta=m_0$. Since we have assumed the unique ergodicity of the minimal measure, $d_c(\zeta,\xi)=0$ for each $\zeta\in\pi(\Gamma)$ and each $\xi\in\mathcal{M}_0(c)$. Thus, (6.2) holds for each $\xi\in\pi(\Gamma)$. q.e.d.

Obviously, $S_c^u(\xi)=S_c^s(\xi)$ for all $\xi\in\pi(\Gamma)$. Thus, in this case we have

$$h_c^\infty(m,m')=S_c^u(m')-S_c^s(m),\qquad \forall\ m,m\in N_\delta. \tag{6.3}$$

We now consider the local stable and unstable manifolds of all invariant circles. Different invariant circle determines different stable and unstable manifolds, i.e. we have a family of these local stable and unstable manifolds. We claim that this family of local stable (unstable) manifolds can be parameterized by some parameter σ so that both S_c^u and S_c^s have $\frac{1}{2}$-Hölder continuity in σ. Indeed we choose one circle and denote it Γ_0 and parameterize another circle Γ_σ by the algebraic area between Γ_σ and Γ_0,

$$\sigma=\int_0^1(\Gamma_\sigma(q)-\Gamma_0(q))dq. \tag{6.4}$$

This integration is in the sense that we pull it back to the standard cylinder by $\Psi\circ\Psi_1\in\mathrm{diff}(\Sigma_0,\Sigma)$. Let $\sigma=A'$ correspond to an invariant circle where the action $p<A$, let $\sigma=B'$ correspond to an invariant circle where the action $p>B$, in the way of (6.4) we obtain an one-parameter family curves Γ: $\mathbb{T}\times\mathbb{S}\to\Sigma$ in which $\mathbb{S}\subset[A',B']$ is a closed set. Clearly, for each $\sigma\in\mathbb{S}$, there is only one $c_q=c_q(\sigma)$ such that $\Gamma_\sigma=\tilde{\mathcal{M}}_0(c)$ where $c=(c_q,0)\in H^1(M,\mathbb{R})$. Clearly, c_q is continuous in σ on $\mathbb{S}$. We can think Γ_σ as a map to function space C^0 equipped with supremum norm Γ: $\mathbb{S}\to C^0(\mathbb{T},\mathbb{R})$,

$$\|\Gamma_{\sigma_1}-\Gamma_{\sigma_2}\|=\max_{q\in\mathbb{T}}|\Gamma(q,\sigma_1)-\Gamma(q,\sigma_2)|.$$

Straight-forward calculation shows

$$|\sigma_1-\sigma_2|\ge\frac{1}{2C_h}\left(\max_{q\in\mathbb{T}}|\Gamma(q,\sigma_1)-\Gamma(q,\sigma_2)|\right)^2,$$

where C_h is the Lipschitz constant for the twist map, it follows that

$$\|\Gamma_{\sigma_1} - \Gamma_{\sigma_2}\| \le C_s|\sigma_1 - \sigma_2|^{\frac{1}{2}}$$

where $C_s = \sqrt{2C_h}$. Since both the stable and the unstable fibers have C^{r-2}-smoothness on their base points on Σ, $r \ge 3$, $(p, y)^{s,u}_\sigma$ is also $\frac{1}{2}$-Hölder continuous in σ. Because we can choose suitably small N_δ such that it can be covered by the stable as well as the unstable manifold of the invariant curve Γ_σ, $\pi(W^{s,u}_{loc}(\Gamma_\sigma)) \supset N_\delta$. Let $S^{s,u}_\sigma = S^{s,u}_{c(\sigma)}$, we have

Lemma 6.3. *Restricted in N_δ the functions $S^s_\sigma(m)$, $S^u_\sigma(m)$ are $\frac{1}{2}$-Hölder continuous in $\sigma \in \mathbb{S}$.*

The Lemma 6.3 is not enough for the proof, because we need to consider the regularity of barrier function defined on the whole configuration manifold. Next, let us consider the dependence of the barrier function on $\sigma \in \mathbb{S}$ and on c. Recall the cohomology classes $c = (c_q, y(c_q))$ under our consideration is in the set

$$\mathbb{H}^1 = \{(c_q, y(c_q)) : \ |c_q| \le \max\{|A|, |B|\} + 1\}.$$

Let us remember $\mathcal{A}(c) \subset N_\delta$. For each $c = (c_q, y(c_q))$, each $m \in M\backslash N_\delta$ and each $\xi \in N_\delta$ there exist $m^+_c, m^-_c \in N_\delta$ and $k^+_c, k^-_c \in \mathbb{Z}^+$ such that

$$h^\infty_c(\xi, m) = h^\infty_c(\xi, m^+_c) + h^{k^+_c}_c(m^+_c, m),$$
$$h^\infty_c(m, \xi) = h^{k^-_c}_c(m, m^-_c) + h^\infty_c(m^-_c, \xi).$$

Clearly, there exists a uniform upper bound $K \in \mathbb{Z}$ such that for each $c = (c_q, y(c_q)) \in \mathbb{H}^1$, each $m \in M\backslash N_\delta$ and each $\xi \in N_\delta$ we have $k^+_c \le K$, $k^-_c \le K$.

