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THE DIFFERENTIABLE-INVARIANCE OF THE ALGEBRAIC MULTIPLICITY OF A HOLOMORPHIC VECTOR FIELD

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Abstract

We prove that the algebraic multiplicity of a holomorphic vector field at an isolated singularity is invariant by topological equivalences which are differentiable at the singular point.

1. Introduction

Given a holomorphic curve $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$, singular at $0 \in \mathbb{C}^2$, we define its *algebraic multiplicity* as the degree of the first nonzero jet of f, that is, $\nu(f) = \nu$ where

$$f = f_\nu + f_{\nu+1} + \cdots$$

is the Taylor development of f and $f_\nu \neq 0$. A well known result by Burau [**2**] and Zariski [**15**] states that ν is a *topological invariant*, that is, given $\widetilde{f} : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ and a homeomorphism $\mathrm{h} : U \to \widetilde{U}$ between neighborhoods of $0 \in \mathbb{C}^2$ such that $\mathrm{h}(f^{-1}(0) \cap U) = \widetilde{f}^{-1}(0) \cap V$ then $\nu(f) = \nu(\widetilde{f})$. Consider now a holomorphic vector field Z in $\mathbb{C}^2$ with a singularity at $0 \in \mathbb{C}^2$. If

$$Z = Z_\nu + Z_{\nu+1} + \cdots, \quad Z_\nu \neq 0$$

we define $\nu = \nu(Z)$ as the *algebraic multiplicity* of Z at $0 \in \mathbb{C}^2$. The vector field Z defines a holomorphic foliation by curves $\mathcal{F}$ with isolated singularity in a neighborhood of $0 \in \mathbb{C}^2$ and the algebraic multiplicity $\nu(Z)$ depends only on the foliation $\mathcal{F}$. A natural question, posed by J.F. Mattei is: is $\nu(\mathcal{F})$ a topological invariant of $\mathcal{F}$? In [**3**], the authors give a positive answer if $\mathcal{F}$ is a *generalized curve*, that is, if the desingularization of $\mathcal{F}$ does not contain complex saddle-nodes. If $\mathcal{F}$ is *dicritical*, that is, after a blow up the exceptional divisor is not invariant by the strict transform of $\mathcal{F}$, the conjecture is also true: in this case, it is not difficult to show that the algebraic multiplicity of $\mathcal{F}$ is equal to the index of $\mathcal{F}$ (as defined in [**3**]) along a generic separatrix. Then the topological invariance of the algebraic multiplicity of a dicritical singularity is a consequence of the topological invariance of the index

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along a curve, which is proved in [**3**]. Thus, in this paper we always assume the non-dicritical case. Given foliations $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ with isolated singularities at $0 \in \mathbb{C}^2$, we say that $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are *topologically equivalent* (at $0 \in \mathbb{C}^2$) if there is a homeomorphism $\mathrm{h} : U \to \widetilde{U}$, $\mathrm{h}(0) = 0$ between neighborhoods of $0 \in \mathbb{C}^2$, taking leaves of $\mathcal{F}$ to leaves of $\widetilde{\mathcal{F}}$. Such a homeomorphism is a *topological equivalence* between $\mathcal{F}$ and $\widetilde{\mathcal{F}}$. In this work we impose conditions on the topological equivalence $\mathrm{h} : U \to \widetilde{U}$ and prove the following.

Theorem 1.1. *Let* $\mathrm{h} : U \to \widetilde{U}$ *be a topological equivalence between* $\mathcal{F}$ *and* $\widetilde{\mathcal{F}}$ *and assume that* h *preserves the orientation of* $\mathbb{C}^2$. *Suppose that* h *is differentiable at* $0 \in \mathbb{C}^2$ *and such that* $d\,\mathrm{h}(0) : \mathbb{R}^4 \to \mathbb{R}^4$ *is a real isomorphism. Then the algebraic multiplicities of* $\mathcal{F}$ *and* $\widetilde{\mathcal{F}}$ *are the same.*

Let $\pi : \widehat{\mathbb{C}^2} \to \mathbb{C}^2$ be the blow up at $0 \in \mathbb{C}^2$. Given a complex line P passing through $0 \in \mathbb{C}^2$, we say that P is *regular for* $\mathcal{F}$, if the strict transform of P by π intersects the divisor E at a regular point of the strict transform of $\mathcal{F}$. The following theorem is a key step in the proof of Theorem 1.1.

Theorem 1.2. *Let* $\mathrm{h} : U \to \widetilde{U}$ *be a topological equivalence between* $\mathcal{F}$ *and* $\widetilde{\mathcal{F}}$ *and assume that* h *preserves the orientation of* $\mathbb{C}^2$. *Let* P *and* $\widetilde{P}$ *be two complex lines passing through* $0 \in \mathbb{C}^2$ *which are regular for* $\mathcal{F}$ *and* $\widetilde{\mathcal{F}}$ *respectively. Suppose that* $P \cap U$ *is homeomorphic to a disc and* $\mathrm{h}(P \cap U) = \widetilde{P} \cap \widetilde{U}$. *Then the algebraic multiplicities of* $\mathcal{F}$ *and* $\widetilde{\mathcal{F}}$ *are equal.*

The paper is organized as follows. In section 2 we prove a weaker version of Theorem 1.2. In section 3 we stay and prove a topological lemma, fundamental for the following sections. We prove Theorem 1.2 in section 4. Finally, in section 5 we prove Theorem 1.1.

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2. A first theorem.

Let $\mathrm{h} : U \to \widetilde{U}$ be a topological equivalence between $\mathcal{F}$ and $\widetilde{\mathcal{F}}$. Let $\mathcal{F}_0$ and $\widetilde{\mathcal{F}}_0$ be the strict transforms of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ respectively. Let W and $\widetilde{W}$ be denote the sets $\pi^{-1}(U)$ and $\pi^{-1}(\widetilde{U})$ respectively. Let

$$h : W \backslash E \to \widetilde{W} \backslash E$$

be the homeomorphism defined by $h = \pi^{-1} \mathrm{h}\, \pi$. We have a natural fibration ρ on $\widehat{\mathbb{C}^2}$ which fibers are the strict transforms of the complex lines passing through $0 \in \mathbb{C}^2$. Consider $p, \widetilde{p} \in E$ and let L_p and $L_{\widetilde{p}}$ be the fibers of ρ passing through p and $\widetilde{p}$ respectively. This section is devoted to prove the following.

Theorem 2.1. *Suppose that p and $\widetilde{p}$ are regular points of $\mathcal{F}_0$ and $\widetilde{\mathcal{F}}_0$ respectively. Let Ω be a neighborhood of p in $\widehat{\mathbb{C}^2}$. Suppose that h extends to $(W\backslash E) \cup \Omega$ as a homeomorphism onto its image, such that $h(L_p \cap W) = L_{\widetilde{p}} \cap \widetilde{W}$. Then the algebraic multiplicities of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are the same.*

Let ν be the algebraic multiplicity of $\mathcal{F}$ at 0 and let $p_1, ..., p_k$ be the singularities of $\mathcal{F}_0$ on E. We have the following relation due to Ven Den Essen (see [**9**], appendix I):

$$\sum_{i=1}^{k} \mu(\mathcal{F}_0, p_i) = \mu(\mathcal{F}, 0) - \nu^2 + \nu + 1,$$

where $\mu(\mathcal{F}, p)$ is the Milnor number of $\mathcal{F}$ at p. Let $s = \sum_{i=1}^{k} \mu(\mathcal{F}_0, p_i)$. In the same way, let $\widetilde{s}$ be the sum of the Milnor numbers of the singularities on E of $\widetilde{\mathcal{F}}_0$. Then, since the Milnor number is a topological invariant, it is sufficient to prove that $s = \widetilde{s}$.

Let $\mathcal{D} \subset E \cap \Omega$ be a closed disc containing p, which does not contain singularities of $\mathcal{F}_0$ and such that $h(\mathcal{D})$ does not contains singularities of $\widetilde{\mathcal{F}}_0$. Let D and $\widetilde{D}$ be the closed discs in E equal to the closure of $E\backslash\mathcal{D}$ and $E\backslash h(\mathcal{D})$ respectively. Then h maps $W\backslash D$ homeomorphically onto $\widetilde{W}\backslash\widetilde{D}$, and the interiors of D and $\widetilde{D}$ contain all the singularities of $\mathcal{F}_0$ and $\widetilde{\mathcal{F}}_0$ respectively. Observe that h is a topological equivalence between $\mathcal{F}_0|_{W\backslash D}$ and $\widetilde{\mathcal{F}}_0|_{\widetilde{W}\backslash\widetilde{D}}$. Since $h(L_p \cap W) = L_{\widetilde{p}} \cap \widetilde{W}$, we have the homeomorphism

$$h : (W\backslash D)\backslash L_p \to (\widetilde{W}\backslash\widetilde{D})\backslash L_{\widetilde{p}}.$$

We know that $W\backslash L_p$ and $\widetilde{W}\backslash L_{\widetilde{p}}$ are isomorphic to $\mathbb{C}^2$, where the divisor can be represented by the vertical line $\{z_1 = 0\}$ and the sets $W\backslash L_p$ and $\widetilde{W}\backslash L_{\widetilde{p}}$ give neighborhoods V and $\widetilde{V}$ of $\{z_1 = 0\}$. Thus, we may think that the foliations $\mathcal{F}_0$ and $\widetilde{\mathcal{F}}_0$ are defined on the sets V and $\widetilde{V}$ in $\mathbb{C}^2$, and that

$$h : V\backslash D \subset \mathbb{C}^2 \to \widetilde{V}\backslash\widetilde{D} \subset \mathbb{C}^2$$

is a topological equivalence between $\mathcal{F}_0$ and $\widetilde{\mathcal{F}}_0$. Observe that $\mathcal{F}_0$ is globally defined by a holomorphic vector field on V and the same holds for $\widetilde{\mathcal{F}}_0$ on $\widetilde{V}$. The disc D is contained in $\{z_1 = 0\}$ and we may assume that $D = \{(0, z_2) : |z_2| \leq r\}$, where $r > 0$.

We proceed now to compute s. Let Z be a holomorphic vector field which generates the foliation $\mathcal{F}_0$ on V. Let B be a neighborhood of D homeomorphic to a ball, such that ∂B is homeomorphic to S^3 and $\overline{B} \subset V$. It is well known that the Milnor number is just the Poincar-Hopf index (considering the holomorphic vector field as a real vector field). Then, since all the singularities of $\mathcal{F}_0$ are contained in B, we have ([**10**],p. 36) that the sum of the Milnor numbers of the singularities of $\mathcal{F}_0$ is equal to the degree of the map

$$\frac{Z}{||Z||} : \partial B \to \mathbb{S}^3,$$

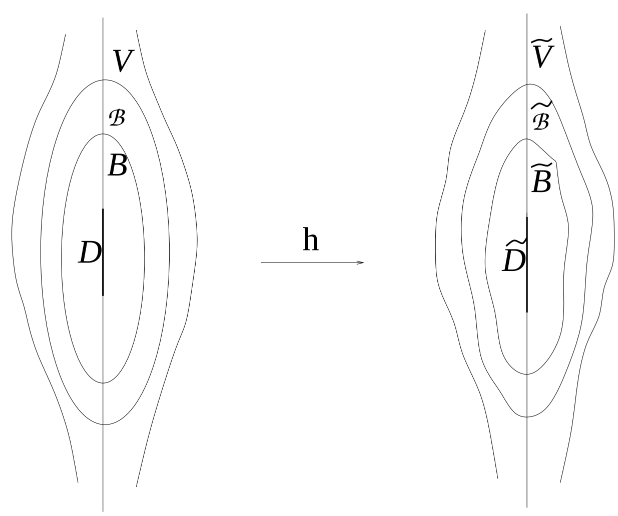
$$\frac{Z}{||Z||}(z) = \frac{Z(z)}{||Z(z)||}.$$

Let $\mathcal{B}$ be a neighborhood of $\overline{B}$ homeomorphic to a ball and such that $\overline{\mathcal{B}} \subset V$. Since V is a neighborhood of $\{z_1 = 0\}$, for $\varepsilon > 0$ small enough, the set $\{|z_1| < 2\varepsilon, |z_2| < 4r\}$, which contains D, is contained in V. Then, we may chose B and $\mathcal{B}$ such that

$$\mathcal{B} \subset \{|z_1| < \varepsilon, |z_2| < 2r\}.$$

The last hypothesis will be only used in the proof of Lemma 2.5.

Consider the sets $\widetilde{B} = h(B \backslash D) \cup \widetilde{D}$, $\widetilde{\mathcal{B}} = h(\mathcal{B} \backslash D) \cup \widetilde{D}$ and $\widetilde{V} = h(V \backslash D) \cup \widetilde{D}$. It is easy to see that $\widetilde{B}$, $\widetilde{\mathcal{B}}$ and $\widetilde{V}$ are neighborhoods of $\widetilde{D}$ in $\mathbb{C}^2$.



Let

$$\varphi : \overline{\mathbb{D}}_\varepsilon \times \overline{\mathcal{B}} \to V \subset \mathbb{C}^2$$

and

$$\widetilde{\varphi} : \overline{\mathbb{D}}_\varepsilon \times \widetilde{\mathcal{B}} \to \widetilde{V} \subset \mathbb{C}^2$$

be the local complex flows of Z and $\widetilde{Z}$ respectively, where $\mathbb{D}_\varepsilon = \{T \in \mathbb{C} : ||T|| < \varepsilon\}$ with ε small enough. Now, we follow the ideas used in [**3**] to prove the topological invariance of the Milnor number.

Lemma 2.2. *There exists continuous functions $\tau : \mathcal{B}\backslash D \to (0, \varepsilon)$ and $\widetilde{\tau} : h(\mathcal{B}\backslash D) \to \mathbb{D}_\varepsilon\backslash\{0\}$ such that for all $z \in \mathcal{B}\backslash D$ we have:*

(i) $\varphi(\tau(z), z) \in \mathcal{B}\backslash D$.

(ii) $\varphi(t\tau(z), z) \neq z$, *for any* $t \in (0, 1]$.

(iii) $h(\varphi(\tau(z), z)) = \widetilde{\varphi}(\widetilde{\tau}(h(z)), h(z))$.

We say that a function $f : U \to \mathbb{R}$ is *lower(upper) semi-continuous* if given $\epsilon > 0$ and $x_0 \in U$, there is a neighborhood Ω of x_0 in U such that $f(x) \geq f(x_0) - \epsilon$ ($f(x) \leq f(x_0) + \epsilon$) for all $x \in \Omega$. We need the following lemma.

Lemma 2.3. *Let U be an open set in $\mathbb{R}^n$ and let $f : U \to \mathbb{R}$ and $g : U \to \mathbb{R}$ be an upper and a lower semicontinuous function respectively. Suppose that $f < g$. Then there exists a continuous function $h : U \to \mathbb{R}$ such that $f < h < g$. In particular, if g is a strictly positive lower semicontinuous function, then there exists a continuous function h such that $0 < h < g$.*

Proof of Lemma 2.2. Clearly, given $z \in \mathcal{B}\backslash D$ there exists $\delta > 0$ such that $\varphi(*, z)$ is injective on $\mathbb{D}_\delta$. Then define $\delta(z) > 0$ as the supremum of $\delta' \leq \varepsilon$ such that $\varphi(*, z)$ is injective on $\mathbb{D}_{\delta'}$.

Assertion 1. *The function $\delta : \mathcal{B}\backslash D \to (0, \varepsilon]$ is lower semicontinuous.*

Proof. Fix $z_0 \in \mathcal{B}\backslash D$ and let $\epsilon > 0$. We will prove that for z close enough to z_0 we have $\delta(z) \geq \delta(z_0) - \epsilon$. Suppose by contradiction that for $z_k \to z_0$ we have that $\varphi(*, z_k)$ is not injective on $\mathbb{D}_{\delta(z_0)-\epsilon}$. Then there are points t_k, t'_k in $\mathbb{D}_{\delta(z_0)-\epsilon}$, with $t_k \neq t'_k$ and such that $\varphi(t_k, z_k) = \varphi(t'_k, z_k)$ for all k. By taking a subsequence we may assume that $t_k \to a$ and $t'_k \to a'$ with $a, a' \in \overline{\mathbb{D}}_{\delta(z_0)-\epsilon} \subset \mathbb{D}_{\delta(z_0)}$. By continuity we have

$$\varphi(a, z_0) = \lim_{k\to\infty} \varphi(t_k, z_k) = \lim_{k\to\infty} \varphi(t'_k, z_k) = \varphi(a', z_0)$$

and, since $\varphi(*, z_0)$ is injective on $\mathbb{D}_{\delta(z_0)}$, we deduce that $a = a'$. Let $z' = \varphi(a, z_0)$ and take a neighborhood Ω of z' and $\delta_0 > 0$ such that $\varphi(*, z)$ is injective on $\mathbb{D}_{\delta_0}$ for all $z \in \Omega$. For k big enough we have that $\varphi(a, z_k) \in \Omega$ and $(t_k - a), (t'_k - a') \in \mathbb{D}_{\delta_0}$. Then, since

$$\varphi(t_k - a, \varphi(a, z_k)) = \varphi(t_k, z_k) = \varphi(t'_k, z_k) = \varphi(t'_k - a', \varphi(a', z_k)),$$

we have that $t_k - a = t'_k - a'$, hence $t_k = t'_k$, which is a contradiction.

Assertion 2. *Consider $\bar{\delta} : \mathcal{B}\backslash D \to (0, \varepsilon]$, where $\bar{\delta}(z)$ is the supremum of $\delta' < \varepsilon$ such that $\varphi(\mathbb{D}_{\delta'}, z) \subset \mathcal{B}\backslash D$. Then $\bar{\delta}$ is a lower semicontinuous function.*

Proof. Fix z_0 and let $\epsilon > 0$. The set $\varphi(\overline{\mathbb{D}}_{\bar{\delta}(z_0)-\epsilon}, z_0)$ is compact and is contained in $\mathcal{B}\backslash D$. If z is close enough to z_0 we have that $\varphi(\overline{\mathbb{D}}_{\bar{\delta}(z_0)-\epsilon}, z)$ is also contained in $\mathcal{B}\backslash D$. Then $\bar{\delta}(z) \geq \bar{\delta}(z_0) - \epsilon$ and it follows that $\bar{\delta}$ is lower semicontinuous.

Consider $\widetilde{\delta} : h(\mathcal{B}\backslash D) \to (0, \varepsilon]$, where $\widetilde{\delta}(w)$ is the supremum of $\delta' < \varepsilon$ such that $\widetilde{\varphi}(*, w)$ is injective on $\mathbb{D}_{\delta'}$. As in Assertion 1, we can prove that $\widetilde{\delta}$ is a lower semicontinuous function.

