

**FLAT SURFACES WITH SINGULARITIES
IN EUCLIDEAN 3-SPACE**

SATOKO MURATA & MASAACKI UMEHARA

Abstract

It is classically known that complete flat (that is, zero Gaussian curvature) surfaces in Euclidean 3-space \mathbf{R}^3 are cylinders over space curves. This implies that the study of global behaviour of flat surfaces requires the study of singular points as well. If a flat surface f admits singularities but its Gauss map ν is globally defined on the surface and can be smoothly extended across the singular set, f is called a *frontal*. In addition, if the pair (f, ν) defines an immersion into $\mathbf{R}^3 \times S^2$, f is called a *front*. A front f is called *flat* if the Gauss map degenerates everywhere. The parallel surfaces and the *caustic* (i.e. focal surface) of a flat front f are also flat fronts. In this paper, we generalize the classical notion of completeness to flat fronts, and give a representation formula for a flat front which has a non-empty compact singular set and whose ends are all immersed and complete. As an application, we show that such a flat front has properly embedded ends if and only if its Gauss map image is a convex curve. Moreover, we show the existence of at least four singular points other than cuspidal edges on such a flat front with embedded ends, which is a variant of the classical four vertex theorem for convex plane curves.

Introduction

In this paper, we shall investigate the global behaviour of flat surfaces with singularities in Euclidean 3-space \mathbf{R}^3 . In fact, for the study of global properties of flat surfaces, considering only immersions is too restrictive, as is already clear from the classical fact (Fact 0.1) below.

Let M^2 be a smooth 2-manifold and $f : M^2 \rightarrow \mathbf{R}^3$ a C^∞ -map. A point $p \in M^2$ is called *regular* if f is an immersion on a sufficiently small neighborhood of p , and is called *singular* if it is not regular. If f is an immersion and has zero Gaussian curvature, it is called a (regular) flat surface. It is classically known that regular flat surfaces have open dense

The second author was partly supported by the Grant-in-Aid for Scientific Research (B), Japan Society for the Promotion of Science.

Received 05/15/2006.

subsets which consist of a cylinder, cone, or tangential developable of space curves. Moreover, the following result is also well-known:

Fact 0.1. *A complete regular flat surface is a plane or a cylinder.*

The first proof of this fact came as a corollary of Hartman and Nirenberg [10]. After that Stoker [25] and Massey [20] gave elementary proofs.

To extend flat surfaces to a larger class that allows singularities, we shall now give the following definitions: A C^∞ -map $f : M^2 \rightarrow \mathbf{R}^3$ is called a *frontal* if there exists a smooth unit vector field ν along f such that ν is perpendicular to $df(TM^2)$. By parallel displacement, ν can be considered as a map into the unit sphere S^2 . In this case, ν is called the *Gauss map* of the frontal f . Moreover, if the map

$$L := (f, \nu) : M^2 \rightarrow \mathbf{R}^3 \times S^2$$

gives an immersion, f is called a *front* or a *wave front*. Using the canonical inner product, we identify the unit tangent bundle $\mathbf{R}^3 \times S^2 = T_1\mathbf{R}^3$ with the unit cotangent bundle $T_1^*\mathbf{R}^3$, which has the canonical contact structure. When f is a front, L gives a Legendrian immersion. Hence, wave fronts are considered as projections of Legendrian immersions.

For a frontal $f : M^2 \rightarrow \mathbf{R}^3$ and a real number δ , a new frontal $f_\delta : M^2 \rightarrow \mathbf{R}^3$ is given by

$$(0.1) \quad f_\delta := f + \delta\nu$$

which is called a *parallel surface* of f . A frontal f is called *flat* if the Gauss map $\nu : M^2 \rightarrow S^2$ degenerates. Since the members of the parallel family $\{f_\delta\}$ have the same Gauss map ν in common, *parallel surfaces of a flat front are also flat*. Moreover, *the caustics (i.e. focal surfaces) of flat fronts are also flat* (Section 1).

Flat surfaces might not have globally defined normal vector fields, in general. A C^∞ -map $f : M^2 \rightarrow \mathbf{R}^3$ is called a *p-frontal* (resp. *p-front*) if for each $p \in M^2$, there exists a neighborhood V such that the restriction $f|_V$ of f to V gives a frontal (resp. front). That is, p-front (i.e. projective front) is locally a front, but the unit normal vector field ν might not extend globally. A p-frontal is called *co-orientable* if it is a front (that is, ν is globally defined on M^2). The class of flat p-fronts is a canonical generalization of that of regular flat surfaces. In fact, the existence of a real analytic flat Möbius strip is known ([30]), which is an example of a flat p-front without singular points.

We will now give two definitions for the completeness of flat fronts.

DEFINITION 0.2. A flat p-front $f : M^2 \rightarrow \mathbf{R}^3$ with first fundamental form $ds^2 = df \cdot df$ is called *complete* (cf. [15, p150]) if there exists a symmetric covariant tensor T on M^2 with compact support such that $ds^2 + T$ gives a complete metric on M^2 . If f is complete and the singular set is non-empty, then the singular set must be compact, by definition.

Moreover, this new definition of completeness for flat fronts is a generalization of the classical one, namely, if f is complete and has no singular points, then completeness coincides with the classical notion of completeness in Riemannian Geometry.

On the other hand, f is called *weakly complete* if the sum of the first fundamental form and the third fundamental form (called the *lift metric* of the p-front f)

$$(0.2) \quad ds_{\#}^2 := df \cdot df + d\nu \cdot d\nu$$

gives a complete Riemannian metric on M^2 , which is the pull-back of the canonical metric on $T_1\mathbf{R}^3$ by $L = (f, \nu)$.

Obviously, *completeness implies weak completeness*. Completeness is convenient to describe the asymptotic behaviour of ends, but is not preserved when lifting to the universal cover. On the other hand, the universal cover of a weakly complete flat front is also weakly complete. Anyhow, using these definitions, we will prove our deepest result (Theorem A below):

Theorem A. *Let M^2 be a connected 2-manifold, and $f : M^2 \rightarrow \mathbf{R}^3$ a complete flat p-front whose singular set is non-empty (namely, the singular set of f is compact and all ends of f are immersed and complete). Then it has no umbilic points and is a front. Moreover, M^2 is diffeomorphic to a circular cylinder.*

In particular, if a complete flat p-front admits an umbilic point, it must be a cylinder. We remark that Theorem A has two surprising consequences:

- A complete flat p-front is a front. In particular, it never contains a Möbius strip as a subset.
- Singular points and umbilics never appear on the same complete flat front.

Moreover, as a consequence of Theorem A, we get the following representation formula for complete flat fronts:

Theorem B. *Let $\hat{\xi} : S^1 \rightarrow S^2$ be a regular curve without inflection points, and $\alpha = a(t)dt$ a 1-form on $S^1 = \mathbf{R}/2\pi\mathbf{Z}$ satisfying the period condition*

$$\left(\int_{S^1} \hat{\xi}\alpha \right) = \int_0^{2\pi} \hat{\xi}(t)a(t)dt = 0,$$

where $\hat{\xi}\alpha$ is considered as a 2π -periodic \mathbf{R}^3 -valued 1-form on \mathbf{R} . Then

$$(0.3) \quad f(t, v) = \hat{\sigma}(t) + v\hat{\xi}(t), \quad \hat{\sigma}(t) := \int_0^t \hat{\xi}(t)a(t)dt$$

gives a complete flat front whose image of the singular set coincides with that of $\hat{\sigma}$. Conversely, any complete flat fronts with non-empty singular set are given in this manner.

A somewhat similar representation theorem for (regular) flat tori in the 3-sphere has been given by Kitagawa [13], that is, a pair of two closed regular spherical curves satisfying a certain compatibility condition correspond to a flat torus in S^3 . As a consequence of Theorem B, outside of a ball of sufficiently large radius centered at the origin, the image of a complete flat front splits into two components, called the *ends* of the front. We shall prove the following:

Theorem C. *An end of a complete flat front f with a singular point is properly embedded if and only if the other end is as well. Moreover, the ends of f are properly embedded if and only if the image of the Gauss map of f is a convex curve.*

It was shown in [16] that a given complete flat front in hyperbolic 3-space H^3 has embedded ends if and only if the sum of the topological degrees of the two hyperbolic Gauss maps of the surface coincides with the number of ends. (In contrast to our case, there are complete flat fronts in H^3 with arbitrary genus and with many ends.) The above theorem is an analogue for \mathbf{R}^3 , though our method is quite different from that of [16]. As an application of Theorem B, Theorem C and the technique of disconjugate operators (cf. [8] and [27]), we prove in this paper the following

Theorem D. *Let $f : M^2 \rightarrow \mathbf{R}^3$ be a complete flat front with non-empty singular set. Suppose that all ends of f are embedded. Then there exist at least four singular points of f on M^2 which are not cuspidal edges.*

The proof of Theorem D is an analogue of that of the classical four vertex theorem for convex plane curves. The embeddedness of ends is truly required in the above statement. In fact, there exists a complete flat front in \mathbf{R}^3 which admits only cuspidal edges. (See Example 5.3.) We show in Section 4 (see Corollary 4.10) that complete flat fronts which admit only cuspidal edges and swallowtails are generic. For such a generic front with embedded ends, Theorem D implies the existence of at least four swallowtails.

In Section 1, we give fundamental properties of flat fronts. In Section 2, we define admissibility of developable frontal and give a representation formula for admissible developable frontals. In Section 3, we extend the representation formula to p-fronts. As an application, we show the existence of umbilic points on a developable Möbius strip. In Section 4, Theorem A and Theorem B are proved. In Section 5, we shall prove Theorem C and Theorem D.

Finally, we mention several related works: [7] for flat fronts in S^3 ; [24], [6] for behaviour of the Gaussian curvature near cuspidal edges, swallowtails, and cuspidal cross caps; [29], [12], [4], [5],[18] for singularity of spacelike maximal surfaces in \mathbf{R}_1^3 and CMC-1 surfaces in de

Sitter 3-space; [20] for improper affine spheres with singularities; [11] for generic singularities of fronts of constant Gaussian curvature in \mathbf{R}^3 .

Acknowledgements. The authors thank Wayne Rossman, Kotaro Yamada and the referees for a careful reading of the first draft and for valuable comments. Their comments were especially helpful for making the revision. The authors also thank Osamu Saeki for valuable comments.

1. Flat surfaces as wave fronts

We fix a 2-manifold M^2 throughout this section. First we recall the following fundamental property of fronts:

Lemma 1.1. *Let M^2 be a 2-manifold and $p \in M^2$ a singular point of a front $f : M^2 \rightarrow \mathbf{R}^3$. Then for any sufficiently small non-zero real number δ , the point p is a regular point of the parallel surfaces f_δ .*

Proof. Since p is a singular point, the differential of the map $(df)_p : T_p M^2 \rightarrow T_{f(p)} \mathbf{R}^3$ is of rank less than 2. When $(df)_p$ is of rank zero, then the Gauss map ν of f is an immersion at p , and the parallel surface $f_\delta := f + \delta\nu$ is an immersion at p for all $\delta \neq 0$. So it is sufficient to consider the case that $(df)_p$ has a 1-dimensional kernel. Then, we can take a local coordinate $(U; u, v)$ such that f_u vanishes at p , where f_u, f_v are partial derivatives of the \mathbf{R}^3 -valued function f with respect to the parameters u, v . Since $f_u(p) = 0$, we have

$$(1.1) \quad 0 = f_u(p) \cdot \nu_v(p) = -f_{uv}(p) \cdot \nu(p) = f_v(p) \cdot \nu_u(p),$$

where ‘ \cdot ’ denotes the canonical inner product on Euclidean 3-space \mathbf{R}^3 . Since f is a front and $(df)_p$ has 1-dimensional kernel, $\nu_u(p), \nu_v(p)$ are both non-vanishing. Thus (1.1) yields that ν_u, ν_v are linearly independent at p , and the assertion follows immediately. q.e.d.

Let $f : M^2 \rightarrow \mathbf{R}^3$ be an immersion with globally defined unit normal vector field ν , and λ_1, λ_2 two (distinct) principal curvature functions. At umbilic points the two functions coincide, and they are continuous functions on M^2 and are C^∞ at non-umbilic points. The following fact is well known:

Fact 1.2. *The principal curvature functions $\lambda_{1,\delta}, \lambda_{2,\delta}$ of the regular set of each parallel surface $f_\delta := f + \delta\nu$ ($\delta \in \mathbf{R}$) satisfy*

$$(1.2) \quad \lambda_j = \lambda_{j,\delta}(1 - \delta\lambda_j) \quad (j = 1, 2).$$

Moreover, the singular set $S(f_\delta)$ of f_δ is given by

$$(1.3) \quad S(f_\delta) := \bigcup_{j=1,2} \{p \in M^2; \delta\lambda_j(p) = 1\}.$$

Now let $f : M^2 \rightarrow \mathbf{R}^3$ be a front. We fix a point $p \in M^2$ arbitrarily. By Lemma 1.1, there exist a neighborhood U of p and a real number c such that the parallel surface f_c gives an immersion on U . A point p is called an *umbilic point* of f if it is the umbilic point of one such f_c . By definition, the set of umbilic points \mathcal{U}_f is common in its parallel family.

Lemma 1.3. *Let $f : M^2 \rightarrow \mathbf{R}^3$ be a front and ν the unit normal vector field. For each non-umbilic point p , there exists a local coordinate $(U; u, v)$ of M^2 centered at p such that ν_u and ν_v are proportional to f_u and f_v on U respectively. In particular, the pair of vectors $\{f_u, \nu_u\}$ (resp. $\{f_v, \nu_v\}$) do not vanish at the same time.*

REMARK 1.4. Such a coordinate $(U; u, v)$ is called a *curvature line coordinate* of the front f . If f is an immersion, the principal curvature functions λ_1, λ_2 satisfy

$$(1.4) \quad \nu_u = -\lambda_1 f_u, \quad \nu_v = -\lambda_2 f_v,$$

which is called the Weingarten formula. The above lemma is a generalization of it for fronts.

Proof. By Lemma 1.1, there exists a parallel surface f_δ , which is regular at p . Since p is not an umbilic point, there exists a curvature line coordinate $(U; u, v)$ satisfying $\nu_u = -\lambda_{1,\delta}(f_\delta)_u$, $\nu_v = -\lambda_{2,\delta}(f_\delta)_v$. Since $f = f_\delta - \delta\nu$, we get the assertion. q.e.d.

DEFINITION 1.5. The direction v in $T_p M^2$ such that $df(v), d\nu(v)$ are linearly dependent is called a *principal direction* of a front f .

It can be easily checked that the two principal directions are common in the parallel family of a front.