Let $\gamma_c : [0, k^+_c] \to M$, with $\gamma_c(0) = m^+_c$ and $\gamma_c(k^+_c) = m$, be the curve which realizes the quantity

$$h^{k^+_c}_c(m^+_c, m) = \int_0^{k^+_c} L_{\eta_c}(d\gamma_c(t), t)dt + k^+_c\alpha(c).$$

Obviously, we have that

$$\begin{aligned} h^\infty_{c'}(\xi, m) \le &\int_0^{k^+_c} L_{\eta_{c'}}(d\gamma_c(t), t)dt + k^+_c\alpha(c') \\ &+ h^\infty_{c'}(\xi, m^+_c) \end{aligned}$$

holds for any other c'. Note $L_{\eta'_c} - L_{\eta_c} = \langle \eta_{c'} - \eta_c, \dot{q}\rangle$, we find

$$\begin{aligned} &\int_0^{k^+_c} L_{\eta_c}(d\gamma_c(t), t)dt + k^+_c\alpha(c) - h^{k^+_c}_c(m^+_c, m) \\ = &\int_0^{k^+_c} \langle \eta_{c'} - \eta_c, \dot{\gamma}_c(t)\rangle dt + k^+_c(\alpha(c') - \alpha(c)) \\ \le &\ |\bar{\gamma}_c(0) - \bar{\gamma}_c(-k^+_c)||c' - c| + k^+_c(\alpha(c') - \alpha(c)), \end{aligned}$$

where $\bar{\gamma}_c$ denotes the lift of γ_c to the universal covering $\mathbb{R}$.

Since c and c' are considered in the bounded set $\mathbb{H}^1$, there exists $C_\alpha > 0$ such that

$$|\alpha(c) - \alpha(c')| \le C_\alpha |c - c'|, \qquad \forall\, c, c' \in \mathbb{H}^1,$$

and there exists $C_r > 0$ such that

$$|\bar{\gamma}_c(0) - \bar{\gamma}_c(-k_c^+)| \le C_r K, \qquad \forall\, c \in \mathbb{H}^1.$$

Note there is a subset in $\mathbb{H}^1$ such that $\sigma \in \mathbb{S}$ has one to one correspondence to this subset, we can write $c = c(\sigma)$ when c is in this subset. Since $m_c^+ \in N_\delta$, we obtain from the Lemma 6.3 and above estimate that

$$\begin{aligned}
&h_{c'}^\infty(\xi, m) - h_c^\infty(\xi, m)\\
\le\ &\int_0^{k_c^+} L_{\eta_{c'}}(d\gamma_c(t), t)dt + k_c^+\alpha(c') - h_c^{k_c^+}(m_c^+, m)\\
&+ h_{c'}^\infty(\xi, m_c^+) - h_c^\infty(\xi, m_c^+)\\
\le\ &C_3(\sqrt{|\sigma - \sigma'|} + |c(\sigma) - c(\sigma')|)
\end{aligned}$$

where $C_3 > 0$ is a positive number depending on $\max\{|A|, |B|\}$, C_α and C_α. By exchanging c with c', in the same way we obtain

$$h_c^\infty(\xi, m) - h_{c'}^\infty(\xi, m) \le C_3(\sqrt{|\sigma - \sigma'|} + |c(\sigma) - c(\sigma')|).$$

Recall the Lemma 3.2. For each $c = (c_q, c_x)$ with $\|c_x - y(c_q)\| < C_q$ ($C_q > 0$), the Mañé set keeps the same. Thus, we can assume that $y = y(c_q)$ is smooth in c_q. Therefore, from the Lemma 6.3 and the formula (6.3) we obtain

Lemma 6.4. *Assume $\sigma, \sigma' \in \mathbb{S}$. Let $c = (\sigma)$, $c' = c(\sigma')$, $m \in M \backslash N_\delta$ and $\xi \in N_\delta$. Then*

$$\left| h_{c(\sigma)}^\infty(\xi, m) - h_{c(\sigma')}^\infty(\xi, m) \right| \le C_3(\sqrt{|\sigma - \sigma'|} + |c - c'|),$$

$$\left| h_{c(\sigma)}^\infty(m, \xi) - h_{c(\sigma')}^\infty(m, \xi) \right| \le C_3(\sqrt{|\sigma - \sigma'|} + |c - c'|).$$

The function $h_{c(\sigma)}^\infty$, defined on $\sigma \in \mathbb{S}$, can be extended to a function $h_{c,\sigma}^\infty$ defined in a neighborhood of the continuous curve $\{\sigma, c_q(\sigma)\} \subset \mathbb{R}^2$ such that $h_{c(\sigma)}^\infty = h_{c,\sigma}^\infty|_{c=c(\sigma)}$ and the above formulas hold for $h_{c,\sigma}^\infty$.

