

DIRAC GEOMETRY, QUASI-POISSON ACTIONS AND D/G -VALUED MOMENT MAPS

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Abstract

We study Dirac structures associated with Manin pairs $(\mathfrak{d}, \mathfrak{g})$ and give a Dirac geometric approach to Hamiltonian spaces with D/G -valued moment maps, originally introduced by Alekseev and Kosmann-Schwarzbach [3] in terms of quasi-Poisson structures. We explain how these two distinct frameworks are related to each other, proving that they lead to isomorphic categories of Hamiltonian spaces. We stress the connection between the viewpoint of Dirac geometry and equivariant differential forms. The paper discusses various examples, including q-Hamiltonian spaces and Poisson-Lie group actions, explaining how presymplectic groupoids are related to the notion of “double” in each context.

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1. Introduction

In this paper, we study Dirac structures [16, 17, 36] associated with Manin pairs $(\mathfrak{d}, \mathfrak{g})$ and develop a theory of Hamiltonian spaces with D/G -valued moment maps based on Dirac geometry. Our approach is parallel to the one originally introduced by Alekseev and Kosmann-Schwarzbach [3] to treat Hamiltonian quasi-Poisson actions, and one of our goals is to explain how these two points of view are related.

This paper is largely motivated by questions, set forth by Weinstein [42], concerning the existence of a unified geometric framework in which recent generalizations of the notion of *moment map* (including [3, 4, 5, 28, 32]) would naturally fit. As it turns out, a fruitful step to address these questions consists in passing from Poisson geometry, which describes classical moment maps, to Dirac structures, and our guiding principle is that generalized moment maps should be seen as morphisms between Dirac manifolds. Building on [10], we illustrate in this paper how Dirac geometry underlies moment map theories arising from Manin pairs, providing a natural arena for their unified treatment.

Our work was also stimulated by the theory of G -valued moment maps, introduced in [4, 5] in order to give a finite-dimensional account of the Poisson geometry of moduli spaces of flat G -bundles over surfaces [1]. A characteristic feature of G -valued moment maps is that they admit two distinct geometrical formulations: the original approach of [5] is based on *twisted 2-forms* and fits naturally into the framework of

Dirac geometry, see [2, 10, 12], whereas the description in [4] involves *quasi-Poisson bivector fields*. Although each approach relies on a different type of geometry, they turn out to be equivalent, see [4, Sec. 10] and [10, Sec. 3.5]. This paper grew out of our attempt to explain the geometric origins of these two formulations of G -valued moment maps as well as the equivalence between them. We prove in this paper that the equivalence between the Dirac geometric and quasi-Poisson approaches to Hamiltonian spaces holds at the more general level of D/G -valued moment maps. This elucidates, in particular, the connection between the Hamiltonian quasi-Poisson spaces of [3] and the symmetric-space valued moment maps, described by closed equivariant 3-forms, studied in [25].

The paper is organized as follows.

In Section 2, we review the basics of Dirac geometry, including the integration of Dirac structures to presymplectic groupoids, and the relationship between Dirac structures and equivariant cohomology [12]. In particular, for a given Dirac manifold S , we recall the general notion of *Hamiltonian space with S -valued moment map* [10].

We consider Dirac structures associated with Manin pairs in Section 3. Given a Manin pair $(\mathfrak{d}, \mathfrak{g})$ integrated by a group pair (D, G) (the definitions are recalled in Section 3.1), we consider, following [3], the homogeneous space $S := D/G$. We view S as a G -manifold with respect to the *dressing action*, induced by the left multiplication of G on D . While the theory of quasi-Poisson actions [3] is based on the additional choice of an isotropic complement of \mathfrak{g} in \mathfrak{d} (not necessarily a subalgebra), making the Manin pair into a Lie quasi-bialgebra [23], our starting point consists of a distinct choice. We instead consider the principal G -bundle $D \rightarrow D/G$, with respect to the action by right multiplication, and choose an *isotropic connection* $\theta \in \Omega^1(D, \mathfrak{g})$, i.e., a principal connection whose horizontal distribution is isotropic in TD (with respect to the invariant pseudo-riemannian metric induced by \mathfrak{d}). As it turns out, such connection θ defines a closed 3-form $\phi_S \in \Omega^3(S)$ as well as a ϕ_S -twisted Dirac structure $L_S \subset TS \oplus T^*S$ on S . This Dirac structure is best understood in terms of Courant algebroids: as observed by Ševera [35] and Alekseev-Xu [6], the trivial bundle $\mathfrak{d}_S := \mathfrak{d} \times S$ over S is naturally an exact Courant algebroid, and fixing θ is in fact equivalent to a choice of identification $\mathfrak{d}_S \cong TS \oplus T^*S$ (under which L_S corresponds to \mathfrak{g}). Upon an extra invariance assumption on θ , the Dirac structure L_S turns out to be determined by a closed equivariant extension of the 3-form ϕ_S .

Starting from a Manin pair $(\mathfrak{d}, \mathfrak{g})$ together with the choice of an isotropic connection $\theta \in \Omega^1(D, \mathfrak{g})$, we investigate in Section 4 the Hamiltonian theory associated with the Dirac manifold (S, L_S, ϕ_S) . In this theory, moment maps are given by suitable morphisms $J : M \rightarrow S$ from

Dirac manifolds M into S [10, 2]. We discuss how specific examples of Manin pairs equipped with particular choices of connections lead to many known moment maps theories, including G -valued and P -valued moment maps [5], G^* -valued moment maps [28], as well as symmetric-space valued moment maps [25]. This section also includes an explicit description of presymplectic groupoids integrating the Dirac manifold (S, L_S, ϕ_S) , explaining how they lead to the appropriate notion of “double” in different examples. A final important observation in this section is that the connection θ determines an interesting 2-form ω_D on the Lie group D . This 2-form makes D into a Morita bimodule (in the sense of [43]) between the Dirac manifold S and its opposite S^{op} . We use this Morita equivalence to define a nontrivial involution in the category of Hamiltonian G -spaces with S -valued moment maps. In the particular cases where $S = G$ or $S = G^*$, this involution agrees with the one induced by the inversion map on G or G^* .

In Section 5, we revisit the quasi-Poisson theory developed in [3]. The main new ingredient in our point of view is the construction of a Lie algebroid describing quasi-Poisson actions (particular examples of this Lie algebroid have appeared in [29] and [10], and an alternative construction was discussed in [11], see also [38]).

The aim of Section 6 is to relate the two approaches to Hamiltonian theories associated with Manin pairs. The set-up is a Manin pair $(\mathfrak{d}, \mathfrak{g})$ together with two extra choices: an isotropic connection $\theta \in \Omega^1(D, \mathfrak{g})$, which leads to a category of Hamiltonian spaces via Dirac geometry, and an isotropic complement \mathfrak{h} of $\mathfrak{g} \subset \mathfrak{d}$, which leads to a category of Hamiltonian spaces described by quasi-Poisson structures. These extra choices can always be made, and they are independent of each other. We give an explicit geometric construction of an isomorphism between the two types of Hamiltonian categories, generalizing [10, Thm. 3.16] (following the methods in [2]). As we will see, the link between Dirac structures and quasi-Poisson bivector fields lies in the theory of Lie quasi-bialgebroids [34]. Under the identification $\mathfrak{d}_S \cong TS \oplus T^*S$ induced by θ , the subspace $\mathfrak{h} \subset \mathfrak{d}$ defines an almost Dirac structure C_S on S transverse to L_S . Given a moment map $J : (M, L) \rightarrow (S, L_S)$ in the Dirac geometric setting, the pull-back image of C_S under J (in the sense of Dirac geometry, see e.g. [2, 14]) defines an almost Dirac structure C on M transverse to L , so the pair $L, C \subset TM \oplus T^*M$ is a Lie quasi-bialgebroid. The bivector field on M naturally induced by this Lie quasi-bialgebroid makes it into a quasi-Poisson space. This procedure can be reversed and establishes the desired equivalence of viewpoints. We note that many features of the Hamiltonian spaces are independent of any of the choices involved, including the construction of reduced spaces. Lastly, the main facts about Courant algebroids and Lie quasi-bialgebroids used in the paper are collected in the Appendix.

There is a more conceptual explanation for the equivalence between the quasi-Poisson and Dirac geometric viewpoints to Hamiltonian spaces associated with Manin pairs: as shown in [13], there is an abstract notion of Hamiltonian space canonically associated with a Manin pair; when additional (noncanonical) choices are made, these abstract Hamiltonian spaces take the two concrete forms studied in this paper.

Notation. Given a Lie group G with algebra \mathfrak{g} (defined by *right*-invariant vector fields), the left/right vector fields in G defined by $u \in \mathfrak{g}$ are denoted by $u^l, u^r \in \mathfrak{X}(G)$. Left and right translations by $g \in G$ are denoted by l_g and r_g , and we write $(u^r)_g = dr_g(u)$, or simply $r_g(u)$ if there is no risk of confusion (similarly for left translations). The left/right Maurer-Cartan 1-forms on G are denoted by $\theta^L, \theta^R \in \Omega^1(G, \mathfrak{g})$ and defined by $\theta^L(u^l) = \theta^R(u^r) = u$.

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2. Dirac geometry and Hamiltonian actions

In this section, we briefly recall the basics of Dirac geometry [16, 17, 36] and describe how to associate a category of Hamiltonian spaces to a given Dirac manifold S , obtaining a general notion of *S-valued moment map*. We will mostly follow [10, 12].

2.1. Dirac geometry. Let M be a smooth manifold. Consider the bundle $\mathbb{T}M := TM \oplus T^*M$ equipped with the symmetric pairing

$$(1) \quad \langle (X_1, \alpha_1), (X_2, \alpha_2) \rangle := \alpha_2(X_1) + \alpha_1(X_2).$$

An **almost Dirac structure** on M is a subbundle $L \subset \mathbb{T}M$ which is *lagrangian* (i.e., maximal isotropic) with respect to (1). Since the pairing has split signature, it follows that $\text{rank}(L) = \dim(M)$. Simple examples of almost Dirac structures include 2-forms $\omega \in \Omega^2(M)$ and bivectors fields $\pi \in \mathfrak{X}^2(M)$, realized as subbundles of $\mathbb{T}M$ via the graphs of the maps $TM \rightarrow T^*M$, $X \mapsto i_X\omega$ and $T^*M \rightarrow TM$, $\alpha \mapsto i_\alpha\pi$.

Let $\phi \in \Omega^3(M)$ be a closed 3-form on M . A ϕ -**twisted Dirac structure** [36] on M is an almost Dirac structure $L \subset \mathbb{T}M$ satisfying the following integrability condition: the space of sections $\Gamma(L)$ is closed under the ϕ -twisted Courant bracket

$$(2) \quad [[(X_1, \alpha_1), (X_2, \alpha_2)]_\phi := ([X_1, X_2], \mathcal{L}_{X_1}\alpha_2 - i_{X_2}d\alpha_1 + i_{X_2}i_{X_1}\phi),$$

where $X_1, X_2 \in \mathfrak{X}(M)$ and $\alpha_1, \alpha_2 \in \Omega^1(M)$. For a 2-form $\omega \in \Omega^2(M)$, the integrability condition amounts to $d\omega + \phi = 0$, and for a bivector field $\pi \in \mathfrak{X}^2(M)$ it gives $\frac{1}{2}[\pi, \pi] = \pi^\sharp(\phi)$ (here $[\cdot, \cdot]$ denotes the Schouten bracket). We will see many other examples later in this paper. We

denote a Dirac manifold by the triple (M, L, ϕ) , or simply (M, L) if the 3-form is clear from the context. Given an ϕ -twisted Dirac structure L on M , we define its **opposite** as

$$(3) \quad L^{\text{op}} := \{(X, \alpha) \in TM \mid (X, -\alpha) \in L\},$$

which is a $-\phi$ -twisted Dirac structure. We often denote $(M, L^{\text{op}}, -\phi)$ simply by M^{op} .

The bracket (2), although not skew-symmetric, becomes a Lie bracket when restricted to the space of sections of a Dirac structure L . The vector bundle $L \rightarrow M$ inherits the structure of a *Lie algebroid* over M , with bracket $[\cdot, \cdot]_{\phi}|_{\Gamma(L)}$ and anchor given by $\text{pr}_{TM}|_L$, where $\text{pr}_{TM} : TM \rightarrow TM$ is the natural projection. As a result, the distribution $\text{pr}_{TM}(L) \subset TM$ is integrable and defines a singular foliation on M . Each leaf $\iota : \mathcal{O} \hookrightarrow M$ inherits a 2-form $\omega_{\mathcal{O}} \in \Omega^2(\mathcal{O})$, defined at each point $x \in \mathcal{O}$ by

$$(4) \quad \omega_{\mathcal{O}}(X_1, X_2) = \alpha(X_2),$$

where $\alpha \in T_x^*M$ is such that $(X_1, \alpha) \in L_x$ (the value of $\omega_{\mathcal{O}}$ is independent of the particular choice of α). The integrability of L implies that $d\omega_{\mathcal{O}} + \iota^*\phi = 0$. This singular foliation, equipped with the leafwise 2-forms, is referred to as the **presymplectic foliation** of L . Note that the leafwise 2-forms are nondegenerate if and only if L is the graph of a bivector field, and the Lie algebroid of L is *transitive* (i.e., with surjective anchor) if and only if L is the graph of a 2-form.

Given manifolds M and S and a smooth map $J : M \rightarrow S$, we say that the elements $(X, \alpha) \in TM_x$ and $(Y, \beta) \in TS_{J(x)}$ are **J -related** at x if

$$Y = (dJ)_x(X) \quad \text{and} \quad \alpha = (dJ)_x^*\beta.$$

A direct calculation shows the following (see [2, Sec. 2] and [37]):

Lemma 2.1. *If (X_i, α_i) and (Y_i, β_i) are J -related at x , $i = 1, 2$, then $\langle (X_1, \alpha_1), (X_2, \alpha_2) \rangle_x = \langle (Y_1, \alpha_1), (Y_2, \alpha_2) \rangle_{J(x)}$. Also, if $(X_i, \alpha_i) \in \Gamma(TM)$ and $(Y_i, \beta_i) \in \Gamma(J^*TS)$ are J -related at all points in a neighborhood of $x \in M$, then $[(X_1, \alpha_1), (X_2, \alpha_2)]_{J^*\phi_S}$ is J -related to $[(Y_1, \alpha_1), (Y_2, \alpha_2)]_{\phi_S}$ at x .*

Here J^*TS denotes the pull-back vector bundle over M .

Consider the subbundle $\Gamma_J \subset J^*TS \oplus TM$ defined by

$$\Gamma_J := \{((Y, \beta), (X, \alpha)) \in J^*TS \oplus TM \mid (X, \alpha) \text{ is } J\text{-related to } (Y, \beta)\}.$$

Then Γ_J is lagrangian in $J^*\overline{TS} \oplus TM$, where \overline{TS} is equipped with minus the pairing (1). We use Γ_J to define morphisms of almost Dirac structures using composition of lagrangian relations, following [20, 42] (c.f. [2, 14]). Let L and L_S be almost Dirac structures on M and S .

We say that L_S is the **forward image** of L if $\Gamma_J \circ L = J^*L_S$ at each $x \in M$, that is, for all $x \in M$,

$$(L_S)_{J(x)} = \{((dJ)_x(X), \beta) \mid X \in T_x M, \beta \in T_{J(x)}^* S, \\ \text{and } (X, (dJ)_x^*(\beta)) \in L_x\}.$$

In this case we call J a **forward Dirac map** (or simply f-Dirac map). Similarly, we say that L is the **backward image** of L_S if $L = (J^*L_S) \circ \Gamma_J$ at each $x \in M$, which amounts to

$$L_x = \{(X, (dJ)_x^*(\beta)) \mid X \in T_x M, \beta \in T_{J(x)}^* S, \\ \text{and } ((dJ)_x(X), \beta) \in (L_S)_{J(x)}\}.$$

When both L and L_S are (graphs of) 2-forms, the notion of backward image reduces to the usual notion of pull-back; on the other hand, when both L and L_S are bivector fields, then the forward image amounts to the push-forward relation.

Just as Poisson manifolds are infinitesimal versions of symplectic groupoids [41] (c.f. [15, 18]), Dirac manifolds also have global counterparts. The objects integrating ϕ -twisted Dirac structures are **ϕ -twisted presymplectic groupoids** [12, 43], i.e., Lie groupoids \mathcal{G} over a base M equipped with a 2-form $\omega \in \Omega^2(\mathcal{G})$ such that:

- i) ω is *multiplicative*, i.e., $m^*\omega = p_1^*\omega + p_2^*\omega$, where $m : \mathcal{G} \times_M \mathcal{G} \rightarrow \mathcal{G}$ is the groupoid multiplication and $p_i : \mathcal{G} \times_M \mathcal{G} \rightarrow \mathcal{G}$, $i = 1, 2$, are the natural projections onto the first and second factors.
- ii) $d\omega = s^*\phi - t^*\phi$, where s, t are source, target maps on \mathcal{G} , and $\phi \in \Omega_{cl}^3(M)$.
- iii) $\dim(\mathcal{G}) = 2\dim(M)$.
- iv) $\ker(\omega)_x \cap \ker(ds)_x \cap \ker(dt)_x = \{0\}$, for all $x \in M$.

If ω satisfies condition ii), then we say that it is **relatively ϕ -closed**. Conditions i) and ii) together are equivalent to $\omega + \phi$ being a 3-cocycle in the *bar-de Rham complex* of the Lie groupoid \mathcal{G} [8] (see also [43]), i.e., the total complex of the double complex $\Omega^p(\mathcal{G}_q)$ (here \mathcal{G}_q denotes the space of composable sequence of q -arrows) computing the cohomology of $B\mathcal{G}$.

As proven in [12], any ϕ -twisted presymplectic groupoid \mathcal{G} over M defines a canonical ϕ -twisted Dirac structure L on M , uniquely determined by the fact that t is an f-Dirac map (whereas s is anti f-Dirac). Moreover, there is an explicit identification (as Lie algebroids) of L with the Lie algebroid of \mathcal{G} . In this context, we say that the presymplectic groupoid is an **integration** of the Dirac manifold (M, L, ϕ) . Conversely, if a ϕ -twisted Dirac structure L on M is integrable (as a Lie algebroid), then the corresponding s -simply-connected groupoid admits a unique ϕ -twisted presymplectic structure integrating L . We will see many concrete examples of presymplectic groupoids in Section 4.3.

2.2. The Hamiltonian category of a Dirac manifold. The fact that moment maps in symplectic geometry are *Poisson maps* indicates that moment maps in Dirac geometry should be represented by *f-Dirac maps*. Indeed, we need a special class of f-Dirac maps to play the role of moment maps.

Given Dirac manifolds (M, L, ϕ) and (S, L_S, ϕ_S) , we call a smooth map $J : M \rightarrow S$ a **strong Dirac map** if

- 1) $\phi = J^* \phi_S$,
- 2) J is an f-Dirac map: $\Gamma_J \circ L = J^* L_S$,
- 3) Denoting $\ker(L) := L \cap TM$, the following transversality condition holds:

$$(5) \quad \ker(dJ)_x \cap \ker(L)_x = \{0\}, \quad \forall x \in M.$$

(This is equivalent to the composition $\Gamma_J \circ L$ being transversal.)

Strong Dirac maps are alternatively called *Dirac realizations* in [10] (see also [2]). In particular, we will refer to a strong Dirac map $J : M \rightarrow S$ for which the Dirac structure on M is a 2-form as a **presymplectic realization** of S . An immediate example of a presymplectic realization is the inclusion $\iota : \mathcal{O} \hookrightarrow S$ of a presymplectic leaf. More generally, since the composition of strong Dirac maps is a strong Dirac map, the restriction of a strong Dirac map $M \rightarrow S$ to leaves of M define presymplectic realizations of S .

Definition 2.1. The **Hamiltonian category** of a Dirac manifold (S, L_S, ϕ_S) is the category $\overline{\mathcal{M}}(S, L_S, \phi_S)$ whose objects are strong Dirac maps $J : M \rightarrow S$ and morphisms are smooth maps $\varphi : M \rightarrow M'$ which are f-Dirac maps and such that $J' \circ \varphi = J$. We denote by $\mathcal{M}(S, L_S, \phi_S)$ the subcategory of presymplectic realizations.

This definition will be justified by general properties of strong Dirac maps as well as concrete examples. First of all, a strong Dirac map $J : M \rightarrow S$ induces a canonical action on M . Indeed, the properties of J define a smooth bundle map [2, 10]

$$(6) \quad \rho_M : J^* L_S \rightarrow TM,$$

where $X = \rho_M(Y, \beta)$ is uniquely determined by the conditions

$$(7) \quad (dJ)_x(X) = Y \quad \text{and} \quad (X, (dJ)_x^*(\beta)) \in L.$$

Let us also consider the bundle map

$$(8) \quad \widehat{\rho}_M : J^* L_S \rightarrow L, \quad (Y, \beta) \mapsto (\rho_M(Y, \beta), (dJ)_x^*(\beta)).$$

A direct computation shows that, at the level of sections, the induced map $\widehat{\rho}_M : \Gamma(L_S) \rightarrow \Gamma(L)$ preserves Lie brackets, and hence so does $\rho_M : \Gamma(L_S) \rightarrow \mathfrak{X}(M)$. As a result, we have

Proposition 2.1. *If $J : M \rightarrow S$ is a strong Dirac map, then the map ρ_M (6) defines a Lie algebroid action of L_S on M .*

More on Lie algebroid actions can be found e.g. in [30].

It immediately follows from (7) that the action ρ_M is tangent to the presymplectic leaves of M . In particular, when L is defined by a 2-form ω on M (i.e., J is a presymplectic realization), then the conditions in (7), relating J and the action ρ_M , take the form

$$dJ(\rho_M(Y, \beta)) = Y \quad \text{and} \quad i_{\rho_M(Y, \beta)}\omega = J^*\beta,$$

which can be interpreted as an *equivariance condition* for J (with respect to the canonical action of L_S on S) together with a *moment map condition*. In this sense, we think of M as carrying a *Hamiltonian action* and $J : M \rightarrow S$ as an *S -valued moment map*. Various properties of usual Hamiltonian spaces are present in the framework of strong Dirac maps. For example, as discussed in [10, Sec. 4.4], there is a natural reduction procedure generalizing Marsden-Weinstein's reduction [31] (a particular case of which will be recalled in Section 4.1).

Let us recall some examples of Hamiltonian spaces defined by strong Dirac maps [10].

Example 2.2. The identity map $\text{Id} : S \rightarrow S$ is always an object in $\overline{\mathcal{M}}(S, L_S, \phi_S)$, whereas inclusions of presymplectic leaves $\iota : \mathcal{O} \hookrightarrow S$ are objects in $\mathcal{M}(S, L_S, \phi_S)$.

Example 2.3. If (\mathcal{G}, ω) is a presymplectic groupoid integrating the Dirac manifold (S, L_S, ϕ_S) , then

$$(t, s) : \mathcal{G} \rightarrow S \times S^{\text{op}}$$

is a strong Dirac map, i.e., it is an object in $\mathcal{M}(S \times S, L_S \times L_S^{\text{op}}, \phi_S \times (-\phi_S))$.

Example 2.4. If L_S is the graph of a Poisson structure π_S and $J : (M, L) \rightarrow (S, \pi_S)$ is a strong Dirac map, then the transversality condition (5) implies that L must be (the graph of) a Poisson structure. Hence $\overline{\mathcal{M}}(S, L_S)$ is simply the category of Poisson maps into S (whereas $\mathcal{M}(S, \pi_S)$ is the category of Poisson maps from *symplectic* manifolds into S). These are the infinitesimal versions of the Hamiltonian spaces studied by Mikami and Weinstein [32] in the context of symplectic groupoid actions. For the specific choice of $S = \mathfrak{g}^*$, equipped with its canonical linear Poisson structure $\pi_{\mathfrak{g}^*}$, then $\overline{\mathcal{M}}(S, L_S)$ (resp. $\mathcal{M}(S, L_S)$) is the category of Poisson (resp. symplectic) Hamiltonian \mathfrak{g} -spaces in the classical sense.

Example 2.5. Let G be a Lie group whose Lie algebra \mathfrak{g} carries an Ad-invariant, symmetric, nondegenerate bilinear form $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$. We consider G equipped with the Cartan-Dirac structure (see e.g. [2, 12, 36])

$$(9) \quad L_G := \{(\rho(v), \sigma(v)) \mid v \in \mathfrak{g}\} \subset \mathbb{T}G,$$

where $\rho(v) = v^r - v^l$ and $\sigma(v) := \frac{1}{2}\langle \theta^L + \theta^R, v \rangle_{\mathfrak{g}}$. This Dirac structure is integrable with respect to $-\phi_G$, where ϕ_G is the bi-invariant Cartan 3-form, defined by

$$\phi_G(u, v, w) := \frac{1}{2}\langle [u, v], w \rangle_{\mathfrak{g}}, \quad u, v, w \in \mathfrak{g}.$$

As shown in [10, 12] (see also [2]), $\mathcal{M}(G, L_G, \phi_G)$ is the category of q-Hamiltonian \mathfrak{g} -spaces in the sense of Alekseev-Malkin-Meinrenken [5] (more general objects in $\overline{\mathcal{M}}(G, L_G, \phi_G)$ correspond to foliated spaces whose leaves are q-Hamiltonian \mathfrak{g} -spaces).

We will return to Example 2.5 in Section 4.

We finally observe the behavior of the Hamiltonian category under gauge transformations. There is a natural action of $\Omega^2(S)$, the abelian group of 2-forms on S , on the set of Dirac structures on S : if L_S is a ϕ_S -twisted Dirac structure and $B \in \Omega^2(S)$, we define

$$\tau_B(L_S) := \{(Y, \beta + i_Y B) \mid (Y, \beta) \in L_S\} \subset \mathbb{T}S,$$

which is a $(\phi_S - dB)$ -twisted Dirac structure on S . We refer to τ_B as a **gauge transformation** by B . We use the notation $\tau_B(S)$ for the Dirac manifold $(S, \tau_B(L_S), \phi_S - dB)$.

Proposition 2.2. *Let $B \in \Omega^2(S)$, and let (M, L, ϕ) , (S, L_S, ϕ_S) be Dirac manifolds. Then $J : M \rightarrow L$ is a strong Dirac map if and only if $J : \tau_{J^*B}(M) \rightarrow \tau_B(S)$ is a strong Dirac map. Moreover, this correspondence defines an isomorphism of Hamiltonian categories*

$$\mathcal{I}_B : \overline{\mathcal{M}}(S, L_S) \xrightarrow{\sim} \overline{\mathcal{M}}(S, \tau_B(L_S)),$$

which restricts to an isomorphism $\mathcal{M}(S, L_S) \cong \mathcal{M}(S, \tau_B(L_S))$.

The proof is a direct verification using the definitions (see also [14]).

Remark 2.6. (Global actions)

We have only defined the Hamiltonian category of a Dirac manifold at the infinitesimal level. The global counterparts of the S -valued Hamiltonian spaces in $\mathcal{M}(S, L_S, \phi_S)$ are manifolds M equipped with a 2-form $\omega_M \in \Omega^2(M)$ and carrying a left \mathcal{G} -action $\rho_M : \mathcal{G} \times_S M \rightarrow M$ (where (\mathcal{G}, ω) is a presymplectic groupoid integrating L_S) along a smooth map $J : M \rightarrow S$ such that $d\omega_M + J^*\phi_S = 0$, $\ker(dJ) \cap \ker(\omega_M) = \{0\}$ and

$$(10) \quad \rho_M^* \omega_M = \text{pr}_M^* \omega_M + \text{pr}_{\mathcal{G}}^* \omega.$$

Here $\text{pr}_M, \text{pr}_{\mathcal{G}}$ are the natural projections from $\mathcal{G} \times_S M$ on M and \mathcal{G} . Condition (10) is the global version of J being an f-Dirac map [12, Sec. 7]. These global Hamiltonian spaces are studied in [43]. The global counterparts of the more general objects in $\overline{\mathcal{M}}(S, L_S)$ are similar, but now M carries a Dirac structure and (10) holds leafwise [10,

Sec. 4.3]. In this paper, we will be mostly concerned with the infinitesimal Hamiltonian category (but all results have global versions that can be obtained by standard integration procedures).

2.3. Dirac structures and equivariant cohomology. Let (\mathcal{G}, ω) be a ϕ -twisted presymplectic groupoid integrating a Dirac manifold (M, L, ϕ) . As recalled in Section 2.1, $\omega + \phi$ defines a 3-cocycle in the bar-de Rham complex of \mathcal{G} . Let us assume that $\mathcal{G} = G \ltimes M$ is an *action groupoid*, relative to an action of a Lie group G on M (i.e., $s(g, x) = x$, $t(g, x) = g.x$ and $m((h, y), (g, x)) = (hg, x)$). In this case, the bar-de Rham complex of \mathcal{G} becomes the total complex of the double complex $\Omega^p(G^q \times M)$, which computes the equivariant cohomology of M in the Borel model (see e.g. [8]). In particular, $\omega + \phi$ defines an equivariant 3-cocycle. We now discuss the infinitesimal counterpart of this picture, which relates Dirac structures to equivariant 3-cocycles in the Cartan model (see [12, Sec. 6.4]).