Let $f : M^2 \rightarrow \mathbf{R}^3$ be a front with the unit normal vector field ν . We fix $p \in M^2 \setminus \mathcal{U}_f$ arbitrarily. By Lemma 1.3, we can take a curvature line coordinate $(U; u, v)$ containing p . Then $\{f_u, \nu_u\}$ and $\{f_v, \nu_v\}$ are linearly dependent respectively. So we define maps $\Lambda_j : M^2 \setminus \mathcal{U}_f \rightarrow P^1(\mathbf{R})$ ($j = 1, 2$) by

$$\Lambda_1 = [-\nu_u : f_u], \quad \Lambda_2 = [-\nu_v : f_v],$$

where $[-\nu_u : f_u]$ and $[-\nu_v : f_v]$ mean the proportional ratio of $\{-\nu_u, f_u\}$ and $\{-\nu_v, f_v\}$ respectively as elements of the real projective line $P^1(\mathbf{R})$. When f is an immersion, then we can write $\nu_u = -\lambda_1 f_u$, $\nu_v = -\lambda_2 f_v$, and we have $\Lambda_j = [-\lambda_j, 1]$ ($j = 1, 2$). If $f_u = \nu_u = 0$ at p , then f cannot be a front at p . So Λ_1 and Λ_2 are both well-defined C^∞ -maps, which are called the *principal curvature maps*. They are canonical generalizations of the principal curvature functions.

Proposition 1.6. *The principal curvature maps Λ_1, Λ_2 can be continuously extended to the entirety of M^2 .*

Proof. We fix $p \in M^2$ arbitrarily. By Lemma 1.1, there exists a parallel surface f_δ , which is regular on a neighborhood U of p . Then there are two principal curvature functions $\lambda_{1,\delta}, \lambda_{2,\delta}$. We set

$$\hat{\Lambda}_j = [-\lambda_{j,\delta} : 1 + \delta\lambda_{j,\delta}] \quad (j = 1, 2).$$

Since $\lambda_{1,\delta}, \lambda_{2,\delta}$ can be taken to be continuous on U , the maps $\hat{\Lambda}_1, \hat{\Lambda}_2$ are both continuous on U . Since p is arbitrary, $\hat{\Lambda}_1, \hat{\Lambda}_2$ are both continuous functions on M^2 . On the other hand, by Fact 1.2, $\hat{\Lambda}_1, \hat{\Lambda}_2$ coincide with the principal curvature maps of f defined on $M^2 \setminus \mathcal{U}_f$, so the assertion is proved. q.e.d.

The following assertion is now obvious from the definition of Λ_1 and Λ_2 .

Proposition 1.7. *Let $f : M^2 \rightarrow \mathbf{R}^3$ be a front, and Λ_1, Λ_2 the principal curvature maps. Then a point $p \in M^2$ is an umbilic point if $\Lambda_1(p) = \Lambda_2(p)$. On the other hand, p is a singular point if and only if $\Lambda_1(p) = [1 : 0]$ or $\Lambda_2(p) = [1 : 0]$.*

As defined in the introduction, a front $f : M^2 \rightarrow \mathbf{R}^3$ is called *flat* if its Gauss map degenerates everywhere. Since the Gauss map is common in the same parallel family, we get the following assertions.

Proposition 1.8. *Let $f : M^2 \rightarrow \mathbf{R}^3$ be a front and ν the unit normal vector field. Then f is flat if and only if, for each point p , there exist a real number δ and a neighborhood U of p such that the parallel surface f_δ is an immersion on U and its Gaussian curvature vanishes identically on U .*

Corollary 1.9. *Let $f : M^2 \rightarrow \mathbf{R}^3$ be a front. Suppose that the regular set $R(f)$ of f is dense in M^2 . Then f is flat if and only if f has zero Gaussian curvature on $R(f)$.*

Proposition 1.10. *Let p be an umbilic point of a flat front $f : M^2 \rightarrow \mathbf{R}^3$, then p is a regular point of f .*

Proof. By the definition of umbilic points for fronts, both of ν_u and ν_v vanish at p . Since (f, ν) is an immersion, f must be an immersion. Thus p is a regular point of f . q.e.d.

Corollary 1.11. *Let $f : M^2 \rightarrow \mathbf{R}^3$ be a flat front and $p \in M^2$ a singular point. Then p is a regular point of f_δ for all $\delta \neq 0$.*

Proof. Since p is a singular point, it is not an umbilic point. Thus the assertion follows from Lemma 1.1 and (1.3), since we may set one of the principal curvature functions to be vanishing identically. q.e.d.

Now we fix a flat front $f : M^2 \rightarrow \mathbf{R}^3$ with a unit normal vector field ν . Then we may assume that $\Lambda_2 = [0 : 1]$ identically. (That is, v -lines

correspond to the asymptotic direction.) So we may set

$$\Lambda := \Lambda_1 : M^2 \rightarrow P^1(\mathbf{R}),$$

which is called the *principal curvature map* of the flat front f . Moreover, the function

$$\rho := \frac{\psi_2}{\psi_1} : M^2 \rightarrow \mathbf{R} \cup \{\infty\} \quad (\Lambda = [-\psi_1 : \psi_2])$$

is called the *curvature radius function*. If f is an immersion, ρ coincides with the usual curvature radius function. The restriction of ρ to $M^2 \setminus \mathcal{U}_f$ is a real valued C^∞ -function.

DEFINITION 1.12. Let $f : M^2 \rightarrow \mathbf{R}^3$ be a flat front, and J an open interval. A curve $\gamma : J \rightarrow M^2$ is called an *asymptotic curve* if $\hat{\gamma}''(t) \cdot \nu$ vanishes identically, where $\hat{\gamma} = f \circ \gamma(t)$ and $\prime = d/dt$.

Now, we fix a non-umbilic point $p \in M^2$ and take a special *curvature line coordinate* $(U; u, v)$ centered at p , such that

- (1) u -curves are non-zero principal curvature lines, and v -curves are asymptotic curves.
- (2) $|f_v| = 1$, that is v is the arc-length parameter of v -curves.

The existence of $(U; u, v)$ for flat immersion is well-known (See [20] or [3, 5-8]). Since $\nu_v = 0$, $|(f_\delta)_v| = |f_v| = 1$ holds for parallel surface, the existence of the coordinate for flat front is also shown. The following assertion can be proved directly.

Proposition 1.13. *On the curvature line coordinate $(U; u, v)$ as above, the derivative of the Gauss map ν_u never vanishes and satisfies $\nu_v = 0$, $f_{vv} = 0$.*

Proof. Since

$$f_v \cdot \nu_u = (f_v \cdot \nu)_u - f_{uv} \cdot \nu = -f_{uv} \cdot \nu = (f_u \cdot \nu)_v - f_u \cdot \nu_v = 0,$$

$\{f_v, \nu, \nu_u\}$ is a local orthogonal frame field on U . Then $|f_v| = 1$ implies that f_{vv} is perpendicular to f_v . Obviously, f_v is orthogonal to ν . Thus, to prove $f_{vv} = 0$, it is sufficient to show that f_{vv} is perpendicular to ν_u . In fact, we have

$$f_{vv} \cdot \nu_u = (f_v \cdot \nu_u)_v - f_v \cdot \nu_{uv} = -f_v \cdot (\nu_v)_u = 0,$$

which proves the assertion. q.e.d.

In the proof of Fact 0.1 in [20], the following important lemma is given (cf. [3, 5-8]):

Fact 1.14. (*Massey's lemma*) *Let $f : M^2 \rightarrow \mathbf{R}^3$ be a flat immersion. Let J be an open interval and $\gamma(v)$ ($v \in J$) be an asymptotic curve on M^2 passing through a non-umbilic point p . Suppose that v is the arc-length parameter on it. Then $\hat{\gamma} := f \circ \gamma$ is contained in a straight line and the restriction $\rho(v)$ of the curvature radius function satisfies the*

equality $\rho''(v) = 0$. Furthermore, the closure of $\gamma(J)$ does not meet the umbilic set \mathcal{U}_f .

The following generalization of Massey’s lemma holds for flat fronts:

Lemma 1.15. *(Generalized Massey’s lemma) The above Massey’s lemma (Fact 1.14) holds for any flat front $f : M^2 \setminus \mathcal{U}_f \rightarrow \mathbf{R}^3$. In particular, the curvature radius function ρ is a smooth function on $M^2 \setminus \mathcal{U}_f$ which satisfies $\rho''(v) = 0$ along each asymptotic curve $\gamma(v)$ with $|\hat{\gamma}'(v)| = 1$.*

Proof. Since the asymptotic curve is a zero curvature line, it is also an asymptotic curve of the parallel surface f_δ . For a given $p \in M^2$, there exists a real number δ and a neighborhood U of p such that f_δ is an immersion on U . Then the image of the asymptotic curve $f_\delta \circ \gamma = \hat{\gamma}(v) + \delta\nu(\gamma(v))$ is contained in a straight line. Since the Gauss map ν is constant along γ , the image of $\hat{\gamma} = f \circ \gamma$ is also contained in a straight line.

On the other hand, by Fact 1.14, the principal curvature radius function ρ_δ (of f_δ) satisfies $\rho_\delta''(v) = 0$ whenever $\gamma(v) \in U$ by Lemma 1.14. Then, we have

$$0 = \rho_\delta'' = (\rho - \delta)'' = \rho'',$$

which proves the first part. (The property that $\iota = d/dv$ is the arclength parameter of f_δ does not depend on δ .) Next we suppose that $\gamma(J)$ accumulates at $q \in \mathcal{U}_f$. Then there exists a sequence $\{p_n\}$ in $\gamma(J)$ converging to q . Since q is a regular point of f , there exists a neighborhood U of q such that f is an immersion on U . Without loss of generality, we may assume that $\{p_n\}$ lies in U . Since $f|_U$ is a flat immersion, by Fact 1.14, $\{p_n\}$ cannot accumulate at the umbilic point q , which is a contradiction. q.e.d.

Let ρ be the curvature radius function of f . Then the map

$$C_f := f + \rho\nu : M^2 \setminus \mathcal{U}_f \rightarrow \mathbf{R}^3$$

is called the *caustic* or the *focal surface* of the front f . The following assertion can be proved by a straight-forward calculation using curvature line coordinates as in Proposition 1.13:

Theorem 1.16. *The caustic $C_f : M^2 \setminus \mathcal{U}_f \rightarrow \mathbf{R}^3$ of a flat front f is also a flat front. Moreover, the caustic has no umbilics.*

Later, we shall reprove the above assertion as an application of our representation formula for flat fronts in Section 2. (See Proposition 2.18.) The caustics of flat fronts in the hyperbolic 3-space H^3 or the 3-sphere S^3 are also flat. Roitman [23] gave a representation formula for caustics of flat fronts in H^3 . (See also [14].)

2. The structure of developable frontals

In this section, we shall investigate a weaker class of flat surfaces: Let M^2 be a 2-manifold. Recall that a C^∞ -map $f : M^2 \rightarrow \mathbf{R}^3$ is called a *frontal* if there exists a unit vector field $\nu : M^2 \rightarrow S^2$ such that $\nu(p)$ is perpendicular to $df(T_p M^2)$ for each $p \in M^2$. (ν is called the *unit normal vector field* of f .) In addition, if the pair $(f, \nu) : M^2 \rightarrow \mathbf{R}^3 \times S^2$ defines an immersion, f is called a *front*.

A frontal $f : M^2 \rightarrow \mathbf{R}^3$ is called a *ruled frontal*, if for each $p \in M^2$, there exists a neighborhood U of p and a vector field ξ on U such that

- (1) $df_q(\xi_q)$ is a unit vector for all $q \in U$,
- (2) the image of each integral curve of ξ by f is a part of straight line in \mathbf{R}^3 .

This ξ is called a (*local*) *unit asymptotic vector field* of f . Moreover, if the Gauss map $\nu : M^2 \rightarrow S^2$ of f is constant on each integral curve of ξ , f is called a *developable frontal*. In the introduction, *flat frontal* is defined. We note that *developable frontals are examples of flat frontals*. In this case, the integral curves associated with ξ are called the *asymptotic lines*. (When f is a front, asymptotic lines coincide with the asymptotic curve as in Definition 1.12.)

A vector field X on a manifold M is called *complete* if each integral curve $\gamma(t)$ of X is defined for $t \in \mathbf{R}$. If X is complete, it generates a 1-parameter group of transformation $\varphi_t : M \rightarrow M$ ($t \in \mathbf{R}$).

On the other hand, we denote by $P(TM)$ the projective tangent bundle, whose fibre is the projective space $P(T_p M)$ of the tangent vector space $T_p M$. A section of $P(TM)$ is called a *projective vector field*. Let ξ be a projective vector field on M which is not a globally defined vector field. Then there exists double covering \hat{M} of M such that ξ can be lifted up as a C^∞ -vector field $\hat{\xi}$ on \hat{M} . In this case, the integral curves of $\hat{\xi}$ correspond to those of ξ , but the orientation of the curves as a flow does not have global meaning on M in general.

DEFINITION 2.1. A ruled frontal $f : M^2 \rightarrow \mathbf{R}^3$ is called *asymptotically complete* or *a-complete* if the unit asymptotic vector field ξ is determined as a complete projective vector field, that is, all of the asymptotic lines of f are complete, namely, each image of them by f consists of the entirety of a straight line.

Proposition 2.2. *Let $f : M^2 \rightarrow \mathbf{R}^3$ be a flat front. If f has no umbilic points, then f is developable.*

Proof. We take a local curvature line coordinate $(U; u, v)$ of f as in Proposition 1.13 such that v -curves correspond to the zero curvature lines satisfying $|f_v| = 1$. Then $\xi_U = \partial/\partial v$ gives the asymptotic direction, and $\nu_v = 0$ holds. q.e.d.

DEFINITION 2.3. A ruled frontal $f : M^2 \rightarrow \mathbf{R}^3$ is called *admissible* if it is a -complete and there exists an embedding $\sigma : J \rightarrow M^2$ such that

- (1) J is an open interval of \mathbf{R} or $J = S^1$,
- (2) σ' is transversal to the unit asymptotic vector field ξ ,
- (3) σ meets each asymptotic line exactly once.

This curve σ is called a *generator* of an admissible developable frontal f , which is not uniquely determined in general (see Remark 2.6).

EXAMPLE 2.4. Let $c(t) : J \rightarrow \mathbf{R}^3$ ($J = \mathbf{R}$ or S^1) be a smooth map and $\xi(t) \in T_{c(t)}\mathbf{R}^3$ a smooth vector field along c such that

$$(2.1) \quad \det(c'(t), \xi(t), \xi'(t)) = 0 \quad (t \in J).$$

Then

$$F_{c,\xi}(t, v) := c(t) + v\xi(t) \quad (t \in J)$$

gives a typical example of a developable frontal, if there is a unit vector field $\nu(t)$ of $F_{c,\xi}$ along c such that $c'(t) \cdot \nu(t) = \xi(t) \cdot \nu(t) = 0$ for all $t \in J$. For example, when $c'(t)$ and $\xi(t)$ are linearly independent, $F_{c,\xi}$ is a developable frontal, since $\nu = c' \times \xi / |c' \times \xi|$ is a unit normal vector field of $F_{c,\xi}$. In this situation, $F_{c,\xi}$ is admissible, since the t -axis in (t, v) -plane meets all of the v -curves exactly once transversally. Later, we shall show that an admissible flat frontal is congruent to one of $F_{c,\xi}$ (see Proposition 2.5). Let $c(t)$ be a regular space curve. Then $\xi(t) := c'(t)$ satisfies (2.1), and the induced map

$$F_{c,\xi}(t, v) := c(t) + vc'(t) \quad (t \in J)$$

is called the *tangential developable*, whose singular set is $c(J)$.

