Coordinate-free characterization of homogeneous polynomials with isolated singularities

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Abstract

The Durfee conjecture, proposed in 1978, relates two important invariants of isolated hypersurface singularities by a famous inequality; however, the inequality in this conjecture is not sharp. In 1995, Yau announced his conjecture which proposed a sharp inequality. The Yau conjecture characterizes the conditions under which an affine hypersurface with an isoalted singularity at the origin is a cone over a nonsingular projective hypersurface; in other words, the conjecture gives a coordinate-free characterization of when a convergent power series is a homogeneous polynomial after a biholomorphic change of variables. In this project, we prove that the Yau conjecture holds for $n = 5$. As a consequence, we have proved that $5!p_g \leq \mu - p(v)$, where $p(v) = (v-1)^5 - v(v-1)\dots(v-4)$ and p_g, μ, and v are, respectively, the geometric genus, the Milnor number, and the multiplicity of the isolated singularity at the origin of a weighted homogeneous polynomial. In the process, we have also defined yet another sharp upper bound for the number of positive integral points within a 5-dimensional simplex.

1 Introduction

Let Δ_n be an n-dimensional real right-angled simplex defined by the inequality

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + ... + \frac{x_n}{a_n} \leq 1$$

where $x_1 \geq 0$, ... , $x_n \geq 0$ and $a_1 \geq a_2 \geq ... \geq a_n \geq 1$. Define P_n to be the number of positive integral points in Δ_n, as shown below:

$$P_n = \#\{(x_1, x_2, ..., x_n) \in \mathbb{Z}_+ | \frac{x_1}{a_1} + \frac{x_2}{a_2} + ... + \frac{x_n}{a_n} \leq 1\}$$

Define Q_n to be the number of nonnegative integral points in Δ_n, as shown below:

$$Q_n = \#\{(x_1, x_2, ..., x_n) \in \mathbb{Z}_+ \cup \{0\} | \frac{x_1}{a_1} + \frac{x_2}{a_2} + ... + \frac{x_n}{a_n} \leq 1\}$$

The problem of obtaining the numbers P_n and Q_n has occupied mathematicians for decades, simply because a sharp upper estimate of the former would benefit those in singularity theory and a sharp upper estimate of the latter would benefit those in number theory. Granville [Gr] has stated that an estimate of Q_n would help with finding large gaps between primes, and research done by Xu and Yau [Xu-Ya] on the Durfee conjecture has shown that an estimate of P_n does aid mathematicians in singularity theory. These two different numbers are tied together through the equation

$$P_n(a_1, a_2, ..., a_n) = Q_n(a_1(1-a), a_2(1-a), ..., a_n(1-a)),$$

where $a = \frac{1}{a_1} + ... + \frac{1}{a_n}$. A simple proof exists that allows one to state with confidence that these two numbers are related, and so if one discovers a new estimate for P_n, a new estimate for Q_n will also be present.

In 1899, Pick [Pi] discovered a formula for Q_2:

$$Q_2 = \text{area}(\triangle) + \frac{|\partial\triangle \cap \mathbb{Z}^n|}{2} + 1$$

where $\partial\triangle$ is the boundary of the simplex and $|\partial\triangle \cap \mathbb{Z}^n|$ is the number of integral points on the boundary. In 1951, Mordell [Mo] discovered the formula for Q_3 using Dedekind sums, but the real breakthrough occurred when Ehrhart [Eh] proved a polynomial of degree n could calculate the number of non-negative lattice points in n-dimensional simplex. However, his formula is only effective when the coefficients of every term are either known or able to be determined from other facts provided. Starting in 1939, attempts were made to find lower and upper bounds for Q_n instead of a concrete equation. It was later discovered by Lehmer [Le] that if $a = a_1 = a_2 = ... = a_n$,

$$Q_n = \binom{[a]+n}{n}.$$

From that formula, a definition for a sharp estimate R_n of Q_n was reached. The estimate is only sharp if

$$R_n|_{a_1=...=a_n=a\in\mathbb{Z}} = \binom{[a]+n}{n}.$$

In other words, any upper or lower bound is only considered to be a sharp estimate if equality is obtained if and only if $a = a_1 = a_2 = ... = a_n \in \mathbb{Z}$.

The search for a sharp estimate also led mthematicians into singularity theory, and Durfee [Du] formed his famous conjecture. The conjecture states that for an isolated singularity $(V, 0)$ defined by a weighted homogeneous polynomial $f(z_0, z_1, z_2, ..., z_n)$,

$$n!p_g \leq \mu$$

where p_g is the geometric genus of V, μ is the Milnor number, and equality holds if and only if $\mu = 0$. A polynomial $f(z_0, z_1, z_2, ..., z_n)$ is a weighted homogeneous polynomial of the type $(w_0, w_1, w_2, ..., w_n)$, where w_0, w_1, w_2, ... , w_n are fixed positive rational numbers, if f can be expressed as a linear combination of monomials $z_0^{i_0} z_1^{i_1} ... z_n^{i_n}$ for which $\frac{i_0}{w_0} + \frac{i_1}{w_1} + ... + \frac{i_n}{w_n} = 1$. Furthermore, the Milnor number μ [Mi-Or] is defined as $(w_0 - 1)(w_1 - 1)...(w_n - 1)$.

The next sharp estimate to be constructed was the GLY conjecture, formulated by Lin, Yau, and Granville [Li-Ya 4]. It has two different parts: the sharp estimate, which varies depending on n, and the rough estimate.

Conjecture 1. *Let* $P_n = \#\{(x_1, x_2, ..., x_n) \in \mathbb{Z}_+^n; \frac{x_1}{a_1} + \frac{x_2}{a_2} + ... + \frac{x_n}{a_n} \leq 1\}$, *and let* $n \geq 3$.

(1) If $a_1 \geq a_2 \geq ... \geq a_n \geq n - 1$, *then the sharp GLY estimate holds.*

(2) Rough estimate: If $a_1 \geq a_2 \geq ... \geq a_n > 1$, *then*

$$n!P_n \leq q_n := \prod_{i=1}^{n} (a_i - 1).$$

The GLY conjecture was the first main step to proving the following conjecture made by Yau in 1995 [Ya-Zh]:

Conjecture 2. *Let* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *be a weighted homogeneous polynomial with an isolated singularity at the origin. Let* μ, p_g, *and* v *be the Milnor number, geometric genus and multiplicty of the singularity* $V = \{z : f(z) = 0\}$, *then*

$$\mu - p(v) \geq n!p_g,$$

where $p(v) = (v - 1)^n - v(v - 1)...(v - n + 1)$, *and equality holds if and only if* f *is a homogeneous polynomial.*

This conjecture is a sharp estimate that holds without the restriction of the sharp GLY estimate, $a_1 \geq a_2 \geq ... \geq a_n \geq n-1$, and it also has some important applications in geometry.

The Yau conjecture was already proven for the cases $n = 3$ [Xu-Ya] and $n = 4$ [Li-Ya 3]. In this paper, we aim to prove the Yau conjecture for $n = 5$, which is stated below.

Theorem 3. *Let $f : (\mathbb{C}^5, 0) \to (\mathbb{C}, 0)$ be a weighted homogeneous polynomial with an isolated singularity at the origin. Let μ, p_g, and v be the Milnor number, geometric genus and multiplicty of the singularity $V = \{z : f(z) = 0\}$, then*

$$\mu - p(v) \geq 5!p_g,$$

where $p(v) = (v-1)^5 - v(v-1)...(v-4)$, and equality holds if and only if f is a homogeneous polynomial.