Let A be a Lie algebroid over M , with bracket $[\cdot, \cdot]_A$ and anchor $\rho : A \rightarrow TM$. As proven in [12], the infinitesimal version of a multiplicative, relatively ϕ -closed 2-form on a Lie groupoid is a pair (σ, ϕ) where $\sigma : A \rightarrow T^*M$ is a bundle map, $\phi \in \Omega^3(M)$ is closed, and such that

$$(11) \quad \langle \sigma(a), \rho(a') \rangle = -\langle \sigma(a'), \rho(a) \rangle,$$

$$(12) \quad \sigma([a, a']_A) = \mathcal{L}_{\rho(a)}\sigma(a') - i_{\rho(a')}d\sigma(a) + i_{\rho(a) \wedge \rho(a')}\phi,$$

for all $a, a' \in \Gamma(A)$. Let us assume that the bundle map $(\rho, \sigma) : A \rightarrow \mathbb{T}M$ has constant rank, and let $L := (\rho, \sigma)(A) \subset \mathbb{T}M$. Then (11) says that L is isotropic, whereas (12) means that the space of section $\Gamma(L)$ is closed under the Courant bracket $[\![\cdot, \cdot]\!]_\phi$. It immediately follows that if $\text{rank}(L) = \dim(M)$, then L is a Dirac structure on M .

In this paper, we will be particularly interested in the following special case of this construction. Suppose that a manifold S carries an action of a Lie algebra \mathfrak{g} , denoted by $\rho : \mathfrak{g} \rightarrow \mathfrak{X}(S)$. Let $A = \mathfrak{g} \ltimes S$ be the associated action Lie algebroid, whose anchor is ρ and Lie bracket on $\Gamma(A) = C^\infty(S, \mathfrak{g})$ is uniquely defined by the bracket on \mathfrak{g} (viewed as constant sections) and the Leibniz rule (see Lemma 3.3). Assume that we are given a bundle map $\sigma : \mathfrak{g} \times S \rightarrow T^*S$ and a closed 3-form $\phi_S \in \Omega^3(S)$ satisfying (11) and (12). Let us also suppose that

$$(13) \quad \dim(\mathfrak{g}) = \dim(S), \quad \text{and} \quad \ker(\rho) \cap \ker(\sigma) = \{0\}.$$

The last two conditions guarantee that $\text{rank}(L_S) = \dim(S)$, hence

$$(14) \quad L_S := \{(\rho(v), \sigma(v)) \mid v \in \mathfrak{g}\}$$

is a ϕ_S -twisted Dirac structure on S . By construction, L_S is isomorphic to $A = \mathfrak{g} \ltimes S$ as a Lie algebroid (via $(\rho, \sigma) : A \rightarrow L_S$).

Proposition 2.3. *Given a strong Dirac map $J : M \rightarrow S$, we have an induced \mathfrak{g} -action $\rho_M : \mathfrak{g} \rightarrow \mathfrak{X}(M)$ uniquely determined by the conditions*

$$dJ \circ \rho_M = \rho, \quad \text{and} \quad (\rho_M(v), J^* \sigma(v)) \in L, \quad \forall v \in \mathfrak{g}.$$

This is a direct consequence of Prop. 2.1: the action ρ_M is just the restriction of (6) to \mathfrak{g} , viewed as constant sections in $\Gamma(L_S) \cong C^\infty(S, \mathfrak{g})$. We also have the associated bracket-preserving map

$$(15) \quad \widehat{\rho}_M : \mathfrak{g} \rightarrow \Gamma(L), \quad \widehat{\rho}_M(v) = (\rho_M(v), J^* \sigma(v)).$$

We can use the action ρ_M to give an alternative description of presymplectic realizations of S , phrased only in terms of the maps $\sigma : \mathfrak{g} \rightarrow \Omega^1(S)$ and $\rho : \mathfrak{g} \rightarrow \mathfrak{X}(S)$, without any explicit reference to Dirac structures:

Proposition 2.4. *Let M be equipped with a 2-form ω . Equip S with the Dirac structure L_S of (14), and let $J : M \rightarrow S$ be a smooth map. Then J is a presymplectic realization of (S, L_S, ϕ_S) if and only if the following is satisfied:*

- i) $d\omega + J^* \phi_S = 0$;
- ii) At each $x \in M$, $\ker(\omega)_x = \{\rho_M(v)_x : v \in \ker(\sigma)\}$;
- iii) The map $J : M \rightarrow S$ is \mathfrak{g} -equivariant and satisfies the moment map condition

$$i_{\rho_M(v)} \omega = J^* \sigma(v), \quad \forall v \in \mathfrak{g}.$$

Proof. The only condition that remains to be checked is ii), which follows from the transversality condition (5). The proof is identical to the one in [12, Thm. 7.6] (c.f. [2, Sec. 5]). q.e.d.

An immediate consequence of conditions i) and iii) in Prop. 2.4 is that

$$(16) \quad \mathcal{L}_{\rho_M(v)} \omega = J^*(d(\sigma(v)) - i_{\rho(v)} \phi_S).$$

Hence the 2-form ω on M will not be \mathfrak{g} -invariant in general unless $\sigma : \mathfrak{g} \rightarrow \Omega^1(S)$ and $\phi_S \in \Omega_{cl}^3(S)$ satisfy the extra condition

$$(17) \quad d(\sigma(v)) = i_{\rho(v)} \phi_S.$$

In this case, it immediately follows from (12) that

$$(18) \quad \sigma([u, v]) = \mathcal{L}_{\rho(u)} \sigma(v), \quad v \in \mathfrak{g}.$$

Note that (11) implies that $i_{\rho(v)} \sigma(v) = 0$ and, by (17), $\mathcal{L}_{\rho(v)} \phi_S = 0$. These conditions together precisely say that $\sigma + \phi_S$ defines an equivariantly closed 3-form, i.e., a 3-cocycle in the Cartan complex

$$(19) \quad \Omega_G^k(S) := \left(\bigoplus_{2i+j=k} S^i(\mathfrak{g}^*) \otimes \Omega^j(S) \right)^G,$$

$$d_G(P) = d(P(v)) - i_{\rho(v)}(P(v)),$$

where P is viewed as a G -invariant $\Omega^\bullet(S)$ -valued polynomial on \mathfrak{g} , and G is a connected Lie group integrating \mathfrak{g} . Conversely, we see that an

equivariantly closed 3-form $\sigma + \phi_S$ on S satisfying (13) defines a particular type of Dirac structure L_S by (14). We will see in this paper many concrete examples of this interplay between Dirac structures and equivariant 3-forms.

Remark 2.7. When a Dirac structure L_S is determined by an equivariantly closed 3-form $\sigma + \phi_S$, then a gauge transformation of L_S by an invariant 2-form B changes the equivariant 3-form by an equivariant coboundary: $\sigma + \phi_S \mapsto (\sigma + i_\rho B) + (\phi_S - dB)$.

As we will see in Section 4.3, one has explicit formulas for the multiplicative 2-forms on $\mathcal{G} = G \times S$ arising via integration of Dirac structures defined by (14), and these formulas are particularly simple when $\sigma + \phi_S$ is an equivariant 3-form. In this case, the integration procedure for Dirac structures gives a concrete realization of the natural map from the cohomology of the Cartan complex (19) into the equivariant cohomology of M in degree three.

3. Manin pairs and isotropic connections

3.1. Manin pairs. This section recalls the basic definitions in [3] and fixes our notation.

A **Manin pair** is a pair $(\mathfrak{d}, \mathfrak{g})$, where \mathfrak{d} is a Lie algebra of dimension $2n$, equipped with an Ad-invariant, nondegenerate, symmetric bilinear form $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$ of signature (n, n) , and $\mathfrak{g} \subset \mathfrak{d}$ is a Lie subalgebra which is also a maximal isotropic subspace. (In Appendix A.2 we discuss the more general notion of Manin pair *over a manifold M* .)

Throughout this paper we assume that a Manin pair $(\mathfrak{d}, \mathfrak{g})$ is integrated by a **group pair** (D, G) , where D is a connected Lie group whose Lie algebra is \mathfrak{d} , and G is a connected, closed Lie subgroup of D whose Lie algebra is \mathfrak{g} . Given a group pair (D, G) , one considers the quotient space

$$S = D/G$$

with respect to the G -action on D by right multiplication. The action of D on itself by left multiplication induces an action of D on S , called the **dressing action**. We denote by

$$(20) \quad \rho_S : \mathfrak{d} \rightarrow \mathfrak{X}(S)$$

the induced infinitesimal action, and by

$$(21) \quad \rho : \mathfrak{g} \longrightarrow \mathfrak{X}(S)$$

its restriction to \mathfrak{g} . The following are two key examples from [3].

Example 3.1. Let \mathfrak{g} be a Lie algebra and consider $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$, with Lie bracket given by

$$[(u, \mu), (v, \nu)]_{\mathfrak{d}} = ([u, v], \text{ad}_u^*(\nu) - \text{ad}_v^*(\mu)),$$

i.e. $\mathfrak{d} = \mathfrak{g} \ltimes \mathfrak{g}^*$ is the semi-direct product Lie algebra with respect to the coadjoint action. If we set the pairing $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$ to be the canonical one,

$$(22) \quad \langle (u, \mu), (v, \nu) \rangle_{\mathfrak{d}} := \langle (u, \mu), (v, \nu) \rangle_{can} = \nu(u) + \mu(v),$$

then $(\mathfrak{d}, \mathfrak{g})$ is a Manin pair. The Lie group integrating \mathfrak{d} is $D = G \ltimes \mathfrak{g}^*$, the semi-direct product Lie group with respect to the coadjoint action of G on \mathfrak{g}^* . Hence $S = \mathfrak{g}^*$, and the infinitesimal dressing action of \mathfrak{g} on S is the coadjoint action.

Example 3.2. Let \mathfrak{g} be a Lie algebra equipped with a symmetric, nondegenerate, ad-invariant bilinear form $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$. Consider the direct sum of Lie algebras $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$, together with the pairing

$$\langle (u_1, v_1), (u_2, v_2) \rangle_{\mathfrak{d}} := \langle u_1, u_2 \rangle_{\mathfrak{g}} - \langle v_1, v_2 \rangle_{\mathfrak{g}}.$$

We also write $\mathfrak{g} \oplus \bar{\mathfrak{g}}$ to denote \mathfrak{d} with the pairing above. If we consider \mathfrak{g} as a subalgebra of \mathfrak{d} through the diagonal embedding $\mathfrak{g} \hookrightarrow \mathfrak{d}$, $v \mapsto (v, v)$, then $(\mathfrak{d}, \mathfrak{g})$ is a Manin pair. The associated group pair is $(D = G \times G, G)$, where G is identified with the diagonal of D . In this case $S = (G \times G)/G \cong G$ via the map $[(a, b)] \mapsto ab^{-1}$. Under this identification, the dressing action of D on S is

$$(a, b) \cdot g = agb^{-1},$$

and, infinitesimally, we have

$$\rho_S : \mathfrak{d} \rightarrow TG, \quad (u, v) \mapsto u^r - v^l,$$

so the dressing action restricted to $G \subset D$ is the action by conjugation.

3.2. Connections and differential forms. We now introduce certain differential forms on $S = D/G$ and D which arise once a connection on the principal G -bundle (with respect to right multiplication)

$$(23) \quad p : D \longrightarrow S$$

is chosen. These differential forms play a central role in the definition of D/G -valued Hamiltonian spaces in Section 4.

A principal connection on the bundle (23) is called **isotropic** if its horizontal spaces are isotropic in TD (with respect to the bi-invariant pseudo-riemannian metric defined by $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$).

Proposition 3.1. *A connection on (23) is equivalent to the choice of a 1-form*

$$s \in \Omega^1(S, \mathfrak{d})$$

satisfying $\rho_S(s(X)) = X$, for all $X \in TS$, and the connection is isotropic if and only if s has isotropic image in \mathfrak{d} .

Proof. A principal connection is a G -equivariant bundle map $H : p^*TS \rightarrow TD$ such that $dp \circ H = \text{Id}$. We relate H and s by trivializing TD using right translations: $H(X, a) = dr_a(s(X))$. Since the dressing

action on S is $\rho_S(u)_{p(a)} = dp(dr_a(u))$, we have $\rho_S \circ s = dp \circ H$. The last assertion in the lemma follows from the invariance of $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$. q.e.d.

A connection on (23) can also be given in terms of a 1-form $\theta \in \Omega^1(D, \mathfrak{g})$ satisfying

$$\theta(dl_a(v)) = v, \quad \theta_{ag} dr_g = \text{Ad}_{g^{-1}} \theta_a,$$

for $a \in D$, $g \in G$ and $v \in \mathfrak{g}$, and it is isotropic if and only if

$$(24) \quad \langle \theta(X), \theta^L(Y) \rangle_{\mathfrak{d}} + \langle \theta(Y), \theta^L(X) \rangle_{\mathfrak{d}} = \langle X, Y \rangle_{\mathfrak{d}}, \quad X, Y \in TD.$$

The 1-forms $\theta \in \Omega^1(D, \mathfrak{g})$ and $s \in \Omega^1(S, \mathfrak{d})$ are related by

$$(25) \quad \theta_a = \theta_a^L - \text{Ad}_{a^{-1}}(p^*s), \quad a \in D.$$

Once an isotropic connection is fixed, we have the following induced differential forms on S :

$$(26) \quad \phi_S \in \Omega^3(S), \quad \phi_S := \frac{1}{2} \langle ds, s \rangle_{\mathfrak{d}} + \frac{1}{6} \langle [s, s]_{\mathfrak{d}}, s \rangle_{\mathfrak{d}},$$

and a \mathfrak{g}^* -valued 1-form

$$(27) \quad \sigma \in \Omega^1(S, \mathfrak{g}^*), \quad \sigma(X)(u) := \langle s(X), u \rangle_{\mathfrak{d}},$$

which we may alternatively view as a map $\mathfrak{g} \rightarrow \Omega^1(S)$. We will also write ϕ_S^s , σ_s if we want to stress the dependence of these forms on the given connection $s \in \Omega^1(S, \mathfrak{d})$.

These forms satisfy many nice properties, as illustrated below.

Proposition 3.2. *The 3-form ϕ_S is closed, and σ satisfies conditions (11) and (12). Moreover, viewing S as a \mathfrak{g} -manifold with respect to the dressing action $\rho : \mathfrak{g} \rightarrow \mathfrak{X}(S)$, conditions (13) hold, and hence $L_S = \{(\rho(u), \sigma(u)) \mid u \in \mathfrak{g}\}$ is a ϕ_S -twisted Dirac structure on S .*

The proof of Prop.3.2 will be postponed to Section 3.4.

We conclude that the choice of an isotropic connection on $p : D \rightarrow S$ places us in the context of Section 2.3, leading to a category of Hamiltonian spaces with D/G -valued moment maps.

An isotropic connection, given by $\theta \in \Omega^1(D, \mathfrak{g})$, also induces an important 2-form $\omega_D \in \Omega^2(D)$,

$$(28) \quad \omega_D := \frac{1}{2} (\langle \theta^R, \text{Inv}^* \theta \rangle_{\mathfrak{d}} - \langle \theta^L, \theta \rangle_{\mathfrak{d}}),$$

where $\text{Inv} : D \rightarrow D$ denotes the inversion on the Lie group D . Since $\text{Inv}^* \theta^R = -\theta^L$, we have $\text{Inv}^* \omega_D = \omega_D$. Let us consider $\bar{p} := p \circ \text{Inv} : D \rightarrow S$. The main property of ω_D , to be proven in Section 4.1, is that

$$(p, \bar{p}) : (D, \omega_D) \rightarrow (S \times S, L_S \times L_S)$$

is a presymplectic realization (i.e., it is an $S \times S$ -valued Hamiltonian space).

In order to prove the various properties of the differential forms introduced in this section, we will resort to the theory of Courant algebroids and Dirac structures (see the Appendix).

3.3. The Courant algebroid of a Manin pair. In this section we recall how a Manin pair $(\mathfrak{d}, \mathfrak{g})$ gives rise to a Courant algebroid over $S = D/G$. This fact goes back to unpublished work of Ševera [35] and Alekseev-Xu [6].

Given a Manin pair $(\mathfrak{d}, \mathfrak{g})$, let $\mathfrak{d}_S := \mathfrak{d} \times S$ be the trivial vector bundle over S with fiber \mathfrak{d} . The space of sections $\Gamma(\mathfrak{d}_S) = C^\infty(S, \mathfrak{d})$ contains \mathfrak{d} as the constant sections. There are several ways to extend the Lie bracket on \mathfrak{d} to $\Gamma(\mathfrak{d}_S)$, and each way produces a different structure on \mathfrak{d}_S . The simplest possibility of extension is to use the bracket $[\cdot, \cdot]_{\mathfrak{d}}$ of \mathfrak{d} pointwise. This makes \mathfrak{d}_S into a *bundle of Lie algebras*. The second possibility takes into account the infinitesimal action ρ_S of \mathfrak{d} on S and makes \mathfrak{d}_S into a *Lie algebroid*. This is described by the following well-known construction.

Lemma 3.3. *There is a unique extension of the Lie bracket of \mathfrak{d} to a Lie bracket on $\Gamma(\mathfrak{d}_S)$, denoted by $[\cdot, \cdot]_{Lie}$, which makes \mathfrak{d}_S into a Lie algebroid over S with anchor ρ_S .*

Proof. Uniqueness follows from Leibniz identity. For the existence, we give the explicit formula at $x \in S$:

$$(29) \quad [u, v]_{Lie}(x) := [u(x), v(x)]_{\mathfrak{d}} + \mathcal{L}_{\rho_S(u(x))}(v)(x) - \mathcal{L}_{\rho_S(v(x))}(u)(x),$$

for $u, v \in C^\infty(S, \mathfrak{d})$. q.e.d.

Finally, as observed in [6, 35], taking into account ρ_S as well as the bilinear form $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$ on \mathfrak{d} , we can view \mathfrak{d}_S as a *Courant algebroid* (see Sec. A.1). Analogously to Lemma 3.3, we have

Lemma 3.4. *There is a unique extension of the Lie bracket of \mathfrak{d} to a bilinear bracket on $\Gamma(\mathfrak{d}_S)$, denoted by $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$, which makes \mathfrak{d}_S into a Courant algebroid over S with anchor $\rho_S : \mathfrak{d}_S \rightarrow TS$ and symmetric pairing $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$.*

Proof. As in the previous lemma, the uniqueness follows from the Leibniz identity (condition C5) in Section A.1). For the existence, we have the explicit formula

$$\langle [u, v]_{\mathfrak{d}}, w \rangle_{\mathfrak{d}} := \langle [u, v]_{Lie}, w \rangle_{\mathfrak{d}} + \langle \mathcal{L}_{\rho_S(w)}(u), v \rangle_{\mathfrak{d}}, \quad u, v, w \in \Gamma(\mathfrak{d}_S).$$

Note that conditions C1)-C4) in Sec. A.1 follow from the fact that each formula is $C^\infty(S)$ -linear in its arguments, and they are clearly satisfied on constant sections. q.e.d.

Let us consider the trivial bundle $\mathfrak{g}_S = \mathfrak{g} \times S$ over S associated with the Lie subalgebra $\mathfrak{g} \subset \mathfrak{d}$.

Proposition 3.3. *The following holds:*

- i) \mathfrak{g}_S is a Dirac structure in the Courant algebroid \mathfrak{d}_S .
- ii) The Courant algebroid \mathfrak{d}_S is exact, i.e., the sequence

$$0 \longrightarrow T^*S \xrightarrow{\rho_S^*} \mathfrak{d}_S \xrightarrow{\rho_S} TS \longrightarrow 0$$

is exact (see Sec. A.5).

Proof. Using the Leibniz rule, one immediately checks that the space of sections $\Gamma(\mathfrak{g}_S) \subset \Gamma(\mathfrak{d}_S)$ is closed under any of the extensions of the Lie bracket on \mathfrak{d} to \mathfrak{d}_S . In particular, \mathfrak{g}_S is a Dirac structure in \mathfrak{d}_S .

To prove ii), note that ρ_S is onto. On the other hand, we have that $\text{Im}(\rho_S^*) \subseteq \text{Ker}(\rho_S)$, see (147). Since $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$ has signature (n, n) , it follows that $\dim(\mathfrak{g}) = n$, so $S = D/G$ has dimension n . The rank of $\text{Ker}(\rho_S)$ is n , which agrees with the rank of $\text{Im}(\rho_S^*) = \rho_S^*(T^*S)$. Hence $\text{Im}(\rho_S^*) = \text{Ker}(\rho_S)$. q.e.d.

3.4. Invariant connections and equivariant 3-forms. Given a Manin pair $(\mathfrak{d}, \mathfrak{g})$, let us consider its associated Courant algebroid \mathfrak{d}_S as in Lemma 3.4. Let us fix an isotropic splitting $s : TS \rightarrow \mathfrak{d}_S$ of the exact sequence

$$0 \longrightarrow T^*S \xrightarrow{\rho_S^*} \mathfrak{d}_S \xrightarrow{\rho_S} TS \longrightarrow 0.$$

(Isotropic splittings always exist, see Sec. A.2.) Since, according to Prop. 3.1, $s \in \Omega^1(S, \mathfrak{d})$ is equivalent to the choice of an isotropic connection on the bundle $p : D \rightarrow S$, we refer to s as a **connection splitting**.

It is a general fact about exact Courant algebroids (see Sec. A.5) that an isotropic splitting s determines a *closed* 3-form $\phi_S^s \in \Omega^3(S)$ by

$$(30) \quad \phi_S^s(X, Y, Z) := \langle [s(X), s(Y)]_{\mathfrak{d}}, s(Z) \rangle_{\mathfrak{d}}, \quad X, Y, Z \in \mathfrak{X}(S).$$

If s is clear from the context, we simplify the notation by just writing ϕ_S for this form.

Lemma 3.5. *The 3-form ϕ_S in (30) agrees with (26).*

Proof. By the definition of $\llbracket \cdot, \cdot \rrbracket_{\mathfrak{d}}$ in terms of the brackets $[\cdot, \cdot]_{\mathfrak{d}}$ and $[\cdot, \cdot]_{Lie}$, we have

$$\begin{aligned} (31) \quad \phi_S(X, Y, Z) &= \langle [s(X), s(Y)]_{Lie}, s(Z) \rangle_{\mathfrak{d}} + \langle \mathcal{L}_Z(s(X)), s(Y) \rangle_{\mathfrak{d}} \\ &= \langle [s(X), s(Y)]_{\mathfrak{d}} + \mathcal{L}_X(s(Y)) - \mathcal{L}_Y(s(X)), s(Z) \rangle_{\mathfrak{d}} \\ &\quad + \langle \mathcal{L}_Z(s(X)), s(Y) \rangle_{\mathfrak{d}}. \end{aligned}$$

Using that s is isotropic, we find the expression

$$(32) \quad \phi_S(X, Y, Z) = \langle [s(X), s(Y)]_{\mathfrak{d}}, s(Z) \rangle_{\mathfrak{d}} + \sum_{cycl} \langle \mathcal{L}_X(s(Y)), s(Z) \rangle_{\mathfrak{d}},$$

where \sum_{cycl} denotes cyclic sum in X, Y and Z .

On the other hand, using again that s is isotropic, we have

$$\begin{aligned}\langle ds(X, Y), s(Z) \rangle_{\mathfrak{d}} &= \langle \mathcal{L}_X(s(Y)), s(Z) \rangle_{\mathfrak{d}} - \langle \mathcal{L}_Y(s(X)), s(Z) \rangle_{\mathfrak{d}} \\ &= \langle \mathcal{L}_X(s(Y)), s(Z) \rangle_{\mathfrak{d}} + \langle \mathcal{L}_Y(s(Z)), s(X) \rangle_{\mathfrak{d}},\end{aligned}$$

and it follows that $\langle ds, s \rangle_{\mathfrak{d}}(X, Y, Z) = 2 \sum_{cycl} \langle \mathcal{L}_X(s(Y)), s(Z) \rangle_{\mathfrak{d}}$. Similarly, using that

$$\langle [s, s]_{\mathfrak{d}}(X, Y), s(Z) \rangle_{\mathfrak{d}} = 2 \langle [s(X), s(Y)]_{\mathfrak{d}}, s(Z) \rangle,$$

we obtain that $\langle [s, s]_{\mathfrak{d}}, s \rangle_{\mathfrak{d}}(X, Y, Z) = 6 \langle [s(X), s(Y)]_{\mathfrak{d}}, s(Z) \rangle$. Now (26) follows from (32). q.e.d.

The previous lemma explains why the 3-form (26) is closed. To finish the proof of Prop. 3.2, note that the connection splitting s induces an identification of Courant algebroids

$$(33) \quad (\rho_S, s^*) : \mathfrak{d}_S \xrightarrow{\sim} TS \oplus T^*S,$$

where $TS \oplus T^*S$ is equipped with the ϕ_S -twisted Courant bracket (see Sec. A.5). In particular, the image of \mathfrak{g}_S under (33) is a ϕ_S -twisted Dirac structure L_S^s on S . Defining

$$(34) \quad \sigma_s := s^*|_{\mathfrak{g}} : \mathfrak{g}_S \rightarrow T^*S,$$

where $s^* : \mathfrak{d} \rightarrow T^*S$ is dual to s after the identification $\mathfrak{d} \cong \mathfrak{d}^*$, we can write

$$(35) \quad L_S^s = \{(\rho(u), \sigma_s(u)) \mid u \in \mathfrak{g}\}.$$

It is clear that the presymplectic leaves of L_S are the dressing \mathfrak{g} -orbits. To simplify the notation, we may omit the dependence on s . Note that (13) holds, and the integrability of L_S implies that σ satisfies (11) and (12), as claimed in Prop. 3.2.

We now discuss when $\sigma + \phi_S$ is an equivariantly closed 3-form with respect to the \mathfrak{g} -action ρ .

Let us suppose that the connection we have fixed on $p : D \rightarrow G$ is invariant with respect to the action of G on D by left multiplication. This is equivalent to the connection splitting $s : TS \rightarrow \mathfrak{d}_S$ being G -equivariant, where the G -action on \mathfrak{d}_S is given by

$$g.(x, u) = (gx, \text{Ad}_g(u)), \quad g \in G, x \in S, u \in \mathfrak{d}.$$

Infinitesimally, the equivariance of s becomes

$$(36) \quad \mathcal{L}_{\rho(v)}(s(X)) + [v, s(X)]_{\mathfrak{d}} - s([\rho(v), X]) = 0, \quad \forall X \in \mathfrak{X}(S), v \in \mathfrak{g}.$$

Proposition 3.4. *Suppose that the connection splitting $s : TS \rightarrow \mathfrak{d}_S$ is equivariant. Then $\sigma + \phi_S$ defines an equivariantly closed 3-form on S .*

Proof. Let us first show that σ is \mathfrak{g} -equivariant, i.e., $\sigma([v, w]) = \mathcal{L}_{\rho(v)}\sigma(w)$. Note that

$$(37) \quad \mathcal{L}_{\rho(v)}\langle\sigma(w), X\rangle_{\mathfrak{d}} = \mathcal{L}_{\rho(v)}\langle s(X), w\rangle_{\mathfrak{d}} = \langle\mathcal{L}_{\rho(v)}(s(X)), w\rangle_{\mathfrak{d}}.$$

On the other hand,

$$\mathcal{L}_{\rho(v)}\langle\sigma(w), X\rangle_{\mathfrak{d}} = \langle\mathcal{L}_{\rho(v)}\sigma(w), X\rangle_{\mathfrak{d}} + \langle w, s([\rho(v), X])\rangle_{\mathfrak{d}}.$$

Using (36) and the invariance of $\langle\cdot, \cdot\rangle_{\mathfrak{d}}$, we obtain

$$\begin{aligned} \mathcal{L}_{\rho(v)}\langle\sigma(w), X\rangle_{\mathfrak{d}} &= \langle\mathcal{L}_{\rho(v)}(\sigma(w)), X\rangle_{\mathfrak{d}} + \langle\mathcal{L}_{\rho(v)}(s(X)), w\rangle_{\mathfrak{d}} \\ &\quad - \langle s(X), [v, w]_{\mathfrak{d}}\rangle_{\mathfrak{d}}. \end{aligned}$$

Comparing with (37), the equivariance of σ follows.