Assertion 3. *Define $\hat{\delta} : \mathcal{B}\backslash D \to (0, \varepsilon]$, where $\hat{\delta}(z)$ is the supremum of $\delta' < \varepsilon$ such that $h(\varphi(\mathbb{D}_{\delta'}, z))$ is contained in $\widetilde{\varphi}(\mathbb{D}_{\widetilde{\delta}(h(z))}, h(z))$. Then $\hat{\delta}$ is a lower semicontinuous function.*

Proof. Fix z_0 and let $\epsilon > 0$. Since $h(\varphi(\mathbb{D}_{\hat{\delta}(z_0)}, z_0))$ is contained in $\widetilde{\varphi}(\mathbb{D}_{\widetilde{\delta}(h(z_0))}, h(z_0))$, there is $\epsilon' > 0$ such that $h(\varphi(\overline{\mathbb{D}}_{\hat{\delta}(z_0)-\epsilon}, z_0))$ is contained in $\widetilde{\varphi}(\mathbb{D}_{\widetilde{\delta}(h(z_0))-\epsilon'}, h(z_0))$. Let Σ be a disc passing through $h(z_0)$ and transverse to the foliation. Since $\widetilde{\delta}$ is lower semicontinuous, we may take Σ small enough such that $\widetilde{\varphi}(*, z)$ is injective on $\overline{\mathbb{D}}_{\widetilde{\delta}(h(z_0))-\epsilon'}$ for all $z \in \Sigma$. Moreover, we may take Σ small enough such that $\widetilde{\varphi}$ is injective on $\mathbb{D}_{\widetilde{\delta}(h(z_0))-\epsilon'} \times \Sigma$. Let M denote the open set $\widetilde{\varphi}(\mathbb{D}_{\widetilde{\delta}(h(z_0))-\epsilon'} \times \Sigma)$ and let $M' = \widetilde{\varphi}(\mathbb{D}_{\epsilon'/2} \times \Sigma)$. We may take a neighborhood Ω of z_0 such that $h(\Omega) \subset M'$ and $\widetilde{\delta}(h(z)) \geq \widetilde{\delta}(h(z_0)) - \epsilon'/2$ for all $z \in \Omega$, because $\widetilde{\delta}$ is lower semicontinuous. Since $h(\varphi(\overline{\mathbb{D}}_{\hat{\delta}(z_0)-\epsilon}, z_0))$ is compact and is contained in M, we may assume Ω small enough such that $h(\varphi(\overline{\mathbb{D}}_{\hat{\delta}(z_0)-\epsilon}, z))$ is contained in M for all $z \in \Omega$. Fix $z \in \Omega$. Since $h(z) \in M'$, there is $w' \in \Sigma$ and t', with $|t'| < \epsilon'/2$, such that $h(z) = \widetilde{\varphi}(t', w')$. Since $h(\varphi(\overline{\mathbb{D}}_{\hat{\delta}(z_0)-\epsilon}, z))$ is contained in M, we deduce that it is contained in $\widetilde{\varphi}(\mathbb{D}_{\widetilde{\delta}(h(z_0))-\epsilon'}, w')$. Then, given w in $h(\varphi(\overline{\mathbb{D}}_{\hat{\delta}(z_0)-\epsilon}, z))$, we have that $w = \widetilde{\varphi}(t'', w')$ with $|t''| < \widetilde{\delta}(h(z_0)) - \epsilon'$. Thus

$$w = \widetilde{\varphi}(t'', w') = \widetilde{\varphi}(t'' - t', \widetilde{\varphi}(t', w')) = \widetilde{\varphi}(t'' - t', h(z)),$$

where $|t''-t'| \leq |t''|+|t'| < \widetilde{\delta}(h(z_0))-\epsilon'+\epsilon'/2 = \widetilde{\delta}(h(z_0))-\epsilon'/2 \leq \widetilde{\delta}(h(z))$. Then $h(\varphi(\overline{\mathbb{D}}_{\hat{\delta}(z_0)-\epsilon}, z))$ is contained in $\widetilde{\varphi}(\mathbb{D}_{\widetilde{\delta}(h(z))}, h(z))$ and it follows that $\hat{\delta}$ is lower semicontinuous.

It is easy to see that the function $g = \min\{\delta, \bar{\delta}, \hat{\delta}\}$ is also lower semicontinuous. Then, by Lemma 2.3, there exists a positive continuous function τ on $\mathcal{B}\backslash D$ such that $\tau < \delta, \bar{\delta}, \hat{\delta}$. By the definition of $\bar{\delta}$, (i) is satisfied. Since $\varphi(*, z)$ is injective on $\mathbb{D}_{\bar{\delta}}$ and $\tau(z) \in \mathbb{D}_{\bar{\delta}}$, we have that (ii) holds. Now, we shall define $\widetilde{\tau}$. Let $w = h(z) \in h(\mathcal{B}\backslash D)$. Since $\tau < \hat{\delta}$, we have that $h(\varphi(\tau(z), z))$ is contained in $\widetilde{\varphi}(\mathbb{D}_{\widetilde{\delta}(h(z))}, h(z))$

and by injectivity there exists a unique $\widetilde{\tau}(h(z))$ in $\mathbb{D}_{\widetilde{\delta}(h(z))}$ such that $h(\varphi(\tau(z),z)) = \widetilde{\varphi}(\widetilde{\tau}(h(z)),h(z))$. Now, it is easy to see that $\widetilde{\tau}$ is continuous and therefore (iii) holds. q.e.d.

Proof of Lemma 2.3. Consider $x \in U$ and $a_x \in \mathbb{R}$, such that $f(x) < a_x < g(x)$. It follows from the definition of lower and upper semicontinuous function that there exists a neighborhood V_x of x in U such that $f(y) < a_x < g(y)$ for all $y \in V_x$. We may take a subset $I \subset U$, such that $U \subset \bigcup_{i\in I} V_i$ and $\{V_i\}_{i\in I}$ is locally finite. Thus, we have $f(x) < a_i < g(x)$ for all $x \in V_i$. Let $\{\psi_i\}_{i\in I}$ be a partition of the unity subordinate to $\{V_i\}_{i\in I}$. Then, we define $h : U \to \mathbb{R}$ by

$$h(x) = \sum_{i\in I} \psi_i(x) a_i.$$

Clearly, h is continuous. If $x \in V_i$, then $f(x) < a_i < g(x)$, hence $\psi_i(x)f(x) < \psi_i(x)a_i < \psi_i(x)g(x)$ and it follows that $f < h < g$. q.e.d.

From Lemma 2.2, we have the maps

$$f : \mathcal{B}\backslash D \to \mathcal{B}\backslash D,$$
$$f(z) = \varphi(\tau(z), z)$$

and

$$\widetilde{f} : \widetilde{\mathcal{B}}\backslash \widetilde{D} \to \widetilde{\mathcal{B}}\backslash \widetilde{D},$$
$$\widetilde{f}(w) = \widetilde{\varphi}(\widetilde{\tau}(w), w)$$

with

$$h \circ f = \widetilde{f} \circ h$$

and such that f and $\widetilde{f}$ are without fixed points.

There exists $\psi, \widetilde{\psi} : \mathbb{C}^2 \to \mathbb{C}^2$ with the following properties:

- (i) $\psi(D) = 0$ and $\widetilde{\psi}(\widetilde{D}) = 0$.
- (ii) $\psi : \mathbb{C}^2\backslash D \to \mathbb{C}^2\backslash\{0\}$ and $\widetilde{\psi} : \mathbb{C}^2\backslash \widetilde{D} \to \mathbb{C}^2\backslash\{0\}$ are homeomorphisms.
- (iii) ψ and $\widetilde{\psi}$ are equal to the identity out of B and $\widetilde{B}$ respectively.

We define

$$\begin{aligned} f' &= \psi f \psi^{-1} : \mathcal{B}\backslash\{0\} \to \mathcal{B}\backslash\{0\} \subset \mathbb{C}^2, \\ \widetilde{f}' &= \widetilde{\psi}\widetilde{f}\widetilde{\psi}^{-1} : \widetilde{\mathcal{B}}\backslash\{0\} \to \widetilde{\mathcal{B}}\backslash\{0\} \subset \mathbb{C}^2, \\ h' &= \widetilde{\psi} h \psi^{-1} : V \to \widetilde{V}. \end{aligned}$$

Then we have the following:

- (i) f' and $\widetilde{f}'$ do not have fixed points.
- (ii) On ∂B, we have $f' = f$ and $\widetilde{f}' = \widetilde{f}$.
- (iii) h' is a homeomorphism with $h'(0) = 0$ and such that $h' \circ f' = \widetilde{f}' \circ h'$.

Thus, there are well defined maps:

$$\begin{aligned}(f' - \mathrm{id}) &: \mathcal{B}\backslash\{0\} \to \mathbb{C}^2\backslash\{0\},\\ (\widetilde{f}' - \mathrm{id}) &: \widetilde{\mathcal{B}}\backslash\{0\} \to \mathbb{C}^2\backslash\{0\}.\end{aligned}$$

Observe that $H_3(\mathcal{B}\backslash\{0\}) \subset H_3(\mathbb{C}^2\backslash\{0\})$ and this inclusion is an isomorphism between the groups. Then $(f' - \mathrm{id})$ induces a map

$$(f' - \mathrm{id})_* : H_3(\mathbb{C}^2\backslash\{0\}) \to H_3(\mathbb{C}^2\backslash\{0\})$$

at the homology level.

Lemma 2.4. $(f' - \mathrm{id})_*$ *is the multiplication by* s.

Proof. We have that $\partial B \subset \mathcal{B}$ is a generator of $H_3(\mathbb{C}^2\backslash\{0\})$. It is known that, homologically:

$$(f' - \mathrm{id})(\mathbb{S}^3) = (f' - \mathrm{id})(\partial B) = n\mathbb{S}^3,$$

where n is the degree of the map:

$$g : \partial B \to \mathbb{S}^3,$$

$$g(z) = \frac{(f' - \mathrm{id})}{||(f' - \mathrm{id})||}(z).$$

Thus, it is sufficient to prove that $\deg(g) = s$. Observe that $g = \frac{(f-\mathrm{id})}{||(f-\mathrm{id})||}$, since $f' = f$ on ∂B. By (ii) of Lemma 2.2 the map

$$G : [0, 1] \times \partial B \to \mathbb{S}^3,$$

$$G(t, z) = \frac{\varphi(t\tau(z), z) - z}{||\varphi(t\tau(z), z) - z||}, \quad t \neq 0,$$

$$G(0, z) = \frac{\tau(z)}{||\tau(z)||} \cdot \frac{Z(z)}{||Z(z)||}$$

is well defined. Evidently, $G(1, z) = g(z)$. On the other hand:

$$\begin{aligned}\lim_{t\to 0} G(t, z) &= \frac{\tau(z)}{||\tau(z)||} \lim_{t\to 0} \left|\left|\frac{\varphi(t\tau(z), z) - z}{t\tau(z)}\right|\right|^{-1} \cdot \lim_{t\to 0} \frac{\varphi(t\tau(z), z) - z}{t\tau(z)}\\ &= \frac{\tau(z)}{||\tau(z)||} \lim_{s\to 0} \left|\left|\frac{\varphi(s, z) - z}{s}\right|\right|^{-1} \cdot \lim_{s\to 0} \frac{\varphi(s, z) - z}{s}\\ &= \frac{\tau(z)}{||\tau(z)||} \cdot \frac{Z(z)}{||Z(z)||}.\end{aligned}$$

It follows that G is continuous and therefore is a homotopy between $g(z)$ and $G(0, z) = \frac{\tau(z)}{||\tau(z)||} \cdot \frac{Z(z)}{||Z(z)||}$. Now, since $\pi_3(\mathbb{S}^1) = \{0\}$, the map $\tau/|\tau| : \partial B \to \mathbb{S}^1$ is homotopic to the constant $1 \in \mathbb{S}^1$ and g is homotopic to $Z/||Z||$. Therefore $\deg(g) = \deg(Z/||Z||) = s$.

In the same way, we have that

$$(\widetilde{f}' - \mathrm{id})_* : H_3(\mathbb{C}^2\backslash\{0\}) \to H_3(\mathbb{C}^2\backslash\{0\})$$

is the multiplication by $\widetilde{s}$.

Let

$$h'_* : H_3(\mathbb{C}^2\backslash\{0\}) \to H_3(\mathbb{C}^2\backslash\{0\})$$

be the isomorphism induced by h'. Clearly, the following lemma implies Theorem 2.1.

Lemma 2.5. *The following diagram commutes:*

$$\begin{array}{ccc} H_3(\mathbb{C}^2\backslash\{0\}) & \xrightarrow{(f'-\mathrm{id})_*} & H_3(\mathbb{C}^2\backslash\{0\}) \\ \downarrow h'_* & & \downarrow h'_* \\ H_3(\mathbb{C}^2\backslash\{0\}) & \xrightarrow{(\widetilde{f}'-\mathrm{id})_*} & H_3(\mathbb{C}^2\backslash\{0\}) \end{array}$$

Proof. Recall that $\mathcal{B}$ was chosen such that

$$\mathcal{B} \subset \{|z_1| < \varepsilon, |z_2| < 2r\} \subset \{|z_1| < 2\varepsilon, |z_2| < 4r\} \subset V.$$

Since $h' \circ f' = \widetilde{f}' \circ h'$ we have $(\widetilde{f}' - \mathrm{id}) \circ h' = \widetilde{f}' \circ h' - h' = h' \circ f' - h'$. It is sufficient to prove that $h' \circ f' - h'$ and $h' \circ (f' - \mathrm{id}) : \mathcal{B}\backslash\{0\} \to \mathbb{C}^2\backslash\{0\}$ are homotopic. For any $z \in \mathcal{B}\backslash\{0\}$ and $t \in [0,1]$ we have that $f'(z), (1-t)z \in \mathbb{D}_\epsilon \times \mathbb{D}_{2r}$. Then $(f'(z) + (1-t)z)$ is contained in $\mathbb{D}_{2\epsilon} \times \mathbb{D}_{4r} \subset V$. Therefore, the map:

$$F : [0,1] \times (\mathcal{B}\backslash\{0\}) \to \mathbb{C}^2\backslash\{0\},$$

$$F(t,z) = h'(f'(z) - (1-t)z) - h'(tz)$$

is well defined. F is continuous and $F(t,z) \neq 0$ for all $(t,z) \in [0,1] \times (\mathcal{B}\backslash\{0\})$ because $F(t,z) = 0$ implies $h'(f'(z) - (1-t)z) = h'(tz)$ and since h' is a homeomorphism $f'(z) - (1-t)z = tz$, hence $f'(z) = z$, which contradicts $f'(z) \neq z$. Thus F is a homotopy between $h' \circ f' - h'$ and $h' \circ (f' - \mathrm{id})$. q.e.d.

3. A topological fact.

Let M be a complex manifold. We say that $\mathcal{D}$ is a complex disc in M, if $\mathcal{D} \subset M$ and there is a map $f : \overline{\mathbb{D}} \to M$, which is a homeomorphism onto $\mathcal{D}$ and is holomorphic on $\mathbb{D}$. Let V be any subset of M containing $\partial\mathcal{D}$. The map $f|_{S^1} : S^1 \to \partial\mathcal{D} \subset M$ defines a 1-cycle in V and represents an element in $H_1(V)$ which does not depend on f. We denote this 1-cycle by $\partial\mathcal{D}$ independently of the set V. For simplicity, we write $\gamma = \gamma'$ in $H_1(M)$ for means that the 1-cycles γ and γ' represents the same element in the group $H_1(M)$. Let $\pi : \widehat{\mathbb{C}^2} \to \mathbb{C}^2$ be the blow up at $0 \in \mathbb{C}^2$ and let $E = \pi^{-1}(0)$. Let $\rho : \widehat{\mathbb{C}^2} \to E$ be the natural projection. (If L is the strict transform by π of a complex line passing through $0 \in \mathbb{C}^2$, then $\rho(L) = L \cap E$.) The following Lemma is a reason for assuming that the topological equivalence h preserves the orientation of $\mathbb{C}^2$.

Lemma 3.1. *Let* $\mathrm{h} : U \to U'$ *be a homeomorphism, where* U *and* U' *are neighborhoods of* $0 \in \mathbb{C}^2$ *homeomorphic to balls. Let* P *and* P' *be two complex lines passing through* $0 \in \mathbb{C}^2$*. Suppose that* $P \cap U$ *is homeomorphic to a disc and* $\mathrm{h}(P \cap U) = P' \cap U'$*. Let* L *and* L' *be the strict transforms by* π *of* P *and* P' *respectively. Let* p *and* p' *be the points of intersection of* L *and* L' *with* E *respectively. Denote by* W *and* W' *the sets* $\pi^{-1}(U)$ *and* $\pi^{-1}(U')$ *in* $\widehat{\mathbb{C}^2}$ *and let* $h : W\backslash E \to W'\backslash E$ *be the homeomorphism defined by* $h = \pi^{-1}\,\mathrm{h}\,\pi$*. Let* $V \subset W$ *be a neighborhood of* p *and let*

$$\varphi : \mathbb{D} \times \mathbb{D} \to V$$

be a biholomorphism such that $\varphi(\{0\} \times \mathbb{D}) = L \cap V$ *and* $\varphi(\mathbb{D} \times \{0\}) = E \cap V$*. Let* r *with* $0 < r < 1$ *and consider the disc* $\mathcal{B}_w = \varphi(w, |z| \le r)$*, where* $w \in \mathbb{D}$*. Let* Ω *be a neighborhood of* p' *in* E*, homeomorphic to a disc. Let* $V' \subset \widehat{\mathbb{C}^2}$ *be the set* $\rho^{-1}(\Omega)$*. Let* $\mathcal{A}' \subset V'\backslash E$ *and* $\mathcal{B}' \subset V'\backslash L'$ *be complex discs transverse to* L' *and* E *respectively. Then, for* $|w|$ *small enough we have the following:*

(*i*) *If* h *preserves the orientation of* $\mathbb{C}^2$*, then*

$$h(\partial\mathcal{B}_w) = \xi\partial\mathcal{B}' \quad in \quad H_1(V'\backslash(L' \cup E)),$$

where $\xi = +1$ *or* -1*.*

(*ii*) *If* h *inverts the orientation of* $\mathbb{C}^2$*, then*

$$h(\partial\mathcal{B}_w) = -2\xi\partial\mathcal{A}' + \xi\partial\mathcal{B}' \quad in \quad H_1(V'\backslash(L' \cup E)),$$

where $\xi = +1$ *or* -1*.*

Remark. With some hypothesis on the foliation $\mathcal{F}$, we have in fact that the topological equivalence h necessarily preserves the orientation of $\mathbb{C}^2$. Precisely, we have the following.