According to [Li-Ya 3], proving our Main Theorem, above, is akin to proving the following theorem about the number of integral points within a five-dimensional simplex.

Theorem 4. *Let $a \geq b \geq c \geq d \geq e \geq 2$ be real numbers and let P_5 be the number of positive integral solutions of $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1$; i.e. $P_5 = \#\{(v, w, x, y, z) \in \mathbb{Z}^5_+ : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1\}$. Define $\mu = (a-1)(b-1)(c-1)(d-1)(e-1)$. Then,*

$$5!P_5 \leq \mu - (5v^4 - 25v^3 + 40v^2 - 19v - 1),$$

where v is the multiplicity - calculated by $v = e$, if e is an integer, or by $v = [e] + 1$, if e is not an integer, where $[e]$ is the integral part of e - and equality holds if and only if $a = b = c = d = e$ are all integers.

In order to do this, we will split up the proof into three main cases, depending on the value of e, and utilize the GLY conjecture, in each case. Case I covers $e \geq 4$, while cases II and III cover $3 \leq e < 4$ and $2 \leq e < 3$ respectively.

2 Proof of the Main Theorem

2.1 CASE IA

We will first analyze the case that occurs when $e = 4$.

Proposition 5. *Let $a \geq b \geq c \geq d \geq 4$ be real numbers and let $e = 4$. Consider $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1$. Let P_5 be the number of positive integral solutions of the above inequality; i.e. $P_5 = \#\{(x, y, z, v, w) \in \mathbb{Z}^5_+ : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1\}$. Then,*

$$120P_5 \leq -15a + \frac{385ab}{24} - \frac{135abc}{32} + \frac{245abcd}{128} - \frac{135abd}{32} + \frac{385ac}{24} - \frac{135acd}{32} - 15b + \frac{385bc}{24} - \frac{135bcd}{32} - 15c$$

We can slice the five-dimensional simplex into the hyperplanes, $w = 1$, $w = 2$, ..., $w = e$. The inequality above is derived from summing up the four-dimensional GLY conjecture estimates from level $w = 1$ to $w = e = 4$. Before we go any further, we should note the interesting properties of P_4 that the lemma below points out.

Lemma 6. *Let $a \geq b \geq c \geq d \geq 1$ be real numbers and let $P_4 = \#\{(x, y, z, w) \in \mathbb{Z}^4_+ : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leq 1\}$. Then, the following statements hold,*

(1) if $b \leq 3$, then $P_4 = 0$
(2) if $c \leq 2$, then $P_4 = 0$.

The theorem below is basically the Yau conjecture for $n = 5$ with the property $e = 4$.

Theorem 7. *Let $a \geq b \geq c \geq d \geq 4$ be real numbers and let $e = 4$. Consider $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1$. Let P_5 be the number of positive integral solutions of $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1$; i.e. $P_5 = \#\{(v, w, x, y, z) \in \mathbb{Z}^5_+ : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1\}$. Define $\mu = (a-1)(b-1)(c-1)(d-1)(e-1)$. Then,*

$$120P_5 \leq \mu - (5v^4 - 25v^3 + 40v^2 - 19v - 1)|_{v=e=4} = -240 - 3a - 3b + 3ab - 3c + 3ac + 3bc - 3abc - 3d + 3ad + 3bd - 3abd + 3cd - 3acd - 3bcd + 3abcd$$

Let Δ_1 be the different between the R.H.S. of Theorem 7 and Proposition 5. Because the GLY conjecture already holds as an effective estimate for the number of integral points in an n-dimensional simplex, we only need to prove that the Yau estimate is larger than the GLY estimate for $n = 5$.

$$\Delta_1 = -240 + 12a - \frac{384ab}{24} + \frac{135abc}{32} - \frac{245abcd}{128} + \frac{135abd}{32} - \frac{385ac}{24} + \frac{135acd}{32} - \frac{385bc}{24} + \frac{135bcd}{32} + 12b + 3ab + 12c + 3ac + 3bc - 3abc - 3d + 3ad + 3bd - 3abd + 3cd - 3acd - 3bcd + 3abcd$$

Let $A = \frac{a}{d}$, $B = \frac{b}{d}$, and $C = \frac{c}{d}$.

$$\Delta_1 = -240 + 12Ad - \frac{385}{24}ABd^2 - \frac{385}{24}ACd^2 - \frac{385}{24}BCd^2 + \frac{135}{32}ABCd^3 + \frac{135}{32}ABd^3 + \frac{135}{32}BCd^3 + \frac{135}{32}ACd^3 - \frac{245}{128}ABCd^4 + 12Bd + 3ABd^2 + 12Cd + 3ACd^2 + 3BCd^2 - 3ABCd^3 - 3d + 3Ad^2 + 3Bd^2 + 3Cd^2 - 3ABd^2 - 3ACd^2 - 3BCd^2 + 3ABCd^4$$

By taking the derivative of this function with respect to A, B, and C, one variable at a time, we found that the following values are all positive: $\frac{\partial}{\partial A}\Delta_1$, $\frac{\partial}{\partial B}\Delta_1$, and $\frac{\partial}{\partial C}\Delta_1$. The actual process of calculating these partial derivatives was quite a bit of work; we had to ensure that $\frac{\partial^3\Delta_1}{\partial A\partial B\partial C}$, $\frac{\partial^2\Delta_1}{\partial A\partial B}$, $\frac{\partial^2\Delta_1}{\partial A\partial C}$, and $\frac{\partial^2\Delta_1}{\partial B\partial C}$ were all positive before we could proceed. This method of calculating the partial derivative with respect to all the variables and then the partial derivative of the function with respect to one less variable for each step will now be called the "partial differentiation test". All of our calculations throughout this entire paper can be found at: http://sites.google.com/site/theyauconjecture/.

From our partial derivatives, we can conclude that Δ_1 is an increasing function of A, B, and C for $A \geq B \geq C \geq 1$ and $d \geq 4$. Our next step is to evaluate the function at the minimum.

$$\Delta_1|_{A=B=C=1} = 1200$$

Evaluated at the minimum, $\Delta_1 = 1200 > 0$, so we can conclude that $\Delta_1 \geq 0$ for $a \geq b \geq c \geq d \geq 4$ and $e = 4$. Therefore, the Yau conjecture is an accurate upper bound of the number of integral points within the simplex on the hyperplane $v = e = 4$.

2.2 CASE IB

We will now analyze the case that occurs when $e > 4$. We must also keep in mind the hypothesis that the non-integral portion of e, β, has to be either $\frac{e}{a}$, $\frac{e}{b}$, $\frac{e}{c}$, or $\frac{e}{d}$. This is important because $\triangle_1$ becomes negative for certain values of a, b, c, d, and e, such as when $a = b = c = d = e = 4(\triangle_1 = -501)$ or $a = b = c = d = e = 4.25(\triangle_1 = -377.737)$, indicating that the above proofs no longer hold. A lemma is needed to show that β is a ratio of the weights, which lemma 8 does for us below.