Now we return to the general situation. Let $f : M^2 \rightarrow \mathbf{R}^3$ be an a -complete ruled frontal. If the unit asymptotic vector field ξ can be globally defined as a single-valued vector field on M^2 , f is called *a-orientable* or *asymptotically orientable*. There are non- a -orientable developable fronts. (See Proposition 3.5 in the next section.) In this section, we shall mainly consider a -orientable developable frontals.

Proposition 2.5. *Let $f : M^2 \rightarrow \mathbf{R}^3$ be an a -complete and a -orientable ruled frontal, and ξ the unit asymptotic vector field on M^2 . Suppose that f is admissible and $\sigma : J \rightarrow M^2$ is a generator (cf. Definition 2.3), where $J = \mathbf{R}$ or S^1 . Then there exists a diffeomorphism $\Phi : J \times \mathbf{R} \rightarrow M^2$ such that*

$$f \circ \Phi(t, v) = f \circ \sigma(t) + vdf(\xi_{\sigma(t)}) \quad (|df(\xi_{\sigma(t)})| = 1).$$

In particular, M^2 is orientable.

Proof. Since f is a -complete, each integral curve $\varphi_v(q)$ of ξ passing through q is defined for all $v \in \mathbf{R}$. In particular, it induces a 1-parameter group of transformations

$$\varphi_v : M^2 \rightarrow M^2 \quad (v \in \mathbf{R}).$$

Now we define a smooth map Φ by

$$\Phi : J \times \mathbf{R} \ni (t, v) \mapsto \varphi_v(\sigma(t)) \in M^2.$$

Since σ' is transversal to ξ , so is $d\varphi_v(\sigma'(t))$, which implies that Φ is locally a diffeomorphism. The bijectivity of Φ follows from the fact that $\sigma(J)$ meets all asymptotic lines exactly once. Thus Φ is a diffeomorphism. By definition, for each $t \in J$

$$f \circ \Phi(t, v) = f(\varphi_v(\sigma(t))) \quad (v \in \mathbf{R})$$

gives the straight line passing through $f(\sigma(t))$. Moreover, since

$$\left| \frac{\partial f \circ \Phi(t, v)}{\partial v} \right| = |df(\xi)| = 1,$$

the parametrization $v \mapsto f \circ \Phi(t, v)$ coincides with the arc-length parameter of the straight line passing through $f(\sigma(t))$ with the initial velocity vector $df(\xi_{\sigma(t)})$. Thus we get

$$f \circ \Phi(t, v) = f \circ \sigma(t) + vdf(\xi_{\sigma(t)}),$$

which proves the assertion. q.e.d.

REMARK 2.6. The above coordinate (t, v) of M^2 is called a *canonical coordinate of M^2 with respect to a generator σ* . In this coordinate, each t -curve $J \ni t \mapsto \Phi(t, v) \in M^2$ (for fixed $v \in \mathbf{R}$) gives also a generator. So, for an arbitrarily given point $p \in M^2$, we can take a canonical coordinate (t, v) such that $\Phi(0, 0) = p$.

Now we fix an admissible flat developable with a canonical coordinate

$$f(t, v) = f \circ \sigma(t) + vdf(\xi_{\sigma(t)}) \quad (t \in J, v \in \mathbf{R}),$$

such that $|df(\xi_p)| = 1$ for all $p \in M^2$. Then we get a smooth curve

$$\hat{\xi} : J \ni t \mapsto df(\xi_{\sigma(t)}) \in S^2,$$

called the *asymptotic Gaussian curve*, which gives a parametrization of the image of the asymptotic Gauss map $M^2 \ni p \mapsto df(\xi_p) \in S^2$.

On the other hand, a smooth curve

$$\hat{\nu} : J \ni t \mapsto \nu_{\sigma(t)} \in S^2$$

is called the *Gaussian curve* of f , which gives a parametrization of the image of the Gauss map $\nu : M^2 \rightarrow S^2$.

The common parameter t of two curves $\hat{\xi}(t)$ and $\hat{\nu}(t)$ depends on the choice of a generator of f , but their spherical images $\hat{\xi}(J)$ and $\hat{\nu}(J)$ do not depend on the choice of generator.

Proposition 2.7. *Let $f : M^2 \rightarrow \mathbf{R}^3$ be an admissible α -orientable developable frontal. Then the Gaussian curve $\hat{\nu}(t)$ and the asymptotic Gaussian curve $\hat{\xi}(t)$ satisfy the following relations*

$$\hat{\nu} \cdot \hat{\xi} = \hat{\nu}' \cdot \hat{\xi} = \hat{\nu} \cdot \hat{\xi}' = 0,$$

that is, $\hat{\nu}$ and $\hat{\xi}$ are dual frontals in S^2 , where $' = d/dt$. (See the appendix.) Moreover, when f is a front, $\hat{\nu}'(t)$ vanishes if and only if $\nu^{-1}(\hat{\nu}(t))$ is an umbilic set. In particular, if f has no umbilics, then $\hat{\nu}(t)$ is a regular curve, and $\hat{\xi}(t)$ is a front (as a spherical curve).

Proof. The first assertion $\hat{\xi} \cdot \hat{\nu} = 0$ obviously holds. We fix $p \in M^2$ arbitrarily. As mentioned in Remark 2.6, we can take a canonical coordinate

$$\Phi : J \times \mathbf{R} \ni (t, v) \mapsto \Phi(t, v) \in M^2$$

such that $\Phi(0, 0) = p$ and $|d(f \circ \Phi)(\partial/\partial v)| = 1$. Then

$$\begin{aligned} -\hat{\xi}' \cdot \hat{\nu} &= \hat{\xi} \cdot \hat{\nu}' = f_v(t, 0) \cdot \nu_t(t, 0) \\ &= -f_{vt}(t, 0) \cdot \nu(t, 0) = f_t(t, 0) \cdot \nu_v(t, 0) = 0. \end{aligned}$$

Next, we suppose that f is a front and $\hat{\nu}'(t) = 0$, then it implies $\nu_t(0, 0) = 0$, and so $d\nu(0, 0) = 0$, that is, p is an umbilic point of f .
q.e.d.

Let $f : M^2 \rightarrow \mathbf{R}^3$ be an admissible a -orientable developable frontal. In the same notations as in the proof of Proposition 2.7, we have the expression

$$f \circ \Phi(t, v) = \hat{\sigma}(t) + v\hat{\xi}(t) \quad ((t, v) \in J \times \mathbf{R}),$$

where $\hat{\sigma}(t) = f \circ \Phi(t, 0)$. Since $\hat{\sigma}'(t)$ is perpendicular to $\hat{\nu}$, we can write

$$\hat{\sigma}' = a\hat{\xi} + b\hat{\eta}, \quad \hat{\eta} := \hat{\xi} \times \hat{\nu},$$

where ' \times ' is the vector product in \mathbf{R}^3 , and $a, b \in C^\infty(J)$. We define 1-forms α, β on J by

$$\alpha := a(t)dt, \quad \beta := b(t)dt \quad (t \in J).$$

Since $d\hat{\sigma} = \alpha\hat{\xi} + \beta\hat{\eta}$ holds, the quadruple $(\alpha, \beta, \hat{\xi}, \hat{\nu})$ is independent of the choice of the parameter t on J as a 1-dimensional manifold. It induces a C^∞ -map

$$f(t, v) = v\hat{\xi} + \int_0^t (\alpha\hat{\xi} + \beta\hat{\eta}), \quad \hat{\eta} := \hat{\xi} \times \hat{\nu},$$

which coincides with $f \circ \Phi$ up to parallel translation. We prove the following:

Theorem 2.8. (A representation formula for developable frontals)

Let α, β be 1-forms on an open interval $J(\subset \mathbf{R})$. Let $\hat{\nu}, \hat{\xi} : J \rightarrow S^2$ be mutually dual frontals. Then a C^∞ -map defined by

$$(2.2) \quad \begin{aligned} f(t, v) &:= v\hat{\xi}(t) + \hat{\sigma}(t), \\ \hat{\sigma}(t) &:= \int_0^t (\alpha\hat{\xi} + \beta\hat{\eta}), \quad \hat{\eta} := \hat{\xi} \times \hat{\nu} \end{aligned}$$

gives an a -orientable admissible developable frontal on $J \times \mathbf{R}$. The singular set of f is the zeros of $\mu_1(t)v + b(t)$, where

$$\hat{\xi}'(t) = \mu_1(t)\hat{\eta}(t), \quad \beta = b(t)dt.$$

Moreover, f is a front if and only if $\hat{\nu}'(t) \neq 0$ or

$$(2.3) \quad \hat{\xi}'(t) = \hat{\nu}'(t) = 0 \text{ and } b(t) \neq 0$$

holds for each $t \in J$. (In particular, if $\hat{\nu}$ is a regular curve, then f is a front.) Conversely, any admissible a -orientable developable frontals defined on $J \times \mathbf{R}$ are given in this manner.

REMARK 2.9. As stated in the theorem, $\hat{\nu}'(t) \neq 0$ ($t \in J$) is a sufficient condition that f is a front. On the other hand, if f is a front, then the condition (2.3) at $t = t_0$ implies that f is an immersion at $\gamma(t_0)$ and the asymptotic line passing through $\gamma(t_0)$ consists of umbilic points. For example, if we set

$$\alpha = 0, \quad \beta = dt, \quad \hat{\xi}(t) = (1, 0, 0), \quad \hat{\nu}(t) = (0, 0, 1),$$

then the data $(\alpha, \beta, \hat{\xi}, \hat{\nu})$ satisfies (2.3), and the corresponding surface is the plane $f(t, v) = (t, v, 0)$. In this case, $\hat{\nu}'(t)$ vanishes identically although f is an immersion.

REMARK 2.10. Quadruples

$$(\alpha, \beta, \hat{\xi}, \hat{\nu}), \quad (\alpha, \beta, -\hat{\xi}, -\hat{\nu}), \quad (-\alpha, -\beta, \hat{\xi}, \hat{\nu})$$

give the same developable frontals up to congruence. Moreover, in our construction, the quadruple

$$(\alpha + d\varphi, \beta + \varphi\mu_1 dt, \hat{\xi}, \hat{\nu})$$

corresponds to a flat developable $f + \varphi\hat{\xi}$ which is congruent to f .

Proof. We have already seen that any a -complete flat frontals defined on $J \times \mathbf{R}$ are given in this manner. So it is sufficient to show that $f(t, v)$ given by (2.2) is an (orientable) a -complete developable frontal. It can be easily checked that $\hat{\nu}$ gives a unit normal vector field of f . Since it does not depend on v , the flatness follows immediately. Since $v \in \mathbf{R}$ is arbitrary, f is obviously a -complete and admissible. (The curve $t \mapsto (t, 0)$ is a generator.)

Finally, we shall examine the condition that f is a front. Suppose that f is a frontal and fix a point $p = (t_0, v_0) \in M^2$ arbitrarily. Then $\hat{\nu}'(t_0) \neq 0$ holds if and only if f is a front at p , and p is not an umbilic point. So it is sufficient to show that (2.3) holds if and only if f is an immersion at p and $\nu'(t_0) = 0$. In fact, if f is a front and p is an umbilical point, we have $\hat{\nu}'(t_0) = 0$, that is, $d\nu$ vanishes at (t_0, v) for all

$v \in \mathbf{R}$. Thus $f(t, v)$ must be an immersion at (t_0, v) ($v \in \mathbf{R}$). On the other hand, we have that

$$(2.4) \quad \begin{aligned} f_t \wedge f_v &= (a(t)\hat{\xi}(t) + b(t)\hat{\eta}(t) + v\hat{\xi}'(t)) \wedge \hat{\xi}(t) \\ &= (b(t) + v\mu_1(t))\hat{\eta}(t) \wedge \hat{\xi}(t), \end{aligned}$$

where $\hat{\xi}'(t) = \mu_1(t)\hat{\eta}(t)$. If $\mu_1(t_0) \neq 0$, $f(t_0, v)$ is not an immersion at $v = -b(t)/\mu_1(t)$, a contradiction. Thus $\mu_1(t_0) = 0$, and then we have $b(t_0) \neq 0$ since $f_t \wedge f_v \neq 0$. Conversely, (2.3) and (2.4) implies that f is an immersion at (t_0, v) for all $v \in \mathbf{R}$. q.e.d.

Next, we consider the case $J = S^1 = \mathbf{R}/2\pi\mathbf{Z}$. The following corollary is immediate from the above theorem.

Corollary 2.11. *(A representation formula for developable frontals with closed generator) Let α, β be 1-forms on S^1 , and $\hat{v}, \hat{\xi} : S^1 \rightarrow S^2$ mutually dual frontals. Suppose that*

$$(2.5) \quad \int_{S^1} (\alpha\hat{\xi} + \beta\hat{\eta}) = 0, \quad \hat{\eta} := \hat{\xi} \times \hat{v}.$$

Then a C^∞ -map defined by

$$(2.6) \quad f(t, v) := \hat{\sigma}(t) + v\hat{\xi}(t), \quad \hat{\sigma}(t) := \int_0^t (\alpha\hat{\xi} + \beta\hat{\eta})$$

is an admissible developable frontal on $S^1 \times \mathbf{R}$. Conversely, any admissible orientable developable frontals defined on $S^1 \times \mathbf{R}$ are given in this manner.

The simplest developable front with closed generator is the circular cylinder

$$f(u, v) = (\cos t, \sin t, v),$$

which corresponds to the data $(0, dt, \mathbf{e}_3, f(t, 0))$, where $\mathbf{e}_3 = (0, 0, 1)$. We give here another example:

EXAMPLE 2.12. (Rectifying developables) Let $J = \mathbf{R}$ or S^1 , and let $c : J \rightarrow \mathbf{R}^3$ be a regular space curve satisfying $c''(t) \neq 0$ for $t \in J$. We may assume that t is the arc-length parameter. Then the envelope f of the rectifying plane of $c(t)$ is called a *rectifying developable* associated with c . It is well-known that f has an expression $f(t, v) = c(t) + v\xi(t)$, where

$$\xi(t) := \frac{1}{\sqrt{\kappa(t)^2 + \tau(t)^2}}(\kappa(t)\mathbf{e}(t) + \tau(t)\mathbf{b}(t)),$$

is called the *Darboux vector field*. (Here $\mathbf{e}(t) = c'(t)$, $\mathbf{b}(t)$ is the unit bi-normal vector, $\kappa(t), \tau(t)$ are the curvature function and the torsion function of $c(t)$ respectively.) When $\kappa(t) \neq 0$, it corresponds to the data

$$\left(\frac{\kappa(t)dt}{\sqrt{\kappa^2 + \tau^2}}, \frac{-\tau(t)dt}{\sqrt{\kappa^2 + \tau^2}}, \xi, \mathbf{n} \right),$$

where $\mathbf{n} = \mathbf{n}(t)$ is the unit principal normal vector of $c(t)$.

We return now to the general situation.