Proposition 3.2. *Let* $\mathcal{F}$ *be a holomorphic foliation by curves on* U *which has* $0 \in \mathbb{C}^2$ *as its unique singularity. Suppose that* $\mathcal{F}$ *has three smooth and transverse separatrices. Suppose that* $\widetilde{\mathcal{F}}$ *is another holomorphic foliation of a neighborhood* $\widetilde{U}$ *of* $0 \in \mathbb{C}^2$ *and let*

$$\mathrm{h} : U \to \widetilde{U}$$

be a topological equivalence between $\mathcal{F}$ *and* $\widetilde{\mathcal{F}}$*. Then* h *preserves the orientation of* $\mathbb{C}^2$*.*

Let $U \subset \mathbb{C}^2$ be an open set homeomorphic to a ball. Let P be a complex line in $\mathbb{C}^2$ and suppose that $U \cap P$ is homeomorphic to a disc. It follows by Alexander's duality theorem that $H_1(U\backslash P) \simeq \mathbb{Z}$. Let $\mathcal{D} \subset \mathbb{C}^2$ be a complex disc transverse to P. The 1-cycle $\partial\mathcal{D}$ represents an element in $H_1(U\backslash P) \simeq \mathbb{Z}$, which does not depends on the disc $\mathcal{D}$. We know that $\partial\mathcal{D}$ is a generator of the group and we say that it is the *positive generator of* $H_1(U\backslash P)$. Given a homeomorphism $f : M \to M'$, where M and M' are oriented manifolds, we define $\deg(f)$ to be 1 or -1 depending on whether f preserves or reverses orientation.

Lemma 3.3. *Let* $\mathrm{h} : U \to U'$ *be a homeomorphism, where* U *and* U' *are neighborhoods of* $0 \in \mathbb{C}^2$ *homeomorphic to balls. Let* P *and* P' *be two complex lines passing through* $0 \in \mathbb{C}^2$. *Suppose that* $P \cap U$ *is homeomorphic to a disc and* $\mathrm{h}(P \cap U) = P' \cap U'$. *Let* a *and* a' *be 1-cycles in* $U \backslash P$ *and* $U' \backslash P'$ *representing the positive generators of* $H_1(U \backslash P)$ *and* $H_1(U' \backslash P')$ *respectively. Then*

$$\mathrm{h}(a) = \deg(\mathrm{h}) \deg(\mathrm{h}\,|_P) a' \quad \textit{in} \quad H_1(U' \backslash P').$$

Proof of Lemma 3.1. If $\mathcal{B}'' \subset V' \backslash L'$ is any complex disc transverse to E, we have that $\partial \mathcal{B}''$ is homologous $\partial \mathcal{B}'$ in $H_1(V' \backslash (L' \cup E))$. Thus, we may change the disc $\mathcal{B}'$ if necessary and assume that it is contained in W'. Let b' be the 1-cycle defined by $b' = \pi(\partial \mathcal{B}')$. Then, since $\pi(\mathcal{B}') \subset U'$ is a complex disc transverse to P' and $\pi(\partial \mathcal{B}') = \partial \pi(\mathcal{B}')$, we have that b' is a positive generator of $H_1(U' \backslash P')$. Analogously, if $b = \pi(\partial \mathcal{B}_w)$, we deduce that b is a positive generator of $H_1(U \backslash P)$. It follows from Lemma 3.3 that:

$$\mathrm{h}(b) = \psi \xi b' \quad \text{in} \quad H_1(U' \backslash P'),$$

where $\psi = \deg(\mathrm{h})$ and $\xi = \deg(\mathrm{h}\,|_P)$. Then, since $\pi^{-1} : U' \backslash P' \to W' \backslash (L' \cup E)$ is well defined, we have that

$$\pi^{-1}(\mathrm{h}(b)) = \psi \xi \pi^{-1}(b') \quad \text{in} \quad H_1(W' \backslash (L' \cup E))$$

and thus

$$h(\partial \mathcal{B}_w) = \psi \xi \partial \mathcal{B}' \quad \text{in} \quad H_1(W' \backslash (L' \cup E)). \tag{1}$$

Observe that $\pi(\mathcal{A}')$ is a complex disc transverse to P'. Then the cycle $\partial \pi(\mathcal{A}') = \pi(\partial \mathcal{A}')$ represents the positive generator of $H_1(U' \backslash P')$. Thus, we deduce that $\pi(\partial \mathcal{A}') = \pi(\partial \mathcal{B}')$ in $H_1(U' \backslash P')$ and therefore

$$\partial \mathcal{A}' = \partial \mathcal{B}' \quad \text{in} \quad H_1(W' \backslash (L' \cup E)). \tag{2}$$

Let $\mathcal{C}$ be the disc $\varphi(0, |z| \leq r)$ in L. Let $\mathcal{C}'$ be a disc in L' containing p'. Since h maps $\mathcal{C}$ homeomorphically into L' with $h(p) = p'$, the cycle $h(\partial \mathcal{C})$ is a generator of the group $H_1(L' \backslash \{p'\})$ and we have $h(\partial \mathcal{C}) = \deg(h|_L) \partial \mathcal{C}'$. Thus, since $h|_L$ preserves orientation if an only if $\mathrm{h}\,|_P$ does, we have that $h(\partial \mathcal{C}) = \xi \partial \mathcal{C}'$ in $H_1(L' \backslash \{p'\})$. Since $L' \backslash \{p'\}$ is contained in $V' \backslash E$, we conclude that

$$h(\partial \mathcal{C}) = \xi \partial \mathcal{C}' \quad \text{in} \quad H_1(V' \backslash E). \tag{3}$$

Observe that $\partial \mathcal{C}' = \partial \mathcal{B}'$ in $H_1(V' \backslash E)$. Moreover, $\partial \mathcal{C} = \varphi(0, |z| = r)$ is homologous to $\partial \mathcal{B}_w = \varphi(w, |z| = r)$ in the set $T = \varphi(|z| \leq |w|, |z| = r)$. It is easy to see that for $|w|$ small enough, the set $h(T)$ is contained in $V'\backslash E$. Then $h(\partial\mathcal{C})$ and $h(\partial\mathcal{B}_w)$ are homologous in $V'\backslash E$. It follows from (3) and the observations above that for $|w|$ small enough:

$$h(\partial\mathcal{B}_w) = \xi\partial\mathcal{B}' \quad \text{in} \quad H_1(V'\backslash E). \tag{4}$$

We know that there exists integers n and m such that

$$h(\partial\mathcal{B}_w) = n\partial\mathcal{A}' + m\partial\mathcal{B}' \quad \text{in} \quad H_1(V'\backslash(L' \cup E)).$$

Then, since $V'\backslash(L' \cup E) \subset V'\backslash E$:

$$h(\partial\mathcal{B}_w) = n\partial\mathcal{A}' + m\partial\mathcal{B}' \quad \text{in} \quad H_1(V'\backslash E),$$

hence

$$h(\partial\mathcal{B}_w) = m\partial\mathcal{B}' \quad \text{in} \quad H_1(V'\backslash E),$$

because $\partial\mathcal{A}' = 0$ in $H_1(V'\backslash E)$. From this and (4) we conclude that $m = \xi$. Then

$$h(\partial\mathcal{B}_w) = n\partial\mathcal{A}' + \xi\partial\mathcal{B}' \quad \text{in} \quad H_1(V'\backslash(L' \cup E))$$

and, since $V'\backslash(E \cup L')$ is contained in $W'\backslash(E \cup L')$, we have that

$$h(\partial\mathcal{B}_w) = n\partial\mathcal{A}' + \xi\partial\mathcal{B}' \quad \text{in} \quad H_1(W'\backslash(L' \cup E)). \tag{5}$$

From (2) we have $\partial\mathcal{A}' = \partial\mathcal{B}'$ in $H_1(W'\backslash(L' \cup E))$. Replacing in (5) we obtain:

$$h(\partial\mathcal{B}_w) = n\partial\mathcal{B}' + \xi\partial\mathcal{B}' \quad \text{in} \quad H_1(W'\backslash(L' \cup E)).$$

Thus, from (1) we have:

$$\psi\xi\partial\mathcal{B}' = n\partial\mathcal{B}' + \xi\partial\mathcal{B}' \quad \text{in} \quad H_1(W'\backslash(L' \cup E))$$

and therefore $n = (\psi - 1)\xi$. This proves the Lemma. q.e.d.

Proof of Proposition 3.2. It is known that the germ of three smooth and transverse curves is equivalent to the germ given by its tangents lines. Therefore we may assume that $\mathcal{F}$ has three transverse complex lines P_1, P_2 and P_3 as separatrices. Then $\mathrm{h}(P_1)$, $\mathrm{h}(P_2)$ and $\mathrm{h}(P_3)$ are smooth and transverse separatrices of $\widetilde{\mathcal{F}}$ and we can also assume that they are contained in complex lines $\widetilde{P}_1$, $\widetilde{P}_2$ and $\widetilde{P}_3$. By reducing U we may assume that $U \cap P_1$, $U \cap P_2$ and $U \cap P_3$ are homeomorphic to discs. We may take a neighborhood $\widetilde{U}' \subset \mathrm{h}(U)$ of $0 \in \mathbb{C}^2$ such that $\widetilde{U}' \cap \widetilde{P}_1$, $\widetilde{U}' \cap \widetilde{P}_2$ and $\widetilde{U}' \cap \widetilde{P}_3$ are homeomorphic to discs and are contained in $\mathrm{h}(U \cap P_1)$, $\mathrm{h}(U \cap P_2)$ and $\mathrm{h}(U \cap P_3)$ respectively. Then if we make $U' = \mathrm{h}^{-1}(\widetilde{U}')$, it is easy to see that $U' \cap P_1$, $U' \cap P_2$ and $U' \cap P_3$ are homeomorphic to discs and $\mathrm{h}(U' \cap P_1) = \widetilde{U}' \cap \widetilde{P}_1$, $\mathrm{h}(U' \cap P_2) = \widetilde{U}' \cap \widetilde{P}_2$, $\mathrm{h}(U' \cap P_3) = \widetilde{U}' \cap \widetilde{P}_3$. We may choose two of the complex lines P_1, P_2 and P_3, say us P_1 and P_2, such that $\deg(\mathrm{h}\,|_{P_1}) = \deg(\mathrm{h}\,|_{P_2})$. Let $\mathcal{D} \subset P_1$ be a disc containing $0 \in \mathbb{C}^2$. Then $\mathrm{h}(\partial\mathcal{D}) = \deg(\mathrm{h}\,|_{P_1})\partial\,\mathrm{h}(\mathcal{D})$ in $H_1(\widetilde{P}_1 \cap \widetilde{U}'\backslash\{0\})$ and, since $\widetilde{P}_1 \cap \widetilde{U}'\backslash\{0\} \subset \widetilde{U}'\backslash\widetilde{P}_2$, we have that

$$\mathrm{h}(\partial\mathcal{D}) = \deg(\mathrm{h}\,|_{P_1})\partial\,\mathrm{h}(\mathcal{D}) \quad \text{in} \quad H_1(\widetilde{U}'\backslash\widetilde{P}_2).$$

On the other hand, since $\partial\mathcal{D}$ and $\partial\,\mathrm{h}(\mathcal{D})$ are positive generators of $H_1(U'\backslash P_2)$ and $H_1(\widetilde{U}'\backslash\widetilde{P}_2)$ respectively, we have by Lemma 3.3 that

$$\mathrm{h}(\partial\mathcal{D}) = \deg(\mathrm{h})\deg(\mathrm{h}\,|_{P_2})\partial\,\mathrm{h}(\mathcal{D}) \quad \text{in} \quad H_1(\widetilde{U}'\backslash\widetilde{P}_2).$$

Finally, since $\deg(\mathrm{h}\,|_{P_1}) = \deg(\mathrm{h}\,|_{P_2})$, it follows from the equations above that $\deg(\mathrm{h}) = 1$ and therefore h preserves orientation. q.e.d.

Proof of Lemma 3.3. We only sketch the proof. Let $\mathcal{D}$ and $\mathcal{D}'$ be complex discs transverse to P and P' respectively. Thus $\partial\mathcal{D}$ and $\partial\mathcal{D}'$ are homologous to a and a' respectively. Clearly $\mathrm{h}(\partial\mathcal{D}) = \xi\partial\mathcal{D}'$, where $\xi = 1$ or -1. Let $L = P \cap U$ and $L' = P' \cap U'$. It follows from the topological invariance of the intersection number (see [**6**], p.200) that

$$\mathrm{h}(L)\cdot\mathrm{h}(\mathcal{D}) = \deg(\mathrm{h})L'\cdot\mathcal{D}'.$$

On the other hand it is easy to see that

$$\mathrm{h}(L)\cdot\mathrm{h}(\mathcal{D}) = (\deg(\mathrm{h}\,|_P)L')\cdot(\xi\mathcal{D}') = \deg(\mathrm{h}\,|_P)\xi L'\cdot\mathcal{D}'.$$

Then $\deg(\mathrm{h}\,|_P)\xi = \deg(\mathrm{h})$ and therefore $\xi = \deg(\mathrm{h}\,|_P)\deg(\mathrm{h})$, which proves the lemma. q.e.d.

4. Proof of theorem 1.2

Let $\rho : \widehat{\mathbb{C}^2} \to \pi^{-1}(0)$ be the projection associated to the natural fibration on a neighborhood of the divisor $\pi^{-1}(0)$. Let $\mathrm{h} : U \to \widetilde{U}$, $\mathcal{F}$, $\widetilde{\mathcal{F}}$, P, and $\widetilde{P}$ be as in Theorem 1.2. We know that the strict transforms of P and $\widetilde{P}$ are fibers of ρ. Let L_p and $L_{\widetilde{p}}$, the fibers passing through p and $\widetilde{p}$, be the strict transforms of P and $\widetilde{P}$ respectively. By the hypothesis on P and $\widetilde{P}$ we have that p and $\widetilde{p}$ are regular points of $\mathcal{F}_0$ and $\widetilde{\mathcal{F}}_0$ respectively. Let W and $\widetilde{W}$ denote the sets $\pi^{-1}(U)$ and $\pi^{-1}(\widetilde{U})$ and let E be the divisor $\pi^{-1}(0)$. Since $\mathrm{h}(P\cap U) = \widetilde{P}\cap\widetilde{U}$, if

$$h : W\backslash E \to \widetilde{W}\backslash E$$

is the homeomorphism given by $h = \pi^{-1}\,\mathrm{h}\,\pi$, we have that

$$h(L_p \cap W\backslash\{p\}) = L_{\widetilde{p}} \cap \widetilde{W}\backslash\{\widetilde{p}\}.$$

Now, it is easy to see that Theorem 1.2 is a direct consequence of the following proposition.

Proposition 4.1. *Let p and $\widetilde{p}$ be points in the divisor which are nonsingular for $\mathcal{F}_0$ and $\widetilde{\mathcal{F}_0}$ respectively. Let L_p and $L_{\widetilde{p}}$ be the fibers through p and $\widetilde{p}$ respectively and suppose that*

$$h(L_p \cap W\backslash\{p\}) = L_{\widetilde{p}} \cap \widetilde{W}\backslash\{\widetilde{p}\}.$$

Then there exists neighborhoods $\mathrm{U} \subset U$ *and* $\widetilde{\mathrm{U}} \subset \widetilde{U}$ *of* $0 \in \mathbb{C}^2$, *and another topological equivalence*

$$\hat{\mathrm{h}} : \mathrm{U} \to \widetilde{\mathrm{U}}$$

between $\mathcal{F}$ *and* $\widetilde{\mathcal{F}}$, *for which the hypothesis of Theorem* 2.1 *holds.*

We need some lemmas. Let $U \subset \mathbb{C}$ be the domain bounded by the Jordan curve J. Let $p \in U$ and $\zeta \in J$. We know that any biholomorphism between $\mathbb{D}$ and U extends as a homeomorphism between $\overline{\mathbb{D}}$ and $\overline{U} = U \cup J$ and there exists a unique biholomorphism $f : \mathbb{D} \to U$ with $f(0) = p$ and such that its extension to $\overline{\mathbb{D}}$ satisfies $f(1) = \zeta$. In other words, $f : \overline{\mathbb{D}} \to \overline{U}$ is the unique orientation preserving homeomorphism, which is conformal on $\mathbb{D}$ and maps 0 to p and 1 to ζ. It is easy to prove that $g : \overline{\mathbb{D}} \to \overline{U}$ defined by $g(z) = f(\bar{z})$ is the unique orientation reversing homeomorphism, which is conformal on $\mathbb{D}$ and maps 0 to p and 1 to ζ. Therefore we have the following.

Lemma 4.2. *Let* $U, U' \subset \mathbb{C}$ *be the domains bounded by the Jordan curves* J *and* J' *respectively. Let* $p \in U$, $\zeta \in J$ *and* $p' \in U'$, $\zeta' \in J'$. *Then there exists exactly two homeomorphisms between* $\overline{U}$ *and* $\overline{U}'$ *which are conformal and maps* p *to* p' *and* ζ *to* ζ'. *The first one preserves orientation and the other one reverses orientation.*

Lemma 4.3. *Let* $J_k : S^1 \to \mathbb{C}$ *be a Jordan curve for all* $k \geq 1$. *Suppose that* J_k *converges uniformly on* S^1 *to the Jordan curve* $J : S^1 \to \mathbb{C}$. *Let* U *and* U_k, $k \geq 1$ *be the domains bounded by* J *and* J_k, $k \geq 1$ *respectively. Let* $p_k \in U_k$ *and* $\zeta_k \in J_k$ *be such that* $p_k \to p \in U$ *and* $\zeta_k \to \zeta \in J$ *as* $k \to \infty$. *Let* $f : \overline{\mathbb{D}} \to \overline{U}$ *and* $f_k : \overline{\mathbb{D}} \to \overline{U}_k$ *be the orientation preserving homeomorphisms which are conformal on* $\mathbb{D}$ *and such that* $f(0) = p$, $f(1) = \zeta$, $f_k(0) = p_k$ *and* $f_k(0) = \zeta_k$. *Then* f_k *converges to* f *uniformly on* $\overline{\mathbb{D}}$. *If under the same hypothesis, we change "orientation preserving homeomorphisms" by "orientation reversing homeomorphisms", the conclusion is also true.*

Lemma 4.4. *Let* $\phi : X \to \mathbb{C}\backslash\{0\}$ *be a continuous function. Suppose that* $\phi_* : \pi_1(X) \to \pi_1(\mathbb{C}\backslash\{0\})$ *is trivial. Then there exists a continuous function* $\log_\phi : X \to \mathbb{C}$ *such that* $\mathrm{e}^{\log_\phi} = \phi$.

Lemma 4.5. *Let* $\phi : S^1 \to S^1$ *be an orientation preserving homeomorphism. Consider* S^1 *as a subset of* $\mathbb{C}$ *and define the closed curve* $\alpha : S^1 \to \mathbb{C}\backslash\{0\}$ *by* $\alpha(\zeta) = \phi(\zeta)/\zeta$. *Then* α *is homotopically trivial in* $\mathbb{C}\backslash\{0\}$.