Remark: We do not know whether the function $\sigma \to c(\sigma)$ has some Hölder continuity in σ.

Let us consider the case when there are several pieces of invariant cylinder. Under such condition, the barrier function may be not continuous at finitely many points if we consider the continuity along the path $\{c = (c_q, y(c_q)) : A-1 < p < B+1\}$. In fact, restricted on each cylinder Σ_j, Φ_H is an area-preserving and twist map. According to the hypothesis (**H2'**), $\partial_p h|_{x=x_j(p)} \ne \partial_p h|_{x=x_{j+1}(p)}$, we find that there exists a unique

$c_q = c_q^j$ with $p_j - \delta < c_q^j < p_j + \delta$ such that the c-minimal measure is uniquely supported on an Aubry-Mather set in Σ_j if $c_q^{j-1} < c_q < c_q^j$ and it has exactly two ergodic components when $c_q = c_q^j$. One component corresponds to an Aubry-Mather set in Σ_k, another one corresponds to an Aubry-Mather set in Σ_{j+1}. When $c = (c_q^j, y(c_q^j))$, the Mañé set contains these two Aubry-Mather sets and the minimal orbits connecting them. Due to the non-degenerate condition (**H2'**), the normally hyperbolic cylinder Σ_j is long enough so that it's interior contains all Aubry-Mather set for those c with $c_q^{j-1} \le c_q \le c_q^j$.

7. Generic property

In this section we also assume that $k = 1$. The task in this section is to show that there is a residual set in $\mathcal{B}_{\epsilon,K}$ such that if P is in this set then there is a generalized transition chain $\{c \in H^1(M,\mathbb{R}) : c = (c_q, y(c_q)), A-1 \le c_q \le B+1\}$. Since there are finitely many invariant cylinders $\{\Sigma_i\}$, it is sufficient to verify the generic property for one of these cylinders.

Let us consider this issue from the Hamiltonian dynamics point of view. Since the system is positive definite in action variable $v = (p, y)$, it has a generating function $G(u, u')$ $(u = (q, x))$

$$G(u, u') = \inf_{\substack{\gamma \in C^1([0,1],\mathbb{R}^{n+1}) \\ \gamma(0)=u, \gamma(1)=u'}} \int_0^1 L(\gamma(s), \dot\gamma(s), s) ds,$$

where u, u' are in the universal covering space $\mathbb{R}^{n+1}$. Clearly, $G(u + 2k\pi, u' + 2k\pi) = G(u, u')$ for each $k \in \mathbb{Z}^{n+1}$. The map Φ_H: $(u, v) \to (u', v')$ is given by

$$v' = \partial_{u'} G(u, u'), \qquad v = -\partial_u G(u, u'),$$

let π_2 be the standard projection from $\mathbb{R}^{n+1} \to \mathbb{T}^{n+1}$, let $c \in \mathbb{R}^{n+1}$ and

$$G_c(u, u') = G(u, u') - \langle c, u' - u \rangle$$

then

$$h_c(m, m') = \min_{\substack{\pi_2(u)=m \\ \pi_2(u')=m'}} G_c(u, u') + \alpha(c).$$

We consider the change of the function h_c^∞ when the generating function is subject to a small perturbation $G \to G + G_1$. Let $m \in M \backslash N_\delta$, $\xi \in \mathcal{M}_0(c)$, $c = (c_q, y(c_q))$. Let $\{k_i\}$ be a subsequence such that

$$\lim_{i\to\infty} h_c^{k_i}(\xi, m) = \liminf_{k\to\infty} h_c^k(\xi, m),$$

let $\{u_1 = \xi, u_2, \cdots, u_{k_i} = m\}$ be the minimal configuration, we claim that there exists $b > 0$ such that $u_i \notin \mathcal{B}_b(m)$ for each $1 \le i \le k_i - 1$, here $\mathcal{B}_b(m)$ denotes a b-ball centered at m. In fact, there is a positive number $A > 0$ such that $h_c^k(m, m) \ge 2A$ for each $c = (c_q, y(c_q)) \in H^1(M, \mathbb{R})$, for each $k \in \mathbb{Z}^+$ and for $m \in M \backslash N_\delta$. If not, there exists a subsequence k_j such that

$$\lim_{k_j \to \infty} h_c^{k_j}(m, m) = 0.$$

It implies that $m \in \mathcal{A}_0(c)$, which contradicts the Lemma 3.1. As $h_c^k(m', m)$ is Lipschitz in m, m', there exists $b > 0$ such that if $m' \in \mathcal{B}_b(m)$ then $h_c^k(m', m) \geq A$ for each $k \in \mathbb{Z}^+$. If there is $u_i \in \mathcal{B}_b(m)$ for some $i \in \{k_i\}$, let m' be an accumulation point of $\{u_i\}$ then there exists some $k \in \mathbb{Z}^+$ such that