Proposition 2.13. *Let $f(t, v)$ be an admissible developable front as in Corollary 2.11. Suppose that f has no umbilics. Then the non-zero principal curvature lines of f are characterized by a graph $v = v(t)$ in tv -plane satisfying $v'(t) + a(t) = 0$. In particular, the non-zero principal curvature lines are all periodic if $\int_{S^1} a(t) dt = 0$.*

Proof. In the tv -plane, each non-zero principal curvature line can be expressed by $w(t) := (t, v(t))$, since it is transversal to the asymptotic lines (i.e. v -lines). Since the principal directions are common in the parallel family and our assertion is local, we may assume that $w(t)$ lies in the regular set of f . Since $f_t - af_v$ is proportional to $\hat{\eta}$, it is perpendicular to the asymptotic direction $\hat{\xi}$. Thus, $w(t)$ gives the non-zero principal direction if and only if

$$\frac{d(f \circ w(t))}{dt} = f_t + v' f_v,$$

is proportional to $f_t - af_v$, which is equivalent to the condition $v' + a = 0$. Now the assertion follows immediately. q.e.d.

REMARK 2.14. The linear map

$$C^\infty(S^1, \mathbf{R}^2) \ni (a, b) \mapsto \int_{S^1} (a(t)\hat{\xi}(t) + b(t)\hat{\eta}(t), a(t)) dt \in \mathbf{R}^4$$

has infinite dimensional kernel corresponding to the set of a -complete developable fronts whose curvature lines are periodic. Thus there are infinitely many examples whose curvature lines are periodic. For example, a cone over a locally convex spherical curve given in Example 4.8 in Section 4 is a complete flat front whose non-zero curvature lines are all closed, since $a(t)$ vanishes identically.

Let $f : S^1 \times J \rightarrow \mathbf{R}^3$ be an admissible developable frontal corresponding to a quadruple $(a(t)dt, b(t)dt, \hat{\xi}(t), \hat{\nu}(t))$. We now define two C^∞ -functions $\mu_1(t)$ and $\mu_2(t)$ on J by

$$(2.7) \quad \hat{\xi}' = \mu_1 \hat{\eta}, \quad \hat{\nu}' = \mu_2 \hat{\eta}, \quad \hat{\eta} := \hat{\xi} \times \hat{\nu} \quad (t = d/dt).$$

Then the singular set is equal to the zeros of $\mu_1(t)v + b(t)$ by Theorem 2.8.

DEFINITION 2.15. A singular point $p = (t, v)$ is *linear* if and only if $\hat{\nu}'(t) = 0$ (i.e. $\mu_1(t) = 0$).

If p is a linear singular point, the asymptotic line passing through p consists of linear singular points. The singular set of the cylinder over a cardioid (cf. Remark 3.8) is a typical example of linear singular points.

A singular point is called a *cuspidal edge* or *swallowtail* if it is locally diffeomorphic to

$$(2.8) \quad f_C(u, v) := (u^2, u^3, v), \quad f_S(u, v) := (3u^4 + u^2v, 4u^3 + 2uv, v)$$

at $(u, v) = (0, 0)$, respectively. These two types of singular points are the generic singularities of fronts. We recall here criteria given in [14] as follows: Let $f : (U; u, v) \rightarrow \mathbf{R}^3$ be a front (not necessarily flat). We denote by $\nu(u, v)$, the unit normal vector field of $f(u, v)$. Then there exists a smooth function $\lambda(u, v)$ defined on U such that $f_u \times f_v = \lambda\nu$. A point $p \in U$ is called *non-degenerate* if and only if $d\lambda \neq 0$. In this case, the implicit function theorem yields that the singular set consists of a regular curve around p , and such a curve is called a *singular curve*. We denote by $\sigma(t)$ the singular curve passing through p . Since σ is a regular curve on U , the direction $\sigma'(t) := d\sigma(t)/dt$ is called the *singular direction*. On the other hand, since $f(u, v)$ is not an immersion along σ , there exists a non-vanishing vector field $Z(t)$ (along σ) such that $df(Z(t)) = 0$. The following criteria are known (see [14]) when f is a front.

- (a) $p = \sigma(t_0)$ is a cuspidal edge if and only if p is a non-degenerate singular point and $Z(t_0)$ is not proportional to $\sigma'(t_0)$, that is, the determinant function $\varphi(t) := \det(Z(t), \sigma'(t))$ does not vanish at $t = t_0$.
- (b) $t = t_0$ is a swallowtail if and only if p is a non-degenerate singular point such that $\varphi(t_0) = 0$ and $\varphi'(t_0) \neq 0$.

Proposition 2.16. *Let $p = (t_0, v_0)$ be a singular point of an admissible developable front $f(t, v)$ as in Theorem 2.8.*

- (1) f is non-degenerate at p if p is a non-linear singular point or a linear singular point satisfying $b'(t_0) + v_0\mu_1'(t_0) \neq 0$.
- (2) Suppose that p is non-linear, namely $\mu_1(t_0) \neq 0$, then
 - (i) a singular point p is locally diffeomorphic to a cuspidal edge if and only if $\mu_2(t_0) \neq 0$ and $a \neq (b/\mu_1)'$ at $t = t_0$,
 - (ii) p is locally diffeomorphic to a swallowtail if and only if $\mu_2(t_0) \neq 0$, $a(t) = (b(t)/\mu_1(t))'$ and $a'(t) \neq (b(t)/\mu_1(t))''$ at $t = t_0$.
- (3) Suppose that p is linear, namely $\mu_1(t_0) = 0$, then $b(t_0)$ must vanish and p is locally diffeomorphic to a cuspidal edge if and only if $\mu_2(t_0) \neq 0$ and $v_0\mu_1'(t_0) + b'(t_0) \neq 0$. Moreover, swallowtails never appear at p .

REMARK 2.17. In [24], the singular curvature κ_s is defined on cuspidal edges. Since the Gaussian curvature of f is non-negative, κ_s is non-positive as shown in [24, Theorem 3.1]. Since $|\kappa_s|$ is equal to the tangential component of the second derivative of the singular curve, the equality of $\kappa_s(p) = 0$ holds if and only if p is a linear singular point.

Proof. Our conclusions follow from the above criteria (a) and (b). In fact, $f_u \times f_v = (v\mu_1 + b)\nu$, and $p = (t_0, v_0)$ is a singular point if and only if $v_0\mu_1(t_0) + b(t_0) = 0$. So the singular point $p = (t_0, v_0)$ is linear i.e. $\mu_1(t_0) = 0$, and then we have $b(t_0) = 0$. On the other hand, p is non-degenerate if and only if $\mu_1(t_0) \neq 0$ or $v\mu_1' + b' \neq 0$ at $t = t_0$. In particular, non-linear singular points are always non-degenerate, and we get (1). Suppose p is a non-degenerate singular point. Then the singular curve σ passing through $p = (t_0, v_0)$ is given by

$$\sigma(t) = \begin{cases} (t, -b(t)/\mu_1(t)) & (\text{if } \mu_1(t_0) \neq 0), \\ (t_0, t) & (\text{if } \mu_1(t_0) = 0). \end{cases}$$

Since $f_t - af_v$ vanishes on this singular curve, the null direction is $Z(t) = (1, -a(t))$. So we have

$$\det(\sigma', Z) = \begin{cases} \det \begin{pmatrix} 1 & 1 \\ -(b/\mu_1)' & -a \end{pmatrix} = -a + (b/\mu_1)' & (\text{if } \mu_1(t_0) \neq 0), \\ \det \begin{pmatrix} 0 & 1 \\ 1 & -a \end{pmatrix} = -1 & (\text{if } \mu_1(t_0) = 0), \end{cases}$$

where ‘det’ denotes the determinant of two column-vectors. The assertions (2) and (3) now follow immediately. When $\mu_1(t_0) = 0$, a non-degenerate singular point is always a cuspidal edge. In particular, a swallowtail never appears. q.e.d.

In the previous section (cf. Theorem 1.16), we pointed out that the caustics of flat fronts are again flat. We give here a refinement of this.

Proposition 2.18. *Let f be a front associated with $(adt, bdt, \hat{\xi}, \hat{\nu})$. Then the quadruple $(adt, (b + \delta\mu_2)dt, \hat{\xi}, \hat{\nu})$ corresponds to the parallel front f_δ ($\delta \in \mathbf{R}$), where $\hat{\nu}' = \mu_2\hat{\eta}$. Suppose that f is a front and has no umbilics (namely $\mu_2(t)$ does not vanish and $\hat{\nu}$ is a regular spherical curve). Then the caustic C_f corresponds to the quadruple*

$$\left(\frac{a\mu_2 + (b/\mu_2)'\mu_1}{\sqrt{(\mu_1)^2 + (\mu_2)^2}} dt, \frac{-a\mu_1 + (b/\mu_2)'\mu_2}{\sqrt{(\mu_1)^2 + (\mu_2)^2}} dt, c_\xi, \hat{\eta} \right),$$

where $\iota = d/dt$, $\hat{\xi}^\iota = \mu_1\hat{\eta}$ and $c_\xi := (\mu_2\hat{\xi} - \mu_1\hat{\nu})/\sqrt{(\mu_1)^2 + (\mu_2)^2}$. In particular, the caustic C_f of an admissible developable front is also admissible.

Proof. The first assertion is obvious. The non-zero principal curvature radius function of $f(t, v)$ is given by

$$\rho(t) = -\frac{b(t) + v\mu_1(t)}{\mu_2(t)}.$$

Since $C_f(t) = f(t, v) + \rho(t)\hat{v}(t)$, we have

$$(C_f)_v = \hat{\xi} - \frac{\mu_1}{\mu_2}\hat{v}, \quad (C_f)_t = a\hat{\xi} - \left(\frac{b + \mu_1\hat{v}}{\mu_2}\right)' \hat{v}.$$

Thus $\hat{\eta}$ is perpendicular to both $(C_f)_v$ and $(C_f)_t$, and gives a unit normal vector of C_f . Since $\hat{\eta}$ depends only on the parameter t , the Gauss map of C_f degenerate everywhere. Moreover, since $\hat{\eta}' = -\mu_1\hat{\xi} - \mu_2\hat{v}$ and $\mu_2 \neq 0$, C_f gives an a -complete flat front. On the other hand, since

$$C_f = \left(\hat{\sigma} - \frac{b}{\mu_2}\hat{v}\right) + v\mu_2\sqrt{(\mu_1)^2 + (\mu_2)^2} c_\xi,$$

$c_\xi(t)$ gives the asymptotic Gaussian curve of C_f . Furthermore, by a straight-forward calculation, we have

$$\left(\hat{\sigma} - \frac{b}{\mu_2}\hat{v}\right)' = \frac{a\mu_2 + (b/\mu_2)'\mu_1}{\sqrt{(\mu_1)^2 + (\mu_2)^2}}c_\xi + \frac{-a\mu_1 + (b/\mu_2)'\mu_2}{\sqrt{(\mu_1)^2 + (\mu_2)^2}}\mathbf{e}_f,$$

where

$$\mathbf{e}_f(t) := c_\xi(t) \times \hat{v}(t) = -\frac{\mu_1(t)\hat{\xi}(t) + \mu_2(t)\hat{v}(t)}{\sqrt{\mu_1(t)^2 + \mu_2(t)^2}}.$$

This proves the assertion.

q.e.d.

REMARK 2.19. By Proposition A.1. in the appendix, $c_\xi(t)$ as above coincides with the caustic of the asymptotic Gaussian curve $\hat{\xi}(t)$ of f .

3. Non-orientable or non-co-orientable developable surfaces.

In this section, we shall generalize the representation formula for admissible developables to those which are not orientable or not admitting a globally defined unit normal vector field.

DEFINITION 3.1. Let M^2 be a 2-manifold. As defined in the introduction, a C^∞ -map $f : M^2 \rightarrow \mathbf{R}^3$ is called a p -frontal (resp. p -front) if for each $p \in M^2$, there exists a neighborhood U of p such that the restriction $f|_U$ gives a frontal (resp. front). A p -frontal f is called *co-orientable* if it is a frontal, that is, there exists a smooth unit normal vector field globally defined on M^2 .

If $f : M^2 \rightarrow \mathbf{R}^3$ is non-co-orientable p -frontal, there exists a double covering $\pi : \hat{M}^2 \rightarrow M^2$ such that $f \circ \pi$ is a frontal.

DEFINITION 3.2. A p -frontal is called *developable* if f is a developable frontal or $f \circ \pi$ is a developable frontal. Similarly, *a-completeness* of a p -frontal is defined using the double covering. A developable p -frontal is called *a-orientable* if and only if there is a globally defined unit asymptotic vector field. Like as the case of *frontal* (cf. Definition 2.3), we define the admissibility for p -frontals: A developable p -frontal $f : M^2 \rightarrow \mathbf{R}^3$ is called *admissible* if it is a -complete, and there

is a curve (called a *generator*) embedded in M^2 satisfying the three conditions in Definition 2.3.

For example, a real analytically immersed flat Möbius strip (cf. [30]) is non-orientable and non-co-orientable at the same time. The following lemma is a key to proving the main results in this section.

Lemma 3.3. (*A criterion of admissibility*) *Let $f : M^2 \rightarrow \mathbf{R}^3$ be an a -complete developable p -frontal. Suppose that M^2 is connected and there exists an embedded closed curve*

$$\sigma : S^1 = \mathbf{R}/2\pi\mathbf{Z} \rightarrow M^2,$$

whose velocity vector $\sigma'(t)$ is transversal to the asymptotic direction of f at $\sigma(t)$ for each $t \in S^1$, and meets each asymptotic line in at most one point. Then f is admissible and σ is a generator.

Proof. Firstly, we suppose that f is a -orientable. Then we can take a unit asymptotic vector field ξ defined on M^2 . Since f is a -complete, each integral curve of vector field ξ is defined on \mathbf{R} . In particular, ξ generates a global 1-parameter group of transformations

$$\varphi_v : M^2 \rightarrow M^2 \quad (v \in \mathbf{R})$$

such that $v \mapsto f(\varphi_v(p))$ is a straight line in \mathbf{R}^3 .

Let C be the image of the curve σ . It is sufficient to show that C meets the asymptotic line $v \mapsto \varphi_v(p)$ for each $p \in M^2$. If not, there exists a point $p_1 \in M^2$ such that $\varphi_v(p_1) \notin C$ for all $v \in \mathbf{R}$. Let p_0 be a point on C . Since M^2 is connected, we can take a continuous curve $p(s)$ ($0 \leq s \leq 1$) in M^2 such that $p(0) = p_0$ and $p(1) = p_1$. Without loss of generality, we may assume that $p(s)$ does not meet C for $s > 0$. We set

$$I_0 := \{s \in [0, 1]; \exists v \in \mathbf{R} \text{ such that } \varphi_v(p(s)) \in C\}.$$

Since I_0 contains $s = 0$, it is a non-empty subset. Since σ' is transversal to ξ , I_0 is an open subset of $[0, 1]$. Next, we show that I_0 is closed. In fact, let $\{s_n\}_{n=1,2,3,\dots}$ be a sequence in I_0 converging to a point $s_\infty \in (0, 1]$. Since $s_n \in I_0$, there exists a point $q_n \in C$ and $v_n \in \mathbf{R}$ such that $\varphi_{v_n}(q_n) = p(s_n)$. Since C is compact, taking a subsequence if necessary, we may assume that $\{q_n\}$ converges to $q_\infty \in C$. Since $|df(\xi)| = 1$, the map $v \mapsto f(\varphi_v(q))$ gives a straight line parametrized by the arc-length parameter. If $v_n \rightarrow 0$, we have $p(s_\infty) = q_\infty \in C$, in this case $s_\infty \in I_0$, which contradicts the fact that $p(s) \notin I_0$ for $s > 0$. So we may assume that v_n does not converge to 0. By replacing ξ by $-\xi$ if necessary, we may also assume that $v_n > 0$ for all n . Thus we have $v_n = |f(q_n) - f(p(s_n))|$. So we have

$$(v_\infty :=) \lim_{n \rightarrow \infty} v_n = |f(q_\infty) - f(p(s_\infty))|,$$

and get the expression

$$\varphi_{v_\infty}(q_\infty) = p(s_\infty),$$

which implies $s_\infty \in C$. Thus I_0 is open and closed in $[0,1]$ and so we can conclude that $I_0 = [0,1]$, which is a contradiction. Hence C is a generator.