Lemma 4.6. *Let* $\phi : S^1 \to S^1$ *be an orientation preserving homeomorphism and let* $\tau : S^1 \to \mathbb{C}$ *be such that* $\mathrm{e}^{\tau(\zeta)} = \phi(\zeta)/\zeta$. *Let* $A \subset \mathbb{C}$ *be the annulus* $\{z \in \mathbb{C} : 1/2 \leq |z| \leq 1\}$. *Then the map*

$$g : A \to A$$

$$g(z) = ze^{(2|z|-1)\tau(z/|z|)}$$

is a homeomorphism. Moreover, $g = \phi$ *on* $\{|z| = 1\}$ *and* $g = \mathrm{id}$ *on* $\{|z| = 1/2\}$.

Lemma 4.7. *Let* $f : \mathbb{D} \to \mathbb{C}$ *be a conformal map. Then there exists* $\delta_0 > 0$ *such that for all* $\delta \leq \delta_0$ *the set* $f(|z| \leq \delta)$ *is convex. (For convenience, we define a set* $\overline{U} \subset \mathbb{C}$ *to be convex if* U *is the domain bounded by a smooth Jordan curve with positive curvature.)*

Lemma 4.8. *Let* $f : \mathbb{D} \to \mathbb{C}$ *be a conformal map. Let* U *be an open set in* $\mathbb{C}$ *and let* $\delta_0 > 0$*. Suppose for all* $\delta \leq \delta_0$ *the set* $f(|z| \leq \delta)$ *is convex and contained in* U*. Then there exists* $\epsilon > 0$ *with the following property: if* $g : \mathbb{D} \to \mathbb{C}$ *is a conformal map with* $||f - g||_{\{|z| \leq \delta_0\}} < \epsilon$*, then for all* $\delta \leq \delta_0$ *the set* $g(|z| \leq \delta)$ *is convex and contained in* U*. (If* K *is compact and* f *is continuous,* $||f||_K$ *is defined as the supremum of* $|f(x)|$ *for* $x \in K$*.)*

Any leaf of $\mathcal{F}_0$ or $\widetilde{\mathcal{F}_0}$ has a natural orientation induced by the complex structure. Thus, given a leaf L of $\mathcal{F}_0$ out of the divisor, we may state if $h|_L : L \to \widetilde{L}$ preserves or reverses orientation. Suppose that $h|_L$ preserves orientation. Then it is not difficult to prove that $h|_{L'}$ preserves orientation of any leaf L' close enough to L. On the other hand, if $h|_L$ reverses orientation, the same holds for $h|_{L'}$ provided the leaf L' is close enough to L. By connectedness we have in fact that: either h preserves orientation for every leaf, or h reverses orientation for every leaf.

Proof of Proposition 4.1. Let V and $\widetilde{V}$ be neighborhoods of p and $\widetilde{p}$ and let $\varphi : \overline{\mathbb{D}} \times \overline{\mathbb{D}} \to V$ and $\widetilde{\varphi} : \overline{\mathbb{D}} \times \overline{\mathbb{D}} \to \widetilde{V}$ be diffeomorphisms with the following properties:

(i) If restricted to $\mathbb{D} \times \mathbb{D}$, the maps φ and $\widetilde{\varphi}$ are biholomorphisms.

(ii) The leaves of $\mathcal{F}_0|_V$ and the leaves of $\widetilde{\mathcal{F}_0}|_{\widetilde{V}}$ are given by the sets $\varphi(\overline{\mathbb{D}} \times \{*\})$ and $\widetilde{\varphi}(\overline{\mathbb{D}} \times \{*\})$ respectively.

(iii) We have $L_p \cap V = \varphi(\{0\} \times \overline{\mathbb{D}})$, $E \cap V = \varphi(\overline{\mathbb{D}} \times \{0\})$, $L_{\widetilde{p}} \cap \widetilde{V} = \widetilde{\varphi}(\{0\} \times \overline{\mathbb{D}})$ and $E \cap \widetilde{V} = \widetilde{\varphi}(\overline{\mathbb{D}} \times \{0\})$.

Let $\varrho : V \to \overline{\mathbb{D}}$ be the projection $\varrho(\varphi(z_1, z_2)) = z_1$ and we also denote by ϱ the projection $\varrho : \widetilde{V} \to \overline{\mathbb{D}}$, $\varrho(\widetilde{\varphi}(z_1, z_2)) = z_1$. Let Σ be the set $L_p \cap V = \varphi(\{0\} \times \overline{\mathbb{D}})$. We have that $h(\Sigma) \subset L_{\widetilde{p}}$ and we may assume V small enough such that $h(\Sigma) \subset \widetilde{V}$ Given $x = \varphi(0, z_2) \in \Sigma$, we denote by D_x the plaque $\varphi(\overline{\mathbb{D}} \times \{z_2\})$ passing through x. We have that D_x is a closed disc in the leaf of $\mathcal{F}_0$ passing through x.

Step 1. Fix a point q in $\partial\mathbb{D} = S^1$ and denote by q_x the unique point in ∂D_x such that $\varrho(q_x) = q$. If h preserves the orientation of the leaves, by Lemma 4.2 we may define $f_x : D_x \to h(D_x)$ as the unique orientation-preserving-homeomorphism which is conformal on the interior of D_x

and such that $f_x(x) = h(x)$ and $f_x(q_x) = h(q_x)$. Otherwise, we define $f_x : D_x \to h(D_x)$ as the unique orientation reversing homeomorphism which is conformal on the interior of D_x and such that $f_x(x) = h(x)$ and $f_x(q_x) = h(q_x)$. Let $\varrho_x^{-1} : \overline{\mathbb{D}} \to D_x$ be the inverse of $\varrho|_{D_x} : D_x \to \overline{\mathbb{D}}$.

Assertion 1. *Let $f : V \backslash E \to \widehat{\mathbb{C}^2}$ be defined by $f|_{D_x} = f_x$ for all $x \in \Sigma\backslash\{p\}$. Then f is continuous.*

Proof. Let $g_x : \overline{\mathbb{D}} \to h(D_x)$ be defined by $g_x = f_x \circ \varrho_x^{-1}$. It is sufficient to prove that g_x varies continuously with x, precisely: fix $x_0 \in \Sigma\backslash\{p\}$ and let $x_k (k \geq 1)$ be such that $x_k \to x_0$ as $k \to \infty$; then we shall prove that $g_{x_k} \to g_{x_0}$ uniformly on $\overline{\mathbb{D}}$. Since $h(D_{x_0})$ is a compact and simply connected subset of a leaf of $\widetilde{\mathcal{F}}_0$, there exits a neighborhood U of $h(D_{x_0})$ and a biholomorphism $\phi = (Z, W) : U \to \mathbb{D} \times \mathbb{D}$ such that the leaves of $\widetilde{\mathcal{F}}_0$ are mapped to the sets $\mathbb{D} \times \{z\}$. We may assume that $h(D_{x_k})$ is contained in U for all $k \geq 0$. Thus, we define $G_k : \overline{\mathbb{D}} \to \mathbb{D} \times \mathbb{D}$ by $G_k = \phi \circ g_{x_k} = (Z \circ g_{x_k}, W \circ g_{x_k})$. Since $g_{x_k}(\overline{\mathbb{D}}) = h(D_{x_k}) \subset U$ is contained in a leaf, there is $z_k \in \mathbb{D}$ such that $G_k(\overline{\mathbb{D}})$ is contained in $\mathbb{D} \times \{z_k\}$. Thus $W \circ g_{x_k} \equiv z_k$ and it is sufficient to show that $F_k = Z \circ g_{x_k} : \overline{\mathbb{D}} \to \mathbb{D}$ converges to $F_0 = W \circ g_{x_0}$ uniformly on $\overline{\mathbb{D}}$. Observe that F_k is a homeomorphism onto its image and is conformal on $\mathbb{D}$. Moreover, we have that

$$F_k(0) = Z \circ g_{x_k}(0) = Z(h(x_k)) \to Z(h(x_0)) = Z \circ g_{x_0}(0) = F_0(0)$$

and

$$F_k(q) = Z \circ g_{x_k}(q) = h(q_{x_k}) \to h(q_{x_0}) = g_{x_0}(q) = F_0(q).$$

Then Assertion 1 follows from Lemma 4.3

Let

$$\theta_x : S^1 \to S^1$$

be the homeomorphism defined by $\theta_x = \varrho f_x^{-1} h \varrho_x^{-1}|_{S^1}$. It is easy to see that θ_x preserves the orientation of S^1.

Define the function

$$\phi : S^1 \times (\Sigma\backslash\{p\}) \to \mathbb{C}\backslash\{0\}$$

$$\phi(\zeta, x) = \frac{\theta_x(\zeta)}{\zeta}.$$

Assertion 2. *At homotopy level, $\phi_* : \pi_1(S^1 \times (\Sigma\backslash\{p\})) \to \pi_1(\mathbb{C}\backslash\{0\})$ is trivial.*

Proof. The generators of $\pi_1(S^1 \times (\Sigma\backslash\{p\}))$ are represented by the paths

$$\alpha, \beta : S^1 \to S^1 \times (\Sigma\backslash\{p\}),$$

defined by $\alpha(\zeta) = (\zeta, x_0)$ and $\beta(\zeta) = (q, \gamma(\zeta))$, where $x_0 \in \Sigma\backslash\{p\}$ and γ is a simple closed curve around p in Σ. Recall that $q \in S^1$, then $|q| = 1$ and we have

$$\begin{aligned}\phi(\beta(\zeta)) &= \phi(q, \gamma(\zeta)) = \frac{\theta_{\gamma(\zeta)}(q)}{q} = \frac{\varrho f^{-1}_{\gamma(\zeta)} h \varrho^{-1}_{\gamma(\zeta)}(q)}{q} \\ &= \frac{\varrho f^{-1}_{\gamma(\zeta)} h(q_{\gamma(\zeta)})}{q} = \frac{\varrho f^{-1}_{\gamma(\zeta)} f_{\gamma(\zeta)}(q_{\gamma(\zeta)})}{q} \\ &= \frac{\varrho(q_{\gamma(\zeta)})}{q} \\ &= 1.\end{aligned}$$

Then $\phi_*(\beta) = 0$. On the other hand, since $\theta_{x_0} : S^1 \to S^1$ is an orientation preserving homeomorphism, we have by Lemma 4.5 that

$$\phi \circ \alpha : S^1 \to \mathbb{C}\backslash\{0\},$$

$$\phi \circ \alpha(\zeta) = \frac{\theta_{x_0}(\zeta)}{\zeta}$$

is homotopically trivial and thus $\phi_*(\alpha) = 0$.

It follows from Assertion 2 and Lemma 4.4 that there exists a continuous function

$$\tau : S^1 \times (\Sigma\backslash\{p\}) \to \mathbb{C}$$

such that $e^\tau = \phi$, that is, $e^{\tau(\zeta,x)} = \theta_x(\zeta)/\zeta$. Consider the annulus $A = \{1/2 \le ||z|| \le 1\} \subset \overline{\mathbb{D}}$ and define the map

$$g : A \times (\Sigma\backslash\{p\}) \to A,$$

$$g(z, x) = ze^{(2|z|-1)\tau(z/|z|,x)}.$$

It follows from Lemma 4.6 that for all x the map

$$g_x : A \to A,$$

$$g_x(z) = g(z, x)$$

is a homeomorphism such that $g_x = \mathrm{id}$ on $\{|z| = 1/2\}$ and $g_x = \theta_x$ on S^1. Let A_x be the annulus $\varrho_x^{-1}(A)$ in D_x and let $\partial A'_x = \varrho_x^{-1}(|z| = 1/2)$ and $\partial A''_x = \varrho_x^{-1}(|z| = 1)$ be the interior and the exterior boundary of A_x respectively. Then the map

$$\bar{g} : A_x \to f_x(A_x)$$

defined by $\bar{g}_x = f_x \varrho_x^{-1} g_x \varrho : A_x \to f_x(A_x)$ is a homeomorphism and it is easy to see that $\bar{g}_x$ coincides with f_x on $\partial A'_x$ and with h on $\partial A''_x$. Then we may define the homeomorphism

$$h_x : D_x \to h(D_x)$$

by

$$\begin{aligned} h_x &= f_x \quad \text{on} \quad \varrho_x^{-1}(|z| \le 1/2), \\ h_x &= g_x \quad \text{on} \quad A_x. \end{aligned}$$

Clearly, h_x coincides with h on ∂D_x and it is easy to see that h_x depends continuously on x. Finally, we define the function h' by

$$\begin{aligned} h'|_{D_x} &= h_x \quad \text{for all} \quad x \in \Sigma\backslash\{p\}, \\ h' &= h, \quad \text{otherwise}. \end{aligned}$$

It is easy to see that h' is injective and take leaves to leaves. Moreover, if we restrict h' to a small enough neighborhood of the divisor, h' is continuous. Hence, h' restricted to a neighborhood of the divisor is a homeomorphism onto its image and is therefore a topological equivalence between $\mathcal{F}_0$ and $\widetilde{\mathcal{F}}_0$. By definition h' is conformal on every plaque $\varrho_x^{-1}(|z| \le 1/2)$, because coincides with f_x. In other words, there is $\epsilon > 0$ such that h' restricted to $\varphi(|z_1| \le 1/2, |z_2| \le \epsilon)$ is conformal along the leaves.

Step 2. From step 1 and by reducing V, we may assume that h restricted to V is conformal along the leaves. Then for all $x \in \Sigma\backslash\{p\}$ the map

$$h\varrho_x^{-1} : \overline{\mathbb{D}} \to h(D_x)$$

is conformal and maps 0 to $h(x)$. Given $x \in \Sigma\backslash\{p\}$, since $h\varrho_x^{-1}(0) = h(x)$ is contained in $L_{\widetilde{p}} \cap \widetilde{V}$, there is $\delta > 0$ such that the disc $\{|z| \le \delta\}$ in $\overline{\mathbb{D}}$ is mapped by $h\varrho_x^{-1}$ into the interior of $\widetilde{V}$. Then the map

$$\varrho h \varrho_x^{-1} : \{|z| \le \delta\} \to \mathbb{D}$$

is well defined and assuming δ be small, by Lemma 4.7 we have that for all $\delta' \le \delta$ the disc $\{|z| \le \delta'\}$ is mapped by $\varrho h \varrho_x^{-1}$ onto a convex subset of $\mathbb{D}$. Define $\delta(x) > 0$ as the supremum of $0 < \delta < 1$ such that for all $\delta' \le \delta$, the disc $\{|z| \le \delta'\}$ in $\overline{\mathbb{D}}$ is mapped by $\varrho h \varrho_x^{-1}$ onto a convex subset of $\mathbb{D}$.

Assertion 3. *The function $\delta : \Sigma\backslash p \to \mathbb{R}^+$ is lower semi-continuous.*

Proof. Fix $x_0 \in \Sigma\backslash p$ and let $\epsilon > 0$. Take δ_0 be such that $\delta(x_0) - \epsilon < \delta_0 < \delta(x_0)$. Then the disc $\{|z| \le \delta_0\}$ is mapped by $\varrho h \varrho_{x_0}^{-1}$ onto a compact subset of $\mathbb{D}$. Then, if Ω is a small enough neighborhood of x_0 in $\Sigma\backslash p$, we have that

$$\varrho h \varrho_x^{-1} : \{|z| \le \delta_0\} \to \overline{\mathbb{D}}$$

is well defined for all $x \in \Omega$. If we write $f = \varrho h \varrho_{x_0}^{-1}$, it follows from the definition of $\delta(x_0)$ that for all $\delta' \le \delta(x_0) - \epsilon$, the set $f(|z| \le \delta')$ is a convex subset of $\mathbb{D}$. Let $\epsilon_0 > 0$ be given by Lemma 4.8 for $f = \varrho h \varrho_{x_0}^{-1}$ and $U = \mathbb{D}$. Then if

$$g : \{|z| \leq \delta_0\} \to \overline{\mathbb{D}}$$

is a conformal map with $||f - g||_{\{|z| \leq \delta(x_0) - \epsilon\}} < \epsilon_0$, we have that for all $\delta' \leq \delta(x_0) - \epsilon$, the set $g(|z| \leq \delta')$ is also convex and contained in $\mathbb{D}$. By reducing the neighborhood Ω of x_0 we may assume that

$$||\varrho h \varrho_{x_0}^{-1} - \varrho h \varrho_x^{-1}||_{\{|z| \leq \delta(x_0) - \epsilon\}} < \epsilon_0$$

for all $x \in \Omega$. Then, for all $\delta' \leq \delta(x_0) - \epsilon$ the set $\varrho h \varrho_x^{-1}(|z| \leq \delta')$ is convex and contained in $\mathbb{D}$. Thus by the definition of $\delta(x)$ we conclude that

$$\delta(x) \geq \delta(x_0) - \epsilon.$$

It follows that δ is a lower semi-continuous function.

Assertion 4. *There exists a positive continuous function*

$$r : \Sigma \backslash \{p\} \to (0, 1)$$

such that for all x the map

$$\varrho h \varrho_x^{-1} : \{|z| \leq r(x)\} \to \overline{\mathbb{D}}$$

is well defined and its image $U_x := \varrho h \varrho_x^{-1}(|z| \leq r(x))$ is a convex subset of $\mathbb{D}$.

Proof. We take any continuous function $r < \delta$ given by Lemma 2.3. Then Assertion 4 is a direct consequence of the definition of δ.

For all $0 < r < 1$ let $\beta_r : [0, 1] \to [0, 1]$ be the homeomorphism defined by

$$\beta_r(t) = t^{\frac{\ln(1/r)}{\ln 2}}.$$

We have that $\beta_r(0) = 0$, $\beta_r(1) = 1$ and it is easy to see that $\beta_r(1/2) = r$. In fact

$$\begin{aligned} \beta_r(1/2) &= (1/2)^{\frac{\ln(1/r)}{\ln 2}} = \left(2^{\frac{1}{\ln 2}}\right)^{-\ln(1/r)} \\ &= \left((e^{\ln 2})^{\frac{1}{\ln 2}}\right)^{\ln(r)} = e^{\ln(r)} = r. \end{aligned}$$

For each $x \in \Sigma \backslash \{p\}$ we define the homeomorphism:

$$f_x : \overline{\mathbb{D}} \to \overline{\mathbb{D}},$$

$$f_x(z) = \beta_{r(x)}(|z|)z.$$

Observe that f_x maps each ratio of $\overline{\mathbb{D}}$ homeomorphically onto itself and this homeomorphism is "given" by $\beta_{r(x)}$. We have that $f_x(0) = 0$, $f_x = \text{id}$ on $\partial\overline{\mathbb{D}}$ and that f_x maps the disc $\{|z| \leq 1/2\}$ onto the disc $\{|z| \leq r(x)\}$. For all $y \in L_{\widetilde{p}} \cap \widetilde{V}$, let $\varrho_y^{-1} : \overline{\mathbb{D}} \to D_y$ be the inverse of

$\varrho|_{D_y} : D_y \to \overline{\mathbb{D}}$.

Assertion 5. *For each $x \in \Sigma\backslash\{p\}$, define the homeomorphism*

$$h_x = h\varrho_x^{-1} f_x \varrho : D_x \to h(D_x).$$

Then h_x coincides with h on ∂D_x and maps the disc $\varrho_x^{-1}(|z| \leq 1/2)$ onto $\varrho_{h(x)}^{-1}(U_x)$. Moreover, h_x depends continuously on x.