$$h_c^\infty(\xi, m) = h_c^\infty(\xi, m') + h_c^k(m', m).$$

Consequently,

$$h_c^\infty(\xi, m') \leq h_c^\infty(\xi, m) - A.$$

On the other hand, from the Lipschitz property we obtain that

$$h_c^\infty(\xi, m') \geq h_c^\infty(\xi, m) - C_L \|m - m'\|.$$

It leads to a contradiction if m' is sufficiently close to m. The contradiction verifies our claim. Consequently, if the generating function subjects to a small perturbation $G(u, u') \to G(u, u') + G_1(u')$, where $\mathrm{supp}(G_1) \subseteq \mathcal{B}_b(m)$, h_c^∞ will also undergo the small perturbation:

$$h_c^\infty(\xi, m') \to h_c^\infty(\xi, m') + G_1(m'), \qquad \forall m' \in \mathcal{B}_b(m),\ \xi \in \mathcal{M}_0(c);$$

while $h_c^\infty(m', \xi)$ remains the unchanged.

Choose $\xi, \zeta \in \mathcal{M}_0(c)$, $m \in M \backslash N_\delta$. The change of $h_{c,e_1}^\infty(\xi, m, \zeta)$ is a little bit complicated when the generating function undergoes the same small perturbation as above. Let $\{k_i\}$ be a subsequence such that

$$\lim_{i \to \infty} h_{c,e_1}^{k_i}(\xi, m, \zeta) = \liminf_{k \to \infty} h_{c,e_1}^k(\xi, m, \zeta).$$

Let $\{u_0 = \xi, u_1, \cdots, u_{l_i} = m, \cdots, u_{k_i} = \zeta\}$ be the minimal configuration that realizes the minimal action $h_{c,e_1}^{k_i}(\xi, m, \zeta)$, denote by γ_i: $[0, k_i] \to M$ the corresponding minimal curve. As the first step, we claim that for each $m \in M \backslash N_\delta$, when k_i is sufficiently large, there exists $b > 0$ such that there is at most one $u_{j_i} \in \mathcal{B}_b(m)$ for some $1 \leq j_i < k_i$, $j_i \neq l_i$. To a curve γ: $[0, k] \to M$ with $\gamma(0), \gamma(k) \in \mathcal{B}_b(m)$ we can associate an element $[\gamma] = ([\gamma]_q, [\gamma]_{x_1}, [\gamma]_{x_2}, \cdots, [\gamma]_{x_n}) \in H_1(M, \mathcal{B}_b(m), \mathbb{Z})$. If there were two other points $u_{j_i}, u_{j_i'} \in \mathcal{B}_b(m)$ (without losing of generality we assume $j_i < j_i' < l_i$), then we would have two alternatives:

1, either $[\gamma|_{[j_i, j_i']}]_{x_1} = 0$, or $[\gamma|_{[j_i', l_i]}]_{x_1} = 0$, or both;

2, both $[\gamma|_{[j_i, j_i']}]_{x_1} \neq 0$ and $[\gamma|_{[j_i', l_i]}]_{x_1} \neq 0$.

In the first case, we can cut off a piece $\gamma|_{[j_i,j_i']}$ from the minimal curve and define a curve γ': $[0,k_i-j_i'+j_i]\to M$ such that

$$\gamma'(t)=\begin{cases}\gamma(t) & t\in[0,j_i],\\ \eta(t) & t\in[j_i,j_i+1],\\ \gamma(t-j_i'+j_i+1) & t\in[j_i+1,k_i-j_i'+j_i],\end{cases}$$

where η: $[j_i,j_i+1]\to M$ is a minimal curve joining $\gamma(j_i)$ with $\gamma(j_i'+1)$. Clearly, $[\gamma']_{x_1}\neq 0$. Since $\gamma(j_i)$ is close to $\gamma(j_i')$, by the Lipschitz property of $h_c(m,m')$ in m,m', we have

$$\int_0^{k_i-j_i'+j_i} L(d\gamma'(t),t)dt+(k_i-j_i'+j_i)\alpha(c)\ \le\ h_{c,e_1}^{k_i}(\xi,m,\zeta)-A. \tag{7.1}$$

To see the absurdity of (7.1), let us observe a simple fact: if some $\{u_0=\xi,\cdots,u_{k_i}=\zeta\}$ is the minimizing sequence and if $u_j\in\mathcal{B}_b(m)$, then $j\to\infty$, $k_i-j\to\infty$ as $i\to\infty$. It implies that $k_i-j_i'+j_i\to\infty$. So (7.1) contradicts the definition of h_c^∞ since we choose $k_i\to\infty$ being such a subsequence that $\lim_{k_i\to\infty}h_{c,e_1}^{k_i}(\xi,m,\zeta)=h_{c,e_1}^\infty(\xi,m,\zeta)$. For the second alternative, by cutting off one piece $\gamma|_{[j_i,j_i']}$ or both $\gamma|_{[j_i,j_i']}$ and $\gamma|_{[j_i',l_i]}$ we can construct a curve γ' such that $[\gamma']_{x_1}\neq 0$ and (7.1) holds for γ'. But it is also absurd.

