J. DIFFERENTIAL GEOMETRY
83 (2009) 501-550

INTEGRATED HARNACK INEQUALITIES ON LIE GROUPS

Bruce K. Driver & Maria Gordina

Abstract

We show that the logarithmic derivatives of the convolution heat kernels on a uni-modular Lie group are exponentially integrable. This result is then used to prove an "integrated" Harnack inequality for these heat kernels. It is shown that this integrated Harnack inequality is equivalent to a version of Wang's Harnack inequality. (A key feature of all of these inequalities is that they are dimension independent.) Finally, we show these inequalities imply quasi-invariance properties of heat kernel measures for two classes of infinite dimensional "Lie" groups.

Contents

The first author was supported in part by NSF Grants DMS-0504608 and DMS-0804472 and the Miller Institute at the University of California, at Berkeley. The second author was supported in part by NSF Grant DMS-0706784.
Received 11/17/2007.

1. Introduction

1.1. Basic setup. Let (M,g) be a connected complete Riemannian manifold, $d : M \times M \to [0,\infty)$ be the Riemannian distance function, dV be the Riemannian volume measure on M, Δ be the Laplace–Beltrami operator acting on the space of smooth differential forms, $\Omega(M)$, over M, and $\Delta_0 := \Delta|_{\Omega_c^0(M)}$, where $\Omega_c^0(M) := C_c^\infty(M)$ is the space of compactly supported smooth functions on M. From Gaffney [**26**], Roelcke [**53**], Chernoff [**11**] and Strichartz [**60**], we know that the $L^2(M,dV)$–closure, $\bar{\Delta}_0$, of Δ_0 is a non-positive self-adjoint operator on $L^2(M,dV)$. Moreover, there exists an associated smooth heat kernel, $(0,\infty) \times M \times M \ni (t,x,y) \to p_t(x,y) \in (0,\infty)$, such that $p_t(x,y) = p_t(y,x)$,

$$\int_M p_t(x,y)dV(y) \le 1 \text{ for all } x \in M, \text{ and} \tag{1.1}$$

$$\left(e^{t\bar{\Delta}_0/2} f\right)(x) = \int_M p_t(x,y)f(y)dV(y) \text{ for all } f \in L^2(M). \tag{1.2}$$

We also let "Ric" denote the Ricci curvature tensor of (M,g). For the bulk of this paper we will be considering the special case, where $M = G$ is a Lie group equipped with a left invariant Riemannian metric as we now describe.

Let G be a connected finite dimensional uni-modular Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$ be its Lie algebra, and suppose that $\mathfrak{g}$ is equipped with an inner product, $(\cdot,\cdot) = (\cdot,\cdot)_{\mathfrak{g}}$. Let $|A|_{\mathfrak{g}} := \sqrt{(A,A)}$ for all $A \in \mathfrak{g}$. We endow G with the unique left invariant Riemannian metric which agrees with $(\cdot,\cdot)_{\mathfrak{g}}$ at $e \in G$, i.e. the unique metric on G such that $L_{g*} : \mathfrak{g} \to T_g G$ is isometric for all $g \in G$. The Riemannian distance between $x, y \in G$ will be denoted by $d(x,y)$.

For $A \in \mathfrak{g}$ let $\tilde{A}$ denote the unique left invariant vector field on G such that $\tilde{A}(e) = A \in \mathfrak{g}$ and let $L = \sum_{i=1}^{\dim \mathfrak{g}} \tilde{A}_i^2$, where $\{A_i\}_{i=1}^{\dim \mathfrak{g}}$ is an orthonormal basis for $\mathfrak{g}$. As is well-known, since G is uni-modular, L is the Laplace-Beltrami operator (for example, see [**22**, Remark 2.2] and Lemma 6.1 below) restricted to $C^\infty(G)$. Since $L_g : G \to G$ is an isometry for all $g \in G$, if $p_t(x,y)$ is the heat kernel on G, then

$p_t(gx, gy) = p_t(x, y)$ for all $x, y, g \in G$. Taking $g = x^{-1}$ then implies that $p_t(x, y) = p_t\left(e, x^{-1}y\right)$. Similarly, $d(gx, gy) = d(x, y)$ for all $x, y, g \in G$ and therefore $d(x, y) = d\left(e, x^{-1}y\right)$.

Notation 1.1. By a slight abuse of notation, let $p_t(x) := p_t(e, x)$ for $x \in G$. We will refer to $p_t(\cdot)$ as the **convolution heat kernel** on G and to the probability measure, $d\nu_t(x) := p_t(x)\,dx$, as the **heat kernel measure on** G. We also write dx for $dV(x)$ and $|x|$ for $d(e, x)$.

The following lemma is an immediate consequence of the comments above and the basic properties of $p_t(x, y)$.