Since $d\phi_S = 0$, in order to check that $\sigma + \phi_S$ is an equivariantly closed 3-form, it remains to prove that

$$(38) \quad i_{\rho(v)}\sigma(v) = 0, \quad \text{and} \quad i_{\rho(v)}\phi_S - d\sigma(v) = 0.$$

The equation on the left is a consequence of the fact that \mathfrak{g}_S sits in $\mathfrak{d}_S \cong TS \oplus T^*S$ as an isotropic subbundle. For the equation on the right, first note that

$$\begin{aligned} (39) \quad &d(\sigma(v))(X, Y) \\ &= \mathcal{L}_X\langle v, s(Y)\rangle_{\mathfrak{d}} - \mathcal{L}_Y\langle v, s(X)\rangle_{\mathfrak{d}} - \langle v, s([X, Y])\rangle_{\mathfrak{d}} \\ &= \langle v, \mathcal{L}_X(s(Y))\rangle_{\mathfrak{d}} - \langle v, \mathcal{L}_Y(s(X))\rangle_{\mathfrak{d}} - \langle v, s([X, Y])\rangle_{\mathfrak{d}}. \end{aligned}$$

Using (31), we have

$$\phi_S(X, Y, \rho(v)) = \langle [s(X), s(Y)]_{Lie}, s(\rho(v))\rangle_{\mathfrak{d}} + \langle \mathcal{L}_{\rho(v)}(s(X)), s(Y)\rangle_{\mathfrak{d}}.$$

Since $s\rho_S = \text{Id} - \rho_S^*s^*$ and s is isotropic, we use (29) to write the previous expression as

$$(40) \quad \begin{aligned} &\langle \mathcal{L}_X(s(Y)), v\rangle_{\mathfrak{d}} - \langle \mathcal{L}_Y(s(X)), v\rangle_{\mathfrak{d}} + \langle [s(X), s(Y)]_{\mathfrak{d}}, v\rangle_{\mathfrak{d}} - \\ &\langle [s(X), s(Y)]_{Lie}, \rho_S^*(s^*(v))\rangle_{\mathfrak{d}} + \langle \mathcal{L}_{\rho(v)}(s(X)), s(Y)\rangle_{\mathfrak{d}}. \end{aligned}$$

Note that pairing (36) with $s(Y)$ and using that s is isotropic and $\langle\cdot, \cdot\rangle_{\mathfrak{d}}$ is invariant, we obtain that $\langle \mathcal{L}_{\rho(v)}(s(X)), s(Y)\rangle_{\mathfrak{d}} + \langle [s(X), s(Y)]_{\mathfrak{d}}, v\rangle_{\mathfrak{d}} = 0$. On the other hand, using that $\rho_S \circ s = \text{Id}$ and that $\rho_S : C^\infty(S, \mathfrak{d}) \rightarrow \mathfrak{X}(S)$ is a Lie algebra homomorphism with respect to $[\cdot, \cdot]_{Lie}$, we have

$$\langle [s(X), s(Y)]_{Lie}, \rho_S^*(s^*(v))\rangle_{\mathfrak{d}} = \langle s([X, Y]), v\rangle_{\mathfrak{d}}.$$

Hence (40) agrees with (39), and this concludes the proof. q.e.d.

The previous proposition could also be derived from the discussion in [9, Sec. 2.2].

For a Manin pair $(\mathfrak{d}, \mathfrak{g})$, we have a short exact sequence associated with the inclusion $\mathfrak{g} \hookrightarrow \mathfrak{d}$,

$$(41) \quad \mathfrak{g} \longrightarrow \mathfrak{d} \longrightarrow \mathfrak{g}^*.$$

Here the map on the right is the projection $\mathfrak{d} \rightarrow \mathfrak{d}/\mathfrak{g}$ after the identification $\mathfrak{d}/\mathfrak{g} \cong \mathfrak{g}^*$ induced by $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$. Let us choose an isotropic splitting $j : \mathfrak{g}^* \rightarrow \mathfrak{d}$ of this sequence, which amounts to the choice of an isotropic complement of \mathfrak{g} in \mathfrak{d} . In general, such a splitting j does not define a connection on $p : D \rightarrow S$, but this happens under additional assumptions (see also [6]).

Proposition 3.5. *An isotropic splitting $j : \mathfrak{g}^* \rightarrow \mathfrak{d}$ satisfying $[\mathfrak{g}, j(\mathfrak{g}^*)] \subseteq j(\mathfrak{g}^*)$ (i.e., $\text{Ad}_g(j(\mathfrak{g}^*)) \subseteq j(\mathfrak{g}^*)$) is equivalent to an equivariant connection splitting $s : TS \rightarrow \mathfrak{d}_S$.*

Proof. The right action of G on D is generated by the vector fields $u \mapsto u_a^l = d\lambda_a(u)$, $a \in D$. So $d\lambda_a(j(\mathfrak{g}^*))$, $a \in D$, defines a horizontal distribution on the bundle $p : D \rightarrow S$ (which is automatically invariant under the action of G on D by left multiplication). This distribution is invariant under the right G -action on D if and only if $j(\mathfrak{g}^*)$ is $\text{Ad}(G)$ -invariant. On the other hand, if a given connection is left G -invariant, its horizontal distribution is left invariant, and we get an $\text{Ad}(G)$ -invariant complement to \mathfrak{g} in \mathfrak{d} by left translation of the horizontal distribution.

q.e.d.

Splittings j with the additional invariance of Prop. 3.5 may not exist in general, but they always exist if e.g. G is compact or semi-simple, see Remark 5.2.

Given a Manin pair $(\mathfrak{d}, \mathfrak{g})$, Propositions 3.4 and 3.5 show that the choice of an isotropic complement \mathfrak{h} of \mathfrak{g} satisfying $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$ determines an equivariantly closed 3-form $\sigma + \phi_S$ on S .

4. D/G -valued moment maps via Dirac geometry

In this section, we discuss a moment map theory associated with a Manin pair $(\mathfrak{d}, \mathfrak{g})$ based on the additional choice of an isotropic connection on $p : D \rightarrow S = D/G$. A different moment map theory [3], based on the choice of splitting j of (41), will be discussed in Section 5.

4.1. The Hamiltonian category. Let us fix a connection splitting $s : TS \rightarrow \mathfrak{d}_S$ of the exact Courant algebroid \mathfrak{d}_S .

The **Hamiltonian category** (or *moment map theory*) associated with $(\mathfrak{d}, \mathfrak{g})$ and s is the Hamiltonian category of the Dirac manifold (S, L_S^s, ϕ_S^s) , in the sense of Section 2.2:

$$(42) \quad \overline{\mathcal{M}}_s(\mathfrak{d}, \mathfrak{g}) := \overline{\mathcal{M}}(S, L_S^s, \phi_S^s).$$

We can similarly consider the subcategory of *presymplectic realizations*, in which Hamiltonian spaces carry 2-forms rather than general Dirac structures:

$$(43) \quad \mathcal{M}_s(\mathfrak{d}, \mathfrak{g}) := \mathcal{M}(S, L_S^s, \phi_S^s).$$

From Prop. 2.4, we obtain an explicit characterization of objects in $\mathcal{M}_s(\mathfrak{d}, \mathfrak{g})$ only in terms of the forms ϕ_S^s, σ_s , with no reference to Dirac structures.

We refer to objects in $\overline{\mathcal{M}}_s(\mathfrak{d}, \mathfrak{g})$ as **S -valued Hamiltonian \mathfrak{g} -spaces** (or **G -spaces** if the action ρ_M of Prop. 2.3 integrates to a G -action).

The reduction procedure for strong Dirac maps in [10, Sec. 4.4] immediately leads to:

Proposition 4.1. *Consider a strong Dirac map $J : (M, L) \rightarrow (S, L_S)$ defining an S -valued Hamiltonian G -space. Suppose $y \in S$ is a regular value of J and the action of the isotropy group G_y on $J^{-1}(y)$ is free and proper, and let $M_y := J^{-1}(y)/G_y$. Then:*

- i) *The backward image of L to $J^{-1}(y)$ is a smooth Dirac structure;*
- ii) *The quotient M_y acquires a Poisson structure uniquely characterized by the fact that the quotient map $J^{-1}(y) \rightarrow M_y$ is f -Dirac;*
- iii) *The reduced Poisson structure on M_y is symplectic if J is a presymplectic realization.*

Given another connection splitting $s' : TS \rightarrow \mathfrak{d}_S$, we have (see Sec. A.5) an induced twist 2-form $B \in \Omega^2(S)$, defined by

$$(44) \quad B(X, \rho_S(v)) = \langle (s - s')(X), v \rangle_{\mathfrak{d}}, \quad X \in TS, v \in \mathfrak{d}.$$

The 2-form B relates L_S^s and $L_S^{s'}$ by a gauge transformation: $\tau_B(L_S^s) = L_S^{s'}$. As an immediate consequence of Prop. 2.2, we have

Proposition 4.2. *If s and s' are isotropic splittings and $B \in \Omega^2(S)$ is as in (44), then the gauge transformation by B defines an isomorphism of categories*

$$\mathcal{I}_B : \overline{\mathcal{M}}_s(\mathfrak{d}, \mathfrak{g}) \xrightarrow{\sim} \overline{\mathcal{M}}_{s'}(\mathfrak{d}, \mathfrak{g}),$$

which restricts to an isomorphism $\mathcal{M}_s(\mathfrak{d}, \mathfrak{g}) \cong \mathcal{M}_{s'}(\mathfrak{d}, \mathfrak{g})$.

It is immediate to check that this functor preserves reduced spaces.

From the general theory of Section 2.2, we know some canonical examples of Hamiltonian spaces associated with the Manin pair $(\mathfrak{d}, \mathfrak{g})$ and s . For example, the inclusion $\mathcal{O} \hookrightarrow S$ of a dressing orbit is a presymplectic realization with respect to the canonical 2-form $\omega_{\mathcal{O}}$,

$$\omega_{\mathcal{O}}(\rho(v), \rho(w)) = \langle \sigma(v), \rho(w) \rangle.$$

On the other hand, presymplectic groupoids $\mathcal{G} = (G \ltimes S, \omega_{\mathcal{G}})$ integrating $L_S = \mathfrak{g} \ltimes S$ define presymplectic realizations $(t, s) : \mathcal{G} \longrightarrow S \times S^{\text{op}}$. These examples will be illustrated in concrete situations in Sections 4.2 and 4.3.

The remaining of this section presents a nontrivial object in $\mathcal{M}_{s \oplus s}(\mathfrak{d} \oplus \mathfrak{d}, \mathfrak{g} \oplus \mathfrak{g})$, i.e, an $S \times S$ -valued Hamiltonian space: the Lie group D , equipped with the 2-form ω_D given by (28).

Theorem 4.1. *Consider the right principal G -bundle $p : D \rightarrow S = D/G$ and let $\bar{p} = p \circ \text{Inv}$, where $\text{Inv} : D \rightarrow D$ is the inversion map. Then*

$$(p, \bar{p}) : (D, \omega_D) \rightarrow (S \times S, L_S \times L_S)$$

is a presymplectic realization, and the induced $\mathfrak{g} \times \mathfrak{g}$ -action on D is given by $(u, v) \mapsto u^r - v^l$.

The proof will follow from three lemmas. First, we compare the pull-back $p^*\phi_S$ with the Cartan 3-form on D ,

$$\phi_D := \frac{1}{12} \langle [\theta^R, \theta^R]_{\mathfrak{d}}, \theta^R \rangle_{\mathfrak{d}}.$$

As in Section 3.2, we denote the connection 1-form on $p : D \rightarrow S$ associated with the connection splitting s by $\theta \in \Omega^1(D, \mathfrak{g})$.

Lemma 4.1. *The following holds:*

$$p^*(\phi_S) = -\phi_D + \frac{1}{2} d \langle \theta^L, \theta \rangle_{\mathfrak{d}}.$$

Proof. Note that using the wedge product, the Lie bracket on \mathfrak{d} defines a graded Lie bracket $[\cdot, \cdot]_{\mathfrak{d}}$ on $\Omega^\bullet(D, \mathfrak{d})$, while the nondegenerate pairing on \mathfrak{d} defines a pairing $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$ on $\Omega^\bullet(D, \mathfrak{d})$. The de Rham differential d also extends to $\Omega^\bullet(D, \mathfrak{d})$. Let us consider the following operation:

$$\Omega^\bullet(D, \mathfrak{d}) \longrightarrow \Omega^\bullet(D, \mathfrak{d}), \quad \eta \longrightarrow \widehat{\eta},$$

where $\widehat{\eta}_a(X_a) = \text{Ad}_{a^{-1}}(\eta_a(X_a))$, $a \in D$. One can check that

$$(45) \quad \widehat{[\eta, \xi]_{\mathfrak{d}}} = [\widehat{\eta}, \widehat{\xi}]_{\mathfrak{d}}, \quad \langle \widehat{\eta}, \widehat{\xi} \rangle_{\mathfrak{d}} = \langle \eta, \xi \rangle_{\mathfrak{d}}, \quad \text{and} \quad \widehat{d\eta} = d\widehat{\eta} - [\theta^L, \widehat{\eta}]_{\mathfrak{d}}.$$

To prove the last formula in (45), one checks it directly when $\eta = f$ is of degree zero, and similarly when $\eta = df$ is exact of degree 1, and use the Leibniz identities to conclude that the formula holds in general.

In order to simplify our notation, we will identify forms on S with forms on D via p^* , so we will often abuse notation and denote $p^*\eta$ simply by η . With these conventions, formula (25) relating θ and s becomes

$$(46) \quad \theta = \theta^L - \widehat{s}.$$

By (26) and the first two properties in (45), we have

$$\phi_S = \frac{1}{2} \langle \widehat{ds}, \widehat{s} \rangle_{\mathfrak{d}} + \frac{1}{6} \langle [\widehat{s}, \widehat{s}]_{\mathfrak{d}}, \widehat{s} \rangle_{\mathfrak{d}}$$

Using (46) and the last property in (45), we can write

$$\frac{1}{2} \langle \widehat{ds}, \widehat{s} \rangle_{\mathfrak{d}} = \frac{1}{2} \left(\langle d\theta^L, \theta^L - \theta \rangle_{\mathfrak{d}} - \langle d\theta, \theta^L - \theta \rangle_{\mathfrak{d}} - \langle [\theta^L, \theta^L - \theta]_{\mathfrak{d}}, \theta^L - \theta \rangle_{\mathfrak{d}} \right).$$

Note that $\langle d\theta, \theta \rangle_{\mathfrak{d}} = 0$ since θ takes values in the isotropic subspace $\mathfrak{g} \subset \mathfrak{d}$. Using the Maurer-Cartan equation $d\theta^L = \frac{1}{2}[\theta^L, \theta^L]_{\mathfrak{d}}$, we obtain

$$(47) \quad \begin{aligned} \frac{1}{2}\langle \widehat{ds}, \widehat{s} \rangle_{\mathfrak{d}} &= \frac{3}{4}\langle [\theta^L, \theta^L]_{\mathfrak{d}}, \theta \rangle_{\mathfrak{d}} - \frac{1}{4}\langle [\theta^L, \theta^L]_{\mathfrak{d}}, \theta^L \rangle_{\mathfrak{d}} \\ &\quad - \frac{1}{2}\langle d\theta, \theta^L \rangle_{\mathfrak{d}} - \frac{1}{2}\langle [\theta^L, \theta]_{\mathfrak{d}}, \theta \rangle_{\mathfrak{d}} \end{aligned}$$

Similarly, we have

$$(48) \quad \frac{1}{6}\langle [\widehat{s}, \widehat{s}], \widehat{s} \rangle_{\mathfrak{d}} = -\frac{1}{2}\langle [\theta^L, \theta^L]_{\mathfrak{d}}, \theta \rangle_{\mathfrak{d}} + \frac{1}{6}\langle [\theta^L, \theta^L]_{\mathfrak{d}}, \theta^L \rangle_{\mathfrak{d}} + \frac{1}{2}\langle [\theta^L, \theta]_{\mathfrak{d}}, \theta \rangle_{\mathfrak{d}}.$$

Adding up (47) and (48), we obtain

$$\begin{aligned} -\phi_D + \frac{1}{4}\langle [\theta^L, \theta^L]_{\mathfrak{d}}, \theta \rangle_{\mathfrak{d}} - \frac{1}{2}\langle d\theta, \theta^L \rangle_{\mathfrak{d}} &= -\phi_D + \frac{1}{2}\langle d\theta^L, \theta \rangle_{\mathfrak{d}} - \frac{1}{2}\langle d\theta, \theta^L \rangle_{\mathfrak{d}} \\ &= -\phi_D + \frac{1}{2}d\langle \theta^L, \theta \rangle_{\mathfrak{d}}, \end{aligned}$$

proving the lemma. q.e.d.

The next result relates ω_D and σ :

Lemma 4.2. *For all $u \in \mathfrak{g}$, we have*

$$(49) \quad i_{u^r}(\omega_D) = p^*(\sigma(u)), \quad \text{and} \quad -i_{u^l}(\omega_D) = \bar{p}^*(\sigma(u)),$$

where $u^l, u^r \in \mathfrak{X}(D)$ are the left, right invariant vector fields determined by u .

Proof. As a first step, we prove the following expression for ω_D :

$$(50) \quad \begin{aligned} \omega_D(X, Y) &= \langle dl_a(\theta(X)) - dr_a(\theta(\text{Inv}(X))) - X, Y \rangle_{\mathfrak{d}}, \\ &\quad \forall a \in D, X, Y \in T_a D. \end{aligned}$$

To prove this formula, we use (24) to write

$$\begin{aligned} \langle \theta^L, \theta \rangle_{\mathfrak{d}}(X, Y) &= \langle dl_{a^{-1}}(X), \theta(Y) \rangle_{\mathfrak{d}} - \langle dl_{a^{-1}}(Y), \theta(X) \rangle_{\mathfrak{d}} \\ &= \langle X, Y \rangle_{\mathfrak{d}} - 2\langle dl_a \theta(X), Y \rangle_{\mathfrak{d}}. \end{aligned}$$

Using that $dr_{a^{-1}}(X) = -dl_a(\text{Inv}(X))$, we can use again (24) to write

$$\begin{aligned} \langle \theta^R, \text{Inv}^* \theta \rangle_{\mathfrak{d}}(X, Y) &= \langle dr_{a^{-1}}(X), \theta(\text{Inv}(Y)) \rangle_{\mathfrak{d}} - \langle dr_{a^{-1}}(Y), \theta(\text{Inv}(X)) \rangle_{\mathfrak{d}} \\ &= -\langle \text{Inv}(X), \text{Inv}(Y) \rangle_{\mathfrak{d}} + \langle dl_a(\text{Inv}(Y)), \theta(\text{Inv}(X)) \rangle_{\mathfrak{d}} \\ &\quad - \langle dr_{a^{-1}}(Y), \theta(\text{Inv}(X)) \rangle_{\mathfrak{d}} \\ &= -\langle X, Y \rangle_{\mathfrak{d}} - 2\langle dr_{a^{-1}}(Y), \theta(\text{Inv}(X)) \rangle_{\mathfrak{d}}. \end{aligned}$$

Comparing with the original expression (28) for ω_D , formula (50) follows.

We now prove the first equation in (49) (the second one follows by applying Inv^*). For $u \in \mathfrak{g}$ and $X = X_a \in T_a D$, we have (using (50)) that $i_{u^r} \omega_D(X)$ equals

$$-\omega_D(X, u^r) = \langle X - dl_a(\theta(X)), dr_a(u) \rangle_{\mathfrak{d}} + \langle dr_a \theta(\text{Inv}(X)), dr_a(u) \rangle_{\mathfrak{d}}.$$

Looking at the r.h.s., we see that the last term vanishes since $\mathfrak{g} \subset \mathfrak{d}$ is isotropic. By (25), we have that $X - dl_a \theta(X) = dr_a s(dp(X))$, which gives us

$$i_{u^r} \omega_D(X) = \langle s(dp(X)), u \rangle_{\mathfrak{d}} = p^* \sigma(u)(X),$$

as desired. q.e.d.

Finally, we will need

Lemma 4.3. *At each $a \in D$, we have*

$$\begin{aligned} \ker(\omega_D) \cap \ker(dp) &= \{dl_a(u) \mid u \in \mathfrak{g}, \theta(dr_a^{-1}(u)) = 0\}, \\ \ker(\omega_D) \cap \ker(d\bar{p}) &= \{dr_a(v) \mid v \in \mathfrak{g}, \theta(dr_a v) = 0\}. \end{aligned}$$

Proof. We prove the second one here. We have $\ker(d\bar{p})_a = dr_a(\mathfrak{g})$. By Lemma 4.2 it follows that, for $u \in \mathfrak{g}$, $dr_a(u) \in \ker(\omega_D)$ if and only if $p^* \sigma(u) = 0$, i.e.,

$$\langle s((dp)(X)), u \rangle_{\mathfrak{d}} = 0, \quad \forall X \in T_a D.$$

Using (25), the previous equation implies that

$$\langle X, dr_a(u) \rangle_{\mathfrak{d}} = \langle \theta(X), dl_a^{-1} dr_a(u) \rangle_{\mathfrak{d}}$$

and, using (24) to re-write the r.h.s of the last equation, we get the identity

$$\langle X, dr_a(u) \rangle_{\mathfrak{d}} = \langle X, dr_a(u) \rangle_{\mathfrak{d}} - \langle \theta(dr_a(u)), dl_a(X) \rangle_{\mathfrak{d}} \quad \forall X \in T_a D,$$

from which the statement follows. q.e.d.

Proof. (of Theorem 4.1) Since $\text{Inv}^* \theta^L = -\theta^R$, $\text{Inv}^* \phi_D = -\phi_D$, and $\bar{p}^* = \text{Inv}^* p^*$, Lemma 4.1 immediately implies that

$$(51) \quad p^* \phi_S + \bar{p}^* \phi_S = -d\omega_D.$$

We now prove that p is an f-Dirac map from D into S (and this automatically implies that the same holds for \bar{p}). It suffices to check that L_S is contained in the forward image of $L = \text{graph}(\omega_D)$ under p (since these bundles have equal rank), i.e.,

$$\{(\rho(u), \sigma(u)) \mid u \in \mathfrak{g}\} \subseteq \{(dp(X), \beta) \mid p^* \beta = i_X \omega_D\}.$$

But this follows since, for $u \in \mathfrak{g}$, $\rho(u) = dp(u^r)$ and, from Lemma 4.2, $i_{u^r}(\omega_D) = p^*(\sigma(u))$.

In order to conclude the proof of the theorem, it remains to check that

$$(52) \quad \ker(\omega_D) \cap \ker(dp) \cap \ker(d\bar{p}) = 0.$$

This is a consequence of Lemma 4.3: If $X \in T_a D$ is in the triple intersection above, then

$$X = dl_a(u) = dr_a(v), \text{ with } \theta(dr_a^{-1}(u)) = 0 \text{ and } \theta(dr_a(v)) = 0.$$

Since θ is a connection 1-form for the right G -action on D , we obtain $u = \theta(dl_a(u)) = \theta(dr_a(v)) = 0$, and hence $X = 0$. q.e.d.

Since a strong Dirac map preserves the kernels of the Dirac structures, it follows from Thm. 4.1 that L_S is a Poisson structure on S if and only if ω_D is a symplectic form on D (which is only the case when $\sigma_s : \mathfrak{g}_S \rightarrow T^*S$ is an isomorphism).

Theorem 4.1 has the following interesting consequence: since D carries principal G -actions on the left and on the right (by left/right multiplication) which commute, the fact that $(p, \bar{p}) : D \rightarrow S \times S$ is a presymplectic realization can be re-stated as saying that (D, ω_D) defines a *Morita equivalence* between the Dirac manifold S and its opposite S^{op} (i.e., a Morita equivalence of their s -simply-connected presymplectic groupoids in the sense of Xu [43, Sec. 4]).

Proposition 4.3. *Let $J : (M, \omega) \rightarrow S$ be a presymplectic realization defining an S -valued Hamiltonian G -space. Then*

- 1) *The quotient $(D \times_{(\bar{p}, J)} M)/G$ by the diagonal G -action is a smooth manifold.*
- 2) *The pull-back of $\omega_D \oplus (-\omega)$ to the submanifold $D \times_{(\bar{p}, J)} M \hookrightarrow D \times M$ is basic with respect to the G -action. The quotient space $(D \times_{(\bar{p}, J)} M)/G$ equipped with the resulting 2-form is denoted by $D \otimes_G M^{\text{op}}$.*
- 3) *The map $D \otimes_G M^{\text{op}} \rightarrow S$, $a \otimes x \mapsto p(a)$ is a presymplectic realization making $D \otimes_G M^{\text{op}}$ into an S -valued Hamiltonian G -space.*

Moreover, this procedure defines a self-equivalence functor \mathcal{F}_D on the category of S -valued Hamiltonian G -spaces satisfying $\mathcal{F}_D \circ \mathcal{F}_D \cong \text{Id}$.

Proof. Since $J : (M, \omega) \rightarrow S$ is a presymplectic realization, so is $J : (M, -\omega) \rightarrow S^{\text{op}}$. Since $S \xleftarrow{p} (D, \omega_D) \xrightarrow{\bar{p}} S^{\text{op}}$ is a Morita bimodule, Xu's Morita theory for presymplectic groupoids [43, Sec. 4] directly implies the assertions in parts 1, 2 and 3. The property $\mathcal{F}_D \circ \mathcal{F}_D \cong \text{Id}$ follows from the fact that the inverse of the bimodule $S \xleftarrow{p} (D, \omega_D) \xrightarrow{\bar{p}} S^{\text{op}}$ is the Morita bimodule $S^{\text{op}} \xleftarrow{\bar{p}} (D, -\omega_D) \xrightarrow{p} S$, which is isomorphic to $S^{\text{op}} \xleftarrow{p} (D, -\omega_D) \xrightarrow{\bar{p}} S$ via the inversion $\text{Inv} : D \rightarrow D$. Let us denote this last bimodule by D^{op} . Then

$$\begin{aligned} \mathcal{F}_D \circ \mathcal{F}_D(M) &= D \otimes_G (D \otimes_G M^{\text{op}})^{\text{op}} \cong D \otimes_G (D^{\text{op}} \otimes_G M) \\ &\cong (D \otimes_G D^{\text{op}}) \otimes_G M \cong M. \end{aligned}$$

q.e.d.

More generally, Proposition 4.3 holds for Hamiltonian spaces given by strong Dirac maps, not necessarily presymplectic realizations. The proposition shows that D induces an *involution* in the category of S -valued Hamiltonian G -spaces, which we illustrate in examples below.

4.2. Examples. We now discuss various concrete examples of D/G -valued moment maps arising from specific choices of Manin pairs $(\mathfrak{d}, \mathfrak{g})$ and connection splittings $s : TS \rightarrow \mathfrak{d}_S$.

Example 4.4 (\mathfrak{g}^* -valued moment maps). Let us consider the Manin pair $(\mathfrak{g} \times \mathfrak{g}^*, \mathfrak{g})$ of Example 3.1. In this case $S = \mathfrak{g}^*$, and we have a canonical equivariant connection splitting $s : T\mathfrak{g}^* \rightarrow (\mathfrak{g} \oplus \mathfrak{g}^*) \times \mathfrak{g}^*$ given by

$$s(\mu_x) = ((0, \mu), x).$$

Then $\sigma = \text{Id} : \mathfrak{g} \times \mathfrak{g}^* \rightarrow T^*\mathfrak{g}^* \cong \mathfrak{g} \times \mathfrak{g}^*$, and $(L_S)_\mu = \{(\text{ad}_u^*(\mu), u) \mid u \in \mathfrak{g}\}$, $\mu \in \mathfrak{g}^*$, is just the graph of the Lie-Poisson structure on \mathfrak{g}^* . As we saw in Example 2.4, $\overline{\mathcal{M}}_s(\mathfrak{g} \times \mathfrak{g}^*, \mathfrak{g})$ is simply the category of Poisson maps into \mathfrak{g}^* , i.e., classical Hamiltonian \mathfrak{g} -spaces.

More generally, one can consider Manin pairs coming from Lie bialgebras (see e.g. [27]).

Example 4.5 (G^* -valued moment maps). Let $(\mathfrak{g}, \mathfrak{g}^*)$ be a Lie bialgebra and \mathfrak{d} be its Drinfeld double (see Section 5.1). We consider the Manin pair $(\mathfrak{d}, \mathfrak{g})$, assuming that an extra *completeness* condition [27, Sec. 2.5] holds, as we now recall.

Let D be the simply-connected Lie group integrating \mathfrak{d} , and G and G^* be the simply-connected Lie groups integrating \mathfrak{g} and \mathfrak{g}^* , respectively. The inclusion of \mathfrak{g} and \mathfrak{g}^* into \mathfrak{d} integrate to Lie group homomorphisms $i_1 : G \rightarrow D$ and $i_2 : G^* \rightarrow D$, and we obtain a local diffeomorphism

$$(53) \quad G \times G^* \rightarrow D, \quad (g, x) \mapsto i_1(g)i_2(x).$$

We further assume that this map is a *global diffeomorphism*. To simplify the notation, we identify G and G^* with their images in D under the maps i_1 and i_2 .