For the second step, let us recall that the support of each $c=(c_q,y(c_q))$-minimal measure is in a small δ-neighborhood of the circle N_δ if the perturbation is sufficiently small. Let $\gamma_c^+(t,m)$ be the forward c-semi-static curve such that $\gamma_c^+(0,m)=m$, let $\gamma_c^-(t,m)$ be the backward c-semi-static curve such that $\gamma_c^-(0,m)=m$. Denote by $W^{s,u}(c)$ the stable and unstable set for the support of the c-minimal measure, which contains forward and backward semi-static orbits respectively, we define

$$S_{t_0,t_1}=\bigcup_{\substack{u\in\partial N_{2k\delta}\\ (u,\dot u)\in W^{s,u}(c)\\ t_0\le t\le t_1}}\pi\phi_L^t(u,\dot u).$$

If δ is suitably small and k is suitably large, then

i, $H_1(N_{3k\delta},N_{k\delta},\mathbb{Z})=0$;

ii,

$$\Big\{\pi\phi_L^t(u,\dot u):\forall\ u\in\partial N_{2k\delta},\ (u,\dot u)\in W^{s,u}(c),\ t\in[-3,3]\Big\}\subset N_{3k\delta}\backslash N_{k\delta};$$

iii, $\gamma_c^+(t,m)$, $\gamma_c^-(-t,m)$ stay in $N_{3k\delta}$ for all $t\ge 0$ if $m\in S_{-2,2}$.

Thus, there exists $b>0$ such that $\gamma_c^+(t,m)$ does not visit $\mathcal{B}_b(m)$ again when $t\ge 1$, and $\gamma_c^-(t,m)$ does not visit $\mathcal{B}_b(m)$ again when $t\le -1$.

We assume the c-minimal measure is uniquely ergodic. Given $u\in S_{-2,2}$, there is a minimizing curve $\gamma(t)$ for $h_{c,e_1}^\infty(\xi,u,\zeta)$ in the sense that there are two sequence $\{k_i^+\}$, $\{k_i^-\}$, such that $\gamma(k_i^+)\to\xi$, $\gamma(-k_i^-)\to\zeta$,

$\gamma(0) = u$ and

$$h^\infty_{c,e_1}(\xi, x, \zeta) = \lim_{\substack{k_i^+\to\infty \\ k_i^-\to\infty}} \int_{-k_i^-}^{k_i^+} (L_c(d\gamma(t,u)) + \alpha(c))dt.$$

Obviously, γ is smooth everywhere except for $t = 0$, and either $\{d\gamma(t, x) : t \le 0\} \subset N_{3k\delta}$ or $\{d\gamma(t, x) : t \ge 0\} \subset N_{3k\delta}$ alternatively.

Let $m \in S_{-1/2,1/2}$. We consider the case that the minimal curve $\{\gamma(t)\}$ of $h^\infty_{c,e_1}(\xi, m, \zeta)$ returns to a small ball $\mathcal{B}_b(m)$ in the sense that there is some $i \in \mathbb{Z}\backslash 0$ such that $\gamma(i) \in \mathcal{B}_b(m)$. Without losing generality, we assume $i > 0$, the case that $i < 0$ can be treated similarly. Let us consider the minimizing curve $\gamma_2(t)$ for $h^\infty_{c,e_1}(\xi, \gamma(2), \zeta)$ and $\gamma_2(0) = \gamma(2)$.

In this case, $\gamma_2(t) = \gamma(t + 2)$ for all $t \ge 0$. We claim that $\gamma_2(t)$ does not return to $\mathcal{B}_b(\gamma(2))$ if we choose suitably small $b > 0$. Otherwise, from the continuity of the solution of ODE on initial value we can see that the minimal curve γ shall return again to a small neighborhood of m at $t = 2i$, but this contradicts to the fact there is at most one point of the minimal configuration falls into the small ball $\mathcal{B}_b(m)$. Obviously, $\{\gamma_1(t) : t \le 0\} \subset N_{3k\delta}$ in this case.

If for some $m \in S_{-5/2,5/2}$, the minimizing curve γ with $\gamma(0) = m$ for $h^\infty_c(\xi, m, \zeta)$ does not return to $\mathcal{B}_{2b}(m)$ for all $k \in \mathbb{Z}\backslash\{0\}$, then for m' suitably close to m and c' suitably close to c, the minimizing curve $\gamma'(t)$ for $h^\infty_{c'}(\xi, m', \zeta)$ with $\gamma'(0) = m'$ does not return to $\mathcal{B}_{2b}(x)$ for all $k \in \mathbb{Z}^+\backslash\{0\}$ either. Therefore, there is an open set $O_c \subset S_{-3,3}$ and a neighborhood for c such that for each $m \in O_c$, each c' in the neighborhood of c, and each minimizing curve γ for $h^\infty_{c',e_1}(\xi, m, \zeta)$ with $\gamma(0) = m$, $\gamma(i) \notin \mathcal{B}_b(x)$ for all $i \in \mathbb{Z}\backslash\{0\}$.