Lemma 1.2. *For all $x, y \in G$*

1) $d(x, y) = \left|x^{-1}y\right|,$
2) $\left|x^{-1}\right| = |x|,$
3) $p_t\left(x^{-1}\right) = p_t(x),$
4) $p_t(x, y) = p_t\left(x^{-1}y\right) = p_t\left(y^{-1}x\right),$
5) *dV is a bi-invariant Haar measure on G,*
6) *for $f \in L^2(G, dV)$,*

$$\begin{aligned}\left(e^{t\bar{\Delta}_0/2}f\right)(x) &= \int_G p_t\left(x^{-1}y\right) f(y)\,dy \\ &= \int_G p_t\left(y^{-1}x\right) f(y)\,dy \\ &= \int_G p_t(yx)\, f\left(y^{-1}\right) dy.\end{aligned}$$

1.2. The main theorems. We may now state the main theorems of this paper.

Theorem 1.3. *If $T > 0$ and $A \in \mathfrak{g}$, then*

$$\int_G \exp\left(-\frac{\tilde{A}p_T(x)}{p_T(x)}\right) p_T(x)\,dx \le \exp\left(\frac{k/2}{e^{kT}-1}|A|_{\mathfrak{g}}^2\right), \tag{1.3}$$

where $k \in \mathbb{R}$ is a lower bound on the Ricci curvature, i.e. $\mathrm{Ric} \ge kI$. *(Here and in what follows we will always use the convention that $k/\left(e^{kT}-1\right) = 1/T$ whenever $k = 0$.)*

The proof of this theorem relies on martingale inequalities applied to the probabilistic representation for $\tilde{A}\ln p_T(x)$ in Theorem 6.4. We also have another related integral bound on $\tilde{A}\ln p_T(x)$.

Theorem 1.4. *Continuing the notation in Theorem 1.5, for any $q \in (1, \infty)$ there is a constant, $C_q < \infty$ such that*

$$\left\|\tilde{A}\ln p_T\right\|_{L^q(\nu_T)} \le C_q\sqrt{\frac{k}{e^{kT}-1}}\,|A| \text{ for all } A \in \mathfrak{g}. \tag{1.4}$$

These theorems will be proved in Sections 5 and 6 below. Also see [**23**, Theorem 5.11] for a version of this theorem valid on a general compact Riemannian manifold and Proposition E.1 in Appendix E where we use a Hamilton type inequality to show that an inequality similar to that in Eq. (1.3) holds on any complete Riemannian manifolds whose Ricci curvature is bounded from below. However, as is noted in Remark E.2, in general, we can not choose the constants appearing in Proposition E.1 to be independent of dimension.

The following theorem is a corollary of Theorem 1.3 above and Theorem 2.5 below. The details will be given in Section 3 below.

Theorem 1.5. *Let $T > 0$ be given and let $k \in \mathbb{R}$ be a lower bound on the Ricci curvature,* $\mathrm{Ric} \geq kI$. *Then for every $y \in G$ and $q \in [1, \infty)$,*

$$\left(\int_G \left[\frac{p_T\left(xy^{-1}\right)}{p_T\left(x\right)}\right]^q p_T\left(x\right)dx\right)^{1/q} \leq \exp\left(\frac{(q-1)\,k/2}{e^{kT}-1}\left|y\right|^2\right). \tag{1.5}$$

From Theorem 1.5 and Lemma 1.2 we have,

$$\begin{aligned}
\left(\int_G \left[\frac{p_T\left(y,x\right)}{p_T\left(z,x\right)}\right]^q p_T\left(z,x\right)dx\right)^{1/q} & \\
&= \left(\int_G \left[\frac{p_T\left(y^{-1}x\right)}{p_T\left(z^{-1}x\right)}\right]^q p_T\left(z^{-1}x\right)dx\right)^{1/q} \\
&= \left(\int_G \left[\frac{p_T\left(y^{-1}zx\right)}{p_T\left(x\right)}\right]^q p_T\left(x\right)dx\right)^{1/q} \\
&\leq \exp\left(\frac{(q-1)\,k/2}{e^{kT}-1}\left|y^{-1}z\right|^2\right) \\
&= \exp\left(\frac{(q-1)\,k/2}{e^{kT}-1}d^2\left(y,z\right)\right) \qquad (1.6)
\end{aligned}$$

for all $y, z \in G$. This form of the integrated Harnack inequality makes sense on any Riemannian manifold. We will show in Corollary D.3 of Appendix D below that Eq. (1.6) does indeed hold when G is replaced by a complete connected Riemannian manifold with $\mathrm{Ric} \geq kI$ for some $k \in \mathbb{R}$. The key point is that the estimate in Eq. (1.6) is equivalent to Wang's dimension free Harnack inequality, see [**66**, **67**] and Theorem D.2 below. We are grateful to Michael Röckner for pointing out the relationship between Wang's inequality and the integrated Harnack inequality in Eq. (1.6).