Proof. If $\zeta \in \partial D_x$, then $\varrho(\zeta) \in S^1$ and since $f_x = \mathrm{id}$ on S^1 we have that $f_x(\varrho(\zeta)) = \varrho(\zeta)$. Then

$$h_x(\zeta) = h\varrho_x^{-1} f_x \varrho(\zeta) = h\varrho_x^{-1}\varrho(\zeta) = h(\zeta).$$

On the other hand,

$$h_x(\varrho_x^{-1}(|z| \leq 1/2)) = h\varrho_x^{-1} f_x \varrho(\varrho_x^{-1}(|z| \leq 1/2)) = h\varrho_x^{-1} f_x(|z| \leq 1/2)$$

and, since $f_x(|z| \leq 1/2) = \{|z| \leq r(x)\}$, we obtain:

$$h_x(\varrho_x^{-1}(|z| \leq 1/2)) = h\varrho_x^{-1}(|z| \leq r(x)).$$

Recall that $U_x = \varrho h \varrho_x^{-1}(|z| \leq r(x))$ and so

$$\varrho_{h(x)}^{-1}(U_x) = h\varrho_x^{-1}(|z| \leq r(x)).$$

therefore

$$h_x(\varrho_x^{-1}(|z| \leq 1/2)) = \varrho_{h(x)}^{-1}(U_x).$$

Finally, h depends continuously on x because β_r depends continuously on r.

We now define the function h' by

$$\begin{aligned} h'|_{D_x} &= h_x \quad \text{for all} \quad x, \\ h' &= h, \quad \text{otherwise.} \end{aligned}$$

It is easy to see that h' is injective and take leaves to leaves. Moreover, if we restrict h' to a small enough neighborhood of the divisor, it is continuous. Hence, h' restricted to a neighborhood of the divisor is a homeomorphism onto its image and is therefore a topological equivalence between $\mathcal{F}_0$ and $\widetilde{\mathcal{F}}_0$. By definition, h' maps each plaque $\varrho_x^{-1}(|z| \leq 1/2)$ onto $\varrho_{h(x)}^{-1}(U_x)$. In other words, any plaque $\varrho_x^{-1}(|z| \leq 1/2)$ is mapped by h' onto a set which projection by ϱ is a convex set U_x in $\mathbb{D}$.

Step 3. From step 2 and by reducing V we may assume that h maps each plaque D_x onto $\varrho_{h(x)}^{-1}(U_x)$. Since $U_x \subset \mathbb{D}$ is convex and contains 0, given $w \in \overline{\mathbb{D}}$ there exists a unique point in the intersection of ∂U_x with the ray $\overrightarrow{0w}$. Let $r_x(w)$ be the norm of this point. It is easy to see that $r_x(w)$ depends continuously on x and w. We define the homeomorphism:

$$f_x : \overline{\mathbb{D}} \to \overline{\mathbb{D}},$$

$$f_x(w) = \beta_{r_x(w)}(|w|)w.$$

Observe that f_x maps the ratio of $\mathbb{D}$ passing through w homeomorphically onto itself and this homeomorphism is "given" by $\beta_{r_x(w)}$. We have that f_x maps the disc $\{|z| \leq 1/2\}$ onto U_x.

Assertion 6. *For each $x \in \Sigma\backslash\{p\}$ define the homeomorphism*

$$g_x = \varrho_{h(x)}^{-1} f_x^{-1} \varrho : D_{h(x)} \to D_{h(x)}.$$

Then $g_x = \mathrm{id}$ on $\partial D_{h(x)}$ and maps $\varrho_{h(x)}^{-1}(U_x)$ onto $\varrho_{h(x)}^{-1}(|z| \leq 1/2)$. Moreover, g_x depends continuously on x.

Proof. If $\zeta \in \partial D_{h(x)}$, then $\varrho(\zeta) \in S^1$ and since $f_x = \mathrm{id}$ on S^1 we have that $f_x^{-1}(\varrho(\zeta)) = \varrho(\zeta)$. Then

$$g_x(\zeta) = \varrho_{h(x)}^{-1} f_x^{-1} \varrho(\zeta) = \varrho_{h(x)}^{-1} \varrho(\zeta) = \zeta.$$

On the other hand:

$$g_x(\varrho_{h(x)}^{-1}(U_x)) = \varrho_{h(x)}^{-1} f_x^{-1} \varrho(\varrho_{h(x)}^{-1}(U_x)) = \varrho_{h(x)}^{-1} f_x^{-1}(U_x).$$

From the definition of f_x, we have that $f_x^{-1}(U_x) = \{|z| \leq 1/2\}$. Then

$$g_x(\varrho_{h(x)}^{-1}(U_x)) = \varrho_{h(x)}^{-1} f_x^{-1}(U_x) = \varrho_{h(x)}^{-1}(|z| \leq 1/2).$$

Finally, g_x depends continuously on x because r_x depends continuously on x.

Now, define the function g by

$$\begin{aligned} g|_{D_{h(x)}} &= g_x \quad \text{for all} \quad x, \\ g &= \mathrm{id}, \quad \text{otherwise.} \end{aligned}$$

It is easy to see that g is injective and maps leaves of $\widetilde{\mathcal{F}}_0$ to leaves of $\widetilde{\mathcal{F}}_0$. Moreover, if we restrict g to a small enough neighborhood of the divisor, g is continuous. Hence, g restricted to a neighborhood of the divisor is a homeomorphism onto its image and is therefore a topological equivalence of $\widetilde{\mathcal{F}}_0$ with itself. Finally we define $h' = g \circ h$. Then h' is a topological equivalence between $\mathcal{F}_0$ and $\widetilde{\mathcal{F}}_0$ and from the definition of g we have

$$h'(D_x) = g(h(D_x)) = g(\varrho_{h(x)}^{-1}(U_x)) = \varrho_{h(x)}^{-1}(|z| \leq 1/2).$$

Thus h' maps each plaque D_x onto the plaque $\varrho_{h(x)}^{-1}(|z| \leq 1/2)$.

Step 4. From step 3 and by redefining $\widetilde{V}$ we may assume that for all $y \in \overline{\mathbb{D}}\backslash\{0\}$ the equivalence h maps the plaque $\varphi(\overline{\mathbb{D}} \times \{y\})$ onto the plaque the $\widetilde{\varphi}(\overline{\mathbb{D}} \times \{f(y)\})$, where $f : \overline{\mathbb{D}} \to \mathbb{D}$ is a homeomorphism onto its image. Therefore $h|_{V\backslash E} : V\backslash E \to \widetilde{V}\backslash E$ is expressed as

$$h(\varphi(x,y)) = \widetilde{\varphi}(h_y(x), f(y)),$$

where $h_y : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is a homeomorphism such that $h_y(0) = 0$ (because $h(\Sigma) \subset L_{\widetilde{p}}$). As a first case we assume that the homeomorphisms h_y preserve orientation. Define the function

$$\phi : S^1 \times (\overline{\mathbb{D}} \backslash \{0\}) \to \mathbb{C} \backslash \{0\}$$

$$\phi(\zeta, y) = \frac{h_y(\zeta)}{\zeta}.$$

Assertion 7. *At homotopy level,* $\phi_* : \pi_1(S^1 \times (\overline{\mathbb{D}} \backslash \{0\})) \to \pi_1(\mathbb{C} \backslash \{0\})$ *is trivial.*

Proof. The generators of $\pi_1(S^1 \times (\overline{\mathbb{D}} \backslash \{0\}))$ are represented by the paths

$$\alpha, \beta : S^1 \to S^1 \times (\overline{\mathbb{D}} \backslash \{0\}),$$

defined as $\alpha(\zeta) = (\zeta, 1)$ and $\beta(\zeta) = (1, \zeta)$. Then we have that

$$\phi \circ \alpha(\zeta) = \phi(\zeta, 1) = \frac{h_1(\zeta)}{\zeta}$$

and, since $h_1|_{S^1} : S^1 \to S^1$ preserves the orientation, we have by Lemma 4.5 that $\phi \circ \alpha$ is homotopically trivial in $\mathbb{C} \backslash \{0\}$. Observe that β is the boundary of the disc $\{(1, y) : |y| \leq 1\}$. Thus, $\varphi(\beta)$ is the boundary of the complex disc $\mathcal{B} = \varphi(1, |y| \leq 1)$. Consider the disc $\mathcal{B}_w = \varphi(w, |y| \leq 1)$, where $w \in \mathbb{D} \backslash \{0\}$. By Lemma 3.1 we may chose w such that the path $h(\partial \mathcal{B}_w)$ in $\widetilde{V}$ does not link the fiber $L_{\widetilde{p}}$. Thus, since $\partial \mathcal{B} = \partial \mathcal{B}_w$ in $H_1(V \backslash (L_p \cup E))$ and $h(V \backslash (L_p \cup E)) \subset \widetilde{V} \backslash (L_{\widetilde{p}} \cup E)$, we have that $h(\partial \mathcal{B})$ does not link the fiber $L_{\widetilde{p}}$. Therefore the path $\widetilde{\varphi}^{-1} h(\partial \mathcal{B})$ in $(\overline{\mathbb{D}} \backslash \{0\}) \times \overline{\mathbb{D}}$ does not link $\{0\} \times \overline{\mathbb{D}}$ and, since

$$\begin{aligned} \widetilde{\varphi}^{-1} h(\partial \mathcal{B}) &= \widetilde{\varphi}^{-1} h(\varphi(\beta)) = \widetilde{\varphi}^{-1} h(\varphi(1, \zeta)) \\ &= \widetilde{\varphi}^{-1} \widetilde{\varphi}(h_\zeta(1), f(\zeta)) = (h_\zeta(1), f(\zeta)), \end{aligned}$$

we conclude that the path $\zeta \to h_\zeta(1) = \phi(\beta(\zeta))$ is homotopically trivial in $\mathbb{C} \backslash \{0\}$.

Assertion 7 and Lemma 4.4 imply that there exists a continuous function

$$\tau : S^1 \times (\overline{\mathbb{D}} \backslash \{0\}) \to \mathbb{C}$$

such that $e^{\tau(\zeta, y)} = h_y(\zeta)/\zeta$. We define the map:

$$h' : V \backslash E \to \widetilde{V} \backslash E$$

by:

$$\begin{aligned} h'(\varphi(x, y)) &= \widetilde{\varphi}(x, f(y)), \quad \text{for} \quad |x| < 1/2, \quad \text{and} \\ h'(\varphi(x, y)) &= \widetilde{\varphi}\left(x e^{(2|x|-1)\tau(x/|x|, y)}, f(y)\right), \quad \text{for} \quad |x| \geq 1/2. \end{aligned}$$

By Lemma 4.6 we have that h' maps the plaque $\varphi(\overline{\mathbb{D}} \times \{y\})$ homeomorphically onto the plaque $\widetilde{\varphi}(\overline{\mathbb{D}} \times \{f(y)\})$. Thus h' is a homeomorphism which preserves the plaques and it is easy to see that h' coincides with h on $\varphi(\partial\mathbb{D}\times(\overline{\mathbb{D}}\backslash\{0\}))$. Moreover h' extends to $\varphi(|x|<1/2, y=0)\subset E$ as $h'(\varphi(x,0))=\widetilde{\varphi}(x,0)$. It is easy to see that this extension is a homeomorphism onto its image. We now define:

$$\begin{aligned} \hat{h} &= h' \quad \text{on} \quad V\backslash E, \\ \hat{h} &= h \quad \text{otherwise.} \end{aligned}$$

As before, on a neighborhood of the divisor, $\hat{h}$ is also a topological equivalence between $\mathcal{F}_0$ and $\widetilde{\mathcal{F}_0}$. Moreover, from above $\hat{h}$ extends to the open set $\varphi(|x|<1/2, |y|<1)$ and Proposition 4.1 is therefore proved in this case. We now suppose that the homeomorphisms h_x inverts orientation. Then we define

$$h' : V\backslash E \to \widetilde{V}\backslash E$$

by:

$$\begin{aligned} h'(\varphi(x,y)) &= \widetilde{\varphi}(\bar{x}, f(y)), \quad \text{for} \quad |x|<1/2, \quad \text{and} \\ h'(\varphi(x,y)) &= \widetilde{\varphi}\left(\bar{x}e^{(2|x|-1)\tau(\bar{x}/|x|,y)}, f(y)\right), \quad \text{for} \quad |x|\geq 1/2. \end{aligned}$$

and the proof follows in the same way. q.e.d.

Proof of Lemma 4.3. This lemma is a direct consequence of a theorem of Rado (see [**11**], p.26). q.e.d.

Proof of Lemma 4.4. Fix $x_0\in X$. There is a neighborhood Ω of $z_0\phi(x_0)$ in $\mathbb{C}\backslash\{0\}$ where a branch of logarithm function is well defined. Then there exist a holomorphic function

$$f:\Omega\to\mathbb{C}$$

such that $e^{f(z)}=z$ for all $z\in\Omega$. We know that f can be analytically continued along any path γ in $\mathbb{C}\backslash\{0\}$ with $\gamma(0)=z_0$ and $\gamma(1)=z\in\mathbb{C}\backslash\{0\}$. This analytic continuation has a value at $\gamma(1)=z$, which we denote by $f_\gamma(z)$. Let $x\in X$. Take a path α in X connecting x_0 to x. Then we define $F_\alpha(x)=f_{\phi\circ\alpha}(\phi(x))$. Let α' be other path in X connecting x_0 to x. Then, since

$$\phi_*:\pi_1(X)\to\pi_1(\mathbb{C}\backslash\{0\})$$

is trivial, it follows that $\phi\circ\alpha$ and $\phi\circ\alpha'$ are homotopic in $\mathbb{C}\backslash\{0\}$. Then

$$f_{\phi\circ\alpha}(\phi(x))=f_{\phi\circ\alpha'}(\phi(x))$$

and so $F_\alpha(x)=F_{\alpha'}(x)$. Therefore we define $\log_\phi(x)=F_\alpha(x)$ for any α. q.e.d.

Proof of Lemma 4.5. It is known that a map $\phi: S^n\to S^n$ is homotopically determined by its degree (Brouwer). Thus, a preserving-orientation homeomorphism of S^1 is homotopic to the identity map id : $S^1 \to S^1$, that is, there exists a map

$$F : S^1 \times [0,1] \to S^1$$

such that $F(\zeta,0) = \phi(\zeta)$ and $F(\zeta,1) = \zeta$ for all $\zeta \in S^1$. Then the map

$$G : S^1 \times [0,1] \to S^1 \subset \mathbb{C}\backslash\{0\}$$

defined by

$$G(\zeta,t) = \frac{F(\zeta,t)}{\zeta}$$

is a homotopy between α and the constant 1. q.e.d.

Proof of Lemma 4.6. We first observe that each circle $\{|z| = r\}$ in A is mapped into itself. Let $z \in A$ with $|z| = r$. Since

$$e^{\tau(\zeta)} = \phi(\zeta)/\zeta \in S^1$$

for all $\zeta \in S^1$, it follows that $\tau(z/|z|) = 2\pi it$ with $t \in \mathbb{R}$. Then

$$|g(z)| = \left|ze^{(2|z|-1)\tau(z/|z|)}\right| = |z|\left|e^{(2|z|-1)(2\pi it)}\right| = |z| = r.$$

Now, it is sufficient to prove that g maps each $\{|z| = r\}$ homeomorphicaly onto itself, which is equivalent to prove that the map $h : S^1 \to S^1$ defined by $h(\zeta) = g(r\zeta)/r$ is a homeomorphism. We have that

$$h(\zeta) = g(r\zeta)/r = (r\zeta)e^{(2|r\zeta|-1)\tau(r\zeta/|r\zeta|)}/r = \zeta e^{(2r-1)\tau(\zeta)},$$

where $1/2 \leq r \leq 1$. Since ϕ is a homeomorphism and preserves the orientation, there exists a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ such that $\phi(e^{2\pi it}) = e^{2\pi if(t)}$ and $f(t+1) = f(t)+1$ for all $t \in \mathbb{R}$. Then, since $e^{\tau(\zeta)} = \phi(\zeta)/\zeta$, we obtain

$$e^{\tau(e^{2\pi it})} = \phi(e^{2\pi it})/e^{2\pi it} = e^{2\pi if(t)}/e^{2\pi it} = e^{2\pi i(f(t)-t)}.$$

Hence $\tau(e^{2\pi it}) = 2\pi i(f(t) - t + N)$, where $N \in \mathbb{Z}$. Then

$$\begin{aligned} h(e^{2\pi it}) &= e^{2\pi it}e^{(2r-1)\tau(e^{2\pi it})} = e^{2\pi it}e^{(2r-1)(2\pi i)(f(t)-t+N)} \\ &= e^{(2\pi i)(t+(2r-1)f(t)-(2r-1)t+(2r-1)N)} \\ &= e^{(2\pi i)((2r-1)f(t)+(2-2r)t+(2r-1)N)} \qquad (6) \end{aligned}$$

and we have therefore

$$h(e^{2\pi it}) = e^{2\pi i\bar{f}(t)},$$

where $\bar{f}(t) = (2r-1)f(t)+(2-2r)t+(2r-1)N$. An easy computation shows that $\bar{f}(t+1) = \bar{f}(t)+1$. Moreover, since f is increasing, it is easy to see that $\bar{f}$ also is. Then $\bar{f} : \mathbb{R} \to \mathbb{R}$ is a homeomorphism and the lemma follows. q.e.d.

Proof of Lemma 4.7. Since the conjugation $z \to \bar{z}$ preserves the convex sets, by replacing f with $\bar{f}$ we may assume that f is holomorphic. For $r > 0$ small enough, define $g_r : \mathbb{D} \to \mathbb{C}$, $g_r(z) = f(rz/a)/r$, where

$a = f'(0)$. It is easy to see that $g_r(z) \to z$ as $r \to 0$ for all z. Then g_r converges uniformly on compact sets to the identity $\mathrm{id} : \mathbb{D} \to \mathbb{D}$ as $r \to 0$. Hence there is r_0 such that for all $r \leq r_0$ we have

$$|| \,\mathrm{id} - g_r||_{\{|z| \leq 1/2\}} < \epsilon,$$

where ϵ is given by Lemma 4.8 for $\delta_0 = 1/2$. Therefore $g_r(|z| \leq 1/2)$ is convex for all $r \leq r_0$. But

$$g_r(|z| \leq 1/2) = g\left(\frac{r\{|z| \leq 1/2\}}{a}\right)/r = g\left(|z| \leq \frac{r}{|2a|}\right)/r,$$

which is convex in and only if the set $g(|z| \leq r/|2a|)$ is convex. Then, if we take $\delta_0 = r_0/(2|a|)$, we have that the set $g(|z| \leq \delta)$ is convex for all $\delta \leq \delta_0$. q.e.d.