Lemma 8. *Let $f(x, y, z, v, w) : (\mathbb{C}^5, 0) \to (\mathbb{C}, 0)$ be a weighted homogeneous polynomial with an isolated singularity at the origin. Define $wt(n)$ as the weight of the variable n in the polynomial. Assume that $wt(x) \geq wt(y) \geq wt(z) \geq wt(v) \geq wt(w) \geq 2$, where $wt(x)$, $wt(y)$, $wt(z)$, $wt(v)$, and $wt(w)$ are all rational numbers. If $wt(w)$ is not an integer, then $[wt(w)] = \frac{wt(x)}{wt(w)}, \frac{wt(y)}{wt(w)}, \frac{wt(z)}{wt(w)}$, or $\frac{wt(v)}{wt(w)}$, where $[wt(w)]$ is the fractional part of $wt(w)$.*

The following proposition is obtained by from the GLY conjecture.

Proposition 9. *Let $a \geq b \geq c \geq d \geq e \geq 4$ be real numbers and let P_5 be the number of positive integral solutions of $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1$; i.e. $P_5 = \#\{(x, y, z, v, w) \in \mathbb{Z}^5_+ : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1\}$. Suppose e is not an integer and $e = [e] + \beta$, where β is either $\frac{e}{a}$, $\frac{e}{b}$, $\frac{e}{c}$, or $\frac{e}{d}$.*

$$120P_5 \leq 5a+5b+5c-\frac{55}{6}ab-\frac{55}{6}ac-\frac{55}{6}bc+\frac{15}{4}abc+\frac{15}{4}abd+\frac{15}{4}acd+\frac{15}{4}bcd-5ae-5be$$
$$-5ce+\frac{55}{9}abe+\frac{55}{9}abe+\frac{55}{9}ace-\frac{15}{8}bce-\frac{15}{8}abde-\frac{15}{8}acde-\frac{15}{8}bcde-\frac{5}{2}abcd+abcde$$
$$-\frac{abcd}{6e^2}+\frac{55ab}{18e}+\frac{55ac}{18e}+\frac{55bc}{18e}-\frac{15abc}{8e}-\frac{15abd}{8e}-\frac{15acd}{8e}-\frac{15bcd}{8e}+\frac{5abcd}{3e}+\frac{abcd\beta}{6e^4}-\frac{55ab\beta}{18e^2}$$
$$-\frac{55ac\beta}{18e^2}-\frac{55bc\beta}{18e^2}+\frac{5a\beta}{e}+\frac{5b\beta}{e}+\frac{5c\beta}{e}+\frac{15abc\beta^2}{8e^3}+\frac{15abd\beta^2}{8e^3}+\frac{15acd\beta^2}{8e^3}+\frac{15bcd\beta^2}{8e^3}$$
$$-\frac{55ab\beta^2}{6e^2}-\frac{55ac\beta^2}{6e^2}-\frac{55bc\beta^2}{6e^2}+\frac{5a\beta^2}{e}+\frac{5b\beta^2}{e}+\frac{5c\beta^2}{e}-\frac{5abcd\beta^3}{3e^4}+\frac{15abc\beta^3}{4e^3}+\frac{15abd\beta^3}{4e^3}$$
$$+\frac{15acd\beta^3}{4e^3}+\frac{15bcd\beta^3}{4e^3}-\frac{55ab\beta^3}{9e^2}-\frac{55ac\beta^3}{9e^2}-\frac{55bc\beta^3}{9e^2}-\frac{5abcd\beta^4}{2e^4}+\frac{15abc\beta^4}{8e^3}+\frac{15abd\beta^4}{8e^3}$$
$$+\frac{15acd\beta^4}{8e^3}+\frac{15bcd\beta^4}{8e^3}-\frac{abcd\beta^5}{e^4}$$

We can slive the five-dimensional simplex into the hyperplanes, $w=1$, $w=2$, ..., $w=e-\beta-1$. The inequality above is derived from summing up the four-dimensional GLY conjecture estimates from level $w=1$ to $w=e-\beta-1$.

Theorem 10. *Let $a \geq b \geq c \geq d \geq e \geq 4$ be real numbers. Consider $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{v}{d}+\frac{w}{e} \leq 1$. Let P_5 be the number of positive integral solutions of the above inequality; i.e. $P_5 = \#\{(v,w,x,y,z) \in \mathbb{Z}_+^5 : \frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{v}{d}+\frac{w}{e} \leq 1\}$. Suppose e is not an integer and $e=[e]+\beta$ where β is either $\frac{e}{a}$, $\frac{e}{b}$, $\frac{e}{c}$, or $\frac{e}{d}$. Define $\mu=(a-1)(b-1)(c-1)(d-1)(e-1)$. Then,*

$$120P_5 \leq \mu-(5v^4-25v^3+40v^2-19v-1)|_{v=e-\beta+1} = abcde-(abcd+abce+abde$$
$$+acde+bcde)+(abc+abd+abe+acd+ace+ade+bcd+bce+bde+cde)$$
$$-(ab+ac+ad+ae+bc+bd+be+cd+ce+de)+(a+b+c+d+e)-1-5e^4+5e^3$$
$$+5e^2-6e-1+\beta(20e^3-15e^2-10e+6)-\beta^2(30e^2+15e+5)+\beta^3(20e-5)-5\beta^4+1$$

Let Δ_2 be the difference between the R.H.S. of Theorem 10 and Proposition 9. Because the GLY conjecture already holds as an efective estimate for the number of integral points in an n-dimensional simplex, we only need to prove that the Yau estimate is larger than the GLY estaimte for $n=5$.

$$\begin{aligned}\Delta_2 = &-1-4a-4b-4c+d-ad-bd-cd+4ae+4be+4ce-de+ade+bde+cde\\ &+\frac{49}{6}ab+\frac{49}{6}ac+\frac{49}{6}bc-\frac{11}{4}abc-\frac{11}{4}abd-\frac{11}{4}acd-\frac{11}{4}bcd-5e+5e^2+5e^3-5e^4\\ &+\frac{3}{2}abcd+\frac{7}{8}abce+\frac{7}{8}abde+\frac{7}{8}acde+\frac{7}{8}bcde+\frac{abcd}{6e^3}-\frac{55ab}{18e}-\frac{55ac}{18e}-\frac{55bc}{18e}\\ &+\frac{15abd}{8e}+\frac{15acd}{8e}+\frac{15bcd}{8e}-\frac{5abcd}{3e}+6\beta-\frac{abcd\beta}{6e^4}+\frac{55ab\beta}{18e^2}+\frac{55ac\beta}{18e^2}+\frac{55bc\beta}{18e^2}\\ &-\frac{5a\beta}{e}-\frac{5b\beta}{e}-\frac{5c\beta}{e}-10e\beta-15e^2\beta+20e^3\beta+5\beta^2-\frac{15abc\beta^2}{8e^3}-\frac{15abd\beta^2}{8e^3}\\ &-\frac{15acd\beta^2}{8e^3}-\frac{15bcd\beta^2}{8e^3}+\frac{55ab\beta^2}{6e^2}+\frac{55ac\beta^2}{6e^2}+\frac{55bc\beta^2}{6e^2}-\frac{5a\beta^2}{e}-\frac{5b\beta^2}{e}\\ &-\frac{5c\beta^2}{e}+15e\beta^2-30e^2\beta^2-5\beta^3+\frac{5abcd\beta^3}{3e^4}-\frac{15abc\beta^3}{4e^3}-\frac{15abd\beta^3}{4e^3}\\ &-\frac{15acd\beta^3}{4e^3}-\frac{15bcd\beta^3}{4e^3}+\frac{55ab\beta^3}{9e^2}+\frac{55ac\beta^3}{9e^2}+\frac{55bc\beta^3}{9e^2}+20e\beta^3-5\beta^4\\ &+\frac{5abcd\beta^4}{2e^4}-\frac{15abc\beta^4}{8e^3}-\frac{15abd\beta^4}{8e^3}-\frac{15acd\beta^4}{8e^3}-\frac{15bcd\beta^4}{8e^3}+\frac{abcd\beta^5}{e^4}\end{aligned}$$

We will consider the following four cases: Case (a) $\beta = \frac{e}{d}$, Case (b) $\beta = \frac{e}{c}$, Case (c) $\beta = \frac{e}{b}$, and Case (d) $\beta = \frac{e}{a}$.