Remarks 1.6. Some of the key features of Theorem 1.5 are:

1) As seen in Example 1.1) below, the estimate in Eq. (1.5) is sharp when $G = \mathbb{R}^n$.

2) For T near zero, $k/\left(e^{kT}-1\right)\cong 1/T$ and for T large, $k/\left(e^{kT}-1\right)\cong \max(0,-k)$.
3) The estimate in Eq. (1.5) is dimension independent and therefore has applications to infinite dimensional settings, see Section 7 below.

Let $R_y : G \to G$ ($L_y : G \to G$) be the operation of right (left) multiplication by $y \in G$, $\nu_T \circ R_y^{-1}$ $\left(\nu_T \circ L_y^{-1}\right)$ be ν_T pushed forward by R_y (L_y), and $d\left(\nu_T \circ R_y^{-1}\right)/d\nu_T$ denote the Radon-Nikodym derivative of $\nu_T \circ R_y^{-1}$ with respect to ν_T. For the infinite dimensional applications of Section 7, it is convenient to rewrite Eq. (1.5) as

$$\left\|\frac{d\left(\nu_T \circ R_y^{-1}\right)}{d\nu_T}\right\|_{L^q(G,\nu_T)} \le \exp\left(\frac{(q-1)\,k/2}{e^{kT}-1}d^2(e,y)\right). \tag{1.7}$$

By Lemma 1.2, Eq. (1.5) may be also be expressed as

$$\left(\int_G\left[\frac{p_T(xy)}{p_T(x)}\right]^q p_T(x)\,dx\right)^{1/q} \le \exp\left(\frac{(q-1)\,k/2}{e^{kT}-1}\,|y|^2\right) \tag{1.8}$$

or as

$$\left(\int_G\left[\frac{p_T\left(y^{-1}x\right)}{p_T(x)}\right]^p p_T(x)\,dx\right)^{1/q} \le \exp\left(\frac{(q-1)\,k/2}{e^{kT}-1}\,|y|^2\right). \tag{1.9}$$

This last equality is equivalent to the left translation analogue of Eq. (1.7), namely

$$\left\|\frac{d\left(\nu_T \circ L_y^{-1}\right)}{d\nu_T(\cdot)}\right\|_{L^q(G,\nu_T)} \le \exp\left(\frac{(q-1)\,k/2}{e^{kT}-1}\,|y|^2\right). \tag{1.10}$$

1.3. Examples and applications.

Definition 1.7. For $A \in \mathfrak{g}$ and $T>0$, let

$$W_A^T(x) := -\left(\tilde{A}\ln p_T\right)(x) = -\frac{\left(\tilde{A}p_T\right)(x)}{p_T(x)}. \tag{1.11}$$

The significance of W_A^T in the above definition stems from the following integration by parts identity;

$$\int_G \tilde{A}f(x)\,p_T(x)\,dx = \int_G f(x)\,W_A^T(x)\,p_T(x)\,dx \;\forall\; f \in C_c^\infty(G). \tag{1.12}$$

Thus W_A^T is the $p_T(x)\,dx$ – divergence of $\tilde{A}$ as described in Definition 2.3 below.

Example 1.1. Suppose $G=\mathbb{R}^n$ so that $\mathfrak{g}\cong\mathbb{R}^n$ which we assume has been equipped with the standard inner product. In this case

$$p_T(x) = \left(\frac{1}{2\pi T}\right)^{n/2}\exp\left(-\frac{|x|^2}{2T}\right),$$

where $|x|^2 := \sum_{i=1}^n x_i^2$. For $A \in \mathfrak{g}$ and $f \in C_c^1(\mathbb{R}^n)$ we have $\tilde{A} = \partial_A$ and

$$\int_{\mathbb{R}^n} \tilde{A} f(x)\, p_T(x)\, dx$$
$$= -\int_{\mathbb{R}^n} f(x)\, \partial_A p_T(x)\, dx = \int_{\mathbb{R}^n} f(x) \frac{x \cdot A}{T} p_T(x)\, dx$$

from which it follows that $W_A^T(x) = \frac{x \cdot A}{T}$. By simple Gaussian integrations,

$$\int_{\mathbb{R}^n} e^{W_A^T(x)} p_T(x)\, dx = \exp\left(\frac{|A|_{\mathfrak{g}}^2}{2T}\right),$$