Proof of Lemma 4.8. We may assume that f is holomorphic. Thus, if the conformal map g is close enough to f, it will be holomorphic too. If $\alpha : (a, b) \to \mathbb{C}$ is a smooth curve, the curvature of α at the point $\alpha(t)$ is given by

$$\begin{aligned} k_\alpha(t) &= \left|\frac{d}{dt}\left(\frac{\alpha'(t)}{|\alpha'(t)|}\right)\right| = \frac{||\alpha''(t)|\alpha'(t)| - \alpha'(t)|\alpha'(t)|'||}{|\alpha'(t)|^2} \\ &= \frac{\left|\left|\alpha''(t)|\alpha'(t)| - \alpha'(t)\left(\frac{\alpha''(t)\overline{\alpha'(t)} + \alpha'(t)\overline{\alpha''(t)}}{2|\alpha'(t)|}\right)\right|\right|}{|\alpha'(t)|^2} \\ &= \frac{||\alpha''(t)|\alpha'(t)|^2 - \overline{\alpha''(t)}(\alpha'(t))^2||}{2|\alpha'(t)|^3}. \end{aligned} \tag{7}$$

Let $r > 0$ and parametrices the boundary of the disc $\{|z| \leq r\}$ by $\gamma_r(t) = re^{it/r}$, $t \in \mathbb{R}$. Let $g : \mathbb{D} \to \mathbb{C}$ be any holomorphic conformal map and let α_{rg} be the curve $\alpha_{rg} = g \circ \gamma_r$. We have $\alpha'_{rg}(t) = g'(\gamma_r(t))\gamma'_r(t)$, $\alpha''_{rg}(t) = g''(\gamma_r(t))(\gamma'_r(t))^2 + g'(\gamma_r(t))\gamma''_r(t)$, $|\gamma'_r(t)| = 1$ and $|\gamma'_r(t)| = 1/r$. Then from (7):

$$\begin{aligned} k_{\alpha_{rg}}(t) &= \frac{|(g''(\gamma_r)(\gamma'_r)^2 + g'(\gamma_r)\gamma''_r)|g'(\gamma_r)| - g'(\gamma_r)\gamma'_r|g'(\gamma_r)|'}{|g'(\gamma_r)|^2} \\ &= \frac{|g'(\gamma_r)\gamma''_r|g'(\gamma_r)| + g''(\gamma_r)(\gamma'_r)^2|g'(\gamma_r)| - g'(\gamma_r)\gamma'_r|g'(\gamma_r)|'}{|g'(\gamma_r)|^2}, \end{aligned}$$

Hence

$$\begin{aligned} k_{\alpha_{rg}}(t) &\geq \frac{|g'(\gamma_r)|^2/r - |g''(\gamma_r)||g'(\gamma_r)| - |g'(\gamma_r)||g'(\gamma_r)|'}{|g'(\gamma_r)|^2} \\ &= \frac{|g'(\gamma_r)|/r - |g''(\gamma_r)| - |g'(\gamma_r)|'}{|g'(\gamma_r)|}. \end{aligned} \tag{8}$$

Observe that

$$|g'(\gamma_r)|' = \frac{g''(\gamma_r)\gamma_r'\overline{g'(\gamma_r)} + g'(\gamma_r)\overline{g''(\gamma_r)\gamma_r'}}{|g'(\gamma_r)|}$$

and thus

$$|g'(\gamma_r)|' \leq \frac{|g''(\gamma_r)||g'(\gamma_r)| + |g'(\gamma_r)||g''(\gamma_r)|}{|g'(\gamma_r)|} \leq 2|g''(\gamma_r)|.$$

Replacing in equation (8) we obtain

$$\begin{aligned} k_{\alpha_{rg}}(t) &\geq \frac{|g'(\gamma_r)|/r - |g''(\gamma_r)| - 2|g''(\gamma_r)|}{|g'(\gamma_r)|} \\ &= 1/r - 3|g''(\gamma_r)|/|g'(\gamma_r)|. \end{aligned}$$

We know that if $g \to f$, then $g''/g' \to f''/f'$ (uniformly on the compact sets). Then there is $\epsilon > 0$ such that $||f-g||_{\{|z|\leq r_0\}} < \epsilon_1$ implies $k_{\alpha_{rg}}(t) \geq 1/r - 3||f''/f'||_{\{|z|\leq \delta_0\}} - 1$ for all $r \in (0, \delta_0)$. Thus we make take $r_0 \in (0, \delta_0)$ such that $k_{\alpha_{rg}}(t) > 0$ whenever $||f-g||_{\{|z|\leq r_0\}} < \epsilon$ and $r < r_0$. On the other hand, clearly if $g \to f$ then $g'(\gamma_r) \to f'(\gamma_r)$ and $g''(\gamma_r) \to f''(\gamma_r)$ uniformly on $\{r_0 \leq |r| \leq \delta_0\}$, $t \in \mathbb{R}$. Consequently, from (8), we have $k_{\alpha_{rg}}(t) \to k_{\alpha_{rf}}(t)$ uniformly on $t \in \mathbb{R}$, $r \in [r_0, \delta_0]$. Then, since $k(\alpha_{rf}(t)) > 0$ for all $t \in \mathbb{R}$, $r \in [r_0, \delta_0]$ (from convexity), we may reduce ϵ in order to have $k_{\alpha_{rg}}(t) > 0$ for all $t \in \mathbb{R}$, $r \in [r_0, \delta_0]$. Thus $g(|z| \leq \delta)$ is convex for all $\delta \leq \delta_0$. Clearly we may assume ϵ small enough such that $g(|z| \leq \delta_0)$ is contained in U, which finishes the proof. q.e.d.

5. The differentiable case.

In this section we prove Theorem 1.1. As before, let $\pi : \widehat{\mathbb{C}^2} \to \mathbb{C}^2$ be the blow up at $0 \in \mathbb{C}^2$ and let E be denote the divisor $\pi^{-1}(0)$. Let $\rho : \widehat{\mathbb{C}^2} \to E$ be the natural projection associated to the fibration on $\widehat{\mathbb{C}^2}$ which fibers are given by the strict transforms of the complex lines passing through $0 \in \mathbb{C}^2$.

Definition 5.1. Let $\{z_k\}$ be a sequence of points in $\mathbb{C}^2 \backslash \{0\}$. Let L be a complex line passing through $0 \in \mathbb{C}^2$. We say that $\{z_k\}$ is tangent to L at 0 if $z_k \to 0$ and every accumulation point of $\{z_k/||z_k||\}$ is contained in L.

Lemma 5.2. *Let $\{x_k\}$ be a sequence of points in $\widehat{\mathbb{C}^2} \backslash E$. Let $x \in E$ and let $P_x = \pi(L_x)$, where L_x is the fiber of ρ through x. Then $x_k \to x \in E$ if and only if $\{\pi(x_k)\}$ is tangent to P_x at 0.*

Let C be an irreducible separatrix (That is: an irreducible holomorphic curve invariant by $\mathcal{F}$) of $\mathcal{F}$ (It exists by Separatrix Theorem, see [**4**]). Then $\widetilde{C} = h(C)$ is an irreducible separatrix of $\widetilde{\mathcal{F}}$. Let P and $\widetilde{P}$ be the tangents lines at $0 \in \mathbb{C}^2$ of C and $\widetilde{C}$ respectively.

Proposition 5.3. *Denote by* A *the derivative* $d\,\mathrm{h}(0) : \mathbb{R}^4 \to \mathbb{R}^4$. *Then* $\mathrm{A}(P) = \widetilde{P}$.

Proof. Given $v \in P\backslash\{0\}$, there exits a path $\gamma : [0,1) \to C$, with $\gamma(0) = 0$ and such that $\gamma'(0) = v$. Then the path $\mathrm{h}\circ\gamma$ is contained in $\widetilde{C}$ and therefore

$$(\mathrm{h}\circ\gamma)'(0) = d\,\mathrm{h}(0)(\gamma'(0)) = \mathrm{A}(v)$$

is contained in $\widetilde{P}$. It follows that $\mathrm{A}(P) \subset \widetilde{P}$, and so $\mathrm{A}(P) = \widetilde{P}$, since A is a isomorphism. q.e.d.

Let L and $\widetilde{L}$ denote the strict transforms by π, of P and $\widetilde{P}$ respectively. Let q and $\widetilde{q}$ be the points of intersection of L and $\widetilde{L}$ with E. We may assume without loss of generality that

$$P = \widetilde{P} = \{(z_1, z_2) \in \mathbb{C}^2 : z_2 = 0\}.$$

Let $\mathcal{U} = \pi^{-1}(z_1 \neq 0)$ and consider holomorphic coordinates (t, x) in $\mathcal{U}$ such that π is given by $\pi(t, x) = (x, tx)$. Then the fibers of ρ are given by the sets $\{t = cte\}$ and , the fibers L and $\widetilde{L}$ are represented by $\{t = 0\}$, that is, $q = \widetilde{q} = (0, 0)$. Since $\widetilde{\mathcal{F}}_0$ has a finite number of singularities on E, we may take $\epsilon > 0$ such that the set $\{(t, 0) : 0 < |t| < 2\epsilon\} \subset E$ does not contain singularities of $\widetilde{\mathcal{F}}_0$. let

$$A : \widehat{\mathbb{C}^2}\backslash E \to \widehat{\mathbb{C}^2}\backslash E$$

be the homeomorphism defined by $A = \pi^{-1}\mathrm{A}\pi$.

Proposition 5.4. *There exists* $\delta > 0$ *such that the set*

$$\{(t, x) : |t| < 2\delta\}\backslash E$$

is mapped by A *into* $\{(t, x) : |t| < 2\epsilon\}$. *Clearly, we may take* δ *such that the set* $\{(t, 0) : 0 < |t| < 2\delta\} \subset E$ *does not contain singularities of* $\mathcal{F}_0$.

Proof. Let $\mathrm{A}(z) = (\mathrm{A}_1(z), \mathrm{A}_2(z))$ for all $z = (z_1, z_2) \in \mathbb{C}^2$. Since $\mathrm{A}(P) = P'$, it follows that $\mathrm{A}_2(z_1, 0) = 0$ for all $z_1 \in \mathbb{C}$. Hence:

$$\frac{\mathrm{A}_2(\zeta, 0)}{\mathrm{A}_1(\zeta, 0)} = 0$$

for all $\zeta \in S^1$. Then there exists $\delta > 0$ such that

$$\frac{\mathrm{A}_2(\zeta, z_2)}{\mathrm{A}_1(\zeta, z_2)} < 2\epsilon \tag{9}$$

for all $\zeta \in S^1$ and all $z_2 \in \mathbb{C}$ with $|z_2| \leq 2\delta$. Since A is real linear:

$$\frac{\mathrm{A}_2(z_1, z_2)}{\mathrm{A}_1(z_1, z_2)} = \frac{|z_1|\mathrm{A}_2(z_1/|z_1|, z_2/|z_1|)}{|z_1|\mathrm{A}_1(z_1/|z_1|, z_2/|z_1|)} = \frac{\mathrm{A}_2(z_1/|z_1|, z_2/|z_1|)}{\mathrm{A}_1(z_1/|z_1|, z_2/|z_1|)} < 2\epsilon$$

and, since $z_1/|z_1| \in S^1$, it follows from (9) that

$$\frac{\mathrm{A}_2(z_1,z_2)}{\mathrm{A}_1(z_1,z_2)} < 2\epsilon \quad \text{whenever} \quad |z_2/z_1| \le 2\delta. \tag{10}$$

If $w \in \{(t,x) : |t| < 2\delta\}\backslash E$, then $\pi(w) = (z_1,z_2)$ with $z_1 \ne 0$ and $|z_2/z_1| < 2\delta$. Therefore

$$\begin{aligned} A(w) &= \pi^{-1}\mathrm{A}\pi(w) = \pi^{-1}\mathrm{A}(z_1,z_2) = \pi^{-1}(\mathrm{A}_1(z_1,z_2), \mathrm{A}_2(z_1,z_2)) \\ &= \left(\frac{\mathrm{A}_2(z_1,z_2)}{\mathrm{A}_1(z_1,z_2)}, \mathrm{A}_1(z_1,z_2)\right), \end{aligned}$$

and it follows from (10) that $A(w)$ is contained in $\{(t,x) : |t| < 2\epsilon\}$.

q.e.d.

Let $p = (\delta, 0) \in E$ and let $L_p = \{t = \delta\}$ (its fiber). Consider the path

$$\beta : S^1 \to L_p,$$

$$\beta(\zeta) = (\delta, \zeta),$$

and let $\beta_A : S^1 \to \{(t,x) : |t| < 2\epsilon\}$ given by $\beta_A = A \circ \beta$.

Proposition 5.5. *The set $\rho(A(L_p\backslash\{p\}))$ is equal to $\rho(\beta_A(S^1))$.*

Proof. Evidently $\rho\beta_A(S^1) \subset \rho(A(L_p\backslash\{p\}))$. On the other hand, let $(\delta, x) \in L_p\backslash\{p\}$, then

$$\begin{aligned} \rho A(\delta,x) &= \rho\pi^{-1}\mathrm{A}\pi(\delta,x) = \rho\pi^{-1}\mathrm{A}(x,\delta x) \\ &= \rho\pi^{-1}(\mathrm{A}_1(x,\delta x), \mathrm{A}_2(x,\delta x)) = \rho\left(\frac{\mathrm{A}_2(x,\delta x)}{\mathrm{A}_1(x,\delta x)}, \mathrm{A}_1(x,\delta x)\right) \\ &= \left(\frac{\mathrm{A}_2(x,\delta x)}{\mathrm{A}_1(x,\delta x)}, 0\right) = \left(\frac{\mathrm{A}_2(x/|x|,\delta x/|x|)}{\mathrm{A}_1(x/|x|,\delta x/|x|)}, 0\right) \\ &= \rho\left(\frac{\mathrm{A}_2(x/|x|,\delta x/|x|)}{\mathrm{A}_1(x/|x|,\delta x/|x|)}, \mathrm{A}_1(x/|x|,\delta x/|x|)\right) \\ &= \rho\pi^{-1}(\mathrm{A}_1(x/|x|,\delta x/|x|), \mathrm{A}_2(x/|x|,\delta x/|x|)) \\ &= \rho\pi^{-1}\mathrm{A}(x/|x|,\delta x/|x|) = \rho\pi^{-1}\mathrm{A}\pi(\delta, x/|x|) \\ &= \rho A(\beta(x/|x|)) = \rho(\beta_A(x/|x|)). \end{aligned}$$

Therefore $\rho(A(L_p\backslash\{p\})) \subset \rho\beta_A(S^1)$. q.e.d.

Define K as the set of points $y \in E$ such that there exists a sequence $\{x_k\}$ in $L_p\backslash\{p\}$ with $h(x_k) \to y$ as $k \to \infty$.

Proposition 5.6. *Given a neighborhood Ω of K in $\widehat{\mathbb{C}^2}$, there exist a disc Σ in L_p containing p, such that the set $h(\Sigma\backslash\{p\})$ is contained in Ω.*

Proof. Is a direct consequence of the definition of K. q.e.d.

Proposition 5.7. *The set K is equal to $\rho\beta_A(S^1)$. Thus, since $\beta_A(S^1) \subset A(L_p\backslash\{p\})$ does not intersect $\widetilde{L}$, the set K is contained in $\{(t,0) : 0 < |t| < 2\epsilon\}$.*

Proof. Let $y \in K$. Then there exist a sequence $\{x_k\}$ in $L_p\backslash\{p\}$ with $h(x_k) \to y$ as $k \to \infty$. Let $P_y = \pi(L_y)$, where L_y is the fiber of ρ through y. It follows from Lemma 5.2 that the sequence $\{\pi(h(x_k))\}$ is tangent to P_y at 0. Since $\pi(x_k) \to 0$ as $k \to \infty$ and A is the derivate of h at 0, we have that

$$\mathrm{h}(\pi(x_k)) = \mathrm{A}(\pi(x_k)) + R(\pi(x_k)),$$

where $R(\pi(x_k))/||\pi(x_k)|| \to 0$ as $k \to \infty$. Therefore

$$\frac{\mathrm{h}(\pi(x_k))}{||\pi(x_k)||} = \frac{\mathrm{A}(\pi(x_k))}{||\pi(x_k)||} + \frac{R(\pi(x_k))}{||\pi(x_k)||}, \tag{11}$$

with $R(\pi(x_k))/||\pi(x_k)|| \to 0$ as $k \to \infty$. Since the sequence $\{\mathrm{h}\,\pi(x_k)\} = \{\pi h(x_k)\}$ is tangent to P_y at 0, we have by definition that any accumulation point of

$$\frac{\mathrm{h}(\pi(x_k))}{||\,\mathrm{h}(\pi(x_k))||}$$

is contained in P_y and the same holds for the sequence

$$\frac{\mathrm{h}(\pi(x_k))}{||\pi(x_k)||} = \frac{\mathrm{h}(\pi(x_k))}{||\,\mathrm{h}(\pi(x_k))||} \cdot \frac{||\,\mathrm{h}(\pi(x_k))||}{||\pi(x_k)||}.$$

Then, it follows from (11) that any accumulation point of the sequence

$$\frac{\mathrm{A}(\pi(x_k))}{||\pi(x_k)||}$$

is contained in P_y and the same property is satisfied by

$$\frac{\mathrm{A}(\pi(x_k))}{||\mathrm{A}(\pi(x_k))||} = \frac{\mathrm{A}(\pi(x_k))}{||\pi(x_k)||} \cdot \frac{||\pi(x_k)||}{||\mathrm{A}(\pi(x_k))||}.$$

Then the sequence

$$\frac{\mathrm{A}(\pi(x_k))}{||\mathrm{A}(\pi(x_k))||} = \frac{\pi(A(x_k))}{||\pi(A(x_k))||}$$

is tangent to P_y at 0. By Lemma 5.2 we have that $A(x_k) \to y$ as $k \to \infty$, hence $\rho(A(x_k)) \to y$ as $k \to \infty$. Then y is a limit point of $\rho(A(L_p\backslash\{p\}))$. But $\rho(A(L_p\backslash\{p\}))$ is equal to $\rho\beta_A(S^1)$ by Proposition 5.5. Then, since $\rho\beta_A(S^1)$ is compact, we have that $y \in \rho\beta_A(S^1)$ and therefore $K \subset \rho\beta_A(S^1)$. On the other hand, let $y \in \rho\beta_A(S^1)$. Then $y = \rho(A(\delta,\zeta))$. For all $k \in \mathbb{N}$ let $x_k = (\delta, s_k\zeta) \in L_p$, where $s_k > 0$

and $s_k \to 0$ as $k \to \infty$. Clearly $x_k \to p = (\delta, 0)$ as $k \to \infty$. Then $\pi(x_k) \to 0 \in \mathbb{C}^2$ as $k \to \infty$ and we have that

$$\mathrm{h}(\pi(x_k)) = \mathrm{A}(\pi(x_k)) + R(\pi(x_k))$$

with $||R(\pi(x_k))||/||\pi(x_k)|| \to 0$ as $k \to \infty$. Therefore

$$\frac{\mathrm{h}(\pi(x_k))}{||\pi(x_k)||} = \frac{\mathrm{A}(\pi(x_k))}{||\pi(x_k)||} + \frac{R(\pi(x_k))}{||\pi(x_k)||}.$$

Hence, since

$$\frac{\mathrm{A}(\pi(x_k))}{||\pi(x_k)||} = \frac{\mathrm{A}(s_k\zeta, s_x\zeta\delta)}{||(s_k\zeta, s_x\zeta\delta)||} = \frac{s_k\,\mathrm{A}(\zeta, \zeta\delta)}{|s_k|||(\zeta, \zeta\delta)||} = \frac{\mathrm{A}(\zeta, \zeta\delta)}{||(\zeta, \zeta\delta)||}$$

and $||R(\pi(x_k))||/||\pi(x_k)|| \to 0$ as $k \to \infty$, we have that

$$\frac{\mathrm{h}(\pi(x_k))}{||\pi(x_k)||} \to \frac{\mathrm{A}(\zeta, \zeta\delta)}{||(\zeta, \zeta\delta)||} \tag{12}$$

as $k \to \infty$. Let L_y be the fiber of ρ through y and let $P_y = \pi(L_y)$. Since $\rho(A(\delta, \zeta)) = y$ we have $A(\delta, \zeta) \in L_y$, hence $\pi A(\delta, \zeta) \in P_y$. Then

$$\frac{\mathrm{A}(\zeta, \zeta\delta)}{||(\zeta, \zeta\delta)||} = \frac{\mathrm{A}(\pi(\delta, \zeta))}{||(\pi(\delta, \zeta)||} = \frac{\pi A(\delta, \zeta)}{||(\pi(\delta, \zeta)||}$$

is contained in P_y and it follows from (12) that any accumulation point of the sequence

$$\frac{\pi(h(x_k))}{||\pi(h(x_k))||} = \frac{\mathrm{h}(\pi(x_k))}{||\pi(x_k)||} \cdot \frac{||\,\mathrm{h}(\pi(x_k))||}{||\pi(x_k)||}$$

is contained in P_y. Then, by Lemma 5.2 we have that $\pi(h(x_k)) \to y$ as $k \to \infty$. Thus $y \in K$ and therefore $\rho\beta_A(S^1) \subset K$. q.e.d.