2.2.1 Case (a): $\beta = \frac{e}{d}$

At level $w = e - \beta - 1$, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{e-\beta-1}{e} \le 1$. If $d = e$, then $\beta = 1$ and $e = [e] + \beta$ is an integer. Hence, $a \ge b \ge c \ge d > e$. Let $A = \frac{a}{e}$, $B = \frac{b}{e}$, $C = \frac{c}{e}$, and $D = \frac{d}{e}$, with $\beta = \frac{e}{d} = \frac{1}{D}$, and $A \ge B \ge C \ge D > 1$. Then, Δ_2 can be written as

$$\Delta_2 = -1-4Ae-4Be-4Ce+De-5e+\frac{1}{6}ABCDe-\frac{55}{18}ABe-\frac{55}{18}ACe-\frac{55}{18}BCe$$
$$+\frac{15}{8}ABCe^2+\frac{15}{8}ABDe^2+\frac{15}{8}ACDe^2+\frac{15}{8}BCDe^2+\frac{49}{6}ABe^2+\frac{49}{6}ACe^2+\frac{49}{6}BCe^2$$
$$-ADe^2-BDe^2-CDe^2+4Ae^2+4Be^2+4Ce^2-De^2+5e^2-\frac{5}{3}ABCDe^3-\frac{11}{4}ABCe^3$$
$$-\frac{11}{4}ABDe^3-\frac{11}{4}ACDe^3-\frac{11}{4}BCDe^3-\frac{46}{9}ABe^3-\frac{46}{9}ACe^3-\frac{46}{9}BCe^3+ADe^3+BDe^3$$
$$+CDe^3+5e^3+\frac{3}{2}ABCDe^4+\frac{7}{8}ABCe^4+\frac{7}{8}ABDe^4+\frac{7}{8}ACDe^4+\frac{7}{8}BCDe^4-5e^4$$
$$-\frac{1}{6}ABC-\frac{5A}{D}-\frac{5B}{D}-\frac{5C}{D}-\frac{10e}{D}-\frac{15e^2}{D}+\frac{20e^3}{D}+\frac{6}{D}+\frac{55AB}{18D}+\frac{55BC}{18D}+\frac{55AC}{18D}$$
$$-\frac{15AB}{8D}-\frac{15AC}{8D}-\frac{15BC}{8D}+\frac{5}{D^2}-\frac{15ABC}{8D^2}+\frac{55AB}{6D^2}+\frac{55AC}{6D^2}+\frac{55BC}{6D^2}-\frac{5A}{D^2}-\frac{5B}{D^2}$$
$$-\frac{5C}{D^2}+\frac{15e}{D^3}-\frac{30e^2}{D^2}-\frac{15AB}{4D^2}-\frac{15AC}{4D^2}-\frac{15BC}{4D^2}+\frac{5ABC}{3D^2}-\frac{5}{D^3}-\frac{15ABC}{4D^3}+\frac{55AB}{9D^3}$$
$$+\frac{55AC}{9D^3}+\frac{55BC}{9D^3}+\frac{20e}{D^3}+\frac{5ABC}{2D^3}-\frac{15AB}{8D^3}-\frac{15AC}{8D^3}-\frac{15BC}{8D^3}-\frac{5}{D^4}-\frac{15ABC}{8D^4}+\frac{ABC}{D^4}$$

Once again, we used partial differentiation to complete our proof. The calculations can be found on the website.

By taking the derivative of this function with respect to A, B, C, and D, one variable at a time, we found that the following values are all positive: $\frac{\partial}{\partial A}\Delta_2$, $\frac{\partial}{\partial B}\Delta_2$, $\frac{\partial}{\partial C}\Delta_2$, and $\frac{\partial}{\partial D}\Delta_2$. Thus, we can conclude that Δ_2 is an increasing function of A, B, C, and D for $A \geq B \geq C \geq D \geq 1$ and $e \geq 4$.

$$\Delta_2|_{A=B=C=D=1,e=4} = 0$$

Evaluated at the minimum, $\Delta_2 = 0$, so we can conclude that $\Delta_2 > 0$ for $a \geq b \geq c \geq d \geq e > 4$.

2.2.2 Case (b): $\beta = \frac{e}{c}$

If $c = e$, then $\beta = 1$ and $e = [e] + \beta$ is an integer. Hence, $a \geq b \geq c > e$. Let $A = \frac{a}{e}$, $B = \frac{b}{e}$, $C = \frac{c}{e}$, and $D = \frac{d}{e}$, with $\beta = \frac{e}{c} = \frac{1}{C}$. Then, Δ_2 can be written as

$$\Delta_2 = -6-4Ae-4Be-4Ce+De-ADe^2-BDe^2-CDe^2+4Ae^2+4Be^2+4Ce^2-De^2$$
$$+ADe^3+BDe^3+CDe^3+\frac{49}{6}ABe^2+\frac{49}{6}ACe^2+\frac{49}{6}BCe^2-\frac{11}{4}ABCe^3-\frac{11}{4}ABDe^3$$
$$-\frac{11}{4}ACDe^3-\frac{11}{4}BCDe^3-5e+5e^2+5e^3-5e^4+\frac{3}{2}ABCDe^4+\frac{7}{8}ABCe^4+\frac{7}{8}ABDe^4$$
$$+\frac{7}{8}ACDe^4+\frac{7}{8}BCDe^4+\frac{1}{6}ABCDe-\frac{55}{18}ABe-\frac{55}{18}ACe-\frac{55}{18}BCe+\frac{15}{8}ABDe^2+\frac{15}{8}ACDe^2$$
$$+\frac{15}{8}BCDe^2-\frac{5}{3}ABCDe^3+\frac{6}{C}-\frac{ABD}{6}+\frac{55AB}{18C}+\frac{55A}{18}+\frac{55B}{18}-\frac{5A}{C}-\frac{5B}{C}-\frac{10e}{C}-\frac{15e^2}{C}$$
$$+\frac{20e^3}{C}+\frac{5}{C^2}-\frac{15AB}{8C}-\frac{15ABD}{8C^2}-\frac{15AD}{8C}-\frac{15BD}{8C}+\frac{55AB}{6C^2}+\frac{55A}{6C}+\frac{55B}{C}-\frac{5A}{C^2}-\frac{5B}{C^2}$$
$$-\frac{5}{C}+\frac{15e}{C^2}-\frac{30e^2}{C^2}-\frac{5}{C^3}+\frac{5ABD}{C^2}-\frac{15AB}{4C^2}-\frac{15ABD}{5C^3}-\frac{15AD}{4C^2}-\frac{15BD}{4C^2}+\frac{55AB}{9C^3}+\frac{55A}{9C^2}$$
$$+\frac{55B}{9C^2}+\frac{20e}{C^3}-\frac{5}{C^4}+\frac{5ABD}{2C^3}-\frac{15AB}{8C^3}-\frac{15ABD}{8C^4}-\frac{15AD}{8C^3}-\frac{15BD}{8C^3}+\frac{ABD}{C^4}$$