Note any minimizing curve γ for $h_{c,e_1}(\xi, m, \zeta)$ must pass through $S_{-1/2,1/2}$, we have proved

Lemma 7.1. *Assume that the c-minimal measure is uniquely ergodic for each $c = (c_q, y(c_q))$ $(A < c_q < B)$. There is a small positive number $\epsilon = \epsilon(A, B, \kappa) > 0$, if $\|L_1\|_{C^r} \le \epsilon$, then:*

i) *an open sets O_c exists such that $\forall\ d\gamma \in \tilde{\mathcal{N}}(c, \tilde{M})\backslash\tilde{\mathcal{N}}(c, M)$, there is $k \in \mathbb{Z}$ such that $\gamma(k) \in O_c$;*

ii), *there is $b > 0$ such that for each $c = (c_q, y(c_q))$ with $A < c_q < B$, if the generating function is subject to a small perturbation $G_c(u, u') \to G_c(u, u') + G_1(u')$, where $\mathrm{supp}(G_1) \subseteq \mathcal{B}_b(u) \subset O_c$, then the barrier function undergoes a small perturbation:*

$$B^*_{c,e_1}(u) \to B^*_{c,e_1}(u) + G_1(u) + \text{a small constant}.$$

iii), *$\tilde{U} = \cup_{A<c_q<B} O_c$ is open in $[A, B] \times M$, $U = \pi\tilde{U} \subset M\backslash N_\delta$, where π is a standard projection from $[A, B] \times M$ to M.*

The next goal is to show that the density of the set $\{P \in C^r : \{u \in M\backslash N_\delta : B^*_{c,e_1}(u) = \min_u B^*_{c,e_1}(u)\}$ is totally disconnected$\}$.

For convenience of notation, we set $x_0 = q$. Let $R_d(u^*) = \{u \in M : |x_i - x_i^*| \le d, \forall\ 0 \le i \le n\} \subset \mathcal{B}_b(u^*)$, $S_{c,\sigma} = B^*_{c(\sigma)} + G_1$, we say a connected set V is non-trivial for R_d if $\Pi_i(V \cap R_d) = \{x_i^* - d \le x_i \le x_i^* + d\}$ for some $0 \le i \le n$. Here Π_i is the standard projection from $\mathbb{T}^{n+1}$ to its i-th component. Let $M_{d,u^*}(S) = \{u : S(u) = \min_{u \in R_d(u^*)} S\}$, we define a set in the function space $\mathfrak{F}(d, u^*) = C^0(R_d(u^*), \mathbb{R})$,

$$\mathfrak{Z}(d, u^*) = \Big\{ S \in \mathfrak{F}(d, u^*) : M_{d,u^*}(S) \supseteq \text{a set non-trivial for } R_d(u^*) \Big\}.$$

We define $\mathfrak{Z}_i$ $(i = 0, 1, \cdots, n)$:

$$\mathfrak{Z}_i = \Big\{ S \in \mathfrak{Z}(d, u^*) : \Pi_i(M_{d,u^*}(S)) = \{x_i^* - d \le x_i \le x_i^* + d\} \Big\}.$$

Clearly:

$$\mathfrak{Z}(d, u^*) = \bigcup_{i=0}^{n} \mathfrak{Z}_i.$$

We claim that for each generating function $G \in C^r(M \times M, \mathbb{R})$ and each $\epsilon > 0$, there is an open and dense set $\mathfrak{H}(d, u^*)$ of $\mathcal{B}_\epsilon(0) \subset C^r(R_d(u^*), \mathbb{R})$, for each $G_1 \in \mathfrak{H}(d, u^*)$, the image of S_σ from $[A', B']$ to $\mathfrak{F}$ has no intersection with the set $\mathfrak{Z}_i$.

Obviously, the set $\mathfrak{Z}_i$ is a closed set and has infinite co-dimension in the following sense, there exists $\mathfrak{N}_i$, an infinite dimensional subspace of $\mathfrak{F}$, such that $(S+F) \notin \mathfrak{Z}_i$ for all $S \in \mathfrak{Z}_i$ and $F \in \mathfrak{N}\backslash\{0\}$. In fact, for each non constant function $F(x_i) \in C^0([x_i^* - d, x_i^* + d], \mathbb{R})$ with $F(x_i^*) = 0$ and each $S \in \mathfrak{Z}_i$ we have $S + F \notin \mathfrak{Z}_i$. Thus, we can choose

$$\mathfrak{N}_i = C^0([x_i^* - d, x_i^* + d], \mathbb{R})/\mathbb{R},$$

which we think as the subspace of $C^0(R_d(u^*), \mathbb{R})$ consisting of those continuous functions independent of other coordinate components x_j $(j \ne i)$.