Next, we suppose that f is not a -orientable. Then ξ is not a globally defined vector field on M^2 , and there exists a double covering $\pi : \hat{M}^2 \rightarrow M^2$ such that ξ can be considered as a C^∞ -vector field of \hat{M}^2 . We set $C = \sigma(S^1)$. We now would like to show that the inverse image $\pi^{-1}(C)$ is connected. If not, there are two disjoint closed curves C_1 and C_2 in \hat{M}^2 such that

$$\pi^{-1}(C) = C_1 \cup C_2.$$

Then C_i ($i = 1, 2$) is a closed regular curve on M^2 satisfying the condition of Lemma 3.3. Since $f \circ \pi$ is a -orientable, C_i ($i = 1, 2$) must meet each asymptotic line. We now fix an asymptotic line $L(\subset \hat{M}^2)$ as an integral curve of ξ . Then there exist two points $p, q \in \hat{M}^2$ such that an arbitrarily fixed integral curve γ of ξ on \hat{M}^2 meets both of C_1 and C_2 at p and q respectively. Then $\pi(\gamma)$ meets σ at $\pi(p)$ and $\pi(q)$. Since σ is a generator, we have $\pi(p) = \pi(q) (= r)$. Then the straight line $f \circ \pi(L)$ passes through the point $f(r)$ twice, which is a contradiction. Thus $\pi^{-1}(C)$ is connected and must be a generator of $f \circ \pi$. Since $\pi^{-1}(C)$ meets each asymptotic line of $f \circ \pi$, C also meets each asymptotic line of f , that is, C is a generator. q.e.d.

Corollary 3.4. *Let $f : M^2 \rightarrow \mathbf{R}^3$ be an admissible developable p -frontal. Then the following three conditions are mutually equivalent:*

- (1) f is a -orientable.
- (2) M^2 is orientable.
- (3) There exists a smooth unit asymptotic vector field, which is globally defined on an generator.

Proof. If f is a -orientable, M^2 is orientable by Proposition 2.5. (Even when f is non-co-orientable p -front, the argument of the proof of Proposition 2.5 works, since we do not use the existence of globally defined unit normal vector ν there.) Thus (1) implies (2). Next we assume that M^2 is oriented. Let $\sigma(t)$ ($0 \leq t \leq 1$) be a generator of f . We can choose $\xi(t)$ for each t such that $(\sigma'(t), \xi(t))$ is positively oriented frame. Then $\xi(0) = \xi(1)$ holds, which implies (3). Finally, we assume (3). Let $\sigma(t)$ ($0 \leq t \leq 1$) be a generator of f . We can choose $\xi(t)$ for each t such that $\xi(0) = \xi(1)$ holds. Suppose f is not a -orientable. Then there exists a double covering $\pi : \hat{M}^2 \rightarrow M^2$ such that $f \circ \pi$ is a -orientable. We denote by C the image of σ . Since $\xi(0) = \xi(1)$, $\pi^{-1}(C)$ consists of a union of two closed curves C_1 and C_2 . However, we have seen in the last-half part of the proof of Lemma 3.3, it makes a contradiction. Thus (3) implies (1). q.e.d.

Here a function $\varphi \in C^\infty(\mathbf{R})$ satisfying $\varphi(t + \pi) = -\varphi(t)$ is called *anti- π -periodic*. Firstly, we consider the non-orientable case:

Proposition 3.5. *Let $f : S^1 \times \mathbf{R} \rightarrow \mathbf{R}^3$ be a front given by*

$$f(t, v) = \hat{\sigma}(t) + v\hat{\xi}(t), \quad \left(\hat{\sigma}(t) := \int_0^t (a(s)\hat{\xi}(s) + b(s)\hat{\eta}(s))ds \right),$$

associated with a data $(a(t)dt, b(t)dt, \hat{\xi}(t), \hat{\nu}(t))$ on $S^1 = \mathbf{R}/2\pi\mathbf{Z}$. Then f is a double covering of a non-orientable p -frontal \bar{f} if and only if $\hat{\xi}(t), a(t)$ are anti- π -periodic. Moreover

- (1) *\bar{f} is co-orientable if and only if $\nu(t)$ is π -periodic and $b(t)$ is anti- π -periodic.*
- (2) *\bar{f} is non-co-orientable if and only if $\nu(t)$ is anti- π -periodic and $b(t)$ is π -periodic.*

Conversely, an arbitrary non-orientable admissible developable p -frontal \bar{f} has such a double covering.

Proof. The first assertion is obvious. So we fix a non-orientable admissible developable p -frontal $\bar{f} : M^2 \rightarrow \mathbf{R}^3$. Since M^2 is non-orientable, f is not a -orientable. Let C be a generator of \bar{f} . Then there exists a double covering $\pi : \hat{M}^2 \rightarrow M^2$ such that ξ can be considered as a C^∞ -vector field on \hat{M}^2 . As seen in the proof of Lemma 3.3, $\pi^{-1}(C)$ is connected and must be a generator of $\bar{f} \circ \pi$. Let $T : \hat{M}^2 \rightarrow \hat{M}^2$ be the covering transformation. We can take a parametrization $\gamma(t)$ ($0 \leq t \leq \pi$) of $\pi^{-1}(C)$ such that $\gamma(t + \pi) = T \circ \gamma(t)$. If f is co-orientable (resp. non-co-orientable) $\nu(t + \pi) = \nu(t)$ (resp. $\nu(t + \pi) = -\nu(t)$) holds. In both of two cases, $\nu(t + 2\pi) = \nu(t)$, and so $\bar{f} \circ \pi$ is a front. Thus, if we take the canonical coordinate (as in Remark 2.6) with respect to $\gamma(t)$, we get the assertion. q.e.d.

Like as in the non-orientable case, the following assertion holds.

Proposition 3.6. *Let $f : S^1 \times \mathbf{R} \rightarrow \mathbf{R}^3$ be a front given by*

$$f(t, v) = \hat{\sigma}(t) + v\hat{\xi}(t), \quad \left(\hat{\sigma}(t) := \int_0^t (a(s)\hat{\xi}(s) + b(s)\hat{\eta}(s))ds \right),$$

associated with a data $(a(t)dt, b(t)dt, \hat{\xi}(t), \hat{\nu}(t))$ on $S^1 = \mathbf{R}/2\pi\mathbf{Z}$. Then f is a double covering of a non-co-orientable p -frontal \bar{f} if and only if $\hat{\nu}(t)$ is anti- π -periodic. Moreover

- (1) *\bar{f} is orientable if and only if $\xi(t), a(t)$ is π -periodic and $b(t)$ is anti- π -periodic.*
- (2) *\bar{f} is non-orientable if and only if $\xi(t), a(t)$ is anti- π -periodic and $b(t)$ is π -periodic.*

Conversely, an arbitrary non-co-orientable admissible developable p -frontal \bar{f} has such a double covering.

Proof. The first assertion is obvious. So we fix a non-co-orientable admissible developable p-frontal $\bar{f} : M^2 \rightarrow \mathbf{R}^3$. Let C be a generator of \bar{f} . Then there exists a double covering $\pi : \hat{M}^2 \rightarrow M^2$ such that ν can be considered as a C^∞ -vector field on \hat{M}^2 . As seen in the proof of Lemma 3.3, $\pi^{-1}(C)$ is connected and must be a generator of $\bar{f} \circ \pi$. Let $T : \hat{M}^2 \rightarrow \hat{M}^2$ be the covering transformation. We can take a parametrization $\gamma(t)$ ($0 \leq t \leq \pi$) of $\pi^{-1}(C)$ such that $\gamma(t + \pi) = T \circ \gamma(t)$. If f is orientable (resp. non-orientable) $\xi(t + \pi) = \xi(t)$ (resp. $\xi(t + \pi) = -\xi(t)$) holds. In both of two cases, $\xi(t + 2\pi) = \xi(t)$, and so $\bar{f} \circ \pi$ is a-orientable. Thus, if we take the canonical coordinate (as in Remark 2.6) with respect to $\gamma(t)$, we get the assertion. q.e.d.

Proposition 3.7. *There exists at least one asymptotic line consisting of umbilic points on an admissible non-orientable developable p-front.*

Proof. Firstly, we consider the case that f is a front. Since M^2 is non-orientable, we can apply Proposition 3.5. Then f can be represented by a quadruple $(a(t)dt, b(t)dt, \hat{\xi}(t), \hat{\nu}(t))$ as in Corollary 2.11, where $t \in S^1 = \mathbf{R}/2\pi\mathbf{Z}$. Suppose that the surface has no umbilics. Then the Gaussian curve $\hat{\nu}$ is a regular curve by Proposition 1.13. So we may assume that $\hat{\nu} = \hat{\nu}(t)$ is parametrized by the arc-length parameter. Then we have $\hat{\xi} = \pm \hat{\nu}' \times \hat{\nu}$. Since \pm -ambiguity does not effect our construction, we may set $\hat{\xi} = \hat{\nu}' \times \hat{\nu}$. Since $\hat{\nu}$ is π -periodic, so is $\hat{\nu}'$. Then we can conclude that $\hat{\xi} = \hat{\nu}' \times \hat{\nu}$ is π -periodic, which contradicts the anti- π -periodicity of $\hat{\xi}$.

Next, we consider the case that f is a non-co-orientable p-front. By Proposition 3.6, there exists a double covering $\pi : \hat{M}^2 \rightarrow M^2$ such that $f \circ \pi$ is an admissible developable front, which is orientable. Then f can be represented by a quadruple $(a(t)dt, b(t)dt, \hat{\xi}(t), \hat{\nu}(t))$ satisfying (2) of Proposition 3.5. Then the Gaussian curve $\hat{\nu}$ is a regular curve by Proposition 1.13. So we may assume that $\hat{\nu} = \hat{\nu}(t)$ is parametrized by the arc-length parameter. We may set $\hat{\xi} = \hat{\nu}' \times \hat{\nu}$. Since $\hat{\nu}$ is anti- π -periodic, so is $\hat{\nu}'$. Then we can conclude that $\hat{\xi} = \hat{\nu}' \times \hat{\nu}$ is π -periodic, which contradicts the anti- π -periodicity of $\hat{\xi}$. q.e.d.

REMARK 3.8. For example, the cylinder over a cardioid

$$f(t, v) = (1 - \sin t) \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} + v \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

gives a developable front, which is non-co-orientable but orientable. The normal vector field of f is given by

$$\nu(t, v) := \begin{pmatrix} (\cos \frac{t}{2} + \sin \frac{t}{2})(1 - 2 \sin t) \\ (\cos \frac{t}{2} - \sin \frac{t}{2})(1 + 2 \sin t) \\ 0 \end{pmatrix}.$$

In this case, the surface has no umbilics. So in the above statement, non-orientability is crucial.

Like as in Remark 2.14, one can see the existence of infinitely many non-orientable p-fronts and non-co-orientable p-fronts. Finally, we remark that by modifying the argument in the proof of Proposition 3.7, one can prove that *a developable Möbius strip (that is, a non-orientable developable immersion on $S^1 \times (-1, 1)$) has at least one umbilic on it.* (See [17].) It should be also remarked that the fundamental properties of flat Möbius strips are given in [9], [2] and [22]. Recently, the existence of real analytic flat Möbius strips of a given isotopy type, whose centerline is a geodesic or a line of curvature, was shown in [17]. Also, in [17], examples of weakly complete flat fronts containing a Möbius strip were given. These fronts cannot be complete, since complete flat fronts are orientable (cf. Theorem B).

4. Completeness

In the introduction, we have defined two types of completeness for flat fronts. In this section, we determine the structure of complete flat fronts. Firstly, we prepare two lemmas.

Lemma 4.1. *A complete flat front $f : M^2 \rightarrow \mathbf{R}^3$ is weakly complete.*

Proof. Let $ds^2 = df \cdot df$ be the first fundamental form. Since f is complete, there exists a symmetric tensor T having compact support $K := \text{supp}(T)$ such that $g := ds^2 + T$ is a complete Riemannian metric on M^2 . It is well-known that any two Riemannian metrics g_1, g_2 on M^2 satisfies $cg_2 < g_1 < (1/c)g_2$ on a given compact set for suitable positive number c . So there exists a constant $0 < c < 1$ such that

$$ds_{\#}^2 := df \cdot df + d\nu \cdot d\nu > cg \quad \text{on } K.$$

On the other hand, it is obvious that $ds_{\#}^2 \geq ds^2$ holds on $M^2 \setminus K$. So $ds_{\#}^2 > cg$ holds on M^2 . Since g is complete, so is $ds_{\#}^2$. q.e.d.

Lemma 4.2. *A weakly complete flat front $f : M^2 \rightarrow \mathbf{R}^3$ which has no umbilics is an a-complete developable front.*

Proof. By Proposition 2.2, f is a developable front. So it is sufficient to show that each asymptotic line is complete. We take an asymptotic line $\gamma(t)$ ($t \in \mathbf{R}$) arbitrarily. Then ν is constant along $\gamma(t)$, that is, $d\nu = 0$ on γ , and $ds_{\#}^2$ -length of γ is equal to that of ds^2 -length. Since $ds_{\#}^2$ -length is greater than or equal to ds^2 -length in general, we can conclude that γ is a geodesic of $(M^2, ds_{\#}^2)$. Since $ds_{\#}^2$ is a complete metric, σ has infinite length and the image of $f \circ \gamma$ must coincide with a complete straight line. q.e.d.