By taking the derivative of this function with respect to A, B, C, and D, one variable at a time, we found that Δ_2 is an increasing function of A, B, and D for $A \geq B \geq C \geq D \geq 1$ and $e \geq 4$. We can also prove that Δ_2 is an increasing function of C for $A \geq B \geq C \geq D \geq 1$ and $e > 5$, but not for $A \geq B \geq C \geq D \geq 1$ and $e \geq 4$. Thus, we will need to consider the following two subcases:

Case (b1): $4 < e < 5$: We will prove Theorem 4 directly in this subcase.
Case (b2): $e > 5$: $\frac{\partial \Delta_2}{\partial C}$ in this subcase.

Case (b1): $4 < e < 5$

In this case, $e = 4 + \beta$, $0 < \beta = \frac{e}{c} < 1$. Also note that $c\beta = 4 + \beta$, and $c = 1 + \frac{4}{\beta} = \frac{e}{e-4} > 5$. Hence, we only need to consider $a \geq b \geq c > 5$, $d \geq e > 4$, and $e = 4 + \beta$. We can also see that at level $w = 4$, the defining inequality of the simplex becomes

$$\frac{x}{a(\frac{\beta}{4+\beta})} + \frac{y}{b(\frac{\beta}{4+\beta})} + \frac{z}{c(\frac{\beta}{4+\beta})} + \frac{v}{d(\frac{\beta}{4+\beta})} \leq 1$$

Because $c\beta = 4 + \beta$, the denominator of the third term becomes 1, and we end up with an inequality with no positive solutions. Thus, we only need to consider the levels $w = 1$, $w = 2$, and $w = 3$ in computing P_5. Let L_1 be the number of positive integral solutions at level $w = 1$. It is bound by the following inequality,

$$24L_1 \leq abcd(\frac{3+\beta}{4+\beta})^4 - \frac{3}{2}(abc+abd+acd+bcd)(\frac{3+\beta}{4+\beta})^3$$
$$+ \frac{11}{3}(ab+ac+bc)(\frac{3+\beta}{4+\beta})^2 - 2(a+b+c)(\frac{3+\beta}{4+\beta})$$

as defined by the following theorem, coming directly from Theorem 2.3 in [Li-Ya 3].

Theorem 11. *Let $a \geq b \geq c \geq d \geq 3$ be real numbers. Consider the inequality $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leq 1$. Let P_4 be the number of positive integral solutions of the above inequality; i.e. $P_4 = \#\{(x,y,z,w) \in \mathbb{Z}_+^4 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leq 1\}$. Then,*

$$24P_4 \leq abcd - \frac{3}{2}(abc + abd + acd + bcd) + \frac{11}{3}(ab + ac + bc) - 2(a + b + c),$$

with equality if and only if $a = b = c = d$ are all integers.

Next, let L_2 be the number of positive lattice points at level $w = 2$; it satisfies

$$\begin{aligned}24L_2 \leq abcd(\frac{2+\beta}{4+\beta})^4 - \frac{3}{2}(abc + abd + acd + bcd)(\frac{2+\beta}{4+\beta})^3 \\ + \frac{11}{3}(ab + ac + bc)(\frac{2+\beta}{4+\beta})^2 - 2(a + b + c)(\frac{2+\beta}{4+\beta})\end{aligned}$$

L_3, the number of positive integral solutions at level $w = 3$, satisfies

$$\begin{aligned}24L_3 \leq abcd(\frac{1+\beta}{4+\beta})^4 - \frac{3}{2}(abc + abd + acd + bcd)(\frac{1+\beta}{4+\beta})^3 \\ + \frac{11}{3}(ab + ac + bc)(\frac{1+\beta}{4+\beta})^2 - 2(a + b + c)(\frac{1+\beta}{4+\beta})\end{aligned}$$

If we take a look at Theorem 4 again, we find that since $e = 4 + \beta$ and $v = e - \beta + 1 = 5$, the R.H.S. of the Yau conjecture becomes

$$\mu - (5v^4 - 25v^3 + 40v^2 - 19v - 1) = (a-1)(b-1)(c-1)(d-1)(3+\beta) - 904$$

Define Δ_3 as following:

$$\begin{aligned}\Delta_3 = {} & [(a-1)(b-1)(c-1)(d-1)(3+\beta) - 904] \\ & - 5[abcd(\frac{3+\beta}{4+\beta})^4 - \frac{3}{2}(abc + abd + acd + bcd)(\frac{3+\beta}{4+\beta})^3 \\ & + \frac{11}{3}(ab + ac + bc)(\frac{3+\beta}{4+\beta})^2 - 2(a + b + c)(\frac{3+\beta}{4+\beta})] \\ & - 5[abcd(\frac{2+\beta}{4+\beta})^4 - \frac{3}{2}(abc + abd + acd + bcd)(\frac{2+\beta}{4+\beta})^3 \\ & + \frac{11}{3}(ab + ac + bc)(\frac{2+\beta}{4+\beta})^2 - 2(a + b + c)(\frac{2+\beta}{4+\beta})] \\ & - 5[abcd(\frac{1+\beta}{4+\beta})^4 - \frac{3}{2}(abc + abd + acd + bcd)(\frac{1+\beta}{4+\beta})^3 \\ & + \frac{11}{3}(ab + ac + bc)(\frac{1+\beta}{4+\beta})^2 - 2(a + b + c)(\frac{1+\beta}{4+\beta})]\end{aligned}$$

Note that $\beta = \frac{4}{c-1}$ and $\beta + 4 = \frac{4c}{c-1}$. Therefore, Δ_3 becomes

$$\Delta_3 = \frac{1}{384c^3}(-4540abc - 400ac^2 - 400bc^2 - 6100abc^2 - 341760c^3 - 3844abc^3 + 4608c^4 - 5008ac^4 - 5008bc^4 + 468abc^4 + 885abd + 1620acd + 1620bcd + 1320abcd + 2700ac^2d + 2700bc^2d + 1170abc^2d + 384c^3d + 2316ac^3d + 2316bc^3d + 624abc^3d + 1152c^4d + 468ac^4d - 2656ac^3 - 2656bc^3 + 468bc^4d + 417abc^4d) = \frac{1}{384c^3}\Delta_4$$

Now all we have to do is partial differentiate to ensure that Δ_4 is positive throughout the domain, and we find that Δ_4 is an increasing function with respect to a, b, c, and d for $a \geq b \geq c \geq d \geq e > 4$.

$$\begin{aligned} \Delta_4|_{a=b=c=1+\frac{4}{\beta}, d=e=4+\beta} &= \tfrac{192(-1+\beta)(4+\beta)^3(-712-750\beta-1238\beta^2+105\beta^3)}{\beta^6} \\ &> 0 \end{aligned}$$

Evaluated at the minimum, $\Delta_4 > 0$, so we can conclude that $\Delta_4 > 0$ for $a \geq b \geq c > 5$, $d \geq e > 4$.

