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ADDENDUM TO "PRINCIPAL BUNDLES ON PROJECTIVE VARIETIES AND THE DONALDSON-UHLENBECK COMPACTIFICATION"

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Abstract

In this addendum we complete the proof of Theorem 7.10 in [**1**].

In the paper [**1**], the proof of Theorem 7.10 is incomplete. This theorem proves the asymptotic non-emptiness of the moduli space of stable G–bundles on smooth projective surfaces, with structure group G which is simple and simply connected. This was pointed to us by Prof. S. Ramanan. In this addendum we complete the proof of this theorem. We retain the notations of [**1**].

The results in [**1**] which need to be modified are Lemma 7.5 and the proof of Proposition 7.6. In the cases which were considered in the proof of [**1**, Proposition 7.6], the case when the monodromy group of a rank 2 stable bundle is the normalizer of a one-dimensional torus in $SL(2)$ was not taken into consideration. When this case is allowed, the loci of bundles in $M_C(SL(2))^s$, where the monodromy groups are proper subgroups of $SL(2)$, is no longer a countable set. In fact, it contains a subvariety whose dimension is $g(C) - 1$, $g(C)$ being the genus of C. We therefore cannot work with the restriction to general curves as was done earlier in [**1**]. Instead, we use a technique due to S. Donaldson (cf. [**2**]) which completes the argument without much difficulty.

Recall from [**1**, Page 393] that we wish to estimate the dimension of subset Z_C of representations of $\pi_1(C)$ in $SU(2)$ which lie entirely in these families of groups up to conjugacy by the diagonal action of $SU(2)$.

Since all the cases except $\mathcal{M}(V) = N(T)$ have been handled in [**1**] we need only take care of the locus of those bundles whose monodromy lies in the *normalizer of the one dimensional torus*. In this case it is easy to see that such a rank two bundle can be realised as a direct image of a line bundle from an unramified two sheeted cover $p : D \to C$ of C, and since we need the structure group to be $SL(2)$, the locus is precisely

$$\{p_*(L) | det(p_*(L)) \simeq \mathcal{O}_C\}.$$

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Consider the "det" map $det : Pic(D) \to Pic(C)$ given by $L \to det(p_*(L))$. This map is surjective with kernel as the Prym variety $Prym(p)$ and $dim(Prym(p)) = g(D) - g(C) = g(C) - 1$. These comments immediately imply the following lemma.

Lemma 1. *(cf. [**1**, Lemma 7.5]) The locus of points Z_C, in the moduli space of stable vector bundles $M_{C,SL(2)}$ of rank 2 and trivial determinant on the curve C whose monodromy is among the set of subgroups listed above has dimension $\leq g(C) - 1$. In particular, if $g \geq 2$, Z_C is a* proper *subset of the moduli space $M^s_{C,SL(2)}$ which does not contain any open subset.*

We now recall the following result, which is a consequence of Donaldson's proof of the generic smoothness theorem of the moduli space of $SL(2)$–bundles on algebraic surfaces ([**2**, page 309]).

Proposition 2. *Let C be any smooth curve on X of genus at least 2. Then for $c = c_2(V) \gg 0$, the restriction map $r_C : M_{SL(2)}(c)^s \longrightarrow M^s_{C,SL(2)}$ is defined on a Zariski open subset $U \subset M_{SL(2)}(c)^s$ and furthermore, it is* differentially surjective.

Using this we have the following cf ([**1**, Prop 7.6]):

Proposition 3. *(cf. [**1**, Prop 7.6]) There exists a rank 2 stable bundle E with $c_2(E) \gg 0$ and trivial determinant on the surface X such that the restriction $E|_C$ to a curve $C \subset X$ has monodromy subgroup to be the whole of $SL(2)$ itself.*

Proof. Consider the rational map defined by restriction to the curve C, $r_C : M_{SL(2)}(c)^s \longrightarrow M^s_{C,SL(2)}$. If the curve C is as in Proposition 2, it implies that $Im(r_C) \subset M^s_{C,SL(2)}$ contains an open subset and hence (by Lemma 1), it is not completely contained in the subset Z_C of stable bundles with finite monodromy groups or with monodromy being the normalizer of T. Therefore there exists at least one stable bundle with the whole of $SL(2)$ as its monodromy subgroup. This proves the proposition. q.e.d.

Now to complete the proof of Theorem 7.1 in [**1**], choose C to be a *high degree curve* in X and E as in Proposition 3. Extending structure group of this bundle to G as in [**1**], it is easy to see that $E(G)$ is stable on X since $E(G)|_C$ is so (see [**1**, Lemma 4.5]).

References

[1] V. Balaji, *Principal bundles on projective varieties and the Donaldson-Uhlenbeck compactification*, J. Differential Geom. **76**, (2007), 351–398. MR 2331525.

[2] R. Friedman, *Algebraic surfaces and holomorphic vector bundles*, Universitext, Springer Verlag (1998). MR 1600388.

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