Proposition 5.8. *Define* $\theta : [0, 1] \to E$ *by* $\theta(s) = \rho\beta_A(e^{\pi is})$ *for all* $s \in [0, 1]$. *Then*

$$\begin{aligned} \rho \circ \beta_A(e^{2\pi is}) &= \theta(2s), \quad \text{if} \quad 0 \le s \le 1/2, \\ \rho \circ \beta_A(e^{2\pi is}) &= \theta(2s-1), \quad \text{if} \quad 1/2 \le s \le 1. \end{aligned}$$

In particular, $\rho\beta(S^1) = \theta([0, 1])$ *and, by Proposition* 5.7*, we have that* $K = \theta([0, 1])$.

Proof. If $s \in [0,1/2]$, then $\rho\beta_A(e^{2\pi i s}) = \rho\beta_A(e^{\pi i(2s)}) = \theta(2s)$. Suppose now that $s \in [1/2,1]$. Then, since A is real linear:

$$\begin{aligned}
w &= \rho A\beta(e^{2\pi i s}) = \rho\pi^{-1}\mathrm{A}\pi(\delta, e^{2\pi i s}) = \rho\pi^{-1}\mathrm{A}(e^{2\pi i s}, \delta e^{2\pi i s}) \\
&= \rho\pi^{-1}(-1)\mathrm{A}((-1)e^{2\pi i s}, (-1)\delta e^{2\pi i s}) \\
&= \rho\pi^{-1}(-1)(\mathrm{A}_1(e^{-\pi i}e^{2\pi i s}, e^{-\pi i}\delta e^{2\pi i s}), \mathrm{A}_2(e^{-\pi i}e^{2\pi i s}, e^{-\pi i}\delta e^{2\pi i s})) \\
&= \rho\pi^{-1}(-\mathrm{A}_1(e^{\pi i(2s-1)}, \delta e^{\pi i(2s-1)}), -\mathrm{A}_2(e^{\pi i(2s-1)}, \delta e^{\pi i(2s-1)})) \\
&= \rho\left(\frac{\mathrm{A}_2(e^{\pi i(2s-1)}, \delta e^{\pi i(2s-1)})}{\mathrm{A}_1(e^{\pi i(2s-1)}, \delta e^{\pi i(2s-1)})}, -\mathrm{A}_1(e^{\pi i(2s-1)}, \delta e^{\pi i(2s-1)})\right) \\
&= \left(\frac{\mathrm{A}_2(e^{\pi i(2s-1)}, \delta e^{\pi i(2s-1)})}{\mathrm{A}_1(e^{\pi i(2s-1)}, \delta e^{\pi i(2s-1)})}, 0\right) \\
&= \rho\left(\frac{\mathrm{A}_2(e^{\pi i(2s-1)}, \delta e^{\pi i(2s-1)})}{\mathrm{A}_1(e^{\pi i(2s-1)}, \delta e^{\pi i(2s-1)})}, \mathrm{A}_1(e^{\pi i(2s-1)}, \delta e^{\pi i(2s-1)})\right) \\
&= \rho\pi^{-1}(\mathrm{A}_1(e^{\pi i(2s-1)}, \delta e^{\pi i(2s-1)}), \mathrm{A}_2(e^{\pi i(2s-1)}, \delta e^{\pi i(2s-1)})) \\
&= \rho\pi^{-1}\mathrm{A}(e^{\pi i(2s-1)}, \delta e^{\pi i(2s-1)}) = \rho\pi^{-1}\mathrm{A}\pi(\delta, e^{\pi i(2s-1)}) \\
&= \rho A(\delta, e^{\pi i(2s-1)}) = \rho A\beta(e^{\pi i(2s-1)}) = \rho\beta_A(e^{\pi i(2s-1)}) \\
&= \theta(2s-1),
\end{aligned}$$

since $(2s-1) \in [0,1]$.

Proposition 5.9. *We have that: either K is a point, or K is equal to a Jordan curve.*

Proof. By Proposition 5.7 and Proposition 5.8, it is sufficient to prove that: either θ is constant or it is a simple closed curve. By Proposition 5.8, we have that $\theta(0) = \theta(2(1/2)-1) = \rho\beta_A(e^{2\pi i(1/2)}) = \theta(2(1/2)) = \theta(1)$. Thus θ defines a closed curve in E. Suppose that θ is not a simple curve, that is, $\theta(s') = \theta(s'')$ for $0 \leq s' < s'' < 1$. Observe that

$$\theta(s') = \rho\pi^{-1}\mathrm{A}\pi(\delta, e^{\pi i s'}) = \rho\pi^{-1}\mathrm{A}(e^{\pi i s'}, \delta e^{\pi i s'}).$$

Writing $\mathrm{A}(e^{\pi i s'}, \delta e^{\pi i s'}) = (\mathrm{A}_1', \mathrm{A}_2')$ we have that

$$\theta(s') = \rho\pi^{-1}(\mathrm{A}_1', \mathrm{A}_2') = \rho\left(\frac{\mathrm{A}_2'}{\mathrm{A}_1'}, \mathrm{A}_1'\right) = \left(\frac{\mathrm{A}_2'}{\mathrm{A}_1'}, 0\right).$$

Analogously, making $\mathrm{A}(e^{\pi i s''}, \delta e^{\pi i s''}) = (\mathrm{A}_1'', \mathrm{A}_2'')$ we obtain

$$\theta(s'') = \left(\frac{\mathrm{A}_2''}{\mathrm{A}_1''}, 0\right).$$

Then $\frac{\mathrm{A}_2'}{\mathrm{A}_1'} = \frac{\mathrm{A}_2''}{\mathrm{A}_1''}$ and we have therefore that

$$\frac{a\mathrm{A}_2' + b\mathrm{A}_2''}{a\mathrm{A}_1' + b\mathrm{A}_1''} = \frac{\mathrm{A}_2'}{\mathrm{A}_1'} = \frac{\mathrm{A}_2''}{\mathrm{A}_1''}$$

for all $a, b \in \mathbb{R}$ such that $a\mathrm{A}_1' + b\mathrm{A}_1'' \neq 0$. Computing as above

$$\rho\pi^{-1}\left(a\mathrm{A}_1' + b\mathrm{A}_1'', a\mathrm{A}_2' + b\mathrm{A}_2''\right) = \left(\frac{a\mathrm{A}_2' + b\mathrm{A}_2''}{a\mathrm{A}_1' + b\mathrm{A}_1''}, 0\right) = \left(\frac{\mathrm{A}_2'}{\mathrm{A}_1'}, 0\right) = \theta(s'),$$

that is,

$$\rho\pi^{-1}(a(\mathrm{A}_1', \mathrm{A}_2') + b(\mathrm{A}_1'', \mathrm{A}_2'')) = \theta(s'). \tag{13}$$

Since $0 \leq s' < s'' < 1$, the vectors $e^{\pi i s'}$ and $e^{\pi i s''}$ are real-linearly independent. Thus, for all $s \in [0,1)$ we have that $e^{\pi i s} = ae^{\pi i s'} + be^{\pi i s''}$ with $a, b \in \mathbb{R}$. Therefore:

$$\begin{aligned}
\theta(s) &= \rho A\beta(e^{\pi i s}) = \rho\pi^{-1}\mathrm{A}\pi(\delta, e^{2\pi i s}) = \rho\pi^{-1}\mathrm{A}(e^{2\pi i s}, \delta e^{2\pi i s}) \\
&= \rho\pi^{-1}\mathrm{A}(ae^{\pi i s'} + be^{\pi i s''}, \delta(ae^{\pi i s'} + be^{\pi i s''})) \\
&= \rho\pi^{-1}\mathrm{A}(a(e^{\pi i s'}, \delta e^{\pi i s'}) + b(e^{\pi i s''}, \delta e^{\pi i s''})) \\
&= \rho\pi^{-1}(a\mathrm{A}(e^{\pi i s'}, \delta e^{\pi i s'}) + b\mathrm{A}(e^{\pi i s''}, \delta e^{\pi i s''})) \\
&= \rho\pi^{-1}(a(\mathrm{A}_1', \mathrm{A}_2') + b(\mathrm{A}_1'', \mathrm{A}_2'')),
\end{aligned}$$

and by using (13):

$$\theta(s) = \theta(s').$$

It follows that θ is constant and the assertion is therefore proved.

We denote by V and $\widetilde{V}$ the sets $\{(t,x) : |t| \leq 2\delta\}$ and $\{(t,x) : |t| \leq 2\epsilon\}$ respectively. Let

$$\widetilde{\beta} : S^1 \to \widetilde{V}$$

be the path defined by $\widetilde{\beta}(\zeta) = (\epsilon, \zeta)$.

Proposition 5.10. *The path β_A is homologous to $\xi\widetilde{\beta}$ in $\widetilde{V}\backslash(\widetilde{L} \cup E)$, where $\xi = 1$ or -1.*

Proof. Let $\mathcal{B}_w$ be the disc $\{(t,x) : t = w, |x| \leq 1\}$ in V. Observe that $\widetilde{\beta}$ is equal to $\partial\widetilde{\mathcal{B}}$, where $\widetilde{\mathcal{B}}$ is the disc $\{(\epsilon, x) : |x| \leq 1\}$ in $\widetilde{V}$. Then, since $\mathrm{A} : \mathbb{R}^4 \to \mathbb{R}^4$ preserves orientation, it follows from Lemma 3.1 that for some $w \neq 0$:

$$A(\partial\mathcal{B}_w) = \xi\partial\widetilde{\mathcal{B}} = \xi\widetilde{\beta} \quad \text{in} \quad H_1(\widetilde{V}\backslash(\widetilde{L} \cup E)). \tag{14}$$

Observe that $\partial\mathcal{B}_w$ is homologous to β in $V\backslash(L \cup E)$. Then, since $A(V\backslash(L \cup E))$ is contained in $\widetilde{V}\backslash(\widetilde{L} \cup E)$, it follows that

$$A(\partial\mathcal{B}_w) = A(\beta) = \beta_A \quad \text{in} \quad H_1(\widetilde{V}\backslash(\widetilde{L} \cup E)). \tag{15}$$

Thus the proposition follows from (15) and (14).

Proposition 5.11. *Suppose that K is a Jordan curve and let $\mathrm{U} \subset \{(t, 0) : |t| < 2\epsilon\}$ be the domain bounded by K. Then $q = (0,0) \notin \mathrm{U}$.*

Proof. Making $C = \{(t,0) : |t| < \epsilon\}$ and since $\rho : \widetilde{V}\backslash(\widetilde{L}\cup E) \to C\backslash\{p'\}$ is well defined, it follows from Proposition 5.10 that

$$\rho(\beta_A) = \xi\rho(\widetilde{\beta}) \quad \text{in} \quad H_1(C\backslash\{p'\}).$$

Then, since $\rho(\widetilde{\beta}) = 0$ in $H_1(C\backslash\{p'\})$, we have that

$$\rho \circ \beta_A = 0 \quad \text{in} \quad H_1(C\backslash\{p'\}). \tag{16}$$

If we consider $\rho \circ \beta_A$ as defined on $[0,1]$ by $s \to \rho\beta_A(e^{2\pi i s})$, it follows from Proposition 5.8 that $\rho \circ \beta_A = \theta * \theta$. Then

$$\rho \circ \beta_A = 2\theta \quad \text{in} \quad H_1(C\backslash\{p'\})$$

and it follows from (16) that

$$\theta = 0 \quad \text{in} \quad H_1(C\backslash\{p'\}),$$

since $H_1(C\backslash\{p'\})$ does not have torsion. Therefore $p' \notin$ U.

Proposition 5.12. *Let Σ be a disc in L_p containing p and such that $\mathcal{A} = h(\Sigma\backslash\{p\})$ is contained in $\widetilde{V}\backslash E$. Let γ be a path in $\mathcal{A}$, which represents a generator of $H_1(\mathcal{A})$. Then γ is homologous to $\xi\widetilde{\beta}$ in $\widetilde{V}\backslash E$ with $\xi = 1$ or -1.*

Proof. Since $\widetilde{V}\backslash(\widetilde{L} \cup E)$ is contained in $\widetilde{V}\backslash E$, it follows from Proposition 5.10 that β_A is homologous to $\xi\widetilde{\beta}$ in $\widetilde{V}\backslash E$ where $\xi = 1$ or -1. Therefore it is sufficient to show that γ is homologous to $\xi\beta_A$ with $\xi = 1$ or -1. Let

$$\vartheta_r : S^1 \to L_p = \{t = \delta\}$$

be the path defined by $\vartheta_r(\zeta) = (\delta, r\zeta)$ with $0 < r < 1$ small enough such that $\{(\delta, x) : |x| \leq r\}$ is contained in Σ. Then ϑ_r is a generator of $H_1(\Sigma\backslash\{p\})$ and consequently $h \circ \vartheta_r$ is a generator of $H_1(\mathcal{A})$. Thus γ is homologous to $\xi h \circ \vartheta_r$ in $\widetilde{V}\backslash E$, where $\xi = 1$ or -1. Therefore it is sufficient to prove that $h \circ \vartheta_r$ is homologous to β_A in $\widetilde{V}\backslash E$. Recall that $\beta(\zeta) = (\delta, \zeta)$. Then β and ϑ_r are homologous in $C = \{(\delta, x) : 0 < |x| \leq 1\} \subset L_p$ and, since $A(C) \subset \widetilde{V}\backslash E$, it follows that the paths $A \circ \beta = \beta_A$ and $A \circ \vartheta_r$ are homologous in $\widetilde{V}\backslash E$. Then, it suffices to show that $h \circ \vartheta_r$ and $A \circ \vartheta_r$ are homologous in $\widetilde{V}\backslash E$ for some $r > 0$.

Let $P' = \pi(L_p)$ and consider the path $\theta_r : S^1 \to P'$ defined by $\theta_r = \pi \circ \vartheta_r$, that is $\theta_r(\zeta) = (r\zeta, \delta r\zeta)$. Recall that $\mathrm{A} : \mathbb{R}^4 \to \mathbb{R}^4$ is an isomorphism, then there exist a constant $c > 0$ such that

$$||\,\mathrm{A}(z)|| > c||z|| \quad \text{for all} \quad z \in \mathbb{C}^2. \tag{17}$$

Since A is the derivate of h at 0, there exists $\varepsilon > 0$ such that

$$\mathrm{h}(z) = \mathrm{A}(z) + R(z), \tag{18}$$

with $|R(z)| < c|z|$ whenever $|z| < \varepsilon$. Now, assume that

$$r < \min\left\{ \frac{\varepsilon}{\sqrt{1+\delta^2}}, c, c/(2\epsilon+1), \frac{\varepsilon_0}{\sqrt{1+\delta^2}} \right\},$$

where the constant $\varepsilon_0 > 0$ will be defined later. Then, since $\theta_r(\zeta) = (r\zeta, \delta r\zeta)$ satisfies

$$||\theta_r(\zeta)|| = r\sqrt{1+\delta^2} < \varepsilon, \tag{19}$$

we have that

$$||R(\theta_r(\zeta))|| < c||\theta_r(\zeta)||. \tag{20}$$

Therefore the map

$$F : S^1 \times [0,1] \to \mathbb{C}^2,$$
$$F(\zeta, s) = \mathrm{A}(\theta_r(\zeta)) + sR(\theta_r(\zeta))$$

is such that

$$\begin{aligned} ||F(\zeta,s)|| &= ||\,\mathrm{A}(\theta_r(\zeta)) + sR(\theta_r(\zeta))|| \\ &\geq ||\,\mathrm{A}(\theta_r(\zeta))|| - ||sR(\theta_r(\zeta))|| \geq c||\theta_r(\zeta)|| - ||R(\theta_r(\zeta))|| > 0. \end{aligned}$$