Case (b2): $e > 5$
In this range

$$\frac{\partial \Delta_2}{\partial C} > 0 \text{ for } A \geq 1, B \geq 1, D \geq 1, e > 5$$

Therefore Δ_2 is an increasing function of A, B, C, and D for $A \geq B \geq C \geq D \geq 1$ and $e > 5$.

Finally, combining the results of cases (b1) and (b2) and knowing that $\Delta_2 = 0$ at the minimum, we can conclude that $\Delta_2 > 0$ for $a \geq b \geq c \geq d \geq e > 4$, where $e = [e] + \frac{e}{c}$.

2.2.3 Case (c): $\beta = \frac{e}{b}$

If $b = e$, then $\beta = 1$ and $e = [e] + \beta$ is an integer. Hence, $a \geq b > e$. Let $A = \frac{a}{e}$, $B = \frac{b}{e}$, $C = \frac{c}{e}$, and $D = \frac{d}{e}$, with $\beta = \frac{e}{b} = \frac{1}{B}$. Then, Δ_2 can be rewritten in terms of A, B, C, and .

To check that Δ_2 is positive throughout the domain, we used the partial differentiation test once again. By taking the derivative of this function with respect to A, B, C, and D, one variable at a time, we found that Δ_2 is an increasing function of A, C, and D for $A \geq B \geq C \geq D \geq 1$ and $e \geq 4$. We can also prove that Δ_2 is an increasing function of B for $A \geq B \geq C \geq D \geq 1$ and $e > 5$, but not for $A \geq B \geq C \geq D \geq 1$ and $e \geq 4$. Thus, we will need to consider the following two subcases, as we did in Case (b):

Case (c1): $4 < e < 5$: We will prove Theorem 4 directly in this subcase.

Case (c2): $e > 5$: $\frac{\partial \Delta_2}{\partial B}$ in this subcase.

Case (c1): $4 < e < 5$

In this case, $e = 4 + \beta$, $0 < \beta = \frac{e}{b} < 1$. Also note that $b\beta = 4 + \beta$, and $b = 1 + \frac{4}{\beta} = \frac{e}{e-4} > 5$. Hence, we only need to consider $a \geq b > 5$, $c \geq d \geq e > 4$, and $e = 4 + \beta$. We can also see that at level $w = 4$, the defining inequality of the simplex becomes

$$\frac{x}{a(\frac{\beta}{4+\beta})} + \frac{y}{b(\frac{\beta}{4+\beta})} + \frac{z}{c(\frac{\beta}{4+\beta})} + \frac{v}{d(\frac{\beta}{4+\beta})} \leq 1$$

Because $b\beta = 4 + \beta$, the denominator of the second term becomes 1, and we end up with an inequality with no positive solutions. Thus, we only need to consider the levels $w = 1$, $w = 2$, and $w = 3$ in computing P_5. Define L_1, L_2, and L_3 in the same way that we did before in subcase (b1) of case (b).

If we take a look at Theorem 4 again, we find that since $e = 4 + \beta$ and $v = e - \beta + 1 = 5$, the R.H.S. of the Yau conjecture becomes

$$\mu - (5v^4 - 25v^3 + 40v^2 - 19v - 1) = (a-1)(b-1)(c-1)(d-1)(3+\beta) - 904$$

Define Δ_3 as following:

$$\begin{aligned}
\Delta_3 = {} & [(a-1)(b-1)(c-1)(d-1)(3+\beta) - 904] \\
& - 5[abcd(\frac{3+\beta}{4+\beta})^4 - \frac{3}{2}(abc + abd + acd + bcd)(\frac{3+\beta}{4+\beta})^3 \\
& + \frac{11}{3}(ab + ac + bc)(\frac{3+\beta}{4+\beta})^2 - 2(a+b+c)(\frac{3+\beta}{4+\beta})] \\
& - 5[abcd(\frac{2+\beta}{4+\beta})^4 - \frac{3}{2}(abc + abd + acd + bcd)(\frac{2+\beta}{4+\beta})^3 \\
& + \frac{11}{3}(ab + ac + bc)(\frac{2+\beta}{4+\beta})^2 - 2(a+b+c)(\frac{2+\beta}{4+\beta})] \\
& - 5[abcd(\frac{1+\beta}{4+\beta})^4 - \frac{3}{2}(abc + abd + acd + bcd)(\frac{1+\beta}{4+\beta})^3 \\
& + \frac{11}{3}(ab + ac + bc)(\frac{1+\beta}{4+\beta})^2 - 2(a+b+c)(\frac{1+\beta}{4+\beta})]
\end{aligned}$$

Note that $\beta = \frac{4}{b-1}$ and $\beta + 4 = \frac{4b}{b-1}$. Therefore, Δ_3 becomes

$$\begin{aligned}
\Delta_3 = {} & \frac{1}{384b^3}(-400ab^2 - 341760b^3 - 2656ab^3 + 4608b^4 - 5008ab^4 - 4540abc - 400b^2c \\
& -6100ab^2c - 2656b^3c - 3844ab^3c - 5008b^4c + 468ab^4c - 4540abd + 11120b^2d - 6100ab^2d \\
& +8864b^3d - 3844ab^3d - 5008b^4d + 468ab^4d + 885acd - 4540bcd + 1320abcd - 6100b^2cd \\
& + 1170ab^2cd - 3844b^3cd + 624ab^3cd + 468b^4cd + 417ab^4cd - 6160bd^2 - 8800b^2d^2 \\
& - 6160b^3d^2 + 4860d^3 + 8100bd^3 + 8100b^2d^3 + 4860b^3d^3) = \frac{1}{384b^3}\Delta_4
\end{aligned}$$

Now all we have to do is partial differentiate to ensure that Δ_4 is positive throughout the domain. We found that Δ_4 is an increasing function with respect to a, b, c, and d for $a \geq b \geq c \geq d \geq e > 4$.

$$\begin{aligned} \Delta_4|_{a=b=1+\frac{4}{\beta},c=d=e=4+\beta} &= \tfrac{64(4+\beta)^3(1352-1468\beta-3710\beta^2+3696\beta^3+1950\beta^4+405\beta^5)}{\beta^5} \\ &> 0 \end{aligned}$$

Evaluated at the minimum, $\Delta_4 > 0$, so we can conclude that $\Delta_4 > 0$ for $a \geq b > 5$, $c \geq d \geq e > 4$.

Case (c2): $e > 5$
In this range

$$\frac{\partial \Delta_2}{\partial B} > 0 \text{ for } A \geq 1, C \geq 1, D \geq 1, e > 5$$

Therefore Δ_2 is an increasing function of A, B, C, and D for $A \geq B \geq C \geq D \geq 1$ and $e > 5$.

Finally, combining the results of cases (b1) and (b2) and knowing that $\Delta_2 = 0$ at the minimum, we can conclude that $\Delta_2 > 0$ for $a \geq b \geq c \geq d \geq e > 4$, where $e = [e] + \frac{e}{b}$.