Any element in D can be written as gx or as $x'g'$, for unique $g, g' \in G$, $x, x' \in G^*$. In this case, let us write $x' = \varphi_g(x)$. The map $\varphi : G \times G^* \rightarrow G^*$, $(g, x) \mapsto \varphi_g(x)$ defines a left action of G on G^* , and it induces a diffeomorphism

$$S = D/G \xrightarrow{\sim} G^*, \quad [(g, x)] \mapsto \varphi_g(x).$$

Under this identification, the action of D on itself by left multiplication induces left actions of G^* and G on $S = G^*$: The G^* -action is by left multiplication, whereas the G -action (i.e., the dressing action) is φ . In particular, we have

$$\rho_S|_{\mathfrak{g}^*} : \mathfrak{g}^* \rightarrow TG^*, \quad \mu \mapsto \mu^r.$$

It follows that there is a canonical choice of connection splitting by

$$(54) \quad s : TG^* \rightarrow \mathfrak{d}_S, \quad V_x \mapsto \theta_{G^*}^R(V_x) = dr_x^{-1}(V_x).$$

Here r_x denotes the right multiplication by x on the Lie group G^* . The induced map $\sigma_s : \mathfrak{g}_S \rightarrow T^*G^*$ is given by

$$(55) \quad \sigma_s(v) = (dr_x^{-1})^*v.$$

Note that $s(TG^*) = \mathfrak{g}_S^* \subset \mathfrak{d}_S$ is transversal to \mathfrak{g}_S , hence the kernel of L_S^s is trivial. It follows that the Dirac structure L_S^s on G^* is the graph of a Poisson structure π_{G^*} ($\phi_S = 0$ since $\mathfrak{g}^* \subset \mathfrak{d}$ is a subalgebra) defined by

$$(56) \quad \pi_{G^*}^\sharp((dr_x^{-1})^*v) = \rho(v), \quad v \in \mathfrak{g}.$$

This Poisson structure makes G^* into the Poisson-Lie group dual to the one integrating $(\mathfrak{g}, \mathfrak{g}^*)$. Since L_S is Poisson, the 2-form ω_D on D is symplectic, and (D, ω_D) is the Heisenberg double (c.f. Example 4.11). The Hamiltonian category $\overline{\mathcal{M}}_s(\mathfrak{d}, \mathfrak{g})$ in this example is the category of Poisson maps into G^* , which are the Hamiltonian Poisson \mathfrak{g} -spaces in the sense of [28]. In particular, when the Hamiltonian space is symplectic, condition *iii*) in Prop. 2.4 becomes Lu's moment map condition

$$i_{\rho_M(v)}\omega = J^*\langle \theta_{G^*}^R, v \rangle.$$

With the identifications $D \cong G \times G^*$ and $S \cong G^*$ (as manifolds), the maps $p, \bar{p} : D \rightarrow G^*$ become

$$p(g, x) = \varphi_g(x), \quad \text{and} \quad \bar{p}(g, x) = x^{-1}.$$

A direct calculation shows that the involution \mathcal{F}_D of Prop. 4.3 takes a Poisson map $J : M \rightarrow G^*$ to $\text{Inv}_{G^*} \circ J : M^{\text{op}} \rightarrow G^*$ (where Inv_{G^*} denotes the inversion map in G^*).

Note that the connection (54) is not equivariant in general, so it does *not* define an equivariant 3-form (we will return to this issue in Section 4.3).

Although a connection splitting s exists even without the extra completeness assumption made in Example 4.5, in general D/G will not be identified with G^* and the choice of s is not canonical.

For a special class of Lie bialgebras, there is a different choice of connection splitting which is equivariant and leads to a gauge equivalent (in the sense of Prop. 4.2) Hamiltonian category:

Example 4.6 (P -valued moment maps). Let G be a connected, simply-connected compact Lie group. We fix an Ad-invariant, nondegenerate, symmetric bilinear form on \mathfrak{g} , and denote by $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ the induced complex-bilinear form on $\mathfrak{g}^{\mathbb{C}}$. We view $(\mathfrak{d} = \mathfrak{g}^{\mathbb{C}}, \mathfrak{g})$ as a Manin pair with

respect to the pairing $\langle \cdot, \cdot \rangle_{\mathfrak{d}} = \mathfrak{Im} \langle \cdot, \cdot \rangle_{\mathbb{C}}$, given by the imaginary part of $\langle \cdot, \cdot \rangle_{\mathbb{C}}$. By the Iwasawa decomposition, we can write

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus \mathfrak{a} \oplus \mathfrak{n},$$

where $\mathfrak{a} = \sqrt{-1}\mathfrak{t}$ (\mathfrak{t} is the Lie algebra of the maximal torus $T \subset G$) and \mathfrak{n} is the sum of positive root spaces. Then $\mathfrak{a} \oplus \mathfrak{n}$ is an isotropic complement of \mathfrak{g} in \mathfrak{d} . Since $\mathfrak{g}^* \cong \mathfrak{a} \oplus \mathfrak{n} \subset \mathfrak{d}$ is a subalgebra, this defines a Lie bialgebra and we are in the context of Example 4.5. At the global level, we have the decomposition $D = G^{\mathbb{C}} = GAN$, and $G^* \cong AN$.

In the present situation, however, one has another choice of isotropic complement of $\mathfrak{g} \subset \mathfrak{d}$, namely $\mathfrak{h} := \sqrt{-1}\mathfrak{g}$. Note that \mathfrak{h} is not a subalgebra, but it satisfies $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$. From Prop. 3.5, we have an induced connection splitting which is *equivariant* (hence distinct from (54)). In order to get a simple explicit formula for the connection, we follow [5, Sec. 10] and choose a different realization of $G^{\mathbb{C}}/G$.

Let ${}^c : G^{\mathbb{C}} \rightarrow G^{\mathbb{C}}$ be the involution given by exponentiating the complex conjugation $v \mapsto \bar{v}$ on $\mathfrak{g}^{\mathbb{C}}$, and consider the map $G^{\mathbb{C}} \rightarrow G^{\mathbb{C}}$ given by $g \mapsto g^\dagger := (g^{-1})^c$. Let

$$P := \{a \in G^{\mathbb{C}} \mid a = a^\dagger\}.$$

Then the map $q : G^{\mathbb{C}} \rightarrow P$, $a \mapsto aa^\dagger$, induces a diffeomorphism from $G^* = D/G$ to P , identifying the dressing G -action on G^* with conjugation on P by G . Using that

$$dq_a = dl_{aa^\dagger}(\text{Ad}_{a^c}(\theta_G^L - \overline{\theta_G^L})),$$

and $\rho_S : \mathfrak{d}_S \rightarrow TP$, $\rho_S(u)_a = dq(dr_a(u)) = dr_{aa^\dagger}(u) - dl_{aa^\dagger}(u)$, one can find the explicit expression for the equivariant connection splitting induced by $\mathfrak{h} = \sqrt{-1}\mathfrak{g}$:

$$(57) \quad s' : TP \rightarrow \mathfrak{d}_S, \quad X \mapsto \frac{1}{2}\theta_P^R(X),$$

where θ_P^R is the pull-back of $\theta_{G^{\mathbb{C}}}^R$ to $P \hookrightarrow G^{\mathbb{C}}$. The equivariant 3-form $\sigma_{s'} + \phi_S^{s'}$ is given by

$$\begin{aligned} \sigma_{s'}(u) &= \frac{1}{2}\langle \theta_P^R, u \rangle_{\mathfrak{d}} = \frac{1}{2} \left(\frac{1}{2\sqrt{-1}} \langle u, \theta_P^R - \overline{\theta_P^R} \rangle_{\mathbb{C}} \right) \\ &= \frac{1}{2} \left(\frac{1}{2\sqrt{-1}} \langle u, \theta_P^R + \theta_P^L \rangle_{\mathbb{C}} \right), \end{aligned}$$

for $u \in \mathfrak{g}$, and, using (26) and the Maurer-Cartan equation for $\theta_{G^{\mathbb{C}}}^R$, we get

$$\phi_S^{s'} = -\frac{1}{2} \left(\frac{1}{12} \mathfrak{Im} \langle [\theta_P^R, \theta_P^R], \theta_P^R \rangle_{\mathbb{C}} \right).$$

The description of Hamiltonian spaces in Prop. 2.4 reproduces the original definition of P -valued moment maps in [5] (up to a factor of 2).

By identifying G^* and P (via q), we get two different connections splittings (54) and (57) for $D = G^{\mathbb{C}} \rightarrow G^*$. By Prop. 4.2, the associated Hamiltonian categories are isomorphic by a gauge transformation, as explicitly shown in [5, Sec. 10.3].

Example 4.7 (G -valued moment maps). Consider the Manin pair $(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g})$ of Example 3.2, where \mathfrak{g} sits in $\mathfrak{g} \oplus \mathfrak{g}$ diagonally. The infinitesimal dressing action is

$$\rho_S : (\mathfrak{g} \oplus \mathfrak{g}) \times G \rightarrow TG, \quad (u, v) \mapsto u^r - v^l.$$

The antidiagonal in $\mathfrak{g} \oplus \mathfrak{g}$ gives an $\text{ad}(\mathfrak{g})$ -invariant isotropic complement of \mathfrak{g} , hence it defines an equivariant connection splitting $s : TG \rightarrow \mathfrak{d}_S$, explicitly given by

$$s(X_g) := \left(\frac{1}{2} dr_g^{-1}(X_g), -\frac{1}{2} dl_g^{-1}(X_g) \right).$$

The associated equivariant 3-form $\sigma^s + \phi_S^s$ is defined by

$$\sigma_s(u) = \frac{1}{2} \langle \theta^R + \theta^L, u \rangle_{\mathfrak{g}}, \quad u \in \mathfrak{g},$$

and the Cartan 3-form $\phi_S^s = -\phi_G := -\frac{1}{12} \langle [\theta^R, \theta^R], \theta^R \rangle_{\mathfrak{g}}$ (using (26)). Note that L_S^s is exactly the Cartan-Dirac structure (9), and the conditions in Prop. 2.4 reproduce the defining axioms of quasi-Hamiltonian spaces in [5].

In this example, $(D = G \times G, \omega_D)$, with $p(a, b) = ab^{-1}$ and $\bar{p}(a, b) = a^{-1}b$, is easily seen to be isomorphic to the AMM-double of [5, Sec. 3.2] (under $(a, b) \mapsto (a, b^{-1})$). The involution \mathcal{F}_D sends a quasi-Hamiltonian space (M, ω, ρ_M, J) to $(M, -\omega, \rho_M, \text{Inv}_G \circ J)$ (c.f. [5, Prop. 4.4]).

Example 4.8 (Symmetric-space valued moment maps). The *symmetric-space valued moment maps* of [25] naturally fit into the Dirac geometric framework of D/G -valued moment maps. The Manin symmetric Lie algebras in [25, Sec. 2] are examples of Manin pairs $(\mathfrak{d}, \mathfrak{g})$ equipped with an $\text{ad}(\mathfrak{g})$ -invariant isotropic complement of \mathfrak{g} , which by Prop. 3.5 define *equivariant* connection splittings. The associated equivariant 3-forms given by Prop. 3.4 agree with the *moment forms* of [25, Sec. 3], and the corresponding *moment spaces* are exactly the objects in $\mathcal{M}_s(\mathfrak{d}, \mathfrak{g})$.

We will explain how all these moment map theories are related to quasi-Poisson geometry in Section 6.

4.3. Presymplectic groupoids and doubles. Let $(\mathfrak{d}, \mathfrak{g})$ be a Manin pair with the choice of a connection splitting $s : TS \rightarrow \mathfrak{d}_S$, and let

$$L_S = \{(\rho(u), \sigma(u)), u \in \mathfrak{g}\}$$

be the associated ϕ_S -twisted Dirac structure on $S = D/G$. In this section, we will discuss the integration of the Dirac manifold (S, L_S, ϕ_S) .

As remarked in Section 2.3, L_S is isomorphic, as a Lie algebroid, to the action algebroid $\mathfrak{g} \ltimes S$ with respect to the dressing action. Hence, as integration of L_S , we can use the action groupoid $\mathcal{G} = G \ltimes S$. The source and target maps are given by $s(g, x) = x$, $t(g, x) = g.x$, and the multiplication is $m((g, x), (h, y)) = (gh, y)$. It remains to describe the 2-form $\omega \in \Omega^2(\mathcal{G})$ making it into a ϕ_S -twisted presymplectic groupoid integrating L_S .

Consider $\lambda \in C^\infty(\mathfrak{g}, \Omega^2(S))$ given by (c.f. Section 2.3)

$$(58) \quad \lambda(v) = d\sigma(v) - i_{\rho(v)}\phi_S.$$

Using that σ satisfies (12),

$$\sigma([u, v]) = \mathcal{L}_{\rho(u)}\sigma(v) - i_{\rho(v)}d\sigma(u) + i_{\rho(u)\wedge\rho(v)}\phi_S, \quad u, v \in \mathfrak{g},$$

it is simple to check that λ satisfies

$$\lambda([u, v]) = \mathcal{L}_{\rho(u)}\lambda(v) - \mathcal{L}_{\rho(v)}\lambda(u),$$

i.e., it is a $\Omega^2(S)$ -valued Lie algebra cocycle. That is, the map $\mathfrak{g} \rightarrow \mathfrak{g} \ltimes \Omega^2(S)$, $v \mapsto (v, \lambda(v))$ is a Lie algebra homomorphism.

Assume now that the cocycle (58) integrates to a Lie group cocycle, i.e. $c \in C^\infty(G, \Omega^2(S))$ satisfying

$$c(gh) = h^*c(g) + c(h),$$

where the pull-back $h^*c(g)$ is with respect to the dressing action of G on S . This happens e.g. if G is simply-connected. It follows from Prop. 3.4 that if s is equivariant, then (58) vanishes and $c \equiv 0$.

The next result follows from [12, Sec. 6.4] and gives an explicit formula for the multiplicative 2-form integrating L_S :

Proposition 4.4. *The 2-form $\omega \in \Omega^2(G \ltimes S)$ integrating L_S is explicitly given by*

$$(59) \quad \begin{aligned} \omega_{g,x}((V, X), (V', X')) &:= \langle \sigma_x(\theta_g^L(V)), \rho_x(\theta_g^L(V')) \rangle \\ &+ \langle \sigma_x(\theta_g^L(V)), X' \rangle - \langle \sigma_x(\theta_g^L(V')), X \rangle + \langle c(g), X \wedge X' \rangle, \end{aligned}$$

where $V, V' \in T_g G$, $X, X' \in T_x S$ and $\theta^L \in \Omega^1(G, \mathfrak{g})$ is the left-invariant Maurer-Cartan 1-form. If s is equivariant, then $c = 0$.

As mentioned in Example 2.3, this 2-form ω on $G \times S$ makes it into a $S \times S^{\text{op}}$ -valued Hamiltonian $G \times G$ -space. This space is closely related to various well-known notions of “double”.

Example 4.9 (Cotangent bundles). Let us consider the Manin pair $(\mathfrak{g}, \mathfrak{g} \ltimes \mathfrak{g}^*)$ with connection s as in Example 4.4. Since the connection is invariant, the cocycle c vanishes and the 2-form on $G \ltimes \mathfrak{g}^*$ of Prop. 59

reads

$$\begin{aligned}\omega_{g,\mu}((V, X), (V', X')) \\ = & \left\langle dl_g^{-1}(V), \text{ad}_{dl_g^{-1}(V')}^*(\mu) \right\rangle + \left\langle dl_g^{-1}(V), X' \right\rangle - \left\langle dl_g^{-1}(V'), X \right\rangle \\ = & \mu([\theta^L(V), \theta^L(V')]) + \left\langle \theta^L(V), X' \right\rangle - \left\langle \theta^L(V'), X \right\rangle,\end{aligned}$$

which is the canonical symplectic structure on $T^*G \cong G \times \mathfrak{g}^*$ (with identification via left translations).

Example 4.10 (AMM groupoid). For the Manin pair $(\mathfrak{g}, \mathfrak{g} \oplus \mathfrak{g})$ with invariant connection s of Example 4.7, $c = 0$ and the 2-form ω on $G \ltimes G$ can be directly computed to be

$$\begin{aligned}\omega_{g,x}((V, X), (V', X')) = & \frac{1}{2} [\left\langle \text{Ad}_x \theta^L(V), \theta^L(V') \right\rangle_{\mathfrak{g}} - \left\langle \text{Ad}_x \theta^L(V'), \theta^L(V) \right\rangle_{\mathfrak{g}} \\ & + \left\langle \theta^L(V), \theta^R(X') \right\rangle_{\mathfrak{g}} - \left\langle \theta^L(V'), \theta^R(X) \right\rangle_{\mathfrak{g}} \\ & + \left\langle \theta^L(V), \theta^L(X') \right\rangle_{\mathfrak{g}} - \left\langle \theta^L(V'), \theta^L(X) \right\rangle_{\mathfrak{g}}],\end{aligned}$$

which can be re-written more concisely as

$$\omega_{g,x} = \frac{1}{2} \left(\left\langle \text{Ad}_x p_1^* \theta^L, p_1^* \theta^L \right\rangle_{\mathfrak{g}} + \left\langle p_1^* \theta^L, p_2^* (\theta^L + \theta^R) \right\rangle_{\mathfrak{g}} \right),$$

where $p_1(g, x) = g$ and $p_2(g, x) = x$ are the natural projections $G \times G \rightarrow G$.

The presymplectic groupoid $(\mathcal{G} = G \ltimes G, \omega)$ is closely related to the double $(D = G \times G, \omega_D)$ of Example 4.7: the change of coordinates $D \rightarrow \mathcal{G}$, $(a, b) \mapsto (g = a, x = b^{-1}a)$ identifies ω_D with ω . (Under this identification, we have $p = t$ and $\bar{p} = s^{-1}$, so D and \mathcal{G} are not identified as bimodules.)

Example 4.11 (Heisenberg groupoid). Let us consider the case of a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ as in Example 4.5, where $S = G^*$ and the connection splitting s is (54). This connection is not equivariant in general, so one has to consider the 1-cocycle of (58): $\lambda(u) = d\sigma(u)$ (in this example, $\phi_S = 0$). Explicitly, we have

$$\begin{aligned}d\sigma(u)(\mu^r, \nu^r) \\ = & \mathcal{L}_{\mu^r} \left\langle (dr_x^{-1})^* u, \nu^r \right\rangle - \mathcal{L}_{\nu^r} \left\langle (dr_x^{-1})^* u, \mu^r \right\rangle - (dr_x^{-1})^* u([\mu^r, \nu^r]) \\ = & -F(v)(\mu, \nu).\end{aligned}$$

Here μ^r, ν^r are the right translations of $\mu, \nu \in \mathfrak{g}^*$ to TG^* , and $F : \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ is the co-bracket (c.f. Section 5.1). The cocycle $\lambda \in C^\infty(\mathfrak{g}, \Omega^2(G^*))$ is then given by

$$\lambda(u)_x = -(dr_x^{-1})^* F(u) \in T_x^* G^*.$$

We know that $F : \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ is a 1-cocycle with respect to the adjoint representation, and there is a unique multiplicative bivector field π_G on

G so that $dl_g^{-1}\pi_G : G \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ is a 1-cocycle integrating F , see e.g. [22]. Hence the following 1-cocycle $c \in C^\infty(G, \Omega^2(G^*))$ integrates λ :

$$c(g)_x = -(dr_x^{-1})^*dl_g^{-1}\pi_G,$$

and we get the following expression for the 2-form on $G \times G^*$ integrating π_{G^*} :

$$\begin{aligned} \omega_{g,x}((V, X), (V', X')) \\ = -\langle dr_x^{-1}\pi_{G^*}, \theta_g^L(V) \wedge \theta_g^L(V') \rangle + \langle \theta_g^L(V), \theta_x^R(X') \rangle \\ - \langle \theta_g^L(V'), \theta_x^R(X) \rangle - \langle dl_g^{-1}\pi_G, \theta_x^R(X) \wedge \theta_x^R(X') \rangle, \end{aligned}$$

where we have used that

$$\langle (dr_x^{-1})^*(\theta_g^L(V)), \rho_x(\theta_g^L(V')) \rangle = -\langle dr_x^{-1}\pi_{G^*}, \theta_g^L(V) \wedge \theta_g^L(V') \rangle.$$

The 2-form ω agrees with ω_D in this example, so, as a symplectic manifold, the groupoid $(G \times G^*, \omega)$ is the Heisenberg double ($D = G \times G^*, \omega_D$).

5. D/G -valued moment maps via quasi-Poisson geometry

In this section we revisit the theory of D/G -valued moment maps in the context of quasi-Poisson actions following [3]. This theory has as starting point a Manin pair $(\mathfrak{d}, \mathfrak{g})$ together with the choice of an isotropic complement of \mathfrak{g} in \mathfrak{d} or, equivalently, an isotropic splitting j of (41). We refer to $(\mathfrak{d}, \mathfrak{g}, j)$ as a **split Manin pair**. We now observe that quasi-Poisson spaces, just as ordinary Poisson manifolds, can be understood in terms of Lie algebroids. This leads to refinements of results in [3].

5.1. Quasi-Poisson \mathfrak{g} -spaces. Let $(\mathfrak{d}, \mathfrak{g})$ be a Manin pair. Following Sections A.2 and A.3, we consider the exact sequence

$$(60) \quad \mathfrak{g} \xrightarrow{\iota} \mathfrak{d} \xrightarrow{\iota^*} \mathfrak{g}^*,$$

where $\iota : \mathfrak{g} \hookrightarrow \mathfrak{d}$ is the inclusion and ι^* is the projection $\mathfrak{d} \rightarrow \mathfrak{d}/\mathfrak{g}$ after the identification $\mathfrak{d}/\mathfrak{g} = \mathfrak{g}^*$ induced by $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$. The choice of an isotropic splitting $j : \mathfrak{g}^* \rightarrow \mathfrak{d}$ defines elements

$$(61) \quad F_j : \mathfrak{g} \longrightarrow \wedge^2 \mathfrak{g}, \quad \text{and} \quad \chi_j \in \wedge^3 \mathfrak{g}$$

by the conditions

$$F_j^*(\mu, \nu) = \iota^*([j(\mu), j(\nu)]_{\mathfrak{d}}), \quad \chi_j(\mu, \nu) = j^*([j(\mu), j(\nu)]_{\mathfrak{d}}), \quad \mu, \nu \in \mathfrak{g}^*,$$

see Sec. A.2. We will omit the subscript j whenever there is no risk of confusion. Using the isometric identification of $(\mathfrak{d}, \langle \cdot, \cdot \rangle_{\mathfrak{d}})$ with $(\mathfrak{g} \oplus \mathfrak{g}^*, \langle \cdot, \cdot \rangle_{can})$ given by (ι, j) , the Lie bracket on \mathfrak{d} takes the form:

$$(62) \quad [(u, 0), (v, 0)]_{\mathfrak{d}} = ([u, v], 0),$$

$$(63) \quad [(v, 0), (0, \mu)]_{\mathfrak{d}} = (i_\mu(F(v)), \text{ad}_v^*\mu),$$

$$(64) \quad [(0, \mu), (0, \nu)]_{\mathfrak{d}} = (\chi(\mu, \nu), F^*(\mu, \nu)),$$

for $u, v \in \mathfrak{g}$ and $\mu, \nu \in \mathfrak{g}^*$.

As in Sec. A.3, we say that a triple (\mathfrak{g}, F, χ) is a **Lie quasi-bialgebra** [7, 19] if the bracket defined by (62), (63), (64) is a Lie bracket, in which case $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g})$ is a split Manin pair. The resulting Lie algebra $\mathfrak{g} \oplus \mathfrak{g}^*$ is the **Drinfeld double** [7] of the Lie quasi-bialgebra (\mathfrak{g}, F, χ) . A Lie quasi-bialgebra with $\chi = 0$ is called a **Lie bialgebra**, in which case F defines a Lie algebra structure on \mathfrak{g}^* . Note that there is a 1-1 correspondence between split Manin pairs $(\mathfrak{d}, \mathfrak{g}, j)$ and Lie quasi-bialgebras.

For a fixed isotropic splitting $j : \mathfrak{g}^* \rightarrow \mathfrak{d}$ for the Manin pair $(\mathfrak{d}, \mathfrak{g})$, and denoting by (\mathfrak{g}, F, χ) the corresponding Lie quasi-bialgebra, we define a **quasi-Poisson \mathfrak{g} -space** [3] as a manifold M endowed with an infinitesimal action of \mathfrak{g} , denoted by $\rho_M : \mathfrak{g} \rightarrow \mathfrak{X}(M)$, and a bivector field $\pi \in \mathfrak{X}^2(M)$, such that

$$(65) \quad \mathcal{L}_{\rho_M(v)}\pi = -\rho_M(F(v)), \quad \text{for all } v \in \mathfrak{g},$$

$$(66) \quad \frac{1}{2}[\pi, \pi] = \rho_M(\chi),$$

where $[\cdot, \cdot]$ is the Schouten bracket. If $F = 0$ and $\chi = 0$, then (M, π) is a Poisson manifold, and π is invariant. In general F controls how π fails to be invariant, whereas χ controls how it fails to be integrable.

Remark 5.1. A different isotropic splitting j' , related to j by a twist $t \in \wedge^2 \mathfrak{g}$ (i.e., $j - j' = t^\sharp$) leads, as discussed in Sec. A.5 (see [3]), to a Lie quasi-bialgebra defined by

$$(67) \quad F' = F + [t, \cdot], \quad \chi' = \chi - d_F(t) + \frac{1}{2}[t, t].$$

If (M, π) is a quasi-Poisson space for (\mathfrak{g}, F, χ) , then (M, π') is a quasi-Poisson space for the Lie quasi-bialgebra $(\mathfrak{g}, F', \chi')$ [3], where

$$(68) \quad \pi' = \pi + \rho_M(t).$$

Remark 5.2. The cobracket F associated with j is a cocycle with values in $\wedge^2 \mathfrak{g}$ (see Q0) in Sec. A.3), hence determines a class in $H^1(\mathfrak{g}, \wedge^2 \mathfrak{g})$. It follows from the first formula in (67) that this class does not depend on the splitting j . By (63), j has the property $[\mathfrak{g}, j(\mathfrak{g}^*)] \subseteq j(\mathfrak{g}^*)$ if and only if $F = 0$, and such j exists if $H^1(\mathfrak{g}, \wedge^2 \mathfrak{g}) = 0$.

Remark 5.3. (*Global actions*) One can similarly consider global quasi-Poisson G -actions, in which case the Lie quasi-bialgebra is replaced by its global counterpart, i.e., a **quasi-Poisson Lie group** [23] (which arise in connection with [19]). Since our main constructions do not require global actions, we will restrict ourselves to the infinitesimal picture.

As shown in [3], the choice of an isotropic splitting j of (60) induces a bivector field π_S on $S = D/G$, which makes S into a quasi-Poisson space with respect to the dressing \mathfrak{g} -action. To define π_S , note that

$(\mathfrak{d}_S, \mathfrak{g}_S)$ is a Manin pair over the manifold S (Sec. A.2), and j induces a splitting of it. As in Sec. A.4, we have an associated bivector field π_S defined by

$$(69) \quad \pi_S^\sharp = \rho(\rho_S j)^*: T^*S \longrightarrow TS,$$

where $\rho = \rho_S \iota$. If j' is another isotropic splitting and t is the associated twist, then $\pi'_S = \pi_S + \rho_S(t)$. Let us consider the map

$$(70) \quad \bar{\sigma}_j = (\rho_S j)^*: T^*S \longrightarrow \mathfrak{g}_S.$$

(As usual, if there is no danger of confusion, we will drop the dependence on j in the notation and write simply $\bar{\sigma}$.)

Remark 5.4. To see that π_S agrees with the bivector defined in [3], consider the **r -matrix** $\mathfrak{r} = \mathfrak{r}_j \in \mathfrak{d} \otimes \mathfrak{d}$, given by $\mathfrak{r}(u^\vee, v^\vee) = \langle j\iota^*(u), v \rangle_{\mathfrak{d}}$ for $u, v \in \mathfrak{d}$ (where $u^\vee, v^\vee \in \mathfrak{d}^*$ are the dual of u, v with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$). Note that \mathfrak{r} is not antisymmetric, since it satisfies

$$(71) \quad \mathfrak{r}(u^\vee, v^\vee) + \mathfrak{r}(v^\vee, u^\vee) = \langle u, v \rangle_{\mathfrak{d}}.$$

But $\rho_S(\mathfrak{r})$ is a bivector field, and $\pi_S = -\rho_S(\mathfrak{r})$,

$$\rho_S(\mathfrak{r})^\sharp = \rho_S \mathfrak{r}^\sharp \rho_S^* = \rho_S j \iota^* \rho_S^* = -\rho_S \bar{\sigma} = -\pi_S^\sharp,$$

in accordance with [3].

Since $\rho \bar{\sigma} = -\bar{\sigma}^* \rho^*$ (which is the skew-symmetry of π_S), it follows that π_S can be restricted to any orbit of the dressing action of G on S : for any such orbit $\mathcal{O} \subset S$ and any $\xi \in T_x^*S$ with $x \in \mathcal{O}$ and $\xi|_{T_x \mathcal{O}} = 0$, then $\rho^*(\xi) = 0$. Hence $\pi_S^\sharp(\xi) = 0$. We denote by

$$\pi_{\mathcal{O}} \in \Gamma(\wedge^2 T\mathcal{O})$$

the resulting bivector field. The next result follows directly from Prop. A.1, and it was first proven in [3].

Proposition 5.1. *(S, π_S) and $(\mathcal{O}, \pi_{\mathcal{O}})$ are quasi-Poisson spaces with respect to the dressing action of G on S .*

Let us compute $\bar{\sigma}$ and π_S in examples.

Example 5.5. For a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, the inclusion $\mathfrak{g}^* \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}^*$ is an obvious choice of isotropic splitting j . Since $\chi = 0$, the induced bivector π_S on $S = D/G$ is a Poisson structure in this case. In the context of Example 4.5, we have $S = G^*$ and $\rho_S|_{\mathfrak{g}^*}(\mu) = \mu^r$, so

$$(72) \quad \bar{\sigma}(\alpha) = (dr_x)^* \alpha, \quad \alpha \in T_x^*G^*$$

and π_S is defined by $\pi_S^\sharp(\alpha)_x = \rho(dr_x^* \alpha)$, which agrees with (56). So the graph of π_S is L_S , the Dirac structure of Example 4.5. The bialgebra in Example 3.1 is a particular case for which $\rho_S|_{\mathfrak{g}^*} = \text{Id}$, so $\bar{\sigma} = \text{Id}$ and $\pi_S^\sharp(u)(\mu) = \rho(u)(\mu) = \text{ad}_u^*(\mu)$ is the usual Lie-Poisson structure.