Let

$$\mathfrak{F}_\sigma = \{B^*_{c(\sigma),e_1} : \sigma \in [A', B']\}.$$

Clearly, this set is determined by the generating function G. Recall the Lemma 6.4, we can extend $B^*_{c(\sigma),e_1}$ to a function $[A, B] \times [A', B'] \to \mathfrak{F}$ that has $\frac{1}{2}$-Hölder continuity, the image of the continuous curve $\{\sigma, c(\sigma)\} \subset [A, B] \times [A', B']$ is compact and its box dimension is not bigger than 4,

$$D_B(\mathfrak{F}_\sigma) \le 4.$$

Let

$$\mathfrak{N} = \bigoplus_{i=0}^{n} \mathfrak{N}_i.$$

Lemma 7.2. *There is an open and dense set $\mathfrak{N}^* \subset \mathfrak{N}$ such that for all $F \in \mathfrak{N}^*$*

$$(\mathfrak{F}_\sigma + F) \cap \mathfrak{Z} = \varnothing. \tag{7.2}$$

Proof. : We show $(\mathfrak{F}_\sigma + F) \cap \mathfrak{Z}_i = \varnothing$ for each $0 \le i \le n$. The open property is obvious. If there was no density property, for any $k \in \mathbb{Z}$, there would be a k-dimensional ϵ-ball $\mathcal{B}_\epsilon \subset \mathfrak{N}_i$ for some $\epsilon > 0$, for each $F \in \mathcal{B}_\epsilon$, there would exist $S \in \mathfrak{F}_\sigma$ such that $F + S \in \mathfrak{Z}_i$.

For each $S \in \mathfrak{F}_\sigma$ there is only one $F \in \mathcal{B}_\epsilon$ such that $S + F \in \mathfrak{Z}_i$, otherwise, there would be $F' \ne F$ such that $S + F' \in \mathfrak{Z}_i$. Since we can write $F' + S = F' - F + F + S$ where $S + F \in \mathfrak{Z}_i$ and $F' - F \in \mathfrak{N}_i \backslash \{0\}$, this contradicts to the definition of $\mathfrak{N}_i$. Given $F \in \mathcal{B}_\epsilon$, there might be more than one element in $\mathfrak{S}_F = \{S \in \mathfrak{F}_\sigma : S + F \in \mathfrak{Z}_i\}$. Given any two $F_1, F_2 \in \mathcal{B}_\epsilon$, for any $S_1 \in \mathfrak{S}_{F_1}$ and any $S_2 \in \mathfrak{S}_{F_2}$ we have

$$\begin{aligned} d(S_1, S_2) &= \max_{u \in R_d(u^*)} |S_1(u) - S_2(u)| \\ &\ge \max_{|x_i - x_i^*| \le d} \Big| \min_{\substack{|x_j - x_j^*| \le d \\ j \ne i}} S_1(u) - \min_{\substack{|x_j - x_j^*| \le d \\ j \ne i}} S_2(u) \Big| \\ &= \max_{|x_i - x_i^*| \le d} |F_1(x_i) - F_2(x_i)| \\ &= d(F_1, F_2) \end{aligned} \tag{7.3}$$

where $d(\cdot,\cdot)$ denotes the C^0-metric. It follows from (7.3) and the definition of box dimension that

$$D_B(\mathfrak{F}_\sigma) \ge D_B(\mathcal{B}_\epsilon) = k,$$

but this is absurd if we choose $k > 4$. q.e.d.

As C^r is dense in C^0, an open and dense set $\mathfrak{H}(d, u^*) \subset C^r(R_d(u^*), \mathbb{R})$ clearly exists such that for each function $G_1 \in \mathfrak{H}(d, u^*)$, we have

$$(\mathfrak{F}_\sigma + G_1) \cap \mathfrak{Z}(d, u^*) = \varnothing, \qquad \forall\ \sigma \in \mathbb{S},$$

where by abuse of terminology we continue to denote S_σ and its restriction to $R_d(u^*)$ by the same symbol.