2.2.4 Case (d): $\beta = \frac{e}{a}$

At level $w = e - \beta - 1$, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{e-\beta-1}{e} \leq 1$. If $a = e$, then $\beta = 1$ and $e = [e] + \beta$ is an integer. Hence, $a > e$. Let $A = \frac{a}{e}$, $B = \frac{b}{e}$, $C = \frac{c}{e}$, and $D = \frac{d}{e}$, with $\beta = \frac{e}{a} = \frac{1}{A}$. Then, Δ_2 can be rewritten in terms of A, B, C, and D. The, use the partial differentiation test to check that Δ_2 is positive throughout the domain; the work can be found on the website. We found that Δ_2 is increasing with respect to A, B, C, and D. Thus, we can conclude that Δ_2 is an increasing function of A, B, C, and D for $A \geq B \geq C \geq D \geq 1$ and $e \geq 4$.

$$\Delta_2|_{A=B=C=D=1} = 0$$

Evaluated at the minimum, $\Delta_2 = 0$, so we can conclude that $\Delta_2 > 0$ for $a \geq b \geq c \geq d \geq e > 4$.

2.3 CASE IIA

We will now analyze the case that occurs when $e = 3$.

Proposition 12. *Let $a \geq b \geq c \geq d \geq 3$ be real numbers and let $e = 3$. Consider $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1$. Let P_5 be the number of positive integral solutions of the above inequality; i.e. $P_5 = \#\{(x, y, z, v, w) \in \mathbb{Z}_+^5 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1\}$. Then,*

$$120P_5 \leq -10(a+b+c) + \frac{275}{27}(ab+ac+bc) + \frac{85}{81}abcd - \frac{5}{2}(abc+abd+acd+bcd)$$

We can slice the five-dimensional simplex into the hyperplanes, $w = 1$, $w = 2$, ..., $w = e$. The inequality above is derived from summing up the four-dimensional GLY conjecture estimates from level $w = 1$ to $w = e = 3$.

Theorem 13. *Let $a \geq b \geq c \geq d \geq 3$ be real numbers and let $e = 3$. Consider $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1$. Let P_5 be the number of positive integral solutions of the above inequality; i.e. $P_5 = \#\{(x, y, z, v, w) \in \mathbb{Z}_+^5 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1\}$. Then,*

$$
\begin{aligned}
120P_5 \leq \mu - (5v^4 - 25v^3 + 40v^2 - 19v - 1)|_{v=e=3} = -30 - 2a - 2b \\
+ 2ab - 2c + 2ac + 2bc - 2abc - 2d + 2ad + 2bd - 2abd + 2cd - 2acd \\
- 2bcd + 2abcd
\end{aligned}
$$

We can use the same procedure we used in Case IA to prove that Theorem 13 is true - by showing that the R.H.S. of Theorem 13 is greater than the L.H.S. of Proposition 12. The resulting calculations from the partial differentiation test can be found on the website, but in summary of our findings, we found that the difference between the two is positive throughout the domain. Therefore, the Yau conjecture is an accurate upper bound of the number of integral points within the simplex on the hyperplane $v = e = 3$.

2.4 CASE IIB

We should consider next the case that occurs when e is in the interval (3, 4). Theorem 14 below sums up what we are trying to prove.

Theorem 14. *Let $a \geq b \geq c \geq d \geq e \geq 3$ be real numbers. Consider $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1$. Let P_5 be the number of positive integral solutions of the above inequality; i.e. $P_5 = \#\{(v, w, x, y, z) \in \mathbb{Z}_+^5 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1\}$. Suppose e is not an integer and $e = [e] + \beta$ where β is either $\frac{e}{a}$, $\frac{e}{b}$, $\frac{e}{c}$, or $\frac{e}{d}$. Define $\mu = (a-1)(b-1)(c-1)(d-1)(e-1)$. Then,*

$$
\begin{aligned}
120P_5 \leq \mu - (5v^4 - 25v^3 + 40v^2 - 19v - 1)|_{v=e-\beta+1} = abcde - (abcd + abce + abde \\
+ acde + bcde) + (abc + abd + abe + acd + ace + ade + bcd + bce + bde + cde) \\
- (ab + ac + ad + ae + bc + bd + be + cd + ce + de) + (a + b + c + d + e) - 1 - 5e^4 + 5e^3 \\
+ 5e^2 - 6e - 1 + \beta(20e^3 - 15e^2 - 10e + 6) - \beta^2(30e^2 + 15e + 5) + \beta^3(20e - 5) - 5\beta^4 + 1
\end{aligned}
$$

The proof entails only the case when e is in the interval (3, 4) and can be divided into four subcases: Case (a) $\beta = \frac{e}{a}$, Case (b) $\beta = \frac{e}{b}$, Case (c) $\beta = \frac{e}{c}$, and Case (d) $\beta = \frac{e}{d}$.

In the interest of keeping this paper within a reasonable length, the full proof has been omitted from this report and uploaded to the website. In summary of our findings, we proved each subcase through methods similar to the ones we used in Case IB, but we also applied Lemma 6 when problems were encountered in the proof of subcases (a), (b), and (c).

2.5 CASE IIIA

This section covers the case that occurs when $e = 2$.

Up until this point, we have been using the sharp GLY estimate to help prove the Yau estimates hold. However, the sharp conjecture is for n-dimensional real right-angled simplices with $a_n \geq n-1$, where a_n is the set of the weights. For Case III, the weights (a, b, c, d, and e) can be less than 3, so we can no longer compare the five-dimensional Yau estimate to the sum of the 4-dimensional sharp GLY esimates. We can, however, use the rough GLY estimate, which is restated below as a theorem.

Theorem 15. *Let $P_n = \#\{(x_1, x_2, ..., x_n) \in \mathbb{Z}_+^n : \frac{x_1}{a_1} + ... + \frac{x_n}{a_n} \leq 1\}$. Let $n \geq 3$. Then, if $a_1 \geq a_2 \geq ... \geq a_n > 1$,*

$$n!P_n < q_n := \prod_{i=1}^{n}(a_1 - 1).$$

The proposition below comes directly from Theorem 15.

Proposition 16. *Let $a \geq b \geq c \geq d \geq 2$ be real numbers and let $e = 2$. Let $P_5 = \#\{(v, w, x, y, z) \in \mathbb{Z}_+^5 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1\}$.* Then,

$$120P_5 \leq 5(\frac{a}{2} - 1)(\frac{b}{2} - 1)(\frac{c}{2} - 1)(\frac{d}{2} - 1).$$

We can slice the five-dimensional simplex into the hyperplanes $w = 1$, $w = 2$, ... , $w = e$. At $w = 2$, the inequality becomes $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{2}{2} \leq 1$, and there are no positive solutions. Thus, the inequality above is derived from taking the four-dimensional rough GLY conjecture estimate for level $w = 1$. We can only use the rough GLY conjecture for this case because the weights are not necessarily greater than 3.

Theorem 16 below sums up what we are trying to prove.