Example 5.6. For Example 3.2, we consider the isotropic splitting given by the anti-diagonal embedding,

$$j(\mu) := \frac{1}{2}(\mu^\vee, -\mu^\vee),$$

where $\mu \in \mathfrak{g}^*$ and $\mu^\vee \in \mathfrak{g}$ is its dual with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$, i.e., $\mu = \langle \mu^\vee, \cdot \rangle_{\mathfrak{g}}$. Then a simple computation shows that $F = 0$ and χ is given by

$$(73) \quad \chi(\mu_1, \mu_2, \mu_3) = \frac{1}{4} \langle [\mu_1^\vee, \mu_2^\vee], \mu_3^\vee \rangle_{\mathfrak{g}},$$

i.e., $\chi \in \wedge^3 \mathfrak{g}$ is the Cartan trivector [3]. In this case, $\bar{\sigma} : T^*G \rightarrow \mathfrak{g}$ is given by

$$\bar{\sigma}(\alpha_g) = \frac{1}{2}((dr_g^* + dl_g^*)(\alpha_g))^\vee = \frac{1}{2}(dr_{g^{-1}} + dl_{g^{-1}})(\alpha_g^\vee),$$

and the bivector field π_S on $S = G$ is

$$\pi_S(dl_{g^{-1}}^*(\mu), dl_{g^{-1}}^*(\nu)) = \frac{1}{2} \langle (\text{Ad}_{g^{-1}} - \text{Ad}_g)\mu^\vee, \nu^\vee \rangle_{\mathfrak{g}},$$

see [4]. Alternatively, if e_i is a basis for \mathfrak{g} and f_j is the dual basis with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$, then $\pi_S = \frac{1}{2} \sum_i e_i^l \wedge f_i^r$.

5.2. The Lie algebroid of a quasi-Poisson \mathfrak{g} -space. We now present the construction of a Lie algebroid associated with any quasi-Poisson space.

If M is a manifold equipped with a bivector field π , one has an induced bracket $[\cdot, \cdot]_\pi$ on the space of 1-forms on M ,

$$(74) \quad [\alpha, \beta]_\pi := \mathcal{L}_{\pi^\sharp(\alpha)}\beta - \mathcal{L}_{\pi^\sharp(\beta)}\alpha - d\pi(\alpha, \beta),$$

for $\alpha, \beta \in \Omega^1(M)$. Then π is a Poisson structure if and only if $[\cdot, \cdot]_\pi$ satisfies the Jacobi identity, making T^*M into a Lie algebroid with anchor $\pi^\sharp : T^*M \rightarrow TM$. The symplectic leaves of a Poisson manifold are precisely the orbits of this Lie algebroid.

Suppose now that M is equipped with a bivector field π as well as an infinitesimal action $\rho_M : \mathfrak{g} \rightarrow TM$. Consider the vector bundle $A := \mathfrak{g} \oplus T^*M$, let

$$(75) \quad r : A \rightarrow TM, \quad r(v, \alpha) := \rho_M(v) + \pi^\sharp(\alpha),$$

and consider the bracket on $\Gamma(A) = C^\infty(M, \mathfrak{g}) \oplus \Omega^1(M)$ defined by

$$(76) \quad [(u, 0), (v, 0)]_A = ([u, v], 0),$$

$$(77) \quad [(v, 0), (0, \alpha)]_A = (-i_{\rho_M^*(\alpha)}(F(v)), \mathcal{L}_{\rho_M(v)}\alpha),$$

$$(78) \quad [(0, \alpha), (0, \beta)]_A = (i_{\rho_M^*(\alpha \wedge \beta)}\chi, [\alpha, \beta]_\pi),$$

for $\alpha, \beta \in \Omega^1(M)$, and $u, v \in \mathfrak{g}$, considered as constant sections in $C^\infty(M, \mathfrak{g})$ (the bracket is extended to general elements by the Leibniz rule). The main result in this section is the following:

Theorem 5.1. $(\mathfrak{g} \oplus T^*M, r, [\cdot, \cdot]_A)$ is a Lie algebroid if and only if (M, π) is a quasi-Poisson \mathfrak{g} -space with respect to the action $\rho_M : \mathfrak{g} \rightarrow \mathfrak{X}^1(M)$.

The Lie algebroid in Theorem 5.1 has as special cases the Lie algebroids previously introduced in [10] and [29], but our general proof follows a different approach. The following result is a direct consequence of Theorem 5.1.

Corollary 5.1. *On a quasi-Poisson \mathfrak{g} -space (M, π) , the generalized distribution*

$$(79) \quad \{\rho_M(v) + \pi^\sharp(\alpha) \mid v \in \mathfrak{g}, \alpha \in T^*M\} \subseteq TM$$

is integrable.

The fact that (79) defines a singular foliation was first observed in [3, 4], but under the additional assumptions of existence of a moment map and that \mathfrak{g} is integrated by a compact Lie group.

In the theory of quasi-Poisson spaces, a particular role is played by those with *transitive* Lie algebroid, i.e.,

$$(80) \quad TM = \{\rho_M(v) + \pi^\sharp(\alpha) \mid v \in \mathfrak{g}, \alpha \in T^*M\}.$$

Note that, if the \mathfrak{g} -orbits are tangent to the distribution $\pi^\sharp(T^*M)$, then this transitivity condition implies that the bivector field π is nondegenerate (but this is not the case in general).

5.2.1. The proof of Theorem 5.1. In this subsection we present the proof of Theorem 5.1; see [11] for an alternative discussion.

As recalled in Section A.1, describing a Lie algebroid structure on $A = \mathfrak{g} \oplus T^*M$ is equivalent to finding a degree-1 derivation $d_A : \Gamma(\wedge^\bullet A^*) \rightarrow \Gamma(\wedge^{\bullet+1} A^*)$ satisfying $d_A^2 = 0$ [39]: the anchor r and bracket $[\cdot, \cdot]_A$ are recovered by the conditions

$$(81) \quad d_A f(a) = \mathcal{L}_{r(a)} f,$$

$$(82) \quad d_A \xi(a, b) = \mathcal{L}_{r(a)} \xi(b) - \mathcal{L}_{r(b)} \xi(a) - \xi([a, b]_A),$$

for $f \in C^\infty(M)$, $a, b \in \Gamma(A)$ and $\xi \in \Gamma(A^*)$. We now present a construction of a differential d_A leading to the bracket defined by (76), (77) and (78).

Let (\mathfrak{g}, F, χ) be a Lie quasi-bialgebra, and let $\mathfrak{X} = \bigoplus_{q \in \mathbb{Z}} \mathfrak{X}^q$ be any graded commutative algebra. We consider the tensor product of graded commutative algebras $\wedge \mathfrak{g}^* \otimes \mathfrak{X}$, which is itself graded commutative with product

$$(\mu \otimes x) \cdot (\nu \otimes y) := (-1)^{qp'} (\mu \wedge \nu) \otimes (x \cdot y),$$

for $\mu \otimes x \in \wedge^p \mathfrak{g}^* \otimes \mathfrak{X}^q$, $\nu \otimes y \in \wedge^{p'} \mathfrak{g}^* \otimes \mathfrak{X}^{q'}$, and grading $(\wedge \mathfrak{g}^* \otimes \mathfrak{X})^k = \bigoplus_{p+q=k} \wedge^p \mathfrak{g}^* \otimes \mathfrak{X}^q$. We assume that \mathfrak{X} is equipped with an operator $d : \mathfrak{X}^\bullet \rightarrow \mathfrak{X}^{\bullet+1}$ which is a derivation of degree 1,

$$d(x \cdot y) = dx \cdot y + (-1)^q x \cdot dy, \quad \text{for } x \in \mathfrak{X}^q, y \in \mathfrak{X}^{q'},$$

but *not* necessarily squaring to zero. (The example to have in mind is when \mathfrak{X}^\bullet is given by multivector fields on M , and $d = [\pi, \cdot]$ for a quasi-Poisson bivector π , where $[\cdot, \cdot]$ is the Schouten bracket). We moreover assume that \mathfrak{g} acts on \mathfrak{X}^\bullet by derivations of degree 0, and that we are given a \mathfrak{g} -equivariant map

$$(83) \quad \varrho : \mathfrak{g} \rightarrow \mathfrak{X}^1$$

with respect to the adjoint action of \mathfrak{g} on itself. The map ϱ induces an equivariant map of graded algebras $\wedge^\bullet \mathfrak{g} \rightarrow \mathfrak{X}^\bullet$ which we also denote by ϱ .

In this framework, one can define various derivations of $\wedge \mathfrak{g}^* \otimes \mathfrak{X}$. First, the Chevalley-Eilenberg operator (i.e., the Lie algebra differential) of \mathfrak{g} with coefficients in \mathfrak{X} ,

$$(84) \quad \partial : \wedge^\bullet \mathfrak{g}^* \otimes \mathfrak{X}^\bullet \rightarrow \wedge^{\bullet+1} \mathfrak{g}^* \otimes \mathfrak{X}^\bullet,$$

is a derivation of $\wedge \mathfrak{g}^* \otimes \mathfrak{X}$ of degree 1. Second, we define

$$(85) \quad d : \wedge^\bullet \mathfrak{g}^* \otimes \mathfrak{X}^\bullet \rightarrow \wedge^{\bullet+1} \mathfrak{g}^* \otimes \mathfrak{X}^{\bullet+1}$$

to be the unique derivation of $\wedge \mathfrak{g}^* \otimes \mathfrak{X}$ extending the operator $d : \mathfrak{X}^\bullet \rightarrow \mathfrak{X}^{\bullet+1}$ and vanishing on $\wedge \mathfrak{g}^*$. Finally, associated with F and χ , we define

$$\partial_F : \wedge^\bullet \mathfrak{g}^* \otimes \mathfrak{X}^\bullet \rightarrow \wedge^\bullet \mathfrak{g}^* \otimes \mathfrak{X}^{\bullet+1} \quad \text{and} \quad \partial_\chi : \wedge^\bullet \mathfrak{g}^* \otimes \mathfrak{X}^\bullet \rightarrow \wedge^{\bullet-1} \mathfrak{g}^* \otimes \mathfrak{X}^{\bullet+2}$$

to be the unique derivations vanishing on \mathfrak{X} and defined on \mathfrak{g}^* by the conditions

$$(86) \quad \partial_F : \mathfrak{g}^* \rightarrow \mathfrak{g}^* \otimes \mathfrak{X}^1, \quad \partial_F \mu(u) = -\varrho(i_\mu(F(u))), \quad u \in \mathfrak{g};$$

$$(87) \quad \partial_\chi : \mathfrak{g}^* \rightarrow \mathfrak{X}^2, \quad \partial_\chi \mu = -\varrho(i_\mu \chi).$$

We now consider the derivation

$$(88) \quad \delta := \partial + \partial_F + \partial_\chi + d : (\wedge \mathfrak{g}^* \otimes \mathfrak{X})^\bullet \rightarrow (\wedge \mathfrak{g}^* \otimes \mathfrak{X})^{\bullet+1}.$$

The following is a key example of this framework in which δ is a differential, i.e., $\delta^2 = 0$.

Lemma 5.7. *Consider $\mathfrak{X} = \wedge \mathfrak{g}$, equipped with the adjoint action of \mathfrak{g} , and let $\varrho = \text{Id} : \mathfrak{g} \rightarrow \mathfrak{g}$ and $d : \wedge^\bullet \mathfrak{g} \rightarrow \wedge^{\bullet+1} \mathfrak{g}$ be the Lie algebra differential of the Lie bracket $-F^* : \mathfrak{g}^* \wedge \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ on \mathfrak{g}^* . Then, under the canonical isomorphism $\wedge \mathfrak{g}^* \otimes \wedge \mathfrak{g} \cong \wedge(\mathfrak{g}^* \oplus \mathfrak{g})$, δ is the Lie algebra differential of the Drinfeld double of the Lie quasi-bialgebra $(\mathfrak{g}, -F, \chi)$.*

To prove the lemma, it suffices to compare how the derivation δ defined by (88) and the Lie algebra differential act on elements in \mathfrak{g}^* and \mathfrak{g} , and a direct computation using the definitions shows that they agree.

Note that to check that $\delta^2 = 0$ in general, it suffices to check that $\delta^2 = 0$ on $\wedge \mathfrak{g}^*$ and \mathfrak{X} separately since, if $\mu \otimes x \in \wedge^p \mathfrak{g}^* \otimes \mathfrak{X}^q$, we have

$$\begin{aligned}\delta^2(\mu \otimes x) &= \delta^2(\mu) \cdot (1 \otimes x) + (-1)^{p+1} \delta(\mu) \cdot \delta(x) + (-1)^p \delta(\mu) \cdot \delta(x) \\ &\quad + (-1)^{2p} (\mu \otimes 1) \cdot \delta^2(x) \\ &= \delta^2(\mu) \cdot (1 \otimes x) + (\mu \otimes 1) \cdot \delta^2(x).\end{aligned}$$

Lemma 5.8. *On \mathfrak{X} , $\delta^2 = 0$ is equivalent to the conditions*

$$(89) \quad d\partial + \partial d + \partial_F \partial = 0$$

$$(90) \quad \partial_\chi \partial + d^2 = 0.$$

Proof. A direct computation shows that

$$(91) \quad \delta^2 = \partial d + d\partial + \partial_F \partial + \partial_\chi \partial + d^2$$

on \mathfrak{X} , where we have used that ∂_F and ∂_χ vanish on \mathfrak{X} and $d^2 = 0$. Splitting (91) according to degrees, one obtains (89) and (90). q.e.d.

Let us now focus on the case where $(\mathfrak{X}, [\cdot, \cdot])$ is a Gerstenhaber algebra (see equations (148), (149) and (150) in Sec.A.1 for conventions) and that $\varrho : \mathfrak{g} \rightarrow \mathfrak{X}^1$ is a Lie algebra homomorphism such that the \mathfrak{g} -action on \mathfrak{X} is given by

$$(92) \quad v \cdot x := [\varrho(v), x], \quad v \in \mathfrak{g}, x \in \mathfrak{X}.$$

Lemma 5.9. *For all $x \in \mathfrak{X}$ and $v \in \mathfrak{g}$, the following holds:*

$$(93) \quad \partial_\chi \partial(x) = -[\varrho(\chi), x];$$

$$(94) \quad \langle (d\partial + \partial d)(x), v \rangle = [\varrho(v), dx] - d[\varrho(v), x]$$

$$(95) \quad \langle \partial_F \partial(x), v \rangle = [\varrho(F(v)), x].$$

Proof. Let $\{e_i\}$ be a basis of \mathfrak{g} , and let $\{e^i\}$ be the dual basis. To prove (93), take $x \in \mathfrak{X}$, and recall that

$$(96) \quad \partial(x) = \sum_l e^l \otimes [\varrho(e_l), x], \quad \text{and} \quad \langle \partial(x), v \rangle = [\varrho(v), x].$$

By definition of ∂_χ , we get

$$\partial_\chi \partial x = - \sum_l \varrho(i_{e^l} \chi) [\varrho(e_l), x] = -3 \sum_{i,j,k} \chi_{ijk} \varrho(e_i) \varrho(e_j) [\varrho(e_k), x],$$

where we have written $\chi = \sum \chi_{ijk} e_i \wedge e_j \wedge e_k$. On the other hand,

$$[\varrho(\chi), x] = \sum_{i,j,k} \chi_{ijk} [\varrho(e_i) \varrho(e_j) \varrho(e_k), x] = 3 \sum_{i,j,k} \chi_{ijk} \varrho(e_i) \varrho(e_j) [\varrho(e_k), x],$$

where the last equality follows from the graded Leibniz identity for $[\cdot, \cdot]$. This proves (93).

From the first formula in (96) and the derivation property of d , it follows that $d\partial x = -\sum e^i \otimes d([\varrho(e_i), x])$, and

$$\langle d\partial x, v \rangle = -\sum v_i d[\varrho(e_i), x] = -d[\varrho(v), x].$$

The second equation in (96) implies that $\langle \partial dx, v \rangle = [\varrho(v), dx]$, so (94) follows.

To prove (95), note that, using (96) and the definition of ∂_F , we have

$$\langle \partial_F \partial(x), v \rangle = \sum_l \varrho(i_{e^l} F(v)) [\varrho(e_l), x] = 2 \sum_{i,j,k} F_{jk}^i v_i \varrho(e_j) [\varrho(e_k), x],$$

where we have written $F(v) = \sum_{i,j,k} F_{jk}^i v_i e_j \wedge e_k$. On the other hand, an application of the Leibniz identity yields

$$[\varrho(F(v)), x] = \sum_{i,j,k} F_{jk}^i v_i [\varrho(e_j) \varrho(e_k), x] = 2 \sum_{i,j,k} F_{jk}^i v_i \varrho(e_j) [\varrho(e_k), x],$$

proving (95). q.e.d.

If we further assume that the derivation $d : \mathfrak{X}^\bullet \rightarrow \mathfrak{X}^{\bullet+1}$ has the form $d_\pi = [\pi, \cdot]$ for some fixed $\pi \in \mathfrak{X}^2$, then combining (94) and (95), and using the graded Jacobi identity for $[\cdot, \cdot]$, one checks that (89) is equivalent to

$$(97) \quad [[\varrho(v), \pi], x] = -[\varrho(F(v)), x] \text{ for all } x \in \mathfrak{X}.$$

Another application of the graded Jacobi identity shows that $d_\pi^2 = [\frac{1}{2}[\pi, \pi], \cdot]$, so, using (93), one sees that (90) can be rewritten as

$$(98) \quad \frac{1}{2}[[\pi, \pi], x] = [\varrho(\chi), x] \text{ for all } x \in \mathfrak{X}.$$

Let us now consider the specific situation where M is a manifold equipped with a bivector field $\pi \in \mathfrak{X}^2(M)$ as well as an infinitesimal \mathfrak{g} -action $\rho_M : \mathfrak{g} \rightarrow \mathfrak{X}^1(M)$ (playing the role of ϱ (83)). We use (92) to define a \mathfrak{g} -action on the Gerstenhaber algebra $\mathfrak{X}^\bullet(M)$ of multivector fields, and consider the derivation (88) in this context,

$$(99) \quad \delta := \partial + \partial_F + \partial_\chi + d : \wedge \mathfrak{g}^* \otimes \mathfrak{X}(M) \rightarrow \wedge \mathfrak{g}^* \otimes \mathfrak{X}(M),$$

with $d = d_\pi = [\pi, \cdot]$.

Lemma 5.10. *The action ρ_M makes (M, π) into a quasi-Poisson \mathfrak{g} -space if and only if $\delta^2 = 0$.*

Proof. To prove the claim, as we have previously remarked, it suffices to show that $\delta^2 = 0$ on $\wedge \mathfrak{g}^*$ and $\mathfrak{X}(M)$ separately. Using Lemma 5.9, we saw that (90) is equivalent to (98), which is the same as

$$(100) \quad \frac{1}{2}[\pi, \pi] = \rho_M(\chi)$$

when $\mathfrak{X} = \mathfrak{X}(M)$. Similarly, (89) is equivalent to

$$(101) \quad [\rho_M(v), \pi] = -\rho_M(F(v)).$$

It then follows from Lemma 5.8 that ρ_M is a quasi-Poisson action if and only if $\delta^2 = 0$ on $\mathfrak{X}(M)$. To finish the proof of the lemma, we will check that if $\delta^2 = 0$ on $\mathfrak{X}(M)$, then it automatically happens that $\delta^2 = 0$ on $\wedge \mathfrak{g}^*$, i.e., (99) squares to zero.

Let δ^0 denote the δ -operator (88) in the specific context of Lemma 5.7, and similarly for the derivations ∂^0 , d^0 , ∂_F^0 and ∂_χ^0 . Consider the homomorphism

$$\Psi : \wedge \mathfrak{g}^* \otimes \wedge \mathfrak{g} \rightarrow \wedge \mathfrak{g}^* \otimes \mathfrak{X}(M), \quad \Psi(\mu \otimes v) = \mu \otimes \rho_M(v).$$

We claim that (101) implies that

$$(102) \quad \Psi \circ \delta^0 = \delta \circ \Psi.$$

Indeed, recall that $d^0 = F$ on \mathfrak{g} , and condition (101) is equivalent to $d_\pi \rho(v) = \rho_M(F(v))$, i.e., $\Psi \circ d^0 = d_\pi \circ \Psi$. One can immediately check that analogous relations automatically hold for ∂ , ∂_F and ∂_χ . For $\mu \otimes v \in \wedge \mathfrak{g}^* \otimes \wedge \mathfrak{g}$, we have

$$\partial^0(\mu \otimes v) = \partial_{\mathfrak{g}}(\mu) \otimes v + \sum_i (e^i \wedge \mu) \otimes e_i \cdot v,$$

where $\{e_i\}$ is a basis of \mathfrak{g} , $\{e^i\}$ is the dual basis, and $\partial_{\mathfrak{g}}$ denotes the Lie algebra differential of \mathfrak{g} (with trivial coefficients). Using the equivariance of ρ_M , we get

$$\begin{aligned} \Psi(\partial^0(\mu \otimes v)) &= \partial_{\mathfrak{g}}(\mu) \otimes \rho_M(v) + \sum_i (e^i \wedge \mu) \otimes e_i \cdot \rho_M(v) \\ &= \partial(\mu \otimes \rho_M(v)) = \partial(\Psi(\mu \otimes v)). \end{aligned}$$

To see that $\Psi \circ \partial_F^0 = \partial_F \circ \Psi$, it suffices to check the equality on \mathfrak{g}^* , and this follows directly from the definitions. One checks that $\Psi \circ \partial_\chi^0 = \partial_\chi \circ \Psi$ similarly.

To check that $\delta^2 = 0$ on $\wedge \mathfrak{g}^*$, note that

$$\delta(\delta(\mu \otimes 1_{\mathfrak{X}})) = \delta(\delta(\Psi(\mu \otimes 1))) = \delta(\psi(\delta^0(\mu \otimes 1))) = \Psi((\delta^0)^2(\mu \otimes 1)) = 0$$

since $(\delta^0)^2 = 0$ by Lemma 5.7. This completes the proof of the lemma.

q.e.d.

Proof of Theorem 5.1. Since $\wedge \mathfrak{g}^* \otimes \mathfrak{X}(M)$ can be identified with $\Gamma(\wedge(\mathfrak{g}^* \oplus TM))$, Lemma 5.10 shows that ρ_M defines a quasi-Poisson action on (M, π) if and only if δ (given by (99)) defines a Lie algebroid structure on $A = (\mathfrak{g}^* \oplus TM)^* = \mathfrak{g} \oplus T^*M$. To complete the proof of Theorem 5.1, it remains to compute the anchor and bracket of A using (81) and (82).

Since ∂_F and ∂_χ vanish on $\mathfrak{X}(M)$, it follows that, for $f \in C^\infty(M) = \mathfrak{X}^0(M)$, we have

$$\delta f = \partial f + d_\pi f \in (\wedge \mathfrak{g}^* \otimes \mathfrak{X}(M))^1 \cong C^\infty(M, \mathfrak{g}^*) \oplus \mathfrak{X}^1(M) = \Gamma(A^*).$$

For $(v, \alpha) \in C^\infty(M, \mathfrak{g}) \oplus \Omega^1(M) = \Gamma(A)$, a direct computation shows that

$$\delta f(v, \alpha) = \mathcal{L}_{\rho_M(v)} f + \mathcal{L}_{\pi^\sharp(\alpha)} f,$$

and, using (81), we conclude that the anchor is given by (75).

We now compute brackets. For $a, b \in \Gamma(A)$, we write

$$[a, b]_A = ([a, b]_1, [a, b]_2),$$

with $[a, b]_1 \in C^\infty(M, \mathfrak{g})$ and $[a, b]_2 \in \Omega^1(M)$. The first case is when $a = (u, 0), b = (v, 0) \in \Gamma(A)$, where $u, v \in \mathfrak{g}$ (considered as constant elements in $C^\infty(M, \mathfrak{g})$). If $\xi = (\mu, 0) \in \Gamma(A^*)$, $\mu \in \mathfrak{g}^*$, then a direct computation shows that $\delta\xi(a, b) = \partial_{\mathfrak{g}}\mu(u, v) = -\mu([u, v])$. From (82), it follows that $\delta\xi(a, b) = -\mu([a, b]_1)$, so $[(u, 0), (v, 0)]_1 = [u, v]$. A similar computation shows that $[(u, 0), (v, 0)]_2 = 0$, proving (76).

Let $u \in \mathfrak{g}, \alpha \in \Omega^1(M)$ and $\mu \in \mathfrak{g}^*$, and consider $a = (u, 0), b = (0, \alpha)$ and $\xi = (\mu, 0)$. Using the definition of δ , we obtain

$$\delta\xi(a, b) = i_\alpha(\partial_F\mu(u)) = -i_\alpha\rho_M(i_\mu F(u)) = \mu(i_{\rho_M^*(\alpha)}F(u)).$$

On the other hand, by (82), we have $\delta\xi(a, b) = -\xi([a, b]) = -\mu([a, b]_1)$, which implies that $[a, b]_1 = -i_{\rho_M^*(\alpha)}F(u)$. To compute $[a, b]_2$, let $\xi = (0, X)$, where $X \in \mathfrak{X}^1(M)$. Then one checks that $\delta\xi(a, b) = \alpha([\rho_M(u), X])$. But, by (82), we have

$$\delta\xi(a, b) = \mathcal{L}_{\rho_M(u)}\alpha(X) - i_X([a, b]_2).$$

So $i_X([a, b]_2) = \mathcal{L}_{\rho_M(u)}i_X\alpha - i_{[\rho_M(u), X]}\alpha = i_X(\mathcal{L}_{\rho_M(u)}\alpha)$. This proves (77).

In order to prove (78), let $a = (0, \alpha), b = (0, \beta)$, where $\alpha, \beta \in \Omega^1(M)$. If $\xi = (\mu, 0)$, $\mu \in \mathfrak{g}^*$, then, by (82), we have $\delta\xi(a, b) = -\mu([a, b]_1)$. On the other hand,

$$\delta\xi(a, b) = \partial_\chi(\mu)(\alpha, \beta) = -i_{\alpha\wedge\beta}(\rho_M(i_\mu\chi)) = -\mu(i_{\rho_M^*(\alpha\wedge\beta)}\chi).$$

It follows that $[a, b]_1 = i_{\rho_M^*(\alpha\wedge\beta)}\chi$. To compute the second component of $[a, b]$, we let $\xi = (0, X)$, $X \in \mathfrak{X}^1(M)$. Then $\delta\xi(a, b) = d_\pi X(\alpha, \beta) = i_{\alpha\wedge\beta}([\pi, X])$. By (82),

$$(103) \quad \begin{aligned} i_X([a, b]_2) &= \mathcal{L}_{\pi^\sharp(\alpha)}i_X\beta - \mathcal{L}_{\pi^\sharp(\beta)}i_X\alpha - i_{\alpha\wedge\beta}([\pi, X]) \\ &= i_X\mathcal{L}_{\pi^\sharp(\alpha)}\beta - i_X\mathcal{L}_{\pi^\sharp(\beta)}\alpha - (i_{\alpha\wedge\beta}([\pi, X]) - i_{[\pi^\sharp(\alpha), X]}\beta + i_{[\pi^\sharp(\beta), X]}\alpha) \end{aligned}$$

On the other hand, $i_Xd(\pi(\alpha, \beta)) = \mathcal{L}_X(\pi(\alpha, \beta))$ equals to

$$\begin{aligned} &(\mathcal{L}_X\pi)(\alpha, \beta) + \pi(\mathcal{L}_X\alpha, \beta) + \pi(\alpha, \mathcal{L}_X\beta) \\ &= i_{\alpha\wedge\beta}[X, \pi] + i_{\pi^\sharp(\mathcal{L}_X\alpha)}\beta - i_{\pi^\sharp(\mathcal{L}_X\beta)}\alpha. \end{aligned}$$

Using that $\pi^\sharp(\mathcal{L}_X\alpha) = [X, \pi^\sharp(\alpha)] - [X, \pi]^\sharp(\alpha)$, it follows that

$$i_Xd(\pi(\alpha, \beta)) = i_{\alpha\wedge\beta}([\pi, X]) - i_{[\pi^\sharp(\alpha), X]}\beta + i_{[\pi^\sharp(\beta), X]}\alpha.$$

From (104), we see that $[a, b]_2 = [\alpha, \beta]_\pi$. q.e.d.

5.3. The Hamiltonian category. Let $(\mathfrak{d}, \mathfrak{g}, j)$ be a split Manin pair, so that \mathfrak{g} acquires the structure of a Lie quasi-bialgebra. Let $\rho_M : \mathfrak{g} \rightarrow \mathfrak{X}(M)$ be an action making (M, π) into a quasi-Poisson \mathfrak{g} -space. A **moment map** [3] for this quasi-Poisson action is a smooth \mathfrak{g} -equivariant map

$$J : M \longrightarrow S = D/G$$

with the property that

$$(104) \quad \pi^\sharp J^* = \rho_M \bar{\sigma},$$

where $\bar{\sigma} = \bar{\sigma}_j : T^*S \rightarrow \mathfrak{g}_S$ is defined in (70). In this case we say that (M, π, ρ_M, J) is a **Hamiltonian quasi-Poisson \mathfrak{g} -space** with respect to the split Manin pair $(\mathfrak{d}, \mathfrak{g}, j)$ (or, equivalently, the Lie quasi-bialgebra $(\mathfrak{g}, F_j, \chi_j)$). If the \mathfrak{g} -action integrates to a G -action, we refer to a Hamiltonian quasi-Poisson G -space.