The authors do not know whether weakly complete flat fronts are developable (more strictly, admissible) or not. Later, we will show that complete flat fronts are all admissible developable. (See the proof of Theorem A in this section.) Conversely, we can prove the following (see also Corollary 4.7):

Proposition 4.3. *Let $f : M^2 \rightarrow \mathbf{R}^3$ be an admissible a -complete developable front which has a closed generator. Then f is weakly complete.*

Proof. Taking the double covering, we may assume f is a -orientable. (See Section 3.) By Corollary 2.11, M^2 can be identified with $S^1 \times \mathbf{R}$, and f is constructed from a quadruple $(a(t)dt, b(t)dt, \hat{\xi}(t), \hat{\nu}(t))$ on $S^1 = \mathbf{R}/2\pi\mathbf{Z}$. Then the lift metric is given by (cf. (0.2))

$$ds_{\#}^2 = df \cdot df + d\nu \cdot d\nu = (a^2 + (b + v\mu_1)^2 + (\mu_2)^2) dt^2 + 2adt dv + dv^2,$$

where $\hat{\xi}'(t) = \mu_1(t)\hat{\eta}(t)$ and $\hat{\nu}'(t) = \mu_2(t)\hat{\eta}(t)$. Since the symmetric matrix

$$\begin{pmatrix} a^2 + (b + v\mu_1)^2 + (\mu_2)^2 & a \\ a & 1 \end{pmatrix}$$

has positive eigenvalues k_1, k_2 ($k_2 > k_1$) such that

$$k_1 + k_2 = 1 + a^2 + (b + v\mu_1)^2 + (\mu_2)^2, \quad k_1 k_2 = (b + v\mu_1)^2 + (\mu_2)^2,$$

we have

$$k_2 \geq k_1 = \frac{(b + v\mu_1)^2 + (\mu_2)^2}{k_2} \geq \frac{(b + v\mu_1)^2 + (\mu_2)^2}{1 + a^2 + (b + v\mu_1)^2 + (\mu_2)^2}.$$

We set $A(t, v) := (b(t) + v\mu_1(t))^2 + \mu_2(t)^2$. If A has a zero at (t_0, v_0) , then we have $\mu_2(t_0) = 0$ and $b(t_0) + v\mu_1(t_0) = 0$. In particular, $\hat{\nu}'(t_0) = \mu_2(t_0)\hat{\eta}(t_0) = 0$. Since f is a front and $\hat{\nu}'(t_0) = 0$, (2.3) holds by Theorem 2.8. Then we have that

$$0 \neq b(t_0) = -v\mu_1(t_0) = 0,$$

which makes a contradiction. Thus $A(t, v) > 0$ on tv -plane. Since $\lim_{v \rightarrow \infty} A(t, v)$ diverges for all $t \in S^1$, the positive number

$$m := \min_{(t,v) \in \mathbf{R}^2} A(t, v)$$

is well-defined. Since $x \mapsto x/(1 + a^2 + x)$ is monotone increasing, we have $k_1 \geq m/(1 + a^2 + m)$. Then we have that

$$ds_{\#}^2 \geq \frac{m}{1 + a^2 + m} (dt^2 + dv^2).$$

Since $dt^2 + dv^2$ gives the complete metric on $S^1 \times \mathbf{R}$, so does $ds_{\#}^2$, which proves the assertion. q.e.d.

Proposition 4.4. $f : M^2 \rightarrow \mathbf{R}^3$ be an a -complete and a -orientable developable front whose singular set is non-empty and compact. Suppose that there are no umbilics on M^2 . Then f is admissible and the singular set of f is a generator.

Proof. There exists a unit asymptotic vector field ξ on M^2 . Since the umbilic set is empty, the principal radius function ρ is a real-valued C^∞ -function on M^2 (see Section 1). Thus we can define a C^∞ -function by $\varphi = d\rho(\xi) : M^2 \rightarrow \mathbf{R}$. We fix a point $p \in M^2$ arbitrarily. We take a curvature line coordinate neighborhood $(U; u, v)$ ($|u| < \varepsilon$, $v \in \mathbf{R}$) centered at p satisfying properties (1)–(2) given in Section 1. Then by Lemma 1.15, there exist C^∞ -functions $A(u), B(u)$ such that

$$(4.1) \quad \rho(u, v) = A(u)v + B(u).$$

By definition, we have

$$(4.2) \quad \varphi(u, v) = A(u).$$

Let Z_φ be the zeros of φ on M^2 . Suppose that Z_φ is non-empty and is not equal to M^2 . Then we may assume $p = (0, 0)$ is on the boundary of Z_φ . Then there exists a sequence $\{(u_n, v_n)\}_{n=1,2,3,\dots}$ in U such that $\lim_{n \rightarrow \infty} (u_n, v_n) = (0, 0)$ and $A(u_n) = \varphi(u_n, v_n) \neq 0$. Then we have

$$\lim_{n \rightarrow \infty} A(u_n) = \lim_{n \rightarrow \infty} \varphi(u_n, v_n) = \varphi(p) = 0.$$

Since $A(u_n) \neq 0$, we can set $w_n := -B(u_n)/A(u_n)$. Then $\rho(u_n, w_n) = 0$ holds, that is, $\{(u_n, w_n)\}$ is a sequence consisting of singular points in U . If $B(0) \neq 0$, then this singular set is unbounded, which contradicts the compactness of the singular set. So we can conclude that $B(0) = 0$. Then $\rho(p) = A(0)v + B(0) = 0$, which implies that the curve $v \mapsto (0, v)$ consists of singular points, which contradicts the compactness of the singular set again. Thus we have $Z_\varphi = M^2$, since we have assumed that Z_φ is non-empty. However, if $Z_\varphi = M^2$, then $\rho(u, v) = B(u)$, that is, ρ does not depend on v . If ρ has zeros, then the singular set contains an asymptotic line. This contradicts the compactness of the singular set. So ρ never vanishes. Then this implies that there are no singular points on M^2 , which contradicts our assumption. Thus Z_φ must be an empty set. In particular, $A(u)$ never vanishes, and so each asymptotic line has a singular point exactly at $v = -B(u)/A(u)$.

Now, we fix a point $p \in M^2$ arbitrarily. We have an expression (4.1). Since we have seen that Z_φ is the empty set, $A(u)$ never vanishes, and so the singular set $S(f)$ on $(U; u, v)$ can be parametrized by $v := -B(u)/A(u)$, which implies that the singular curve is transversal to the asymptotic direction (i.e. v -direction) and the singular point is unique in each asymptotic line. Since we have already seen that each asymptotic line contains a singular point, we can conclude that $S(f)$ is a generator.

q.e.d.

Corollary 4.5. *Let f be as in Proposition 4.4. Then the Gaussian curve and the asymptotic Gaussian curve are both regular curves without inflection points.*

Proof. We may assume that M^2 is connected, and can take the singular set $S(f)$ as a generator. Then M^2 is diffeomorphic to $S^1 \times \mathbf{R}$ by Proposition 2.5, and we can take a quadruple $(a(t)dt, b(t)dt, \hat{\xi}, \hat{\nu})$ which represents f . Here $b(t)$ vanishes identically. In fact, if $b(t_0) \neq 0$,

$$(4.3) \quad f_t(t, v) \times f_v(t, v) = (b(t) + v\mu_1(t))\hat{\eta}(t) \times \hat{\xi}(t)$$

does not vanish at $(t_0, 0)$, which contradicts the fact that $t \mapsto (t, 0)$ is the singular curve. Since f has no umbilics, $\hat{\nu}(t)$ is a regular curve. Suppose that $\hat{\xi}'(t_0) = 0$, that is, $\mu_1(t_0) = 0$. Then (4.3) implies that $v \mapsto (t_0, v)$ is a singular curve, which contradicts the compactness of the singular set. So $\hat{\xi}$ must be a regular curve. q.e.d.

Theorem 4.6. *Let M^2 be a connected 2-manifold, and $f : M^2 \rightarrow \mathbf{R}^3$ an orientable weakly complete flat front whose singular set is non-empty and compact. Then f has no umbilics.*

Proof. We suppose that \mathcal{U}_f is non-empty. Since any asymptotic line passing through a non-umbilic point never meets the umbilic set \mathcal{U}_f (cf. Lemma 1.15), Lemma 4.2 yields that the restriction $\hat{f} := f|_{M^2 \setminus \mathcal{U}_f}$ is an a -complete developable front. Since \mathcal{U}_f does not contain any singular points, \hat{f} is a -complete having compact singular set. If $M^2 = \mathcal{U}_f$, then f has no singular points, a contradiction.

So there exists a boundary point p on \mathcal{U}_f . Then there exists a sequence $\{q_n\}$ on $M^2 \setminus \mathcal{U}_f$ converging to p . Let L_n be the asymptotic line on M^2 passing through q_n . By Proposition 4.4 for \hat{f} , the singular set is a generator, and the condition (3) of Definition 2.3 yields that there exists a unique singular point s_n on L_n . Since the singular set is compact, we may assume that s_n converges to a point s . Since s is a singular point, $s \notin \mathcal{U}_f$, and there exists an asymptotic line L passing through s . Since L_n must converge to L and $q_n \in L_n$, the asymptotic line L meets an umbilic point p , which contradicts Lemma 1.15 again. Thus \mathcal{U}_f must be the empty set. q.e.d.

Proof of Theorem A : Since f is complete, it is weakly complete by Lemma 4.1. First, we consider the case that f is an orientable front. By Theorem 4.6, f has no umbilics. Then by Theorem 4.4, f is admissible and the singular set is a generator. Since M^2 is connected, M^2 is diffeomorphic to $S^1 \times \mathbf{R}$ by Proposition 2.5.

Next we consider the case that f is a non-orientable front or a non-orientable p-front. By Proposition 3.5, we can take a double covering $\pi : \hat{M}^2 \rightarrow M^2$ such that \hat{M}^2 is orientable and $f \circ \pi$ is a complete flat front. Then, by Theorem 4.6, $f \circ \pi$ (and also f) has no-umbilics. On

the other hand, Proposition 3.7 yields that f (and so $f \circ \pi$ also) has an umbilical point, which makes a contradiction. So this case is impossible.

Finally we consider the remaining case that f is a non-co-orientable (but orientable) p-front. Then we can take a double covering $\pi : \hat{M}^2 \rightarrow M^2$ such that $f \circ \pi$ is a complete flat front. Since M^2 is orientable, so is \hat{M}^2 . Then f is a -orientable by Corollary 3.4. Hence, by Theorem 4.6, $f \circ \pi$ has no umbilics, and is an admissible developable front such that the singular set is a generator of $f \circ \pi$. Thus by Corollary 2.11, \hat{M}^2 is diffeomorphic to $S^1 \times \mathbf{R}$. Moreover, since the singular curve is a generator, the proof of Corollary 4.5 yields that $f \circ \pi$ can be represented by a quadruple $(a(t)dt, 0, \hat{\xi}, \hat{\nu})$ such that $a(t), \hat{\xi}(t)$ are π -periodic and $\hat{\nu}(t)$ is anti- π -periodic, that is, $\hat{\nu}(t + \pi) = -\hat{\nu}(t)$ (cf. Proposition 3.6). Then f satisfies

$$(4.4) \quad f(t, v) = f(t + \pi, v) \quad (t, v \in \mathbf{R}).$$

In particular, the curve

$$\sigma_1 : S^1 \ni t \mapsto (t, 1) \in S^1 \times \mathbf{R} (= \hat{M}^2)$$

is π -periodic. Since $b(t)$ vanishes identically and $\mu_1(t)$ never vanishes by Corollary 4.5, the equation $v = 0$ characterizes the singular set (cf. Theorem 2.8). Thus $\sigma_1(t)$ does not meet the singular set of f , and $\hat{\sigma}_1(t) = f \circ \sigma_1(t)$ is a regular space curve. Then $\hat{\nu}(t)$ or $-\hat{\nu}(t)$ must coincide with

$$\hat{\xi}(t) \times \hat{\sigma}'_1(t) / |\hat{\xi}(t) \times \hat{\sigma}'_1(t)|.$$

Since $\hat{\xi}(t), \hat{\sigma}_1(t)$ are both π -periodic, so is $\hat{\nu}(t)$, which contradicts the anti- π -periodicity of $\hat{\nu}(t)$. So this case also never happens. q.e.d.

Corollary 4.7. $f : M^2 \rightarrow \mathbf{R}^3$ be a weakly complete front whose singular set is non-empty and compact. Then f is complete.

Proof. Without loss of generality, M^2 is connected. By Theorem 4.6, f has no umbilics. Then by Theorem 4.4, f is admissible and the singular set is a generator. Since M^2 is connected, Corollary 2.11 yields that f can be represented by a quadruple $(a(t)dt, 0, \hat{\xi}, \hat{\nu})$ ($t \in S^1$). Thus the first fundamental form ds^2 of f can be given by

$$ds^2 = (a(t)^2 + v^2\mu_1(t)^2) dt^2 + 2a(t)dt dv + dv^2,$$

where $\hat{\xi}' = \mu_1 \hat{\eta}$ ($\hat{\eta} = \hat{\xi} \times \hat{\nu}$). Let $\varphi : \mathbf{R} \rightarrow [0, 1]$ be a smooth function such that $\varphi(v) = 1$ for $v \in [-1, 1]$ and $\varphi(v) = 0$ for $|v| \geq 2$. It is sufficient to show that the metric

$$g := ds^2 + \varphi(v)^2 \mu_1(t)^2 dt^2$$

is a complete Riemannian metric on $S^1 \times \mathbf{R}$. Since the singular set is compact and the zeros of μ_1 correspond to the linear singularity (cf.

Definition 2.15), $\mu_1(t)$ never vanishes on S^1 , and we may set

$$m := \min_{t \in S^1} |\mu_1(t)| > 0.$$

Then we have

$$(4.5) \quad g \geq (a^2 + m^2)dt^2 + 2adtdv + dv^2.$$

Since the two eigenvalues k_1, k_2 ($k_2 \geq k_1$) of the matrix

$$\begin{pmatrix} a^2 + m^2 & a \\ a & 1 \end{pmatrix}$$

satisfy $k_1 k_2 = m^2$ and $k_2 \leq k_1 + k_2 = a^2 + m^2$, we have

$$\min_{t \in S^1} k_1(t) = \frac{m^2}{\max_{t \in S^1} k_2(t)} \geq \frac{m^2}{A^2 + m^2} > 0,$$

where $A := \max_{t \in S^1} |a(t)|$. In particular, the right hand side of (4.5) gives a complete Riemannian metric on the tv -plane. Thus g is also complete. q.e.d.

Proof of Theorem B: Let M^2 be a connected 2-manifold, and $f : M^2 \rightarrow \mathbf{R}^3$ a complete flat front. Then as seen in the proof of Theorem A, M^2 is orientable and f is admissible, where the singular set is a generator. Then f can be represented by a quadruple $(a(t)dt, b(t)dt, \hat{\xi}, \hat{\nu})$ ($t \in S^1$) associated with the singular set as a generator. Then as seen in the proof of Corollary 4.5, $b(t)$ vanishes identically. Thus, (2.6) reduces to the formula in Theorem B.

Conversely, let f be a map given by the formula in Theorem B. Since $\hat{\xi}(t)$ is a regular curve without inflection points, so is the dual spherical curve $\hat{\nu}(t)$. (The singular point of $\hat{\xi}(t)$ (resp. $\hat{\nu}(t)$) corresponds to the inflection point of $\hat{\nu}(t)$ (resp. $\hat{\xi}(t)$)). In particular $\nu'(t)$ does not vanish, namely f is a front without umbilical points. Moreover, f is weakly complete by Proposition 4.3. By the construction of f , its singular set is compact, and f is complete by Corollary 4.7. q.e.d.