Theorem 17. *Let $a \geq b \geq c \geq d \geq 2$ be real numbers and let $e = 2$. Consider $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1$. Let P_5 be the number of positive integral solutions of the above inequality; i.e. $P_5 = \#\{(v, w, x, y, z) \in \mathbb{Z}_+^5 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1\}$. Define $\mu = (a-1)(b-1)(c-1)(d-1)(e-1)$. Then,*

$$\begin{aligned}120P_5 \leq \mu - (5v^4 - 25v^3 + 40v^2 - 19v - 1)|_{v=e=2} = -a - b + ab \\ - c + ac + bc - abc - d + ad + bd - abd + cd - acd - bcd + abcd\end{aligned}$$

From here, we proceed in a similar fashion as we did in all of the other subcases; we apply the partial differentiation test on the difference between the R.H.S. of Theorem 17 and the R.H.S. of Proposition 16.

2.6 CASE IIIB

This section covers the case when e is in the interval (2, 3).

The proof here is a mix between the methodology seen in Case IIB and Case IIIA. There are four subcases; however, the rough GLY conjecture is used in lieu of the sharper estimate. The calculations can be found on the website. In summary of our findings, we found that the Yau conjecture is indeed an accurate upper bound of the number of positive integral points when e is in the interval (2, 3), thus concluding our proof of Theorem 4 and the Main Theorem.

Theorem 18 below sums up what we are trying to prove.

Theorem 18. *Let $a \geq b \geq c \geq d \geq e \geq 2$ be real numbers. Consider $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1$. Let P_5 be the number of positive integral solutions of the above inequality; i.e. $P_5 = \#\{(x, y, z, v, w) \in \mathbb{Z}_+^n : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1\}$. Suppose e is not an integer and $e = [e] + \beta$ where β is either $\frac{e}{a}$, $\frac{e}{b}$, $\frac{e}{c}$, or $\frac{e}{d}$. Define $\mu = (a-1)(b-1)(c-1)(d-1)(e-1)$. Then,*

$$120P_5 \leq \mu - (5v^4 - 25v^3 + 40v^2 - 19v + 1)|_{v=e-\beta+1} =$$
$$abcde - (abcd + abce + abde + acde + bcde) + (abc + abd + abe + acd + ace + ade$$
$$+ bcd + bce + bde + cde) - (ab + ac + ad + ae + bc + bd + be + cd + ce + de) + (a + b +$$
$$c + d + e) - 1 - 5e^4 + 5e^3 + 5e^2 - 6e - 1 + \beta(20e^3 - 15e^2 - 10e + 6) - \beta^2(30e^2 + 15e + 5)$$
$$+ \beta^3(20e - 5) - 5\beta^4 + 1$$

We will utilize the rough GLY estimate to aid with the proof. The proof entails only the case when e is in the interval (2,3) and can be divided into four subcases: Case (a) $\beta = \frac{e}{a}$, Case (b) $\beta = \frac{e}{b}$, Case (c) $\beta = \frac{e}{c}$, and Case (d) $\beta = \frac{e}{d}$.

2.6.1 Case (a): $\beta = \frac{e}{a}$

In this case, $e = 2 + \beta$, where $0 < \beta = \frac{e}{a} < 1$. Also note that $a\beta = 2 + \beta$, and $a = 1 + \frac{2}{\beta} = \frac{e}{e-2} > 3$. We can also see that at level $w = 2$, the defining inequality of the simplex becomes

$$\frac{x}{a(\frac{\beta}{2+\beta})} + \frac{y}{b(\frac{\beta}{2+\beta})} + \frac{z}{c(\frac{\beta}{2+\beta})} + \frac{v}{d(\frac{\beta}{2+\beta})} \leq 1.$$

Since $a\beta = 2 + \beta$, the denominator of the first term, $a(\frac{\beta}{2+\beta})$, becomes 1, and the inequality becomes one that has no positive integral solutions. Thus, we can see that we only need to consider the level $w = 1$ in computing P_5. Let L_1 be the number of positive integral solutions at level $w = 1$. It is bound by the following inequality as defined in Theorem 15,

$$24L_1 \leq (a(\frac{1+\beta}{2+\beta}) - 1)(b(\frac{1+\beta}{2+\beta}) - 1)(c(\frac{1+\beta}{2+\beta}) - 1)(d(\frac{1+\beta}{2+\beta}) - 1).$$

Next, we should establish the minimum values of a, b, c, and d. We already know that a has to be greater than 3. In consideration of Lemma 6, the denominator of the second term, $b(\frac{1+\beta}{2+\beta})$, has to be greater than 3, and the denominator of the third term, $c(\frac{1+\beta}{2+\beta})$, has to be greater than 2. Thus, we obtain the inequalities $b(\frac{1+\beta}{2+\beta}) > 3$ and $c(\frac{1+\beta}{2+\beta}) > 2$. Those two restraints lead to the minimums $b > 3 + \frac{3}{1+\beta}$ and $c > 2 + \frac{2}{1+\beta}$.

If we take a look at Theorem 4 again, we find that since $e = 2 + \beta$ and $v = e - \beta + 1 = 3$, the R.H.S. of the Yau conjecture becomes

$$\mu - (5v^4 - 25v^3 + 40v^2 - 19v - 1) - (a-1)(b-1)(c-1)(d-1)(1+\beta) - 32.$$

Define Δ_2 as following:

$$\Delta_2 = [(a-1)(b-1)(c-1)(d-1)(1+\beta) - 32] - \\ 5[(a(\frac{1+\beta}{2+\beta}) - 1)(b(\frac{1+\beta}{2+\beta}) - 1)(c(\frac{1+\beta}{2+\beta}) - 1)(d(\frac{1+\beta}{2+\beta}) - 1)]$$

Note that $\beta = \frac{2}{a-1}$ and $\beta + 2 = \frac{2a}{a-1}$. Therefore, Δ_2 becomes

$$\Delta_2 = \frac{1}{16a^3}(-568a^3 + 24a^4 + 20a^2b + 16a^3b - 4a^4b + 20a^2c + 16a^3c - 4a^4c - 10abc \\ -10a^2bc - 6a^3bc - 6a^4bc + 20a^2d + 16a^3d - 4a^4d - 10abd - 10a^2bd - 6a^3bd - 6a^4bd \\ -10acd - 10a^2cd - 6a^3cd - 6a^4cd + 5bcd + 10abcd + 6a^3bcd + 11a^4bcd) = \frac{1}{48a^3}\Delta_3$$

Now all we have to do is partial differentiate to ensure that Δ_3 is positive throughout the domain.

In the interest of keeping this paper within a reasonable length, the rest of the proof has been omitted from this report and uploaded to the website. In summary of our findings, we proved the remaining subcases through methods similar to the ones we used in Case (a) of Case IIIB.

3 Discussion

As a result of this investigation, we have successfully proved the Yau conjecture for $n = 5$. The Yau conjecture is more efficient than the GLY conjecture as a method of obtaining an upper bound for the number of integral points within a 5-dimensional simplex because it does not require $a_1 \geq a_2 \geq ... \geq a_n \geq n-1$. Furthermore, the Yau conjecture has numerous applications in the fields of number theory, applied mathmeatics, algebraic geometry, and singularity theory.

The next logical step to take after the complete proof of the Yau conjecture for $n = 5$ would be to prove the conjecture for the general case n using mathematical induction. This would propel the mathematical community another step closer to classifying affine varieties as cones over projective nonsingular varieties, something mathematicians have been trying to do since Durfee formulated his famous conjecture.

Other future work includes exploring the geometric applications of the Yau conjecture, proofreading my work for typos or errors, and beginning the induction proof of the Yau conjecture for all n.

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