Combining the equivariance of J and (104), we have that $dJ\pi^\sharp J^* = \rho\bar{\sigma} = \pi_S^\sharp$, i.e., $J_*\pi = \pi_S$.

Remark 5.11. If j' is another isotropic splitting and $t \in \wedge^2 \mathfrak{g}$ is the associated twist, we saw in Remark 5.1 that (M, π') , where $\pi' = \pi + \rho_M(t)$, is a quasi-Poisson space for $(\mathfrak{g}, F_{j'}, \chi_{j'})$. Since

$$\bar{\sigma}_{j'} = (\rho_S \circ j')^* = \bar{\sigma}_j + t^\sharp \circ \rho_S^*,$$

it follows that $(\pi')^\sharp dJ^* = (\pi^\sharp + \rho_M(t)^\sharp) dJ^* = (\rho_M \bar{\sigma}_j + \rho_M t^\sharp \rho_M^* dJ^*) = \rho_M \bar{\sigma}_{j'}$, so the fact that ρ_M is Hamiltonian is independent of the choice of isotropic splitting.

Remark 5.12. The moment map condition (104) is equivalent to the one in [3] using the notion of *admissibility*. An isotropic splitting j is called **admissible** [3, Sec. 3.4] if the restriction $\rho_S|_{\mathfrak{h}_S} : \mathfrak{h}_S \rightarrow TS$ is an isomorphism, where $\mathfrak{h} = j(\mathfrak{g}^*)$. This is equivalent to the bundle map $\bar{\sigma} : T^*S \rightarrow \mathfrak{g}_S$ being an isomorphism, in which case the moment map condition (104) can be written as

$$(105) \quad \pi^\sharp(J^* \bar{\sigma}^{-1}(v)) = \rho_M(v), \quad \forall v \in \mathfrak{g}.$$

Since admissible sections always exist locally, a quasi-Poisson action is Hamiltonian if and only if it satisfies, possibly after a twist, (105) locally, which is the original definition in [3, Def. 5.1.1].

Example 5.13. A canonical example of a Hamiltonian quasi-Poisson action is given by the dressing G -action on S , in which case the moment map is the identity $S \rightarrow S$. This restricts to Hamiltonian actions on dressing orbits, with moment maps given by the inclusion maps $\mathcal{O} \hookrightarrow S$.

We now have definitions parallel to those in Section 4.1: the **Hamiltonian category** (or *moment map theory*) associated with a split Manin pair $(\mathfrak{d}, \mathfrak{g}, j)$ is the category $\overline{\mathcal{M}}_j(\mathfrak{d}, \mathfrak{g})$ whose objects are Hamiltonian quasi-Poisson \mathfrak{g} -spaces (M, π, ρ_M, J) (with respect to the Lie quasi-bialgebra $(\mathfrak{g}, F_j, \chi_j)$) and morphisms are smooth maps $f : M \rightarrow M'$ satisfying $f_*(\pi) = \pi'$ and $J'f = J$. We also consider the subcategory $\mathcal{M}_j(\mathfrak{d}, \mathfrak{g})$ consisting of Hamiltonian quasi-Poisson spaces with transitive Lie algebroids, i.e., satisfying (80).

Example 5.14. Let us consider a Lie bialgebra as in Example 4.5, where $S = G^*$ has the Poisson structure π_{G^*} . In this case, j is the inclusion $\mathfrak{g}^* \hookrightarrow \mathfrak{d}$, and it is admissible (in the sense of Remark 5.12). As a result, the moment map condition (104) completely determines the action ρ_M , and one can check that $\overline{\mathcal{M}}_j(\mathfrak{d}, \mathfrak{g})$ is the category of Poisson maps into G^* . The subcategory $\mathcal{M}_j(\mathfrak{d}, \mathfrak{g})$ consists of Poisson maps $M \rightarrow G^*$ for which M carries a *nondegenerate* Poisson structure, i.e., M is symplectic. Comparing with the Hamiltonian categories associated with the connection s in Example 4.5, we see that $\overline{\sigma}_j = \sigma_s^{-1}$ and we have natural identifications

$$\overline{\mathcal{M}}_s(\mathfrak{d}, \mathfrak{g}) = \overline{\mathcal{M}}_j(\mathfrak{d}, \mathfrak{g}) \quad \text{and} \quad \mathcal{M}_s(\mathfrak{d}, \mathfrak{g}) = \mathcal{M}_j(\mathfrak{d}, \mathfrak{g}).$$

Example 5.15. For the split Manin pair of Example 5.6, the objects in $\overline{\mathcal{M}}_j(\mathfrak{d}, \mathfrak{g})$ are the Hamiltonian quasi-Poisson \mathfrak{g} -manifolds of [4, Sec. 2], i.e., manifolds M equipped with a \mathfrak{g} -action, an invariant bivector field π satisfying $\frac{1}{2}[\pi, \pi] = \rho_M(\chi)$ (where χ is the Cartan trivector (73)), and an equivariant map $J : M \rightarrow G$ satisfying the moment map condition (104), which reads

$$(106) \quad \pi^\sharp J^*(\alpha) = \frac{1}{2}\rho_M((\theta^R + \theta^L)\alpha^\vee),$$

where $\alpha \in \Omega^1(G)$ and $\alpha^\vee \in \mathfrak{X}(G)$ is dual to α with respect to the metric. Note that objects in $\mathcal{M}_j(\mathfrak{d}, \mathfrak{g})$ may still carry degenerate bivector fields, and the relationship between Hamiltonian spaces in $\overline{\mathcal{M}}_j(\mathfrak{d}, \mathfrak{g})$ and $\overline{\mathcal{M}}_s(\mathfrak{d}, \mathfrak{g})$ is now less evident. Nevertheless, as proven in [4, 10] (see also [2]), there is a (nontrivial) isomorphism $\overline{\mathcal{M}}_j(\mathfrak{d}, \mathfrak{g}) \cong \overline{\mathcal{M}}_s(\mathfrak{d}, \mathfrak{g})$, which will be explained and generalized in Section 6.

Just as different choices of connections give rise to gauge transformations (see Prop. 4.2), different choices of isotropic splittings j, j' define a twist $t \in \wedge^2 \mathfrak{g}$ and, following Remarks 5.1 and 5.11, the operation $\pi \mapsto \pi + \rho_M(t)$ induces a functor $\overline{\mathcal{M}}_j(\mathfrak{d}, \mathfrak{g}) \rightarrow \overline{\mathcal{M}}_{j'}(\mathfrak{d}, \mathfrak{g})$.

Proposition 5.2. *If j and j' are two isotropic splittings of \mathfrak{d} , then the associated twist $t = j - j'$ defines a natural isomorphism:*

$$(107) \quad \mathcal{I}_t : \overline{\mathcal{M}}_j(\mathfrak{d}, \mathfrak{g}) \cong \overline{\mathcal{M}}_{j'}(\mathfrak{d}, \mathfrak{g}),$$

which restricts to an isomorphism of subcategories $\mathcal{M}_j(\mathfrak{d}, \mathfrak{g}) \cong \mathcal{M}_{j'}(\mathfrak{d}, \mathfrak{g})$.

Example 5.16. The Manin pair $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{g})$ of Example 4.6 admits two natural splittings j and j' : one corresponds to the Lagrangian complement $\mathfrak{g}^* \cong \mathfrak{a} \oplus \mathfrak{n}$ and the other to $\sqrt{-1}\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$. For the splitting j one has $\chi_j = 0$, whereas, for j' , $F_{j'} = 0$. Prop. 5.2 establishes an isomorphism between the corresponding Hamiltonian categories. Notice that Hamiltonian spaces associated with j are Poisson manifolds for which the Poisson structure is generally not \mathfrak{g} -invariant, whereas for j' Hamiltonian spaces carry \mathfrak{g} -invariant bivector fields which generally fail to be Poisson.

We close this section with remarks about Hamiltonian reduction for D/G -valued moment maps in quasi-Poisson geometry. As proven in [4, Thm. 4.2.2], if (M, π) is a quasi-Poisson G -space, then conditions (66), (65) directly imply that the bracket defined by π makes the space of G -invariant functions $C^\infty(M)^G$ into a Poisson algebra. In particular, the orbit space of M by the G -action is a Poisson manifold whenever the action is free and proper.

Let us assume that we are in the Hamiltonian situation, i.e., there is a moment map $J : M \rightarrow S = D/G$.

Proposition 5.3. *Let (M, π, ρ_M, J) be a Hamiltonian quasi-Poisson G -space associated with $(\mathfrak{d}, \mathfrak{g}, j)$. Let $y \in S$ be a regular value for J , and let \mathcal{O} be the dressing orbit through y . Then the G -invariant functions on $J^{-1}(\mathcal{O})$ form a Poisson algebra. (In particular, the quotient $J^{-1}(\mathcal{O})/G$ is a Poisson manifold whenever the G -action on $J^{-1}(\mathcal{O})$ is free and proper.)*

Proof. Let $f, g \in C^\infty(J^{-1}(\mathcal{O}))^G$, and let \tilde{f}, \tilde{g} be arbitrary extensions of f and g to M . It suffices to check that $\pi(df, dg)|_{J^{-1}(\mathcal{O})}$ does not depend on the extensions. Since $d\tilde{f}(\rho_M(v)) = 0$ over $J^{-1}(\mathcal{O})$, we use the (adjoint of the) moment map condition, $dJ\pi^\sharp = -\sigma^*\rho_M^*$, to see that $\pi^\sharp(d\tilde{f})|_{J^{-1}(\mathcal{O})} \in TJ^{-1}(\mathcal{O})$. Hence if $\tilde{g}|_{J^{-1}(\mathcal{O})} \equiv 0$, we must have $\pi(d\tilde{f}, d\tilde{g})|_{J^{-1}(\mathcal{O})} \equiv 0$. The fact that the bracket $\{f, g\} := \pi(df, dg)$ is a Poisson bracket on $C^\infty(J^{-1}(\mathcal{O}))^G$ is a direct consequence of conditions (66) and (65). \square

It is immediate to check that twists (107) keep reduced spaces unchanged.

The proof of Prop. 5.3 shows that the restriction $C^\infty(M)^G \rightarrow C^\infty(J^{-1}(\mathcal{O}))^G$ is a Poisson map. If the G -action on M is free and proper, it follows that $J^{-1}(\mathcal{O})/G$ sits in M/G as a Poisson submanifold. We will see in Section 6.4 that the symplectic leaves of $J^{-1}(\mathcal{O})/G$ are the projections of the leaves of the Lie algebroid associated with (M, π) (in particular, the quotient is symplectic if the Lie algebroid of (M, π) is transitive).

6. The equivalence of Hamiltonian categories

We saw in Sections 4 and 5 two possible D/G -valued moment map theories associated with a Manin pair $(\mathfrak{d}, \mathfrak{g})$: one depends on the choice of an isotropic connection s on the G -bundle $D \rightarrow G$ and leads to Dirac geometry (and equivariant 3-forms when s is invariant), and the other depends on the choice of an isotropic splitting j of (41) and leads to quasi-Poisson geometry. In this section we will show that, when both s and j are chosen, there is an isomorphism between the corresponding Hamiltonian categories:

$$(108) \quad \overline{\mathcal{M}}_s(\mathfrak{d}, \mathfrak{g}) \xrightarrow{\sim} \overline{\mathcal{M}}_j(\mathfrak{d}, \mathfrak{g}).$$

6.1. The linear algebra of the equivalence. We now recall the linear algebra underpinning the isomorphism (108), following [2, Sec. 1].

We consider the following set-up: V and W are vector spaces, and $J : V \rightarrow W$ is a linear map. The vector space W is equipped with two transversal Dirac structures, i.e., two maximal isotropic subspaces $L_W, C_W \subset \mathbb{W} := W \oplus W^*$ such that $L_W \cap C_W = \{0\}$. In particular, $\mathbb{W} = L_W \oplus C_W$. Let us consider the following two sets of additional data:

- i) A Dirac structure L on V so that $J : V \rightarrow W$ is a strong Dirac map with respect to L_W :

$$L_W = \{(J(v), \beta) \mid (v, J^*\beta) \in L\}, \quad \ker(J) \cap L \cap V = \{0\}.$$

- ii) A bivector $\pi \in \wedge^2 V$ and a linear map $\rho_V : L_W \rightarrow V$ such that

$$(109) \quad J \circ \rho_V = \text{pr}_W, \quad \pi^\sharp \circ J^* = \rho_V \circ \overline{\sigma},$$

where $\text{pr}_W : \mathbb{W} \rightarrow W$ is the natural projection, and $\overline{\sigma} : W^* \rightarrow C_W^* \cong L_W$ is the linear map dual to $\text{pr}_W|_{C_W} : C_W \rightarrow W$.

Recall that, given a strong Dirac map $J : (V, L) \rightarrow (W, L_W)$, there is an induced linear map (see Section 2.2)

$$(110) \quad \rho_V : L_W \rightarrow V, \quad \rho_V(w, \beta) = v,$$

where v is uniquely defined by the properties $J(v) = w$, and $(v, J^*\beta) \in L$. Also, a pair of transversal Dirac structures $L, C \subset V \oplus V^*$ defines a bivector $\pi \in \wedge^2 V$ (see Section A.4) by

$$(111) \quad \pi^\sharp = \text{pr}_V \circ (\text{pr}_V|_C)^*.$$

Theorem 6.1. *Let L_W, C_W be transversal Dirac structures on W , and let $J : V \rightarrow W$ be a linear map. The following holds:*

- 1) Consider L as in i), and let $C \subset V \oplus V^*$ be the backward image of C_W by J . Then L, C are transversal Dirac structures on V , and the induced map ρ_V (110) and bivector π (111) satisfy (109).

2) Consider ρ_V and π as in ii). Then the image of the map

$$L_W \oplus V^* \rightarrow V \oplus V^*,$$

$$(l, \alpha) \mapsto (\pi^\sharp(\alpha) + \rho_V(l), J^* \text{pr}_{W^*}(l) + \alpha - J^* \text{pr}_{W^*} \rho_V^*(\alpha))$$

is a Dirac structure L on V for which $J : (V, L) \rightarrow (W, L_W)$ is a strong Dirac map.

Moreover, these constructions are inverses of each other.

A proof of Thm. 6.1 can be found in [2, Sec. 1.8].

The correspondence in Theorem 6.1 is also functorial:

Proposition 6.1. *Let $J_i : (V, L_i) \rightarrow (W, L_W)$ be a strong Dirac map, $i = 1, 2$, and let $f : V_1 \rightarrow V_2$ be a linear map which is f-Dirac and satisfies $J_1 = J_2 \circ f$. Then $f \circ \rho_{V_1} = \rho_{V_2}$ and $f_* \pi_1 = \pi_2$.*

Conversely, suppose that $\pi_i \in \wedge^2 V_i$ and $\rho_{V_i} : L_W \rightarrow V_i$ satisfy (109), $i = 1, 2$. Then if $f : V_1 \rightarrow V_2$ is such that $f_ \pi_1 = \pi_2$, $J_2 \circ f = J_1$ and $f \circ \rho_{V_1} = \rho_{V_2}$, then f is an f-Dirac map.*

Proof. For the first part, the property $f \circ \rho_{V_1} = \rho_{V_2}$ is a direct consequence of f being an f-Dirac map. Let C_i be the backward image of C_W by J_i , $i = 1, 2$. Then L_2 is the forward image of L_1 by f while C_1 is the backward image of C_2 by f , using that $J_1 = J_2 \circ f$. This implies that $f_* \pi_1 = \pi_2$, see [2, Sec. 1.8].

To prove the second part, we use that $f \circ (\pi_1)^\sharp \circ f^* = (\pi_2)^\sharp$ and $f \circ \rho_{V_1} = \rho_{V_2}$ to conclude that

$$(112) \quad (\pi_2)^\sharp(\alpha) + \rho_{V_2}(l) = f \circ (\pi_1^\sharp f^*(\alpha) + \rho_{V_1}(l)).$$

On the other hand, the conditions $J_1^* = f^* J_2$ and $\rho_{V_1}^* f^* = \rho_{V_2}^*$ imply that

$$(113) \quad f^*(J_2^* \text{pr}_{W^*}(l) + \alpha - J_2^* \text{pr}_{W^*} \rho_{V_2}^*(\alpha)) = J_1^* \text{pr}_{W^*}(l) + f^* \alpha - J_1^* \text{pr}_{W^*} \rho_{V_1}^*(f^* \alpha).$$

Note that (112) and (113) together say that L_2 is contained in the forward image of L_1 . Since both have the same dimension, they must coincide, so f is an f-Dirac map. \square

6.2. Combining splittings. Let $(\mathfrak{d}, \mathfrak{g})$ be a Manin pair, and fix splittings $s : TS \rightarrow \mathfrak{d}_S$ and $j : \mathfrak{g}^* \rightarrow \mathfrak{d}$. We denote by (\mathfrak{g}, F, χ) the Lie quasi-bialgebra determined by j . We have the induced maps

$$\sigma = \sigma_s : \mathfrak{g}_S \rightarrow T^* S, \quad \sigma = s^* \circ \iota \quad \text{and} \quad \overline{\sigma} = \overline{\sigma}_j : T^* S \rightarrow \mathfrak{g}, \quad \overline{\sigma} = j^* \circ \rho_S^*,$$

already considered in Sections 4 and 5. We have another map, which depends on both s and j , given by

$$(114) \quad \overline{\rho} = \overline{\rho}_{s,j} : TS \longrightarrow \mathfrak{g}_S, \quad \overline{\rho} = j^* \circ s.$$

We represent the two short exact sequences associated with $(\mathfrak{d}, \mathfrak{g})$ and the maps defined by the splittings in the diagram below:

(115)

$$\begin{array}{ccccc}
 & & \mathfrak{g}_S & & \\
 & \swarrow \bar{\sigma} & \downarrow \iota & \searrow \bar{\rho} & \\
 T^*S & \longrightarrow & \mathfrak{d}_S & \xleftarrow{s} & TS \\
 & & \uparrow j & & \\
 & & \mathfrak{g}_S^* & &
 \end{array}$$

Decomposing ρ_S and s (and their duals) with respect to $\mathfrak{d} \cong \mathfrak{g} \oplus \mathfrak{g}^*$, we get:

$$s = (\bar{\rho}, \sigma^*), s^* = (\sigma, \bar{\rho}^*), \rho_S = (\rho, \bar{\sigma}^*), \rho_S^* = (\bar{\sigma}, \rho^*),$$

where we always identify $\mathfrak{d} \cong \mathfrak{d}^*$ via the inner product. Using that the vertical and horizontal sequences are short exact, one obtains algebraic identities relating the various maps. In particular, we have

$$(116) \quad \sigma\bar{\rho} = -\bar{\rho}^*\sigma^*, \quad \bar{\sigma}\bar{\rho}^* = -\bar{\sigma}\bar{\rho}^*, \quad \bar{\sigma}\sigma + \bar{\rho}\rho = \text{Id}_{\mathfrak{g}}, \quad \sigma\bar{\sigma} + (\rho\bar{\rho})^* = \text{Id}_{T^*S}.$$

Since $(\rho_S, s^*) : \mathfrak{d}_S \rightarrow \mathbb{T}S$ is an isomorphism of Courant algebroids (where $\mathbb{T}S$ is equipped with the ϕ_S -twisted Courant bracket, and ϕ_S is given by (30)), we immediately obtain (besides (12)) the differential-geometric identities:

(117)

$$\mathcal{L}_{\rho(v)}(\bar{\rho}^*\mu) - i_{\bar{\sigma}^*\mu}d\sigma(v) + i_{\rho(v)\wedge\bar{\sigma}^*\mu}(\phi_S) = \sigma(i_\mu(F(v))) + \bar{\rho}^*(\text{ad}_v^*(\mu)),$$

(118)

$$\mathcal{L}_{\bar{\sigma}^*\mu}(\bar{\rho}^*\nu) - i_{\bar{\sigma}^*\nu}d(\bar{\rho}^*\mu) + i_{\bar{\sigma}^*\mu\wedge\bar{\sigma}^*\nu}(\phi_S) = \sigma(i_{\mu\wedge\nu}(\chi)) + \bar{\rho}^*(F^*(\mu, \nu)),$$

where $u, v \in \mathfrak{g}$, $\mu, \nu \in \mathfrak{g}^*$.

We know that s defines an isomorphism $\mathfrak{d}_S \cong \mathbb{T}S$, and the Dirac structure L_S on S is just \mathfrak{g} under this identification. The additional splitting j defines an isotropic complement $\mathfrak{g}^* \subset \mathfrak{d}$ to \mathfrak{g} , and we let C_S be its image in $\mathbb{T}S$:

$$(119) \quad C_S := \{((\rho_S \circ j)(\mu), (s^* \circ j)(\mu)), \mu \in \mathfrak{g}^*\}.$$

Note that C_S is an almost Dirac structure which depends both on s and j . The pair $L_S, C_S \subset \mathbb{T}S$ defines a Lie quasi-bialgebroid structure, which determines elements $\chi_S \in \Gamma(\wedge^3 L_S)$ and $F_S^* : \Gamma(C_S) \wedge \Gamma(C_S) \rightarrow \Gamma(C_S)$ (see Sec. A.3). Under the identification $\mathfrak{g}_S^* \cong C_S$, it is clear that

$$\chi_S(\mu_1, \mu_2, \mu_3) = \chi(\mu_1, \mu_2, \mu_3), \quad F_S^*(\mu_1, \mu_2) = F^*(\mu_1, \mu_2),$$

where $\mu_1, \mu_2, \mu_3 \in \mathfrak{g}^*$ (viewed as constant sections of $C_S \cong (L_S)^*$), and this completely determines F_S and χ_S . The bivector field associated with L_S, C_S (as in Sec. A.4) is just π_S (69).

6.3. The equivalence theorem. Let $(\mathfrak{d}, \mathfrak{g})$ be a Manin pair with fixed $s : TS \rightarrow \mathfrak{d}_S$ and $j : \mathfrak{g}^* \rightarrow \mathfrak{d}$, as in Section 6.2. As we saw, $S = D/G$ inherits a Dirac structure L_S and an almost Dirac complement C_S . We now use the linear construction of Section 6.1 pointwise to define the isomorphism (108).

Given a Hamiltonian quasi-Poisson \mathfrak{g} -space (M, π, ρ_M, J) (with respect to the Lie quasi-bialgebra (\mathfrak{g}, F, χ) defined by j), let us consider the maps

$$(120) \quad \widehat{\rho}_M : \mathfrak{g}_M = \mathfrak{g} \times M \longrightarrow \mathbb{T}M, \quad v \mapsto (\rho_M(v), J^* \sigma(v)),$$

$$(121) \quad h : T^*M \longrightarrow \mathbb{T}M, \quad \alpha \mapsto (\pi^\sharp(\alpha), (1 - T^*)\alpha),$$

where $T = \rho_M \overline{\rho}(dJ) : TM \rightarrow TM$. The next result generalizes [10, Thm. 3.16], using the techniques in [2].

Theorem 6.2. *The following holds:*

- i) *Let $J : (M, L) \rightarrow (S, L_S)$ be a strong Dirac map, and let $\rho_M : \mathfrak{g} \rightarrow \mathfrak{X}(M)$ be the induced \mathfrak{g} -action (as in Prop. 2.3). Then the pull-back image $C \subset \mathbb{T}M$ of C_S under J is a smooth almost Dirac structure transversal to L , and the bivector field $\pi \in \mathfrak{X}^2(M)$ associated with L and C is such that (M, π, ρ_M, J) is a Hamiltonian quasi-Poisson \mathfrak{g} -space.*
- ii) *Let (M, π, ρ_M, J) be a Hamiltonian quasi-Poisson \mathfrak{g} -space and consider the maps $\widehat{\rho}_M$ and h from (120) and (121). Then*

$$(122) \quad L := \{\widehat{\rho}_M(v) + h(\alpha) \mid v \in \mathfrak{g}_M, \alpha \in T^*M\} \subset \mathbb{T}M,$$

is a Dirac structure for which $J : (M, L) \rightarrow (S, L_S)$ is a strong Dirac map.

Moreover, one construction is the inverse of the other.

Proof. Let us prove i). The fact that the backward image C of C_S under J is smooth and transversal to L is shown in [2, Sec. 2.3]. Hence the pair $L, C \subset \mathbb{T}M$ defines a Lie quasi-bialgebroid over M . We denote the associated 3-tensor by $\chi_M \in \Gamma(\wedge^3 L)$ and cobracket by $F_M : \Gamma(L) \rightarrow \Gamma(L) \wedge \Gamma(L)$, and let d_C be the degree-1 derivation on $\Gamma(\wedge L)$ defined by $C \cong L^*$ (see Sec. A.2).

Lemma 6.1. *Let (\mathfrak{g}, F, χ) be the Lie quasi-bialgebra determined by j . Then*

$$(123) \quad \widehat{\rho}_M(\chi) = \chi_M, \quad \text{and} \quad \widehat{\rho}_M(F(v)) = -d_C(\widehat{\rho}_M(v)), \quad \forall v \in \mathfrak{g},$$

where $\widehat{\rho}_M : \wedge \mathfrak{g} \rightarrow \Gamma(\wedge L)$ is the extension of (15) to exterior algebras.

Proof of Lemma 6.1. The proof of the equation relating χ and χ_M can be found in [2, Sec. 2]. We give here an alternative argument which also proves the second equation in (123).

Let us consider the bundle map (8), $\widehat{\rho}_M : J^*L_S \rightarrow L$, induced by J , and its dual $\widehat{\rho}_M^* : C \rightarrow J^*C_S$, where we identify $L^* \cong C$ and $L_S^* \cong C_S$. It

is clear from the definitions that $(Y, \beta) \in J^*L_S$ is J -related to $\widehat{\rho}_M(Y, \beta)$; similarly, given $(X, \alpha) \in C$ at a point $x \in M$, there exists (a unique) $\mu \in \mathfrak{g}^*$ such that $\alpha = J^*s^*(\mu)$, and $\widehat{\rho}_M^*(X, \alpha) = (\rho_S(\mu), s^*(\mu))$, so (X, α) and $\widehat{\rho}_M^*(X, \alpha)$ are J -related. Given a section ζ' of C extending (X, α) , then $\widehat{\rho}_M^*(\zeta')$ is a section of $J^*C_S \cong J^*\mathfrak{g}_S^*$, but if J has locally constant rank at x , we can extend $\widehat{\rho}_M^*(\zeta')$ to a (local) section $\zeta = \mu$ of $C_S \cong \mathfrak{g}_S^*$, which is necessarily J -related to ζ' . It directly follows from Lemma 2.1 that $\chi_S(\zeta_1, \zeta_2, \zeta_3) = \chi_M(\zeta'_1, \zeta'_2, \zeta'_3)$, which means that $\widehat{\rho}_M(\chi) = \chi_M$ at x . Similarly,

$$\begin{aligned} & \widehat{\rho}_M(F(v))(\zeta'_1(x), \zeta'_2(x)) \\ &= \langle v, [\mu_1(J(x)), \mu_2(J(x))]_{\mathfrak{d}} \rangle_{\mathfrak{d}} \\ &= \langle v, [\zeta_1, \zeta_2]_{\mathfrak{d}} - \mathcal{L}_{\rho_S(\mu_1(J(x)))}\mu_2 + \mathcal{L}_{\rho_S(\mu_2(J(x)))}\mu_1 \rangle_{\mathfrak{d}} - \langle \mathcal{L}_{\rho(v)}\mu_1, \mu_2 \rangle_{\mathfrak{d}} \\ &= \langle v, [\zeta_1, \zeta_2]_{\mathfrak{d}} \rangle_{\mathfrak{d}} - \mathcal{L}_{\rho_S(\mu_1(J(x)))}(\mu_2(v)) + \mathcal{L}_{\rho_S(\mu_2(J(x)))}(\mu_1(v)), \end{aligned}$$

where we used that $\mathfrak{g}^* \subset \mathfrak{d}$ is isotropic to conclude that $\langle \mathcal{L}_{\rho(v)}\mu_1, \mu_2 \rangle_{\mathfrak{d}} = 0$. On the other hand,

$$\begin{aligned} & d_C(\widehat{\rho}_M(v))(\zeta'_1, \zeta'_2) \\ &= \mathcal{L}_{X_1} \langle \zeta'_2, \widehat{\rho}_M(v) \rangle - \mathcal{L}_{X_2} \langle \zeta'_1, \widehat{\rho}_M(v) \rangle - \langle \widehat{\rho}_M(v), [\zeta'_1, \zeta'_2]_{J^*\phi_S} \rangle, \end{aligned}$$

and the fact that $\widehat{\rho}_M(F(v)) = -d_C(\widehat{\rho}_M(v))$ at x is again a direct consequence of Lemma 2.1. Since the points $x \in M$ where J has locally constant rank forms an open, dense subset, we conclude that the equalities in (123) hold everywhere in M . q.e.d.