EXAMPLE 4.8. (A cone over a locally convex spherical curve) Let $\hat{\xi} : S^1 \rightarrow S^2$ be a spherical closed regular curve which has no inflection points. By setting $a(t) = 0$ in Theorem B, the map

$$f(t, v) := v\hat{\xi}(t)$$

gives a complete flat front. The image of the singular curve $\{v = 0\}$ in the (t, v) -plane consists of an isolated cone-like singular point. We remark that non-zero principal curvature lines on f are all periodic (cf. Proposition 2.13).

DEFINITION 4.9. Let J be an open interval and $\gamma : J \rightarrow S^2$ a regular curve. Then $\gamma(t_0)$ is called a *vertex* if $t = t_0$ is a critical point of the geodesic curvature function $\kappa_g(t)$ of $\gamma(t)$. A vertex $t = t_0$ is called *generic* if $\kappa_g''(t_0) \neq 0$.

As an application of Theorem B and Proposition 2.16, we prove the following:

Corollary 4.10. *Let $f : S^1 \times \mathbf{R} \rightarrow \mathbf{R}^3$ be a complete flat front whose singular set is non-empty, and $\hat{\nu}$ its Gaussian curve. Let Γ be the set of real numbers δ so that f_δ has a singular point which is not a cuspidal edge nor a swallowtail.*

- (a) *If f is real analytic, then Γ is discrete.*
- (b) *If $\hat{\nu}$ admits only generic vertices, then Γ is bounded.*

In particular, Γ is finite if f satisfies both (a) and (b).

Proof. Since f has a singular point, we may assume that f is associated with the quadruple $(a(t)dt, 0, \hat{\xi}, \hat{\nu})$. Since $\hat{\nu}(t)$ is a regular curve (cf. Corollary 4.5), we may assume that t is the arc-length parameter. Now, we use the notations as in (2.7). Then $\mu_2(t) = 1$ identically, and $-\mu_1(t)$ is just the non-vanishing geodesic curvature function of the spherical curve $\hat{\nu}(t)$, since $\hat{\nu}' = \hat{\eta}$. We set

$$\varphi_\delta(t) := a(t) - \delta \left(\frac{1}{\mu_1(t)} \right)' \quad (t = d/dt).$$

Since $(a(t)dt, \delta dt, \hat{\xi}, \hat{\nu})$ represents f_δ (cf. Proposition 2.16), a singular point $(t, v) \in S^1 \times \mathbf{R}$ of f_δ which is neither a cuspidal edge nor a swallowtail appears only when $\varphi_\delta(t) = \varphi_\delta'(t) = 0$. In this case, $(t, -\delta/\mu_1(t))$ gives such a singular point.

We shall now prove (a). Suppose that f is real analytic and there exists a bounded sequence $\{\delta_n\}_{n=1,2,3,\dots}$ consisting of distinct points such that f_{δ_n} has a singular point $(t_n, v_n) \in S^1 \times \mathbf{R}$ which is neither a cuspidal edge nor a swallowtail. Then we have $v_n = -\delta_n/\mu_2(t_n)$. Since $t_n \in S^1$, we may assume that $\{t_n\}_{n=1,2,3,\dots}$ converges to a point $t_\infty \in [0, 2\pi)$. We set

$$r(t) := 1/\mu_1(t) > 0.$$

Since $\varphi_{\delta_n}(t_n) = \varphi_{\delta_n}'(t_n) = 0$, the function

$$\psi(t) := r'(t)a'(t) - r''(t)a(t)$$

vanishes at $t = t_n$ ($n = 1, 2, 3, \dots$). Since $\psi(t)$ is real analytic, this implies that there exists a positive integer N such that $t_n = t_N$ for $n \geq N$. Then, for $n \geq N$ we have

$$0 = \varphi_{\delta_n}(t_n) = \varphi_{\delta_n}(t_N) = a(t_N) - \delta_n \left(\frac{1}{\mu_1(t_N)} \right)' \Big|_{t=t_N},$$

which contradicts that $\{\delta_n\}$ consists of mutually distinct points. This proves (a). Next, suppose now that \hat{v} has only generic vertices. Then $|r'(t)| + |r''(t)|$ never vanishes for all $t \in S^1$, since $-1/r(t)$ is the geodesic curvature of $\hat{v}(t)$. If f_δ admits a singular point (t_0, v_0) which is neither a cuspidal edge nor a swallowtail, we have $a(t_0) = \delta r'(t_0)$ and $a'(t_0) = \delta r''(t_0)$. In particular,

$$|\delta| = \frac{|a(t_0)| + |a'(t_0)|}{|r'(t_0)| + |r''(t_0)|} \leq \frac{\max_{t \in S^1} (|a(t)| + |a'(t)|)}{\min_{t \in S^1} (|r'(t)| + |r''(t)|)},$$

which implies that Γ is bounded. q.e.d.

REMARK 4.11. A similar result for flat fronts in H^3 is given in [14].

5. Embeddedness of ends

In this section, we shall prove Theorem C and Theorem D in the introduction. Let f be a complete flat front associated with a pair $(a(t)dt, \hat{\xi}(t))$ on S^1 as in Theorem B. Without loss of generality, we may assume that t is the arclength parameter of $\xi(t)$ (cf. Corollary 4.5). Then $\xi(t)$ and $\xi'(t)$ are linearly independent. First, we suppose $f(t, v)$ ($t \in S^1 = \mathbf{R}/2\pi\mathbf{Z}$, $v \in \mathbf{R}$) has an end with self-intersection. There are two ends $\{v > 0\}$ and $\{v < 0\}$. Without loss of generality, we may assume that the end having self-intersections is $\{v > 0\}$. Then there exist $(u_n, v_n), (x_n, y_n) \in S^1 \times \mathbf{R}$ such that

$$(5.1) \quad f(u_n, v_n) = f(x_n, y_n) \quad (x_n \neq u_n, 0 < v_n < y_n),$$

and the sequence $\{v_n\}$ diverges to ∞ . Taking a subsequence if necessary, we may assume that $u_n \rightarrow u_\infty$ and $x_n \rightarrow x_\infty$, where $u_\infty, x_\infty \in [0, 2\pi)$. Using the expression (0.3), the singular set is parametrized by a curve

$$\hat{\sigma}(t) := \int_0^t \hat{\xi}\alpha = f(t, v) - v\hat{\xi}(t).$$

Since $|\hat{\xi}(u_n)| = |\hat{\xi}(x_n)| = 1$ and $|\hat{\xi}(u_n) \cdot \hat{\xi}(x_n)| \leq 1$, the function $v \mapsto |\hat{\xi}(u_n) - v\hat{\xi}(x_n)|^2$ is a monotone increasing function for $v \geq 1$. Since $y_n/v_n \geq 1$, we have by (5.1),

$$\left| \frac{\hat{\sigma}(x_n) - \hat{\sigma}(u_n)}{v_n} \right| = \left| \hat{\xi}(u_n) - \frac{y_n}{v_n} \hat{\xi}(x_n) \right| \geq |\hat{\xi}(u_n) - \hat{\xi}(x_n)|.$$

Since the singular curve $\hat{\sigma}$ is bounded, if we let $n \rightarrow \infty$, then $v_n \rightarrow \infty$ and the right hand side converges to 0. Thus we have $\hat{\xi}(u_\infty) = \hat{\xi}(x_\infty)$.

On the other hand, it holds that

$$\begin{aligned} \frac{1}{v_n} \left(\frac{\hat{\sigma}(x_n) - \hat{\sigma}(u_n)}{x_n - u_n} \right) + \frac{\hat{\xi}(x_n) - \hat{\xi}(u_n)}{x_n - u_n} &= \frac{f(x_n, v_n) - f(x_n, y_n)}{v_n(x_n - u_n)} \\ &= \left(\frac{1 - y_n/v_n}{x_n - u_n} \right) \hat{\xi}(x_n). \end{aligned}$$

Suppose now that $u_\infty = x_\infty$. Since the singular curve $\hat{\sigma}$ is bounded, the left hand side converges to $\hat{\xi}'(u_\infty)$ when $n \rightarrow \infty$, and the right hand side also must converge. In particular, $(1 - y_n/v_n)/(x_n - u_n)$ must converge, and $\hat{\xi}'(u_\infty)$ must be proportional to $\hat{\xi}(u_\infty)$, which contradicts the fact that $\hat{\xi}(u_\infty), \hat{\xi}'(u_\infty)$ are linearly independent. Thus we have $u_\infty \neq x_\infty$, and so $\hat{\xi}$ must have a self-intersection. By the duality, the Gaussian curve $\hat{\nu}$ also has a self-intersection, since the dual of a convex spherical curve is also convex. (Here we used the fact that $\hat{\xi}$ and $\hat{\nu}$ have no inflections).

Next, we consider the intersection of the front f with the sphere $S^2(r)$ of (sufficiently large) radius r centered at the origin. Rescaling the surface f , it is equivalent to consider the intersection between the unit sphere $S^2(1)$ and the flat front

$$F_\delta(t, v) := \delta\hat{\sigma}(t) + v\hat{\xi}(t) \quad (t \in S^1, v \in \mathbf{R}),$$

corresponding to the data $(\delta\alpha, \hat{\xi})$, where $\delta = 1/r$. Let Γ_δ be the intersection of the image of F_δ with the unit sphere $S^2(1)$.

Let k be a constant so that $0 < k < 1/\sqrt{2}$. Now we assume

$$|\delta| < k/m \quad (m := \max_{t \in S^1} |\hat{\sigma}(t)|).$$

Since

$$|\hat{\sigma} \cdot \hat{\xi}| \leq m, \quad \left| |\hat{\sigma}|^2 - (\hat{\sigma} \cdot \hat{\xi})^2 \right| = |\hat{\sigma} \times \hat{\xi}| \leq m,$$

we have that

$$\begin{aligned} -\delta\hat{\sigma} \cdot \hat{\xi} + \sqrt{1 - \delta^2 \left(|\hat{\sigma}|^2 - (\hat{\sigma} \cdot \hat{\xi})^2 \right)} &> -k + \sqrt{1 - k^2} (> 0), \\ -\delta\hat{\sigma} \cdot \hat{\xi} - \sqrt{1 - \delta^2 \left(|\hat{\sigma}|^2 - (\hat{\sigma} \cdot \hat{\xi})^2 \right)} &< k - \sqrt{1 - k^2} (< 0). \end{aligned}$$

Then we have

$$\begin{aligned} C_{k,\delta} &:= F_\delta^{-1}(\Gamma_\delta) \cap \{v > 0\} \\ &= \left\{ (t, v); v = -\delta\hat{\sigma}(t) \cdot \hat{\xi}(t) + \sqrt{1 - \delta^2 \left(|\hat{\sigma}(t)|^2 - (\hat{\sigma}(t) \cdot \hat{\xi}(t))^2 \right)} \right\}. \end{aligned}$$

This expression tells us that $C_{k,\delta}$ is a simple closed regular curve on $S^1 \times \mathbf{R}$. (This proves also that a complete flat front is a proper mapping.) Now, we suppose that f has an embedded end $\{v > 0\}$. Then for sufficiently large fixed k , the image $F_\delta(C_{k,\delta})$ ($0 < \delta < k$) is a simple closed regular curve on $S^2(1)$. By taking the limit $\delta \rightarrow 0$ in this expression, $C_{k,\delta}$ converges to a curve $\{v = 1\}$, and we have

$$\lim_{\delta \rightarrow 0} F_\delta(C_{k,\delta}) = \hat{\xi}(S^1).$$

Here, the above expression of $C_{k,\delta}$ and its image $F_\delta(C_{k,\delta})$ are both meaningful also for $\delta \leq 0$, and depend on δ smoothly at $\delta = 0$. Since $\hat{\xi}$ has

non-vanishing geodesic curvature, so does $F_\delta(C_{k,\delta})$ for sufficiently small δ . Since it is well-known that a simple closed spherical curve having non-vanishing geodesic curvature is a convex curve on an open hemisphere, we can conclude that $F_\delta(C_{k,\delta})$ is a convex curve on the unit sphere. Thus the limit curve $\hat{\xi}(S^1)$ must be also a convex curve. Since $\hat{\nu}$ is the dual of $\hat{\xi}$, $\hat{\nu}$ is also a convex curve. q.e.d.

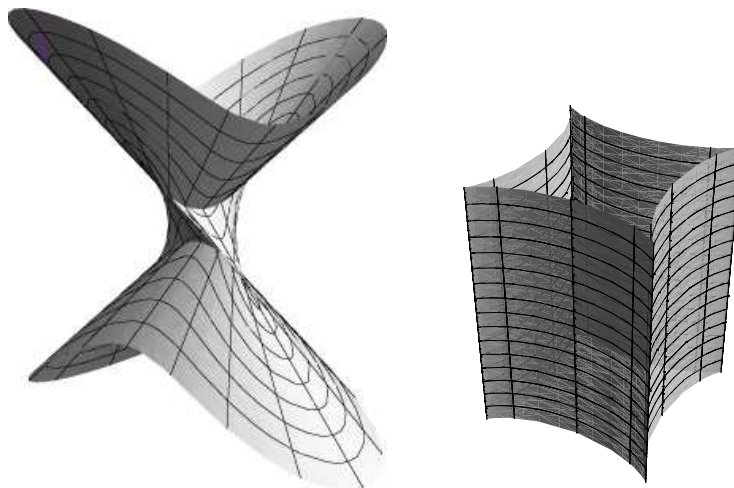


Figure 1. The flat surface for $n = 2$, $\varphi = \pi/4$ and its caustic.

Finally, we give a non-trivial example with embedded ends.

EXAMPLE 5.1. Let φ be a real number such that $|\varphi| < \pi/2$. We set

$$\xi_\varphi(t) = (\cos t \cos \varphi, \sin t \cos \varphi, \sin \varphi) \quad (0 \leq t \leq 2\pi),$$

which is a circle in S^2 . For a positive integer $n \geq 2$, we set $\alpha = (\cos nt)dt$. Then, a flat front f constructed from the pair (α, ξ_φ) by Theorem B is complete, since $\cos nt$ is L_2 -orthogonal to $\sin t$ and $\cos t$. Since ξ_φ is a convex curve, f has embedded ends. All of the principal curvature lines are periodic (cf. Proposition 2.13). When $n = 2$, there are four swallowtails and the other singular points are cuspidal edges. (See Figure 1.) The caustic C_f is a cylinder over a closed planar curves with $3/2$ -cusps, which follows from the fact that ν is a circle.

We now prove Theorem D in the introduction.

(Proof of Theorem D.) Let

$$\hat{\xi} = (\xi_1, \xi_2, \xi_3) : S^1 := \mathbf{R}/2\pi\mathbf{Z} \rightarrow S^2$$

be a convex spherical curve, that is, a simple closed regular curve in the unit sphere with non-vanishing geodesic curvature. Let $f : S^1 \times \mathbf{R} \rightarrow \mathbf{R}^3$

be a complete flat front with embedded ends whose asymptotic Gauss map is $\hat{\xi}(t)$. Then there exists a function $a(t) \in C^\infty(S^1)$ such that

$$\int_0^{2\pi} a(t)\hat{\xi}(t)dt = 0$$

and the pair $(a(t)dt, \hat{\xi})$ represents f via the formula (0.3) in the introduction.