Observe that $F(\zeta,0) = \mathrm{A}(\theta_r(\zeta))$ and $F(\zeta,1) = \mathrm{A}(\theta_r(\zeta)) + R(\theta_r(\zeta)) = \mathrm{h}(\theta_r(\zeta))$. Then F defines a homotopy between $\mathrm{A}(\theta_r)$ and $\mathrm{h}(\theta_r)$ in $\mathbb{C}^2\backslash\{0\}$. Thus, since $\pi^{-1}\,\mathrm{A}(\theta_r) = A(\vartheta_r)$ and $\pi^{-1}\,\mathrm{h}(\theta_r) = h(\vartheta_r)$, it follows that $\pi^{-1}\circ F$ defines a homotopy between $A\circ\vartheta_r$ and $h\circ\vartheta_r$ in $\widehat{\mathbb{C}^2}\backslash E$. Therefore, in order to prove that $A\circ\vartheta_r = h\circ\vartheta_r$ in $H_1(\widetilde{V}\backslash E)$, it suffices to show that $\pi^{-1}\circ F(\zeta,s)$ belongs to $\widetilde{V}$ for all $s\in[0,1]$, $\zeta\in S^1$. We write $F(\zeta,s) = (x_F, y_F)$, $\mathrm{A}(\theta_r(\zeta)) = (x_\mathrm{A}, y_\mathrm{A})$ and $R(\theta_r(\zeta)) = (x_R, y_R)$, then

$$(x_F, y_F) = (x_\mathrm{A}, y_\mathrm{A}) + s(x_R, y_R). \tag{21}$$

Observe that

$$\left(\frac{y_\mathrm{A}}{x_\mathrm{A}}, x_\mathrm{A}\right) = \pi^{-1}(x_\mathrm{A}, y_\mathrm{A}) = \pi^{-1}\,\mathrm{A}(\theta_r(\zeta)) = A\pi^{-1}\theta_r(\zeta) = \mathrm{A}\circ\vartheta_r(\zeta),$$

hence $(y_\mathrm{A}/x_\mathrm{A}, 0) = \rho\,\mathrm{A}\,\vartheta_r(\zeta)$. Then, since $A\vartheta_r(\zeta)$ is contained in $A(L_p\backslash\{p\})$, it follows from Proposition 5.5 and Proposition 5.7 that $(y_\mathrm{A}/x_\mathrm{A}, 0)$ is contained in K. Thus, since K a compact subset of $\{(t,0) : |t| < 2\epsilon\}$, we have that

$$\frac{|y_\mathrm{A}|}{|x_\mathrm{A}|} + \varepsilon_1 < 2\epsilon \tag{22}$$

for some $\varepsilon_1 > 0$ small enough. Take $\varepsilon_2 > 0$ be such that

$$\frac{\varepsilon_2(1+2\epsilon)}{(c/(1+2\epsilon) - \varepsilon_2)} < \varepsilon_1. \tag{23}$$

Now, we chose ε_0 be such that

$$||R(z)|| < \varepsilon_2||z|| \tag{24}$$

whenever $||z|| < \varepsilon_0$. Observe that $\pi^{-1} \circ (x_F, y_F)$ belongs to

$$\widetilde{V} = \{(t,x) : |x| < 2\epsilon\}$$

if and only if $\frac{y_F}{x_F} < 2\epsilon$, and by (21), if and only if

$$\frac{y_{\rm A} + sy_R}{x_{\rm A} + sy_R} < 2\epsilon. \tag{25}$$

An easy computation shows that

$$\frac{y_{\rm A} + sy_R}{x_{\rm A} + sy_R} = \frac{y_{\rm A}}{x_{\rm A}} + \frac{sy_R - sy_R(y_{\rm A}/x_{\rm A})}{x_{\rm A} + sy_R}.$$

Thus, in view of (22), it is sufficient to prove that

$$\frac{|sy_R - sy_R(y_{\rm A}/x_{\rm A})|}{|x_{\rm A} + sy_R|} \leq \epsilon_1. \tag{26}$$

Since that $||\theta_r(\zeta)|| = r\sqrt{1+\delta^2} < \varepsilon_0$, it follows from (24) that $||(y_R, y_R)|| = ||R(\theta_r(\zeta))|| < \varepsilon_2||\theta_r(\zeta)||$, hence $|y_R| < \varepsilon_2||\theta_r(\zeta)||$. Then

$$\begin{aligned} |sy_R - sy_R(y_{\rm A}/x_{\rm A})| &= |sy_R| \cdot |1 - y_{\rm A}/x_{\rm A}| \\ &< \varepsilon_2||\theta_r(\zeta)||(1 + |y_{\rm A}|/|x_{\rm A}|) \end{aligned}$$

and, by using (22), we obtain

$$|sy_R - s(y_{\rm A}/x_{\rm A})y_R| < \varepsilon_2(1+2\epsilon)||\theta_r(\zeta)||. \tag{27}$$

On the other hand, also from (22) we have that $|y_{\rm A}| < 2\epsilon|x_{\rm A}|$, hence

$$(1+2\epsilon)|x_{\rm A}| \geq |x_{\rm A}| + |y_{\rm A}| \geq ||(x_{\rm A}, y_{\rm A})|| = ||A(\theta_r(\zeta))|| \geq c||\theta_r(\zeta)||$$

and therefore

$$|x_{\rm A}| \geq \frac{c}{1+2\epsilon} \cdot ||\theta_r(\zeta)||.$$

Then

$$|x_{\rm A} + sy_R| \geq |x_{\rm A}| - |sy_R| \geq |x_{\rm A}| - |y_R| \geq \frac{c}{1+2\epsilon}||\theta_r(\zeta)|| - \epsilon_2||\theta_r(\zeta)||$$

and so

$$|x_{\rm A} + sy_R| \geq (c/(1+2\epsilon) - \epsilon_2)||\theta_r(\zeta)||.$$

From this and (27) we obtain

$$\frac{|sy_R - s(y_{\rm A}/x_{\rm A})y_R|}{|x_{\rm A} + sy_R|} \leq \frac{\varepsilon_2(1+2\epsilon)||\theta_r(\zeta)||}{(c/(1+2\epsilon) - \epsilon_2)||\theta_r(\zeta)||} = \frac{\varepsilon_2(1+2\epsilon)}{(c/(1+2\epsilon) - \epsilon_2)}$$

and from (23):

$$\frac{|sy_R - sy_{\rm A}/x_{\rm A}y_R|}{|x_{\rm A} + sy_R|} \leq \varepsilon_1,$$

which finishes the proof. q.e.d.

It follows from Proposition 5.7 and Proposition 5.9 that there exists a subset D of the divisor E with the following properties:

(i) D is diffeomorphic to a closed disc.

(ii) D is contained in $\{(t,0) : 0 < |t| < 2\epsilon\}$
(iii) K is contained in the interior of D.

Let $\widetilde{p}$ be a point in the interior of D and let $L_{\widetilde{p}}$ be the fiber of ρ through $\widetilde{p}$. Since D is contained in a leaf of $\widetilde{\mathcal{F}}_0$, there is a disc Σ' in $L_{\widetilde{p}}$ containing $\widetilde{p}$ with the following property: if $z \in \Sigma'$, then there exists a closed disc D_z in the leaf of $\widetilde{\mathcal{F}}_0$ passing through z, such that ρ maps D_z diffeomorphically onto D. Let W denote the set $\bigcup_{z\in\Sigma'} D_z$. By Proposition 5.6, there exists a disc Σ in L_p containing p, such that the set $\mathcal{A} = h(\Sigma\backslash\{p\})$ is contained in the interior of W. We assume Σ be small enough such that $\mathcal{F}_0$ is transverse to Σ.

Proposition 5.13. *There exists a disc $\widetilde{\Sigma} \subset \Sigma'$ containing $\widetilde{p}$, with the following property. Given $x \in \widetilde{\Sigma}\backslash\{\widetilde{p}\}$, the disc D_x intersects $\mathcal{A}$ in a unique point $f(x)$. Moreover, the map $f : \widetilde{\Sigma}\backslash\{\widetilde{p}\} \to \mathcal{A}$ is continuous.*

Proof. The foliation $\widetilde{\mathcal{F}}_0$ induces a complex structure in $\mathcal{A}$ as follows. Let $y \in \mathcal{A}$ and $x \in \Sigma\backslash\{p\}$ with $h(x) = y$. Since Σ is transverse to $\mathcal{F}_0$, there exists a neighborhood W_x of x in $\widehat{\mathbb{C}^2}\backslash E$ such that each leaf of $\mathcal{F}_0|_{W_x}$ intersects Σ only one time. Let W_y be a neighborhood of y where $\widetilde{\mathcal{F}}_0$ is trivial. Thus, there exists a disc $\widetilde{\Sigma}_y$ (complex sub-manifold of W_y) such that each leaf of $\widetilde{\mathcal{F}}_0|_{W_y}$ intersects $\widetilde{\Sigma}_y$ at a unique point. We may assume that $h^{-1}(W_y)$ is contained in W_x. Let $\Sigma_x \subset \Sigma \cap W_x$ be a disc with $x \in \Sigma_x$ and such that the closure of $\Sigma_y = h(\Sigma_x) \subset \mathcal{A}$ is contained in W_y. If w is a point contained in Σ_y, the leaf of $\widetilde{\mathcal{F}}_0|_{W_y}$ passing through it intersects $\widetilde{\Sigma}_y$ in a unique point $\psi_y(w)$. Clearly, ψ_y is continuous and we claim that ψ_y is a homeomorphism of Σ_y onto its image. Since $\overline{\Sigma}_y$ is compact, it suffices to prove that ψ_y is injective on $\overline{\Sigma}_y$. Suppose that w_1 and w_2 are two different points in $\overline{\Sigma}_y$ contained in the same leaf L of $\widetilde{\mathcal{F}}_0|_{W_y}$. Then, since $\pi_y^{-1}(W_y) \subset W_x$, we have that $\pi_y^{-1}(L)$ is contained in a leaf L' of $\mathcal{F}_0|_{W_x}$. Then $h^{-1}(w_1)$ and $h^{-1}(w_2)$ are two different points in the intersection of L' with $\overline{\Sigma}_0$, which is a contradiction. Then we consider $\psi_y : \Sigma_y \to \widetilde{\Sigma}_y$ as a local chart of $\mathcal{A}$. We may assume the sets Σ_y be small enough such that, if $\Sigma_y \cap \Sigma_{y'} \neq \emptyset$, then $\Sigma_y \cup \Sigma_{y'}$ is contained in an open set where $\widetilde{\mathcal{F}}_0$ is trivial. Then it is easy to see that the map $\psi_{y'} \circ \psi_y^{-1}$, which preserves the leaves, is a holonomy map and therefore holomorphic.

Given $y \in \mathcal{A}$, denote by $g(y)$ the point in $\Sigma'\backslash\{\widetilde{p}\}$ such that $y \in D_{g(y)}$. It is not difficult to see that the map $g \circ \psi_y^{-1} : \widetilde{\Sigma}_y \to \Sigma'$ is a holonomy map. Therefore $g : \mathcal{A} \to \Sigma'$ is holomorphic and regular. It is known (see [**1**]) that there exists a biholomorphism

$$\varphi : A_r = \{z \in \mathbb{C} : 0 \leq r < |z| < 1\} \to \mathcal{A}$$

and we may take φ such that $\varphi(z) \to E$ as $|z| \to r$. Hence $g \circ \varphi(z) \to \widetilde{p}$ as $|z| \to r$. Then the map $g \circ \varphi : A_r \to \Sigma'$ extends as $g \circ \varphi \equiv \widetilde{p}$ on $|z| = r$. This implies that $r = 0$. Then $g \circ \varphi$ extends holomorphicaly to $\mathbb{D}$ with $g \circ \varphi(0) = \widetilde{p}$.

Assertion. The map $g \circ \varphi$ is regular at 0.

Proof. Let γ be a path in $\mathbb{D}\backslash\{0\}$ which winds once around 0. It is sufficient to prove that the path $g \circ \varphi(\gamma)$ in Σ' winds once around $\widetilde{p}$. Let β' be a path in $\Sigma'\backslash\{\widetilde{p}\}$ such that

$$\beta' = \widetilde{\beta} \quad \text{in} \quad H_1(\widetilde{V}\backslash E). \tag{28}$$

Clearly β' represents generators in $H_1(\Sigma'\backslash\{\widetilde{p}\})$ and $H_1(W\backslash E)$. Let N and N' be integers such that

$$g \circ \varphi(\gamma) = N\beta' \quad \text{in} \quad H_1(\Sigma'\backslash\{\widetilde{p}\}) \tag{29}$$

and

$$\varphi(\gamma) = N'\beta' \quad \text{in} \quad H_1(W\backslash E). \tag{30}$$

We shall prove that $N = 1$ or -1. Observe that g is the restriction of the map

$$G : W\backslash E \to \Sigma'\backslash\{\widetilde{p}\}$$

defined by $G(D_x) = \{x\}$ for all $x \in \Sigma'\backslash\{\widetilde{p}\}$. Then, since $g(\beta') = \beta'$, it follows from (30) that

$$g \circ \varphi(\gamma) = N'\beta' \quad \text{in} \quad H_1(\Sigma'\backslash\{\widetilde{p}\})$$

and, in view of (29), we conclude that $N' = N$. Thus, since $W\backslash E \subset \widetilde{V}\backslash E$, equation (30) gives:

$$\varphi(\gamma) = N\beta' \quad \text{in} \quad H_1(\widetilde{V}\backslash E).$$

Then, by (28), we have that

$$\varphi(\gamma) = N\widetilde{\beta} \quad \text{in} \quad H_1(\widetilde{V}\backslash E).$$

Thus, since $\varphi(\gamma)$ is a generator of $H_1(\mathcal{A})$, Proposition 5.12 implies that $N = 1$ or -1.

Now, since $g \circ \varphi$ is regular at 0, there exists a disc Ω in $\mathbb{D}$ containing 0, such that $g \circ \varphi|_\Omega$ is a homeomorphism onto its image. Then, since φ is a diffeomorphism, it follows that $\bar{g} = g|_{\varphi(\Omega\backslash\{0\})}$ is a homeomorphism onto its image. Thus we take a disc $\widetilde{\Sigma} \subset g\varphi(\Omega) \subset \Sigma'$ containing $\widetilde{p}$ and define $f = \bar{g}^{-1}$ on $\widetilde{\Sigma}\backslash\{\widetilde{p}\}$. Let $x \in \widetilde{\Sigma}\backslash\{\widetilde{p}\}$. Clearly $f(x) \in \mathcal{A}$ and since $g(f(x)) = x$, we have that $f(x) \in D_x$ and so $f(x) \in D_x \cap \mathcal{A}$. If $y \in D_x \cap \mathcal{A}$, then $g(y) = x$ and therefore $y = f(x)$. Then $f(x)$ is the unique point in the intersection of D_x and $\mathcal{A}$. This proves the proposition. q.e.d.

We need the following lemma.

Lemma 5.14. *For each $x \in \mathbb{D}$, we may take a homeomorphism $h_x : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ such that:*

(i) $h_x(x) = 0$ *for all* $x \in \mathbb{D}$.
(ii) $h_x = \mathrm{id}$ *on* S^1.
(iii) h_x *depends continuously on* x.

Proof of Theorem 1.1. From Lemma 5.14, for each $x \in \widetilde{\Sigma}$ we may take a homeomorphism $h_x : D \to D$ such that:

(*i*) $h_x(\rho(f(x))) = \widetilde{p}$
(*ii*) $h_x = \mathrm{id}$ on ∂D
(*iii*) h_x depends continuously on x.

Then the homeomorphism $g_x : D_x \to D_x$ defined by

$$\rho \circ g_x = h_x \circ \rho \tag{31}$$

depends continuously on $x \in \widetilde{\Sigma} \subset L_{\widetilde{p}}$. Consider the map g defined (g is not the same function that one in previous pages) as

$$\begin{aligned} g &= g_x \quad \text{on} \quad D_x, \\ g &= \mathrm{id} \quad \text{otherwise}. \end{aligned}$$

We have that g is univalent and preserves the leaves of $\widetilde{\mathcal{F}}_0$. Moreover, in a small enough neighborhood of the divisor, g is continuous. Thus, if restricted to a small enough neighborhood of the divisor, g is a topological equivalence between $\widetilde{\mathcal{F}}_0$ and itself. Then, in a neighborhood of the divisor, $g \circ h$ gives a topological equivalence between $\mathcal{F}_0$ and $\widetilde{\mathcal{F}}_0$. Therefore for some neighborhoods U and $\widetilde{\mathrm{U}}$ of $0 \in \mathbb{C}^2$, the map

$$\hat{\mathrm{h}} = \pi g h \pi^{-1} : \mathrm{U} \to \widetilde{\mathrm{U}}$$

is a topological equivalence between $\mathcal{F}$ and $\widetilde{\mathcal{F}}$. Let $P = \pi(L_p)$ and $\widetilde{P} = \pi(L_{\widetilde{p}})$.

Assertion. There exists a disc $\mathcal{D}$ in P containing $0 \in \mathbb{C}^2$, such that $\hat{\mathrm{h}}(\mathcal{D})$ is contained in $\widetilde{P}$.

Proof. If $y \in \mathcal{A}$ is close enough to E, we have that $y \in D_x$ for some $x \in \widetilde{\Sigma}$. Thus, there is a disc $\Sigma_0 \subset \Sigma$ containing p, such that for all y in $h(\Sigma_0 \backslash \{p\}) \subset \mathcal{A}$ we have $y = f(x)$ for some $x \in \widetilde{\Sigma}$. Then, from (31) and (*i*) we have that

$$\rho \circ g(y) = \rho \circ g(f(x)) = h_x \circ \rho(f(x)) = \widetilde{p}.$$

Thus $g(y) \in L_{\widetilde{p}}$ for all $y \in h(\Sigma_0 \backslash \{p\})$ and therefore

$$g \circ h(\Sigma_0 \backslash \{p\}) \subset L_{\widetilde{p}}.$$

Then, if $\mathcal{D} \subset \pi(\Sigma_0) \subset P$, we have that $\hat{\mathrm{h}}(\mathcal{D}) \subset \widetilde{P}$.

Consider a neighborhood $\mathrm{U}' \subset \mathrm{U}$ of $0 \in \mathbb{C}^2$ homeomorphic to a ball and such that $\mathrm{U}' \cap P \subset \mathcal{D}$. We take U' small enough such that $\hat{\mathrm{h}}(\mathrm{U}') \cap \widetilde{P}$ is contained in $h(\mathcal{D})$. Thus, making $\widetilde{\mathrm{U}}' = \hat{\mathrm{h}}(\mathrm{U}')$, it is easy to see that

$$h(\mathrm{U}' \cap P) = \widetilde{\mathrm{U}}' \cap \widetilde{P}.$$

Then,

$$\hat{\mathrm{h}}|_{\mathrm{U}'} : \mathrm{U}' \to \widetilde{\mathrm{U}}'$$

is a topological equivalence between $\mathcal{F}_0$ and $\widetilde{\mathcal{F}}_0$, which satisfies the hypothesis of Theorem 1.2. Therefore Theorem 1.1 is proved. q.e.d.

Proof of Lemma 5.14. Let $\psi : \overline{\mathbb{D}} \to [0,1]$ be such that $\psi = 1$ on $\{|z| \leq 1/2\}$ and $\psi = 0$ on S^1. Let

$$\beta_r(t) : [0,1] \to [0,1]$$

be a diffeomorphism with $\beta_r(0) = 0$, $\beta(1) = 1$, $\beta(r) = 1/2$ and such that β_r depends continuously on $r \geq 0$. Given $x \in \mathbb{D}$, define the vector field

$$V_x : \overline{\mathbb{D}} \to \mathbb{C}$$

$$V_x(z) = -\psi(\beta_{|x|}(|z|))x,$$

and let φ_x the flow associated to V_x. Then define $h_x : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ by $h_x(z) = \varphi_x(1, z)$. It is easy to see that h_x satisfy the conditions of Lemma 5.14. q.e.d.

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