Let π be the bivector field on M associated with the Lie quasi-bialgebroid defined by L, C (as in Sec. A.4), and consider the \mathfrak{g} -action ρ_M induced by J . Then Lemma 6.1 and Prop. A.1 give

$$\frac{1}{2}[\pi, \pi] = \text{pr}_{TM}(\chi_M) = \text{pr}_{TM}(\widehat{\rho}_M(\chi)) = \rho_M(\chi)$$

and, for $v \in \mathfrak{g}$,

$$\mathcal{L}_{\rho_M(v)}\pi = \text{pr}_{TM}(d_C(\widehat{\rho}_M(v))) = -\text{pr}_{TM}(\widehat{\rho}(F(v))) = -\rho_M(F(v)).$$

The moment map condition and the \mathfrak{g} -equivariance of J follow from Thm. 6.1, part 1. Hence (M, π, ρ_M, J) is a Hamiltonian quasi-Poisson \mathfrak{g} -space, finishing the proof of *i*.

We now prove *ii*). Given a Hamiltonian quasi-Poisson \mathfrak{g} -space (M, π, ρ_M, J) , it follows from part 2 of Thm. 6.1 that the subbundle $L \subset TM$ defined in (122) is an almost Dirac structure, and $(dJ)_x : \mathbb{T}M_x \rightarrow \mathbb{T}S_{J(x)}$ is a strong Dirac map relative to L and L_S at all $x \in M$. We let C be the almost Dirac structure given by the pullback image of C_S under J . Then L and C are transversal almost Dirac structures (see e.g. [2, Sec. 1.7]). A direct computation shows that the bundle map h of (121) agrees with the dual of $\text{pr}_{TM}|_C : C \rightarrow TM$,

$$(124) \quad h = (\text{pr}_{TM}|_C)^* : T^*M \rightarrow C^* \cong L,$$

and hence the bivector field associated with L, C , defined by $(\text{pr}_{TM}|_L) \circ (\text{pr}_{TM}|_C)^* : T^*M \rightarrow TM$, is π . To prove *ii*), it remains to check that L is a $J^*\phi_S$ -twisted Dirac structure, i.e., that the associated 3-tensor $\chi'_M \in \Gamma(\wedge^3 C)$, $\chi'_M(l_1, l_2, l_3) = \langle [\![l_1, l_2]\!]_{J^*\phi_S}, l_3 \rangle$, vanishes for all $l_1, l_2, l_3 \in \Gamma(L)$.

Lemma 6.2. *Let $[\![\cdot, \cdot]\!]$ denote the $J^*\phi_S$ -twisted Courant bracket on $\mathbb{T}M$. Then*

$$(125) \quad [\![\widehat{\rho}_M(u), \widehat{\rho}_M(v)]\!] = \widehat{\rho}_M([u, v]), \quad \text{for } u, v \in \mathfrak{g}.$$

$$(126) \quad \langle [\![h(\alpha_1), h(\alpha_2)]\!], h(\alpha_3) \rangle = 0, \quad \text{for } \alpha_i \in \Omega^1(M), i = 1, 2, 3.$$

$$(127) \quad [\![\widehat{\rho}_M(v), h(\alpha)]\!] = -\widehat{\rho}_M(i_{\rho_M^*(\alpha)} F(v)) + h(\mathcal{L}_{\rho_M(v)} \alpha), \\ \text{for } \alpha \in \Omega^1(M), v \in \mathfrak{g}.$$

Proof of Lemma 6.2. To prove (125), we have to show that

$$J^*\sigma([u, v]) = \mathcal{L}_{\rho_M(u)}(J^*\sigma(v)) - i_{\rho_M(v)} d(J^*\sigma(u)) + i_{\rho_M(u) \wedge \rho_M(v)} (J^*\phi_S).$$

Using the equivariance of J , $dJ \circ \rho_M = \rho$, we see that this equation is just the pull-back by J of condition (12) for σ .

To prove (126), we use (124) to see that (126) is equivalent to the condition $\text{pr}_{TM}(\chi'_M) = 0$. A computation as in Prop. A.1 (see [2, Sec. 2] for an alternative argument) shows that the bivector field associated with the transversal almost Dirac structures L and C , which is just π , satisfies $\frac{1}{2}[\pi, \pi] = \text{pr}_{TM}(\chi_M) + \text{pr}_{TM}(\chi'_M)$. It follows that $\text{pr}_{TM}(\chi'_M) = 0$ since, by assumption, $\frac{1}{2}[\pi, \pi] = \rho_M(\chi) = \text{pr}_{TM}(\chi_M)$.

We now prove equation (127). The TM -component of this equation gives

$$(128) \quad \mathcal{L}_{\rho_M(v)} \pi^\sharp(\alpha) = \rho_M(-i_{\rho_M^*(\alpha)} F(v)) + \pi^\sharp(\mathcal{L}_{\rho_M(v)} \alpha).$$

Using the condition $\mathcal{L}_{\rho_M(v)} \pi = -\rho_M(F(v))$, we see that the identity

$$\mathcal{L}_{\rho_M(v)}(\pi(\alpha, \beta)) = (\mathcal{L}_{\rho_M(v)} \pi)(\alpha, \beta) + \pi(\mathcal{L}_{\rho_M(v)} \alpha, \beta) + \pi(\alpha, \mathcal{L}_{\rho_M(v)} \beta),$$

can be re-written as

$$\mathcal{L}_{\rho_M(v)}(\pi(\alpha, \beta)) = -i_\beta \rho_M(i_{\rho_M^*(\alpha)} F(v)) + i_\beta \pi^\sharp(\mathcal{L}_{\rho_M(v)} \alpha) + i_{\pi^\sharp(\alpha)} \mathcal{L}_{\rho_M(v)} \beta.$$

Using the identity $i_{\pi^\sharp(\alpha)} \mathcal{L}_{\rho_M(v)} \beta = -i_\beta \mathcal{L}_{\rho_M(v)} \pi^\sharp(\alpha) + \mathcal{L}_{\rho_M(v)} i_\beta \pi^\sharp(\alpha)$ (which is an application of the general identity $i_{[X, Y]} = \mathcal{L}_X i_Y - i_Y \mathcal{L}_X$), equation (128) immediately follows.

The T^*M -component of equation (127) is equivalent to

$$(129) \quad \begin{aligned} & T^* \mathcal{L}_{\rho_M(v)}(\alpha) - \mathcal{L}_{\rho_M(v)}(T^* \alpha) \\ &= i_{\pi^\sharp(\alpha)} J^* d(\sigma(v)) - J^* \sigma(i_{\rho_M^*(\alpha)} F(v)) - i_{\rho_M(v) \wedge \pi^\sharp(\alpha)} (J^* \phi_S) \\ &= J^*(-i_{\bar{\sigma}^*\mu} d\sigma(v) - \sigma(i_\mu F(v)) + i_{\rho(v) \wedge \bar{\sigma}\mu} \phi_S), \end{aligned}$$

where, for the second equality, we used the \mathfrak{g} -equivariance of ρ_M , the moment map condition $dJ\pi^\sharp = -\bar{\sigma}^*\rho_M^*$ and the notation $\mu = \rho_M^*(\alpha) \in C^\infty(M, \mathfrak{g}^*)$.

Evaluating the left-hand side of (129) on a vector field $X \in \mathfrak{X}(M)$, we obtain

$$\begin{aligned} & -\langle \alpha, [\rho_M(v), T(X)] \rangle + \langle \alpha, T([\rho_M(v), X]) \rangle \\ &= -\langle \alpha, \rho_M([v, \bar{\rho}dJ(X)] + \mathcal{L}_{\rho(v)}(\bar{\rho}dJ(X))) \rangle + \langle \mu, \bar{\rho}dJ([\rho_M(v), X]) \rangle, \end{aligned}$$

where we have used that ρ_M preserves the Lie algebroid bracket on \mathfrak{g}_S . So, at each point, (129) evaluated at X becomes:

$$\begin{aligned} & \langle \mu, -[v, \bar{\rho}(dJ(X))]_{\mathfrak{g}} - \mathcal{L}_{\rho(v)}\bar{\rho}dJ(X) + \bar{\rho}(dJ([\rho_M(v), X])) \rangle \\ &= \langle -i_{\bar{\sigma}^*\mu}d\sigma(v) - \sigma(i_\mu F(v)) + i_{\rho(v)\wedge\bar{\sigma}(\mu)}\phi_S, dJ(X) \rangle. \end{aligned}$$

Since this equation makes sense for all μ and is $C^\infty(M)$ -linear on μ , it suffices to assume $\mu \in \mathfrak{g}^*$ to be constant in order to prove this identity. Using (117), the identity to be proven becomes:

$$\begin{aligned} (130) \quad & \langle \mu, -[v, \bar{\rho}dJ(X)]_{\mathfrak{g}} - \mathcal{L}_{\rho(v)}\bar{\rho}dJ(X) + \bar{\rho}dJ([\rho_M(v), X]) \rangle \\ &= \langle \bar{\rho}^*\text{ad}_v^*\mu - \mathcal{L}_{\rho(v)}(\bar{\rho}^*(\mu)), dJ(X) \rangle. \end{aligned}$$

Let us consider the left-hand side of (130). Noticing that $\bar{\rho} \in \Omega^1(S, \mathfrak{g})$, we have the identity

$$J^*\bar{\rho}([\rho_M(v), X]) = \mathcal{L}_{\rho_M(v)}J^*\bar{\rho}(X) - \mathcal{L}_XJ^*\bar{\rho}(\rho_M(v)) - d(J^*\bar{\rho})(\rho_M(v), X).$$

Using that $dJ(\rho_M(v)) = \rho(v)$, it follows from this identity that the left-hand side of (130) can be re-written as

$$\langle \mu, -[v, \bar{\rho}dJ(X)]_{\mathfrak{g}} - \mathcal{L}_{dJ(X)}\bar{\rho}(\rho(v)) - d\bar{\rho}(\rho(v), dJ(X)) \rangle,$$

from where it becomes clear that it depends on $dJ(X)$ only pointwise, not locally. In particular, it makes sense to replace $dJ(X)$ by an arbitrary vector field V on S . So in order to prove (130), it suffices to prove the identity

$$(131) \quad \langle \mu, -[v, \bar{\rho}(V)]_{\mathfrak{g}} - \mathcal{L}_V\bar{\rho}(\rho(v)) - d\bar{\rho}(\rho(v), V) \rangle = \langle \bar{\rho}^*\text{ad}_v^*\mu - \mathcal{L}_{\rho(v)}(\bar{\rho}^*\mu), V \rangle,$$

for all $V \in \mathfrak{X}(S)$. Now note that $\langle \bar{\rho}^*\text{ad}_v^*(\mu), V \rangle = \langle \mu, [\bar{\rho}(V), v]_{\mathfrak{g}} \rangle$ and

$$\begin{aligned} (132) \quad \langle \mathcal{L}_{\rho(v)}(\bar{\rho}^*\mu), V \rangle &= \mathcal{L}_{\rho(v)}\langle \mu, \bar{\rho}(V) \rangle - \langle \bar{\rho}^*(\mu), [\rho(v), V] \rangle \\ &= \langle \mu, \mathcal{L}_{\rho(v)}\bar{\rho}(V) - \bar{\rho}([\rho(v), V]) \rangle \\ &= \langle \mu, \mathcal{L}_V\bar{\rho}(\rho(v)) + d\bar{\rho}(\rho(v), V) \rangle, \end{aligned}$$

where for the last equality we used that $d\bar{\rho}(U, V) = \mathcal{L}_U\bar{\rho}(V) - \mathcal{L}_V\bar{\rho}(U) - \bar{\rho}([U, V])$. Now (131) follows directly. q.e.d.

To conclude that L is integrable with respect to the $J^*\phi_S$ -twisted Courant bracket, we must check that $\chi'_M(l_1, l_2, l_3) = \langle [l_1, l_2], l_3 \rangle$ vanishes for all $l_1, l_2, l_3 \in \Gamma(L)$. Clearly, it suffices to check this condition when each l_i is of the form $\widehat{\rho}(v_i)$ or $h(\alpha_i)$, for $v_i \in \mathfrak{g}$ and $\alpha_i \in T^*M$.

From (125), we obtain that $\langle [l_1, l_2], l_3 \rangle = 0$ if any two of the l_i 's are of the form $\widehat{\rho}_M(v_i)$. Equation (126) gives $\langle [l_1, l_2], l_3 \rangle = 0$ when each l_i is of the form $h(\alpha_i)$. The case where only two of the l_i 's are of type $h(\alpha_i)$ follows from (127). This concludes the proof of part *ii*) of Thm. 6.2.

q.e.d.

The constructions in parts *i*) and *ii*) of Thm. 6.2 are functorial as a consequence of Prop. 6.1. In particular, Thm. 6.2, part *i*), defines a functor

$$(133) \quad \mathcal{I} : \overline{\mathcal{M}}_s(\mathfrak{d}, \mathfrak{g}) \rightarrow \overline{\mathcal{M}}_j(\mathfrak{d}, \mathfrak{g}),$$

which establishes the desired isomorphism of Hamiltonian categories; its inverse is given by the functor $\overline{\mathcal{M}}_j(\mathfrak{d}, \mathfrak{g}) \rightarrow \overline{\mathcal{M}}_s(\mathfrak{d}, \mathfrak{g})$ constructed in part *ii*).

We have the following characterization of the quasi-Poisson bivector field π constructed in Thm. 6.2, part *i*) (c.f. [10, Prop. 3.20]):

Corollary 6.1. *Let $J : (M, L) \rightarrow (S, L_S)$ be a strong Dirac map, and let $\rho_M : \mathfrak{g} \rightarrow \mathfrak{X}(M)$ be the induced \mathfrak{g} -action. The associated quasi-Poisson bivector field π is uniquely determined by the following conditions: given $\alpha \in T^*M$, then*

$$(134) \quad dJ(\pi^\sharp(\alpha)) = -\bar{\sigma}^* \rho_M^*(\alpha), \quad (\pi^\sharp(\alpha), (\text{Id} - T^*)\alpha) \in L.$$

Proof. The first condition in (134) is just (the dual of) the moment map condition for the quasi-Poisson action, whereas the second condition is just saying that L contains the image of h given by (121). These conditions uniquely define $\pi^\sharp(\alpha)$ as a direct consequence of $\ker(dJ) \cap L \cap TM = \{0\}$.
q.e.d.

6.4. Properties and examples. We keep considering a Manin pair $(\mathfrak{d}, \mathfrak{g})$ together with the choice of splittings s and j . We now discuss several properties of the functor \mathcal{I} given in (133).

Foliations. Given a Hamiltonian quasi-Poisson \mathfrak{g} -space (M, π, ρ_M, J) , its associated Dirac structure L is given by (122). The presymplectic foliation of L is tangent to the distribution

$$\text{pr}_{TM}(L) = \{\rho_M(v) + \pi^\sharp(\alpha), v \in \mathfrak{g}, \alpha \in T^*M\} = \text{Im}(r),$$

where r , given in (75), is the anchor of the Lie algebroid associated with the quasi-Poisson action. In other words, the presymplectic foliation of (M, L) coincides with the orbit foliation of the Lie algebroid of the quasi-Poisson structure. In particular, the functor \mathcal{I} takes presymplectic realizations to quasi-Poisson spaces with transitive Lie algebroids:

Corollary 6.2. *The functor \mathcal{I} restricts to an isomorphism of subcategories*

$$\mathcal{I} : \mathcal{M}_s(\mathfrak{d}, \mathfrak{g}) \xrightarrow{\sim} \mathcal{M}_j(\mathfrak{d}, \mathfrak{g}).$$

Example 6.3. We saw that S has a bivector field π_S , $\pi_S^\sharp = \rho\bar{\sigma}_j$ (depending on j) which makes it into a Hamiltonian quasi-Poisson \mathfrak{g} -space with respect to the dressing action and with $J = \text{Id}$ as moment map. It is easy to check from the explicit formula (122) that the associated Dirac structure is just L_S defined in (14). Moreover, the functor \mathcal{I} takes each dressing orbit $(\mathcal{O}, \omega_{\mathcal{O}})$, viewed as a presymplectic leaf of L_S , to $(\mathcal{O}, \pi_{\mathcal{O}})$, where $\pi_{\mathcal{O}}$ is the restriction of π_S to \mathcal{O} .

Trivial equivalences. Given a Hamiltonian quasi-Poisson \mathfrak{g} -space (M, π, ρ_M, J) , it may happen that the graph of π already defines a Dirac structure, in such a way that the functor \mathcal{I} is just the identity. From (122), we see that this is the case if and only if the following two conditions hold:

$$(135) \quad \pi^\sharp \circ (J^* \sigma) = \rho_M, \quad \text{and} \quad \pi^\sharp \circ T^* = 0.$$

Example 6.4. Let us consider the G^* -valued moment maps of Example 4.5. By (55) and (72), we know that $\sigma_s = \bar{\sigma}_j^{-1}$, so the moment map condition (104) is exactly the first equation in (135). In this example, $\bar{\rho} = 0$ (hence $T = 0$), so the second condition in (135) is also fulfilled. So the functor \mathcal{I} produces no changes on the geometrical structures, as already remarked in Example 5.14.

Note that the conditions in (135) do *not* hold in the case of G -valued moment maps; in this case, the functor \mathcal{I} is nontrivial, and the correspondence it establishes recovers [4, Thm. 10.3] and [10, Thm. 3.16].

Dependence on splittings. The functor \mathcal{I} is determined by the splittings s and j , and we write $\mathcal{I}^{(s,j)}$ to make this dependence explicit. Let s' be another connection splitting, and consider the 2-form $B \in \Omega^2(S)$, defined in (44), and the associated gauge transformation functor \mathcal{I}_B of Prop. 4.2. Similarly, given another splitting j' , let $t \in \wedge^2 \mathfrak{g}$ be the associated twist: $t^\sharp = j - j' : \mathfrak{g}^* \rightarrow \mathfrak{g}$. Then we have the functor \mathcal{I}_t of Prop. 5.2.

Proposition 6.2. *The dependence of the functor \mathcal{I} on the choice of splittings is as follows:*

$$\mathcal{I}^{(s,j)} = \mathcal{I}^{(s',j)} \circ \mathcal{I}_B, \quad \mathcal{I}^{(s,j')} = \mathcal{I}_t \circ \mathcal{I}^{(s,j)}.$$

Proof. It follows from the definition of B that $\sigma^s - \sigma^{s'} = -i_{\rho_S(v)} B$, hence

$$L_S^{s'} = \tau_B(L_S^s), \quad \text{and} \quad C_S^{s',j} = \tau_B(C_S^{s,j}).$$

On the other hand, the pull-back images of $C_S^{s,j}$ and $\tau_B(C_S^{s,j})$ under J , denoted by C and C' , satisfy $C' = \tau_{J^* B}(C)$. A direct computation

shows that the bivector field associated with L, C is the same as the bivector field associated with $\tau_{J^*B}(L), \tau_{J^*B}(C)$, and this proves that $\mathcal{I}^{(s,j)} = \mathcal{I}^{(s',j)} \circ \mathcal{I}_B$.

For the second identity, we note that if $t \in \wedge^2 \mathfrak{g} \cong \wedge^2 L_S$ is a twist relating $C_S^{s,j}$ and $C_S^{s,j'}$, then the twist relating their pull-back images under J is $\widehat{\rho}_M(t)$. The result now follows from part 3 of Prop. A.2.

q.e.d.

Hamiltonian vector fields and reduction. We now discuss the behavior of Hamiltonian vector fields and reduced spaces under the equivalence functor \mathcal{I} .

Given a Dirac manifold (M, L) , we call a smooth function f on M **admissible** [16] if there exists a vector field $X \in \mathfrak{X}(M)$ such that $(X, df) \in L$. In this case X is a **Hamiltonian vector field** for f , though X is not uniquely defined by this property in general. The set of admissible functions is a Poisson algebra, with Poisson bracket $\{f, g\}_L := \mathcal{L}_{X_f} g$, where X_f is any Hamiltonian vector field for f . We now consider Hamiltonian spaces in $\overline{\mathcal{M}}_s(\mathfrak{d}, \mathfrak{g})$.

Proposition 6.3. *Let $J : (M, L) \rightarrow (S, L_S)$ be a strong Dirac map, let ρ_M be the induced \mathfrak{g} -action. Then any \mathfrak{g} -invariant function f is admissible and has a distinguished Hamiltonian vector field X_f uniquely determined by the extra condition $dJ(X_f) = 0$. In particular, $C^\infty(M)^\mathfrak{g}$ is a Poisson algebra.*

Proof. Using the isotropic splitting j of $(\mathfrak{d}, \mathfrak{g})$, let π be the quasi-Poisson bivector associated with L and j via \mathcal{I} . If f is \mathfrak{g} -invariant, then $T^*df = 0$, so the vector field $X_f := \pi^\sharp(df) \in \mathfrak{X}(M)$ satisfies $h(df) = (X_f, df) \in L$ (where h is defined in (121)), i.e., it is a Hamiltonian vector field. Also, $dJ(X_f) = -\bar{\sigma}^* \rho_M^*(df) = 0$. Finally, note that there is at most one vector field with these properties, since $\ker(L) \cap \ker(dJ) = \{0\}$ (in particular, X_f is *independent* of the splitting j defining π). If f and g are \mathfrak{g} -invariant, property (65) for the quasi-Poisson bivector field π directly implies that the function $\{f, g\}_L = \mathcal{L}_{X_f} g = \pi(df, dg)$ is again \mathfrak{g} -invariant, so $C^\infty(M)^\mathfrak{g}$ is a Poisson algebra. q.e.d.

It immediately follows from the previous proof that the Poisson algebra of Prop. 6.3 (using Dirac geometry) agrees with the one of Sec. 5.3 (using quasi-Poisson geometry). The previous proposition recovers [5, Prop. 4.6] in the case of G -valued moment maps.

As we have discussed, one can perform moment map reduction either in the framework of Hamiltonian quasi-Poisson spaces or Dirac geometry. We observe that the functor \mathcal{I} preserves the reduction procedures:

Proposition 6.4. *The functor \mathcal{I} commutes with moment map reduction.*

Proof. Let (M, π, ρ_M, J) be the Hamiltonian quasi-Poisson G -space associated with a strong Dirac map J via \mathcal{I} . Let us fix a dressing orbit \mathcal{O} in S , and a point $y \in \mathcal{O}$ which is regular for J . As we saw in Prop. 5.3, if $f, g \in C^\infty(J^{-1}(\mathcal{O}))^G$, then we have a well-defined Poisson bracket $\{f, g\}_\pi := \pi(d\tilde{f}, d\tilde{g})|_{J^{-1}(\mathcal{O})}$, independent of the extensions \tilde{f}, \tilde{g} of f and g . Since $\pi^\sharp(d\tilde{f})|_{J^{-1}(\mathcal{O})}$ does not depend on the extension \tilde{f} of f and lies in the kernel of dJ , it gives a well-defined vector field X_f on $J^{-1}(\mathcal{O})$ (which is tangent to $J^{-1}(y)$).

Suppose that the isotropy subgroup of y , denoted by G_y , acts freely and properly on $J^{-1}(y)$. We have a natural identification $J^{-1}(\mathcal{O})/G \cong J^{-1}(y)/G_y$, which gives an identification of $C^\infty(J^{-1}(\mathcal{O}))^G$ with $C^\infty(J^{-1}(y))^{G_y}$. If L' denotes the Dirac structure on $J^{-1}(y)$ given by the pull-back image of L under the inclusion $\iota : J^{-1}(y) \hookrightarrow M$, then the Poisson structure on $J^{-1}(y)/G_y$ given in Prop. 4.1 is defined by the identification of $C^\infty(J^{-1}(y))^{G_y}$ with admissible functions of L' [10, Sec. 4.4]. Let $f \in C^\infty(J^{-1}(y))^{G_y} \cong C^\infty(J^{-1}(\mathcal{O}))^G$ and \tilde{f} be any local extension of f to M . Since \tilde{f} is \mathfrak{g} -invariant at each point on $J^{-1}(\mathcal{O})$, it follows that $(\pi^\sharp(d\tilde{f}), d\tilde{f}) \in L$ over $J^{-1}(\mathcal{O})$. By definition of backward image, it directly follows that $X_f = \pi^\sharp(d\tilde{f})|_{J^{-1}(\mathcal{O})}$ (which is tangent to $J^{-1}(y)$) satisfies $(X_f, df = \iota^*d\tilde{f}) \in L'$. Hence X_f is a Hamiltonian vector field for f with respect to L' . By definition, we have

$$\{f, g\}_{L'} = X_f \cdot g = \{f, g\}_\pi,$$

which shows that we get the same reduced Poisson structure by using Dirac reduction or quasi-Poisson reduction q.e.d.

For G -valued moment maps with transitive Lie algebroids, Prop. 6.4 recovers [4, Prop. 10.6]. Using Prop. 4.1, part *iii*), Cor. 6.2 and Prop. 6.4, we see that the reduction of quasi-Poisson spaces with transitive Lie algebroids is symplectic.

The double (D, ω_D) . Let us consider the Lie group D equipped with the 2-form $\omega_D = \omega_D^s$ given by (28). As proven in Thm. 4.1, $(p, \bar{p}) : D \rightarrow S \times S$ is a strong Dirac map (i.e., a presymplectic realization), where $S \times S$ is equipped with the product Dirac structure $L_S \times L_S$. The choice of splitting j of $(\mathfrak{d}, \mathfrak{g})$ induces a splitting $j \times j$ of $(\mathfrak{d} \times \mathfrak{d}, \mathfrak{g} \times \mathfrak{g})$, and we know from Thm. 6.2 that there is an associated bivector field making D into a Hamiltonian quasi-Poisson $\mathfrak{g} \times \mathfrak{g}$ -space with moment map $J = (p, \bar{p})$. We consider the maps $\bar{\sigma} : T^*(S \times S) \rightarrow \mathfrak{g} \times \mathfrak{g}$ and $\bar{\rho} : T(S \times S) \rightarrow \mathfrak{g} \times \mathfrak{g}$ (as in Sec. 6.2) associated with the Manin pair $(\mathfrak{d} \times \mathfrak{d}, \mathfrak{g} \times \mathfrak{g})$ and the splittings $s \times s$ and $j \times j$.

Let us consider the bivector field $\pi_D^j = \pi_D \in \mathfrak{X}^2(D)$, depending on j , given by

$$(136) \quad \pi_D(\alpha, \beta) := \langle \alpha^\vee, \beta^\vee \rangle_{\mathfrak{d}} - (\mathfrak{r}^r + \mathfrak{r}^l)(\alpha, \beta),$$

where $\alpha, \beta \in \Omega^1(D)$, $\alpha^\vee, \beta^\vee \in \mathfrak{X}(D)$ are the dual vector fields via $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$ and $\mathfrak{r} \in \mathfrak{d} \otimes \mathfrak{d}$ is the r -matrix of Remark 5.4. Note that the skew symmetry of π_D follows from (71).

Proposition 6.5. *The quasi-Poisson bivector field corresponding to ω_D via \mathcal{I} is π_D .*

Proof. We must show that the two conditions in (134) hold, i.e.,

$$(137) \quad (dp, d\bar{p})_a(\pi^\sharp(\alpha)) = -\bar{\sigma}^* \rho_D^*(\alpha),$$

$$(138) \quad i_{\pi^\sharp(\alpha)} \omega_D = (1 - T^* \alpha),$$

$\forall \alpha \in T_a D$, where $\rho_D(u, v) = u^r - v^l$, $u, v \in \mathfrak{g}$, and $T = \rho_D \circ \bar{\rho} \circ (dp, d\bar{p}) : TD \rightarrow TD$.

It suffices to prove the equations for $\alpha = (w^r)^\vee = \langle w, \theta_D^R \rangle_{\mathfrak{d}}$, for $w \in \mathfrak{d}$. Let us start with the r.h.s. of (137). To simplify the notation, we always identify $\mathfrak{d} \cong \mathfrak{d}^*$ via $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$. A direct computation shows that

$$(139) \quad \rho_D^*(\alpha_a) = (\iota^*(w), -\iota^*(\text{Ad}_{a^{-1}}(w))), \quad a \in D,$$

where ι^* is an in Section 5.1. Using that $\rho_S = dp \circ dr_a$, we find

$$(140) \quad -\bar{\sigma}^* \rho_D^*(\alpha_a) = (-dp(r_a j \iota^*(w)), dp(r_a j \iota^*(\text{Ad}_{a^{-1}}(w)))),$$

where $\alpha_a = (dr_a(w))^\vee$. (We may use r_a, l_a for dr_a, dl_a in order to simplify the notation.) On the other hand, a direct computation using the definition of π_D gives:

$$(141) \quad \begin{aligned} \pi^\sharp(\alpha_a) &= r_a(w) - l_a j \iota^*(\text{Ad}_{a^{-1}}(w)) - r_a j \iota^* w \\ &= l_a(\iota j^* \text{Ad}_{a^{-1}}(w)) - r_a j \iota^* w, \end{aligned}$$

where we used that $\iota j^* + j \iota^* = 1$. Since $dp_a(dl_a(u)) = 0$ and $dp_a(dr_a(u)) = \rho_S(u)$ if $u \in \mathfrak{g}$, we obtain $dp_a(\pi^\sharp(\alpha_a)) = -dp_a(r_a j \iota^*(w))$. Similarly, one checks that $d\bar{p}_a(\pi^\sharp(\alpha_a)) = dp_a(r_a j \iota^*(\text{Ad}_{a^{-1}}(w)))$. Comparing with (140), (137) follows.