Suppose that $a(t)$ does not change sign. We may assume that $a(t) > 0$. Since $\hat{\xi}(t)$ is a convex curve in S^2 , we may also assume that $\hat{\xi}(t)$ lies in the half-plane $D_+ := \{(x, y, z) \in \mathbf{R}^3; z > 0\}$, which yields that $a(t)\hat{\xi}(t) \in D_+$ and the integral never vanishes, a contradiction.

Next, suppose that $a(t)$ changes sign exactly twice at $t = t_1, t_2$ ($0 < t_1 < t_2 < 2\pi$) where $S^1 = \mathbf{R}/(2\pi\mathbf{Z})$. We define a subspace \mathcal{A} of $C^\infty(S^1)$ by

$$\mathcal{A} := \{c_1\xi_1 + c_2\xi_2 + c_3\xi_3 \in C^\infty(S^1); c_1, c_2, c_3 \in \mathbf{R}\}.$$

Since $\hat{\xi}(t)$ is a convex curve, it meets each 2-dimensional linear subspace

$$\{(x, y, z) \in \mathbf{R}^3; c_1x + c_2y + c_3z = 0, (c_1)^2 + (c_2)^2 + (c_3)^2 \neq 0\}$$

at most twice. Thus the linear map

$$L : \mathcal{A} \ni \varphi \mapsto (\varphi(0), \varphi(t_1), \varphi(t_2)) \in \mathbf{R}^3$$

must be a linear isomorphism. So there exists a unique $\varphi(t) \in \mathcal{A}$ such that

$$(\varphi(0), \varphi(t_1), \varphi(t_2)) = (a(0), a(t_1), a(t_2)) = (a(0), 0, 0).$$

Since $\varphi(t)$ changes sign at most two points, the sign of $\varphi(t)$ coincides with that of $a(t)$, which implies that $a(t)\varphi(t) > 0$ holds for all $t \neq t_1, t_2$. Thus we have $\int_0^{2\pi} a(t)\hat{\xi}(t)dt > 0$, a contradiction. So $a(t)$ changes signs at least four times.

On the other hand, since $\hat{\xi}$ is a regular curve, f has no linear singular points (see Proposition 2.16). Thus, a singular point which is not a cuspidal edge contains in the zeros of $a(t)$ by (1) of Proposition 2.16. Thus there are four singular points which are not cuspidal edges.

q.e.d.

The conclusion of Theorem D is optimal, as the flat front with embedded end as in Figure 1 has exactly four swallowtails. On the other hand, there exists a closed space curve $c(t)$ with non-zero torsion function. (See Example 5.3 below.) So we cannot drop the assumption of embeddedness of ends in Theorem D. We also remark here that the above proof based on an analog of the proof of the classical four vertex theorem for periodic functions (see [8] or [27, Theorem A.4] for details). There are several refinements of the classical four vertex theorem. See, for example, [21], [28] and [26]. We give here the following application:

Corollary 5.2. *Let $c : S^1 \rightarrow \mathbf{R}^3$ be an embedded space curve with non-vanishing curvature function κ . Then the tangential developable*

$$f(t, v) := c(t) + vc'(t)$$

gives an admissible developable frontal. Moreover, f is a complete front if and only if the torsion function τ never vanishes. In this case, f has non-embedded ends.

Proof. Without loss of generality, we may assume that t is the arclength parameter of $c(t)$. The unit bi-normal vector $\mathbf{b}(t)$ of the space curve $c(t)$ must be the normal vector of f . Since f is complete, f has no linear singularity. Then by Theorem 2.8, the fact that f is a front implies that τ has no zeros, since $\mathbf{b}' = \tau\mathbf{n}$, where $\mathbf{n} := c''(t)$. In this case, as pointed out in Example 2.4, f is an admissible orientable developable front. Moreover, by Proposition 4.3, it is weakly complete. Then by Corollary 4.7, f is complete. Suppose that f has embedded ends. Then Theorem D yields that the curve $c(t)$ cannot be a regular curve, a contradiction. q.e.d.

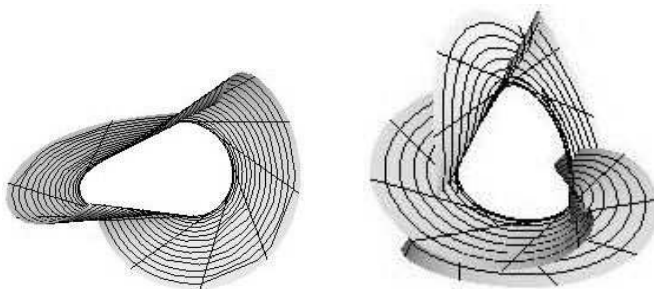


Figure 2. $f(t, v)$ for $0 \leq v \leq 7$ and $|v| \leq 7$ in Example 5.3.

EXAMPLE 5.3. We set

$$c(t) := ((4 + \cos 2t) \cos t, (4 + \cos 2t) \sin t, \sin 2t) \quad (0 \leq t \leq 2\pi),$$

which gives a space curve of non-vanishing curvature and of non-vanishing torsion. By Corollary 5.2, the tangential developable $f(t, v)$ of the curve gives a complete flat front with non-embedded ends (see Figure 2 left). Since c has non-vanishing curvature function, all singular points are non-linear. Moreover, since c is a regular curve, Proposition 2.16 yields that the singular set $c([0, 2\pi])$ consists of cuspidal edges (see Figure 2, right).

Appendix: Fronts in S^2 and their caustics.

In this appendix, we shall define fronts in S^2 and explain their caustics. ([1] is a good reference.) Let J be an open interval. A C^∞ -map $\gamma : J \rightarrow S^2$ is called a *frontal* if there exists a C^∞ -map $n : J \rightarrow S^2$ such that $n(t)$ is perpendicular to $\gamma(t)$ and $\gamma'(t)$ for each $t \in J$. The curve $\pm n(t)$ is called the *dual* of $\gamma(t)$. By definition, $n(t)$ is also a frontal and $\gamma(t)$ is a dual of $n(t)$. Moreover, since $\gamma'(t)$ and $n'(t)$ do not vanish at the same time, $\gamma(t)$ is called a *wave front* or a *front*. Suppose now that $\gamma(t)$ is a front. Then the pair

$$L = (\gamma(t), n(t)) : J \rightarrow \{(x, y) \in S^2 \times S^2; x \cdot y = 0\} = T^*S^2$$

gives a Legendrian immersion with respect to the canonical contact structure, and the fronts can be interpreted as projections of Legendrian immersions. (Here $x \cdot y$ is the inner product of x, y in \mathbf{R}^3 .) Now we set $\gamma_\theta(t) := \gamma(t) \cos \theta + n(t) \sin \theta$, which is called the *parallel curve* of γ . It can be directly checked that $n_\theta(t) := -\gamma(t) \sin \theta + n(t) \cos \theta$ gives the dual of $\gamma_\theta(t)$, that is, *parallel curves of a front are also fronts*. Since $n(t) = \gamma_{\pi/2}(t)$, the dual front is in the parallel family of γ .

Now we assume that $\gamma(t)$ ($t \in J$) is a regular curve. Then we may assume that t is the arc-length parameter of γ . We set $n'(t) = -\kappa_g(t)\gamma'(t)$, then $\kappa_g(t)$ is the geodesic curvature of $\gamma(t)$. The parallel curve $\gamma_\theta(t)$ has a singular point at $t = c$ if and only if $\cos \theta - \kappa_g(c) \sin \theta = 0$. We define a smooth function $A(t) : J \rightarrow \mathbf{R}/\pi\mathbf{Z}$ such that

$$(A.1) \quad \cos A(t) - \kappa_g \sin A(t) = 0.$$

Then $c(t) := \gamma(t) \cos A(t) + n(t) \sin A(t)$ is called the *caustic* (or *evolute*) of $\gamma(t)$, which has \pm ambiguity, since $A(t) + \pi$ also satisfies (A.1). The caustics $\pm c(t)$ give a parametrization of the singular set of the parallel family of γ . The following assertion was applied in Remark 2.19, and can be proved directly.

Proposition A.1. *Suppose that $\gamma(t)$ is a regular spherical curve with arc-length parameter. Then γ' is a regular spherical curve and the dual of γ' coincides with the caustic of γ .*

References

- [1] V. I. Arnold, *The geometry of spherical curves and the algebra of quaternions*, Uspekhi Mat. Nauk **50** (1995) 3–68, English transl. in Russian Math. Surveys **50** (1995) 1–68, MR 1331356, Zbl 0848.58005.
- [2] C. Chicone & N.J. Kalton, *Flat embeddings of the Möbius strip in \mathbf{R}^3* , Comm. Appl. Nonlinear Anal. **9** (2002) 31–50, MR 1896658, Zbl 1035.53006.
- [3] M. P. do Carmo, *Differential geometry of curves and surfaces*, Translated from the Portuguese. Prentice-Hall, Inc., Englewood Cliffs, N.J. (1976), MR 0394451, Zbl 0326.53001.

- [4] S. Fujimori, *Spacelike CMC 1 surfaces with elliptic ends in de Sitter 3-Space*, Hokkaido Math. J. **35** (2006) 289–320, MR 2254654, Zbl 1110.53006.
- [5] S. Fujimori, W. Rossman, M. Umehara, K. Yamada & S.-D. Yang, *Spacelike mean curvature one surfaces in de Sitter 3-space*, to appear in Communications in Analysis and Geometry, arXiv:0706.0973.
- [6] S. Fujimori, K. Saji, M. Umehara & K. Yamada, *Singularities of maximal surfaces*, Math. Z. **259** (2008) 827–848, MR 2403743, Zbl 1145.57026.
- [7] J. A. Gálvez & P. Mira, *Isometric immersions of \mathbf{R}^2 into \mathbf{R}^4 and perturbation of Hopf tori*, to appear in Math. Z.
- [8] L. Guieu, E. Moure & V. Yu. Ovsienko, *Theorem on six vertices of a plane curve via Sturm theory*. The Arnold-Gel'fand Mathematical Seminars, pp. 257–266, Birkhäuser, Boston, 1997, MR 1429895, Zbl 1076.53500.
- [9] B. Halpern & C. Weaver, *Inverting a cylinder through isometric immersions and isometric embeddings*, Trans. Amer. Math. Soc. **230** (1977) 41–70, MR 0474388, Zbl 0305.53003.
- [10] P. Hartman & L. Nirenberg, *On spherical image maps whose Jacobians do not change signs*, Amer. J. Math. **81** (1959) 901–920, MR 0126812, Zbl 0094.16303.
- [11] G. Ishikawa & Y. Machida, *Singularities of improper affine spheres and surfaces of constant Gaussian curvature*, Int. J. Math. **17** (2006) 269–293, MR 2215151, Zbl 1093.53067.
- [12] Young Wook Kim & Seong-Deog Yang, *A family of maximal surfaces in Lorentz-Minkowski three-space*, Proc. Amer. Math. Soc. **134** (2006) 3379–3390, MR 2231923, Zbl 1106.53005.
- [13] Y. Kitagawa, *Periodicity of the asymptotic curves on flat tori in S^3* , J. Math. Soc. Japan, **40** (1988) 457–476, MR 0945347, Zbl 0642.53059.
- [14] M. Kokubu, W. Rossman, K. Saji, M. Umehara & K. Yamada: *Singularities of flat fronts in hyperbolic 3-space*, Pacific J. Math. **221** (2005) 303–351, MR 2196639, Zbl 1110.53044.
- [15] M. Kokubu, W. Rossman, M. Umehara & K. Yamada: *Flat fronts in hyperbolic 3-space and their caustics*, J. Math. Soc. Japan **59** (2007) 265–299, MR 2302672, Zbl 1120.53036.
- [16] M. Kokubu, M. Umehara & K. Yamada, *Flat fronts in hyperbolic 3-space*, Pacific J. Math. **216** (2004) 149–175, MR 2094586, Zbl 1078.53009.
- [17] Y. Kurono & M. Umehara, *Flat Möbius strips of given isotopy type in \mathbf{R}^3 whose centerline are geodesic or lines of curvature*, Geometriae Dedicata (2008) 109–130, MR 2399653, Zbl pre05294712.
- [18] Sung Wook Lee & Seong-Deog Yang, *Spacelike surface of constant mean curvature 1 trinions in de Sitter three-space*, Osaka J. Math. **43** (2006) 641–663, MR 2283414, Zbl 1109.53062.
- [19] A. Martínez, *Improper Affine maps*, Math. Z. **249** (2005) 755–766, MR 2126213, Zbl 1132.53304.
- [20] W.S.Massey, *Surfaces of Gaussian Curvature Zero in Euclidean Space*, Tohoku Math. J. **14** (1962) 73–79, MR 0139088, Zbl 0114.36903.
- [21] V. Ovsienko & S. Tabachnikov, *Sturm theory, Ghys theorem on zeros of the Schwarzian derivative and flattening of Legendrian curves*, Selecta Math. (New Series) **2** (1996) 297–307, MR 1414890, Zbl 0972.58003.

- [22] T. Randrup & P. Røgen, *Side of the Möbius strip*, Arch. Math. **66** (1996) 511–521, MR 1388102, Zbl 0864.53002.
- [23] P. Roitman, *Flat surfaces in hyperbolic 3-space as normal surfaces to a congruence of geodesics*, Tohoku Math. J. **59** (2007) 21–37, MR 2321990, Zbl 1140.53004.
- [24] K. Saji, M. Umehara & K. Yamada, *Geometry of fronts*, Ann. of Math. **169** (2009) 491–529.
- [25] J.J. Stoker, *Developable Surface in the Large*, Comm. Pure and Appl. Math. **14** (1961) 627–635, MR 0137035, Zbl 0114.36902.
- [26] G. Thorbergsson & M. Umehara, *A unified approach to the four vertex theorems II*, Differential and symplectic topology of knots and curves, pp. 229–252, American Math. Soc. Transl. (Series 2) **190**, Amer. Math. Soc., Providence, R.I., 1999, MR 1738398, Zbl 1068.53004.
- [27] G. Thorbergsson & M. Umehara, *A global theory of flexes of periodic functions*, Nagoya Math. J. **173** (2004) 85–138, MR 2041757, Zbl 1066.51007.
- [28] M. Umehara, *A unified approach to the four vertex theorems I*, Differential and symplectic topology of knots and curves, pp. 185–228, American Math. Soc. Transl. (Series 2) **190**, Amer. Math. Soc., Providence, R.I., 1999, MR 1738397, Zbl 1068.53005.
- [29] M. Umehara & K. Yamada, *Maximal surfaces with singularities in Minkowski space*, Hokkaido Math. J. **35** (2006) 13–40, MR 2225080, Zbl 1109.53016.
- [30] W. Wunderlich, *Über ein abwickelbares Möbiusband*, Monatsh. Math. **66** (1962) 276–289, MR 0143115, Zbl 0105.14802.

DEPARTMENT OF MATHEMATICS,
KYOTO MUNICIPAL SAIKYO SENIOR HIGH SCHOOL,
KYOTO 604-8437, JAPAN
E-mail address: oz548-murata@edu.city.kyoto.jp

DEPARTMENT OF MATHEMATICS,
GRADUATE SCHOOL OF SCIENCE,
OSAKA UNIVERSITY,
TOYONAKA, OSAKA 560-0043, JAPAN
E-mail address: umehara@math.sci.osaka-u.ac.jp