Let U be an open set, $M_U(S) = \{u : S(u) = \min_{u \in U} S\}$ and

$$\mathfrak{Z}^c = \Big\{ S \in C^0(U, \mathbb{R}) :\ M_U(S) \text{ is totally disconnected } \Big\}.$$

Given $d_i > 0$, there are finitely many u_{ij} such that $\cup_j R_{d_i}(u_{ij}) \supseteq U$. Thus there exists a sequence $d_i \to 0$ and a countable set $\{u_{ij}\}$ such that for each $G_1 \in \bigcap_{i,j=1}^\infty \mathfrak{H}(d_i, u_{ij})$

$$\mathfrak{F}_\sigma + G_1 \subset \mathfrak{Z}^c.$$

Recall the Lemma 3.4, we have the following

Lemma 7.3. *There exists a residual set $\mathcal{S}_\epsilon \subset \mathcal{B}_\epsilon \subset C^r(U, \mathbb{R})$ ($r = \omega, \infty, \text{or} \geq 3$), for each $G_1 \in \mathcal{S}_\epsilon$*

$$\pi_1 \mathcal{N}_0(c(\sigma), \tilde{M}) \backslash (\mathcal{N}_0(c(\sigma), M) + \delta) \text{ is totally disconnected}$$

when $\sigma \in \mathbb{S}$.

Proof. : For $r = \infty$ or $r =$ finite, we can write $G_1 = \sum_i G_{1i}$ where G_{1i} has its support at $R_{d_i}(u)$ and there exists $d_i^q > 0$ such that $R_{d_i}(u) \times \{|c_q - c_q^*| \leq d_i^q\} \subset \tilde{U}$ and $\cup_i R_{d_i}(u) \times \{|c_q - c_q^*| \leq d_i^q\} = \tilde{U}$, taking the intersection of countably many open-dense sets of C^r-function space, we obtain the residual property.

The perturbation to the generating function G can be achieved by perturbing the Hamiltonian function $H \to H' = H + \delta H$. To do that, let us introduce a differentiable function κ: $M \to \mathbb{R}$ such that $0 \leq \kappa(u - u') \leq 1$, $\kappa(u-u') = 1$ if $|u-u'| \leq K$ and $\kappa(u-u') = 0$ if $|u-u'| \geq K+1$. We choose sufficiently large K so that $\{\|v\| \leq \max(|A|, |B|) + 1\}$ is contained in the set where $|u - u'| \leq K$. Let Φ' be the map determined by the generating function $G + \kappa G_1$, the symplectic diffeomorphism $\Psi = \Phi' \circ \Phi^{-1}$ is closed to identity. We choose a smooth function $\rho(s)$ with $\rho(0) = 0$ and $\rho(1) = 1$, let Φ'_s be the symplectic map determined by $G + \rho(s)\kappa G_{1i}$, let $\Psi_s = \Phi'_s \circ \Phi^{-1}$. Clearly, Ψ_s defines a symplectic isotopy between identity map and Ψ. Thus, there is a unique family of symplectic vector field X_s: $T^*M \to TT^*M$ such that

$$\frac{d}{ds}\Psi_s = X_s \circ \Psi_s.$$

By the choice of perturbation, there is a simply connected and compact domain D_K such that $\Psi_s|_{T^*M \backslash D_K} = id$. It follows that there is a hamiltonian $H_1(u, v, s)$ such that $dH_1(Y) = dv \wedge du(X_s, Y)$ holds for any vector field Y. Re-parameterizing s by t we can make H_1 smoothly and periodically depend on t. To see that dH_1 is also small, let us mention a theorem of Weinstain [**W**]. A neighborhood of the identity map in the symplectic diffeomorphism group of a compact symplectic manifold $\mathbf{M}$ can be identified with a neighborhood of the zero in the vector space of closed 1-forms on $\mathbf{M}$. Since Hamiltomorphism is a subgroup of symplectic diffeomorphism, there is a function H', sufficiently close to H, such that $\Phi_{H_1} \circ \Phi_H = \Phi_{H'}^t|_{t=1}$.

The perturbation made to H does not change the dynamics around the cylinder, it means that the set of invariant circles remains unchanged if H is subject to the perturbation constructed this way.

In the case of twist map, each co-homology class corresponds to a unique rotation number. Obviously, for each rotation number $p/q \in \mathbb{Q}$, there is an open and dense set in the space of area-preserving twist maps such that there is only one minimal (p, q)-periodic orbit without homoclinic loop. Take the intersection of countably open dense sets it is a generic property that there is only one minimal (p, q)-periodic orbit without homoclinic loop for all $p, q \in \mathbb{Z}$. Recall that the minimal measure is always uniquely ergodic when the rotation number is irrational, there is a residual set in $\mathcal{B}_{\epsilon,K}$, such that if L_1 is in this set, then there is a generalized transition chain Γ: $[0,1] \to H^1(M,\mathbb{R}) \cap \{c_x = 0\}$ which connects $\{c_q \le A\}$ with $\{c_q \ge B\}$. For each c in a transition piece, $\mathcal{M}(c)$ is uniquely ergodic, thus the conditions of the Theorem 5.1 are satisfied. Therefore, there is a residual set of small perturbations in $\mathcal{B}_{\epsilon,K}$, such that the perturbed systems has an orbit connecting $\{p < A\}$ to $\{p > B\}$.

Therefore, the proof of the Theorem 1.2 is completed. The Theorem 1.1 is a special case of the Theorem 1.2. q.e.d.

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