In order to prove (138), it suffices to show that

$$(142) \quad \omega_D(\pi^\sharp(\alpha_a), X_a) = \alpha_a(X_a) - \alpha_a(TX_a),$$

for $\alpha_a = (dr_a(w))^\vee$ and $X_a = dr_a(v)$, where $w, v \in \mathfrak{d}$. Using the identity $j \iota^* = 1 - \iota j^*$ in (141), we get:

$$(143) \quad \pi_D^\sharp(\alpha_a) = l_a(\iota j^* \text{Ad}_{a^{-1}}(w)) - r_a j \iota^* w = r_a \iota j^* w - l_a j \iota^* \text{Ad}_{a^{-1}}(w).$$

Using (50), we find

$$\begin{aligned} \omega_D(\pi^\sharp(\alpha_a), X_a) &= \left\langle -l_a \theta_a(X_a) + r_a(\theta_{a^{-1}} \text{Inv}(X_a)) + X_a, \pi_D^\sharp(\alpha_a) \right\rangle_{\mathfrak{d}} \\ &= -\left\langle l_a \theta_a(r_a(v)), \pi_D^\sharp(\alpha_a) \right\rangle_{\mathfrak{d}} - \left\langle r_a \theta_{a^{-1}} l_a v, \pi_D^\sharp(\alpha_a) \right\rangle_{\mathfrak{d}} \\ &\quad + \left\langle r_a(v), \pi_D^\sharp(\alpha_a) \right\rangle_{\mathfrak{d}}, \end{aligned}$$

where we have used that $\text{Inv}(r_a(v)) = -l_{a^{-1}}v$. Using (141) and that θ is isotropic, we have

$$\begin{aligned} \left\langle l_a \theta_a(r_a(v)), \pi_D^\sharp(\alpha_a) \right\rangle_{\mathfrak{d}} &= -\langle l_a \theta_a(r_a(v)), r_a j \iota^* w \rangle_{\mathfrak{d}} \\ &= -\langle \text{Ad}_a \theta_a(r_a(v)), j \iota^* w \rangle_{\mathfrak{d}}, \end{aligned}$$

and, using (143), we similarly obtain

$$\left\langle r_a \theta_{a^{-1}} l_{a^{-1}}(v), \pi_D^\sharp(\alpha_a) \right\rangle_{\mathfrak{d}} = -\langle \text{Ad}_{a^{-1}} \theta_{a^{-1}} l_{a^{-1}}(v), j \iota^* \text{Ad}_{a^{-1}}(w) \rangle_{\mathfrak{d}}.$$

Using (141), we get

$$\left\langle r_a(v), \pi_D^\sharp(\alpha_a) \right\rangle_{\mathfrak{d}} = -\langle w, \iota j^* v \rangle_{\mathfrak{d}} - \langle \iota j^* \text{Ad}_{a^{-1}}(v), \text{Ad}_{a^{-1}}(w) \rangle_{\mathfrak{d}} + \langle v, w \rangle_{\mathfrak{d}}.$$

Combining the last three equations, we find that $\omega_D(\pi_D^\sharp(\alpha_a), X_a)$ equals

$$(144) \quad \begin{aligned} &\langle \iota j^* \text{Ad}_a \theta_a(r_a(v)), w \rangle_{\mathfrak{d}} + \langle \iota j^* \text{Ad}_{a^{-1}} \theta_{a^{-1}} l_{a^{-1}}(v), \text{Ad}_{a^{-1}}(w) \rangle_{\mathfrak{d}} \\ &- \langle w, \iota j^* v \rangle_{\mathfrak{d}} - \langle \iota j^* \text{Ad}_{a^{-1}}(v), \text{Ad}_{a^{-1}}(w) \rangle_{\mathfrak{d}} + \langle v, w \rangle_{\mathfrak{d}}. \end{aligned}$$

We now consider the r.h.s. of (142). Using that $d\bar{p}(r_a(v)) = -dp(l_{a^{-1}}(v))$, we see that $\bar{p} \circ dJ(X_a) = (j^* s, j^* s) \circ (dp, d\bar{p})(r_a(v)) = (j^* sdp(dr_a(v)), -j^* sdp(dl_{a^{-1}}(v)))$, so

$$T(X_a) = \rho_D(\bar{p} \circ dJ(X_a)) = r_a j^* sdp(r_a(v)) + l_a j^* sdp(l_{a^{-1}}(v)).$$

Using (25) to express s in terms of θ , we get

$$(145) \quad \begin{aligned} &\alpha_a(T(X_a)) \\ &= (w, j^* sdp(r_a(v)) + \text{Ad}_a(j^* s_{a^{-1}} dp(l_{a^{-1}}(v)))) \\ &= \langle w, j^* v \rangle_{\mathfrak{d}} - \langle w, j^* \text{Ad}_a \theta_a r_a(v) \rangle_{\mathfrak{d}} + \langle \text{Ad}_{a^{-1}} w, j^* \text{Ad}_{a^{-1}} v \rangle_{\mathfrak{d}} \\ &\quad - \langle \text{Ad}_{a^{-1}} w, j^* \text{Ad}_{a^{-1}} \theta_{a^{-1}} l_{a^{-1}}(v) \rangle_{\mathfrak{d}}. \end{aligned}$$

Using that $\alpha_a(X_a) = \langle w, v \rangle_{\mathfrak{d}}$ and (145), we see that the r.h.s of (142) agrees with (144), and this concludes the proof. q.e.d.

In the case of G -valued moment maps, π_D recovers the quasi-Poisson structure on $G \times G$ of [4, Ex. 5.3], and the previous proposition generalizes [4, Ex. 10.5].

A result analogous to Prop. 6.5, relating the presymplectic structure on the Lie groupoid $G \times S$ (integrating L_S) to quasi-Poisson bivectors is discussed in [13].

Appendix A. Appendix

A.1. Courant algebroids and Dirac structures. A **Courant algebroid** [26] over a manifold M is a (real) vector bundle $E \rightarrow M$ equipped with the following structure: a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on the bundle, a bundle map $\rho_E : E \rightarrow TM$ (called the *anchor*) and a bilinear bracket $[\![\cdot, \cdot]\!]$ on $\Gamma(E)$, so that the following axioms are satisfied:

- C1) $\llbracket e_1, \llbracket e_2, e_3 \rrbracket \rrbracket = \llbracket \llbracket e_1, e_2 \rrbracket, e_3 \rrbracket + \llbracket e_2, \llbracket e_1, e_3 \rrbracket \rrbracket, \quad \forall e_1, e_2, e_3 \in \Gamma(E);$
- C2) $\llbracket e, e \rrbracket = \frac{1}{2} \mathcal{D} \langle e, e \rangle, \quad \forall e \in \Gamma(E),$ where $\mathcal{D} : C^\infty(M) \rightarrow \Gamma(E)$ is defined by $\langle \mathcal{D}f, e \rangle = \mathcal{L}_{\rho_E(e)} f.$
- C3) $\mathcal{L}_{\rho_E(e)} \langle e_1, e_2 \rangle = \langle \llbracket e, e_1 \rrbracket, e_2 \rangle + \langle e_1, \llbracket e, e_2 \rrbracket \rangle, \quad \forall e, e_1, e_2 \in \Gamma(E);$
- C4) $\rho_E(\llbracket e_1, e_2 \rrbracket) = [\rho_E(e_1), \rho_E(e_2)], \quad \forall e_1, e_2 \in \Gamma(E);$
- C5) $\llbracket e_1, fe_2 \rrbracket = f \llbracket e_1, e_2 \rrbracket + (\mathcal{L}_{\rho_E(e_1)} f) e_2, \quad \forall e_1, e_2 \in \Gamma(E), f \in C^\infty(M).$

Note that the bracket $\llbracket \cdot, \cdot \rrbracket$ is *not* skew-symmetric, but rather satisfies

$$(146) \quad \llbracket e_1, e_2 \rrbracket = -\llbracket e_2, e_1 \rrbracket + \mathcal{D} \langle e_1, e_2 \rangle$$

as a consequence of C2). (This is the non-skew-symmetric version of the Courant bracket studied, e.g., in [33]; the original notion of Courant bracket [26] is obtained by skew-symmetrization.) It also follows from C2) that, upon the identification $E \cong E^*$ via $\langle \cdot, \cdot \rangle$, we have

$$(147) \quad \rho_E \circ \rho_E^* = 0.$$

The model example of a Courant algebroid is the following:

Example A.1. Consider $E = T^*M \oplus TM$ equipped with symmetric pairing $\langle (X, \alpha), (Y, \beta) \rangle_{can} := \beta(X) + \alpha(Y)$. Any closed 3-form ϕ on M determines a Courant algebroid structure on E with bracket

$$\llbracket (X, \alpha), (Y, \beta) \rrbracket_\phi := ([X, Y], \mathcal{L}_X \beta - i_Y d\alpha + i_Y i_X \phi).$$

A detailed discussion about Courant brackets with original references can be found in [24].

A subbundle $L \subset E$ which is Lagrangian (i.e., maximal isotropic) with respect to $\langle \cdot, \cdot \rangle$ is called an **almost Dirac structure**. It is a **Dirac structure** if it is *integrable* in the sense that

$$\llbracket \Gamma(L), \Gamma(L) \rrbracket \subseteq \Gamma(L).$$

For a Dirac structure L , (146) implies that the restriction $[\cdot, \cdot]_L := \llbracket \cdot, \cdot \rrbracket|_{\Gamma(L)}$ is a skew-symmetric bracket on $\Gamma(L)$, and axioms C1) and C5) imply that this bracket makes L into a *Lie algebroid* with anchor $\rho_L := \rho_E|_L$. The bracket $[\cdot, \cdot]_L$ can be extended to a bilinear bracket on $\Gamma(\wedge L)$, $[\cdot, \cdot]_L : \Gamma(\wedge^p L) \times \Gamma(\wedge^q L) \rightarrow \Gamma(\wedge^{p+q-1} L)$, by the conditions

$$(148) \quad [l_1, l_2]_L = -(-1)^{(p-1)(q-1)} [l_2, l_1]_L,$$

$$(149) \quad [l_1, l_2 l_3]_L = [l_1, l_2]_L l_3 + (-1)^{(p-1)q} l_2 [l_1, l_3]_L,$$

for $l_1 \in \Gamma(\wedge^p L)$, $l_2 \in \Gamma(\wedge^q L)$, and $l_3 \in \Gamma(\wedge^r L)$. The Jacobi identity on $\Gamma(L)$ translates into the graded Jacobi identity for the extended bracket:

$$(150) \quad [l_1, [l_2, l_3]_L]_L + (-1)^{(p-1)(q+r)} [l_2, [l_3, l_1]_L]_L + (-1)^{(r-1)(p+q)} [l_3, [l_1, l_2]_L]_L = 0,$$

In other words, $\Gamma(\wedge L)$ becomes a *Gerstenhaber algebra*.

The bracket $[\cdot, \cdot]_L$ and anchor ρ_L also define a degree-1 derivation d_L on the graded commutative algebra $\Gamma(\wedge L^*)$,

$$(151) \quad d_L(\xi_1 \xi_2) = d_L(\xi_1) \xi_2 + (-1)^p \xi_1 d_L(\xi_2),$$

for $\xi_1 \in \Gamma(\wedge^p L^*)$ and $\xi_2 \in \Gamma(\wedge^q L^*)$, by the conditions:

$$(152) \quad d_L f(l) = \mathcal{L}_{\rho_L(l)} f, \quad l \in \Gamma(L), f \in C^\infty(M);$$

$$(153) \quad d_L \xi(l_1, l_2) = \mathcal{L}_{\rho_L(l_1)} \xi(l_2) - \mathcal{L}_{\rho_L(l_2)} \xi(l_1) - \xi([l_1, l_2]_L), \\ l_1, l_2 \in \Gamma(L), \xi \in \Gamma(L^*).$$

In this case the Jacobi identity of $[\cdot, \cdot]_L$ translates into $d_L^2 = 0$.

A.2. Manin pairs over manifolds and isotropic splittings. A **Manin pair over a manifold M** is a pair (E, L) consisting of a Courant algebroid E over M for which $\langle \cdot, \cdot \rangle$ has signature (n, n) , and a Dirac structure $L \subset E$. It follows from the signature condition that $\text{rank}(L) = n = \frac{1}{2}\text{rank}(E)$. When M is a point, we recover the notion discussed in Section 3.1.

Given a Manin pair (E, L) over M , there is an associated exact sequence of vector bundles given by

$$(154) \quad 0 \longrightarrow L \xrightarrow{\iota} E \xrightarrow{\iota^*} L^* \longrightarrow 0,$$

where $\iota : L \hookrightarrow E$ is the inclusion and $\iota^*(e)(l) = \langle e, \iota(l) \rangle$. We consider henceforth the identification $E \cong E^*$ induced by $\langle \cdot, \cdot \rangle$. The map ι^* coincides with the projection $E \rightarrow E/L$ after the identification $E/L \cong L^*$ induced by $\langle \cdot, \cdot \rangle$, proving the exactness of the sequence (154).

An **isotropic splitting** of the exact sequence (154) is a linear splitting $s : L^* \rightarrow E$ of (154) whose image is isotropic in E , i.e., $\langle \cdot, \cdot \rangle|_{s(L^*)} = 0$. A Manin pair over M together with the choice of an isotropic splitting is referred to as a **split Manin pair** over M .

Lemma A.2. *Let (E, L) be a Manin pair over M . Then the exact sequence (154) admits an isotropic splitting. Moreover, any isotropic splitting $s : L^* \rightarrow E$ defines an isomorphism*

$$(155) \quad (\iota, s) : L \oplus L^* \xrightarrow{\sim} E$$

with inverse (s^, ι^*) , which identifies the pairing $\langle \cdot, \cdot \rangle$ in E with the canonical symmetric pairing in $L \oplus L^*$ given by*

$$(156) \quad \langle (l_1, \xi_1), (l_2, \xi_2) \rangle_{can} := \xi_2(l_1) + \xi_1(l_2).$$

Proof. If $s : L^* \rightarrow E$ is any linear splitting of (154), then a direct computation shows that $s' = s - \frac{1}{2}\iota s^* s$ is an isotropic splitting. It is straightforward to check that (155) is an isometric isomorphism with respect to $\langle \cdot, \cdot \rangle_{can}$ and $\langle \cdot, \cdot \rangle$. q.e.d.

An immediate consequence of Lemma A.2 is that the following identities hold:

$$(157) \quad s^*s = 0, \quad \iota^*s = 1, \quad s^*\iota = 1, \quad s\iota^* + \iota s^* = 1.$$

Let us fix an isotropic splitting $s : L^* \rightarrow E$. Under the induced identification $E \cong L \oplus L^*$, the maps s^* and ι^* become the natural projections $\text{pr}_L : L \oplus L^* \rightarrow L$ and $\text{pr}_{L^*} : L \oplus L^* \rightarrow L^*$, respectively. Then s induces the following geometrical structures:

i) A cobracket

$$(158) \quad F_s : \Gamma(L) \rightarrow \Gamma(L) \wedge \Gamma(L),$$

ii) A 3-tensor

$$(159) \quad \chi_s \in \Gamma(\wedge^3 L),$$

iii) A bundle map

$$(160) \quad \rho_{L^*}^s := \rho_E|_{s(L^*)} : L^* \rightarrow TM.$$

We will omit the s dependence in the notation whenever there is no risk of confusion.

The cobracket F is defined in terms of its dual, $F^* : \Gamma(L^*) \wedge \Gamma(L^*) \rightarrow \Gamma(L^*)$, by

$$(161) \quad F^*(\xi_1, \xi_2) := \text{pr}_{L^*}([\![s(\xi_1), s(\xi_2)]\!]).$$

We also denote the skew-symmetric bracket F^* on $\Gamma(L^*)$ by $[\cdot, \cdot]_{L^*}$ (the skew-symmetry of (161) is a consequence of $s(L^*) \subset E$ being isotropic).

Similarly, we define $\chi : \Gamma(L^*) \wedge \Gamma(L^*) \rightarrow \Gamma(L)$ by the condition

$$i_{\xi_2} i_{\xi_1} \chi = \text{pr}_L([\![s(\xi_1), s(\xi_2)]\!]).$$

By axiom C3) in the definition of a Courant algebroid, the expression

$$i_{\xi_3} \text{pr}_L([\![s(\xi_1), s(\xi_2)]\!]) = \langle [\![s(\xi_1), s(\xi_2)]\!], s(\xi_3) \rangle$$

is skew-symmetric in ξ_1, ξ_2, ξ_3 . Since it is clearly $C^\infty(M)$ -linear in ξ_3 , it is $C^\infty(M)$ -trilinear and therefore defines (159).

We also have an extension of $[\cdot, \cdot]_{L^*}$ to a bilinear bracket on $\Gamma(\wedge L^*)$ satisfying (148), (149) as well as a degree 1 derivation d_{L^*} on $\Gamma(\wedge L)$ defined by $[\cdot, \cdot]_{L^*}$ and ρ_{L^*} via (152), (153). In general, $[\cdot, \cdot]_{L^*}$ does not satisfy the graded Jacobi identity and d_{L^*} is not a differential, as a consequence of the failure of integrability of $L^* \subset E$.

A.3. Lie quasi-bialgebroids. Let us consider a split Manin pair, identified with $(L \oplus L^*, L)$, where $L \oplus L^*$ is equipped with the symmetric pairing $\langle \cdot, \cdot \rangle_{can}$ (as in Lemma A.2). Fixing this identification, one obtains a formula for the Courant bracket $[\cdot, \cdot]$ on $L \oplus L^*$ in terms of F^*, χ

and ρ_{L^*} :

$$(162) \quad \llbracket(l_1, 0), (l_2, 0)\rrbracket = [l_1, l_2]_L,$$

$$(163) \quad \llbracket(l, 0), (0, \xi)\rrbracket = (-i_\xi d_{L^*} l, \mathcal{L}_l \xi),$$

$$(164) \quad \llbracket(0, \xi_1), (0, \xi_2)\rrbracket = (\chi(\xi_1, \xi_2), F^*(\xi_1, \xi_2)),$$

where $[\cdot, \cdot]_L = \llbracket \cdot, \cdot \rrbracket|_{\Gamma(L)}$, $l, l_1, l_2 \in \Gamma(L)$, $\xi, \xi_1, \xi_2 \in \Gamma(L^*)$ and $\mathcal{L}_l = d_L i_l + i_l d_L$.

Conversely, one may start with a Lie algebroid $(L, [\cdot, \cdot]_L, \rho_L)$ together with a skew-symmetric bracket F^* on $\Gamma(L)$, an element $\chi \in \Gamma(\wedge^3 L)$ and a bundle map $\rho_{L^*} : L^* \rightarrow TM$. This set of data is called a **Lie quasi-bialgebroid** [33, 34] if the bracket defined by (162), (163) and (164) makes $L \oplus L^*$ into a Courant algebroid with pairing $\langle \cdot, \cdot \rangle_{can}$ and anchor $\rho_E = \rho_L + \rho_{L^*}$. This requirement is equivalent to the following explicit compatibility conditions [34]:

$$\text{Q0)} \quad d_{L^*}[l_1, l_2]_L = [d_{L^*}l_1, l_2]_L + [l_1, d_{L^*}l_2]_L, \text{ for all } l_1, l_2 \in \Gamma(L).$$

(Using the Leibniz identity for $[\cdot, \cdot]_L$, one can check that d_{L^*} is actually a derivation of $[\cdot, \cdot]_L$ on $\Gamma(\wedge L)$: $d_{L^*}[l_1, l_2]_L = [d_{L^*}l_1, l_2]_L + (-1)^{p-1}[l_1, d_{L^*}l_2]$, $l_1 \in \Gamma(\wedge^p L)$, $l_2 \in \Gamma(\wedge^q L)$.)

$$\text{Q1)} \quad \rho_{L^*}(F^*(\xi_1, \xi_2)) = [\rho_{L^*}(\xi_1), \rho_{L^*}(\xi_2)] - \rho_L(i_{\xi_2} i_{\xi_1}(\chi)), \text{ for all } \xi_1, \xi_2 \in \Gamma(L^*).$$

$$\text{Q2)} \quad F^*(\xi_1, f\xi_2) = fF^*(\xi_1, \xi_2) + \mathcal{L}_{\rho_{L^*}(\xi_1)}(f)\xi_2, \text{ for all } \xi_1, \xi_2 \in \Gamma(L^*), f \in C^\infty(M).$$

$$\text{Q3)} \quad \text{For all } \xi_1, \xi_2, \xi_3 \in \Gamma(L^*),$$

$$F^*(\xi_1, F^*(\xi_2, \xi_3)) + c.p. =$$

$$d_L \chi(\xi_1, \xi_2, \xi_3) + i_{\chi(\xi_2, \xi_3)} d_L \xi_1 - i_{\chi(\xi_1, \xi_3)} d_L \xi_2 + i_{\chi(\xi_1, \xi_2)} d_L \xi_3.$$

$$\text{Q4)} \quad d_{L^*}\chi = 0.$$

The resulting Courant algebroid $L \oplus L^*$ is called the **double** of the Lie quasi-bialgebroid. Hence we see that there is a natural correspondence between split Manin pairs and Lie quasi-bialgebroids over M .

A.4. Bivector fields. Given a Courant algebroid E over M and a pair of transversal almost Dirac structures L, C , with $E = L \oplus C$, it follows from (147) that

$$\rho_L \circ (\rho_C)^* + \rho_C \circ (\rho_L)^* = 0,$$

where $\rho_L := \rho_E|_L$, $\rho_C := \rho_E|_C$, and we identify $C \cong L^*$ via the pairing on E . Hence the bundle map $\pi^\sharp := \rho_L \circ (\rho_C)^* : T^*M \rightarrow TM$ defines a bivector field on π on M , depending on L and C . In particular, any Lie quasi-bialgebroid over M defines a bivector field $\pi \in \mathfrak{X}^2(M)$ [21].

Proposition A.1. *For a given Lie quasi-bialgebroid $E = L \oplus L^*$, the bivector field $\pi \in \mathfrak{X}^2(M)$ defined by $\pi^\sharp = \rho_L \circ (\rho_{L^*})^*$ satisfies*

$$(165) \quad \frac{1}{2}[\pi, \pi] = \rho_L(\chi),$$

$$(166) \quad \mathcal{L}_{\rho_L(l)}\pi = \rho_L(d_{L^*}(l)), \quad \forall l \in \Gamma(L).$$

Proof. For $f, g, h \in C^\infty(M)$, let $\text{Jac}(f, g, h) = \{f, \{g, h\}\} + c.p.$, where $\{\cdot, \cdot\}$ is the bracket defined by π . It then follows that (see e.g. [10, Sec. 2.2])

$$(167) \quad \begin{aligned} \frac{1}{2}[\pi, \pi](df, dg, dh) &= \text{Jac}(f, g, h) \\ &= \left\langle [\pi^\sharp(df), \pi^\sharp(dg)] - \pi^\sharp(d\{f, g\}), dh \right\rangle. \end{aligned}$$

Using Q1) we see that $\rho_L(\chi)(df, dg, dh)$ equals

$$\begin{aligned} &\left\langle \rho_L(i_{\rho_L^*(dg)} i_{\rho_L^*(df)} \chi), dh \right\rangle \\ &= \left\langle [\pi^\sharp(df), \pi^\sharp(dg)], dh \right\rangle - \langle \rho_{L^*}(F^*(\rho_L^*(df), \rho_L^*(dg))), dh \rangle, \end{aligned}$$

and, by (167), this last expression equals

$$(168) \quad \text{Jac}(f, g, h) + \{\{f, g\}, h\} - \langle F^*(\rho_L^*(df), \rho_L^*(dg)), \rho_{L^*}^*(dh) \rangle.$$

Using the identity (153) for the bracket F^* , we can rewrite (168) as

$$\begin{aligned} &(d_{L^*}(\rho_{L^*}^* dh))(\rho_L^* df, \rho_L^* dg) + \{\{g, h\}, f\} + \{\{h, f\}, g\} \\ &\quad + \text{Jac}(f, g, h) + \{\{f, g\}, h\}, \end{aligned}$$

which equals $(d_{L^*}(\rho_{L^*}^* dh))(\rho_L^* df, \rho_L^* dg)$. Hence

$$(169) \quad \rho_L(\chi)(df, dg, dh) = (d_{L^*}(\rho_{L^*}^* dh))(\rho_L^* df, \rho_L^* dg) = (d_{L^*}^2 h)(\rho_L^* df, \rho_L^* dg).$$

On the other hand, since $d_{L^*}[d_{L^*}f, g]_L = [d_{L^*}^2 f, g]_L + [d_{L^*}f, d_{L^*}g]_L$ (by Q0)), applying ρ_L and using the definition of π we get

$$\mathcal{L}_{\rho_L(d_{L^*}[d_{L^*}f, g]_L)} h = \{\{f, g\}, h\} = \mathcal{L}_{\rho_L([d_{L^*}^2 f, g])} h + \{f, \{g, h\}\} + \{g, \{h, f\}\}.$$

A direct computation shows that $\mathcal{L}_{\rho_L([d_{L^*}^2 f, g])} h = -\rho_L(d_{L^*}^2 f)(dg, dh)$, hence

$$\text{Jac}(f, g, h) = \rho_L(d_{L^*}^2 f)(dg, dh) = d_{L^*}^2 f(\rho_L^* dg, \rho_L^* dh).$$

Using the skew-symmetry of Jac and (169), equation (165) follows.

To prove (166), we use that $d_{L^*}[l, f]_L = [d_{L^*}l, f]_L + [l, d_{L^*}f]_L$ for all $l \in \Gamma(L)$. Applying ρ_L to this expression, it follows that

$$\begin{aligned} \{\mathcal{L}_{\rho_L(l)} f, g\} &= \mathcal{L}_{\rho_L([d_{L^*}l, f]_L)} g + \mathcal{L}_{[\rho_L(l), \pi^\sharp(df)]} g \\ &= \mathcal{L}_{\rho_L([d_{L^*}l, f]_L)} g + \mathcal{L}_{\rho_L(l)} \{f, g\} - \{f, \mathcal{L}_{\rho_L(l)} g\}. \end{aligned}$$

Hence

$$(\mathcal{L}_{\rho_L(l)} \pi)(df, dg) = \mathcal{L}_{\rho_L(l)} \{f, g\} - \{\mathcal{L}_{\rho_L(l)} f, g\} - \{f, \mathcal{L}_{\rho_L(l)} g\}$$

$$= -\mathcal{L}_{\rho_L([d_{L^*}l,f]_L)}g.$$

Using the general identity $\mathcal{L}_{\rho_L([\lambda,f])}g = -\rho_L(\lambda)(df,dg)$, for $f,g \in C^\infty(M)$ and $\lambda \in \Gamma(\wedge^2 L)$, we conclude that

$$-\mathcal{L}_{\rho_L([d_{L^*}l,f]_L)}g = \rho_L(d_{L^*}l)(df,dg),$$

as desired. q.e.d.

A.5. Twists and exact Courant algebroids. Let (E,L) be a Manin pair over M , and suppose that we have two splittings of (154), s and s' . The image of the difference $s - s' : L^* \rightarrow E$ lies in L , hence it defines an element $t \in \wedge^2 L$, called a **twist**, by

$$s - s' = t^\sharp : L^* \rightarrow L \subset E,$$

where $t^\sharp(\xi_1)(\xi_2) = t(\xi_1, \xi_2)$. A direct calculation shows the following:

Proposition A.2. *The following holds:*

- 1) Let $d_{L^*}^s$ be the derivation on $\Gamma(\wedge L)$ associated with the bracket F_s^* on $\Gamma(L^*)$ and bundle map $\rho_{L^*}^s$. Then

$$d_{L^*}^s = d_{L^*}^{s'} + [t, \cdot]_L.$$

- 2) $\chi_s = \chi_{s'} + d_{L^*}^s t - \frac{1}{2}[t, t]_L$.
- 3) $\pi^{s'} = \pi^s + \rho_L(t)$.

An important class of examples of Manin pairs is given by exact Courant algebroids. Following P. Ševera [35], a Courant algebroid is called **exact** if the sequence

$$(170) \quad 0 \longrightarrow T^*M \xrightarrow{\rho_E^*} E \xrightarrow{\rho_E} TM \longrightarrow 0$$

is exact (by (147), it is always true that $\rho_E \rho_E^* = 0$). Viewing T^*M as a subbundle of E via ρ_E^* , $(E,L = T^*M)$ is a Manin pair. Using axioms C3) and C4) in Sec. A.1, one can check that $[\cdot, \cdot]_L = [\cdot, \cdot]|_{\Gamma(T^*M)} \equiv 0$. Since $\rho_L = 0$, we must have $d_L = 0$. Once we choose an isotropic splitting s and identify E with $TM \oplus T^*M$, it is simple to check that the bracket F_s^* on $\Gamma(L^*) = \Gamma(TM)$ is just the Lie bracket of vector fields. From Q4), we see that χ_s is a closed 3-form, and the general bracket (162), (163), (164) becomes the bracket of Example A.1. From Prop. A.2, part 2, we see that a different splitting changes χ_s by an exact 3-form. These observations lead to the following result of Ševera:

Corollary A.1. *Exact Courant algebroids over M are classified by $H^3(M, \mathbb{R})$.*

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