$$\left(\int_{\mathbb{R}^n} \left[\frac{p_T(x-y)}{p_T(x)}\right]^q p_T(x)\, dx\right)^{1/q}$$
$$= \left(\int_{\mathbb{R}^n} \left[e^{-\frac{1}{2T}|y|^2 + \frac{1}{T} x \cdot y}\right]^q p_T(x)\, dx\right)^{1/q} = \exp\left(\frac{(q-1)}{2T}|y|^2\right),$$

and

$$\int_{\mathbb{R}^n} \left|W_A^T(x)\right|^q p_T(x)\, dx = \int_{\mathbb{R}^n} \left|\frac{x \cdot A}{T}\right|^q p_T(x)\, dx = T^{q/2} |A|^q \tilde{C}_q^q, \tag{1.13}$$

where

$$\tilde{C}_q^q := \int_{\mathbb{R}^n} |x|^q p_1(x)\, dx.$$

The first two results show the estimates in Eqs. (1.3) and (1.5) are sharp. The identity in Eq. (1.13) shows the form of Eq. (1.4) is sharp. We do not know if, in general, the constant C_q appearing in Eq. 1.4 can be taken to be $\tilde{C}_q$ defined above.

Our main interest in Theorem 1.5 is in its application to proving that certain "heat kernel measures" on infinite dimensional Lie groups, G, are quasi-invariant under left and right translations by elements of a certain subgroup, G_0. We will postpone our discussion of this application to Section 7. For now let us give a couple of finite dimensional applications of Theorems 1.5 and 1.4.

Proposition 1.8. *Suppose that $T > 0$, $q > 1$, and $f \in L^q(\nu_T)$ is a harmonic function, i.e. $\Delta f = 0$. Then*

$$\int_G p_T(y, x)\, f(x)\, dx = f(y) \quad \text{for all} \quad y \in G. \tag{1.14}$$

At an informal level we expect

$$\int_G p_t(y, x)\, f(x)\, dx = \left(e^{t\bar{\Delta}_0/2} f\right)(y)$$

and hence

$$\frac{d}{dt}\int_G p_t(y, x)\, f(x)\, dx = \frac{d}{dt}\left(e^{t\bar{\Delta}_0/2} f\right)(y) = \left(e^{t\bar{\Delta}_0/2} \frac{\bar{\Delta}_0}{2} f\right)(y) = 0.$$

Therefore it is reasonable to conclude that

$$\int_G p_T(y,x) f(x)\,dx = \left(e^{T\bar{\Delta}_0/2} f\right)(y) = \left(e^{0\bar{\Delta}_0/2} f\right)(y) = f(y).$$

However, this argument is not rigorous as f is only square–integrable relative to the rapidly decaying measure, ν_T, rather than to Haar measure on G. The rigorous proof of Proposition 1.8 will be given in Section 7.

The following corollary is a simple consequence of Proposition 1.8, Eq. (7.4) in the proof of this proposition, and Theorem 1.5 in the form of Eq. (1.9).

Corollary 1.9. *Suppose that* $q \in (1,\infty)$. *Under the hypothesis of Theorem* 1.5*, if* $f \in L^q(\nu_T)$ *and* f *is harmonic (i.e.* $\Delta f = 0$*), then*

$$|f(y)| \le \|f\|_{L^q(\nu_T)} \exp\left(\frac{1}{q-1}\frac{k/2}{e^{kT}-1}|y|^2\right). \tag{1.15}$$

In particular, if G *is further assumed to be a complex Lie group and* $f \in L^q(\nu_T)$ *is assumed to be holomorphic, then the pointwise bound in Eq. (*1.15*) is still valid.*

Remark 1.10. When f is holomorphic, $q = 2$, $T = 1/2$, and $G = \mathbb{C}^d$, the inequality in Eq. (1.15) is Bargmann's pointwise bound in [**4**, (Eq. (1.7)] except that the constant in the exponent is off by a factor of two. More generally, when G is a general complex Lie group and f is holomorphic, it has been shown in [**22**, Corollary 5.4] that

$$|f(y)| \le \|f\|_{L^2(\nu_{t/2})} e^{|y|^2/2t} \text{ for all } y \in G.$$

The reason for the discrepancy in the coefficients in the exponents between these inequalities is that $p_{t/2}(x,y)$ is not the reproducing kernel for the holomorphic functions in $L^2(\nu_{t/2})$ in that $y \to p_{t/2}(x,y)$ is not holomorphic. The coefficient in the exponent of Eq. (1.15) is also not sharp since $y \to p_T(x,y)$ is not harmonic.

Acknowledgements. We are grateful to Alexander Grigor′yan and Laurent Saloff-Coste for their comments and suggestions on the heat kernel bounds used in this paper. The first author would also like to thank the Berkeley mathematics department and the Miller Institute for Basic Research in Science for their support of this project in its latter stages.

2. L^q – Jacobian estimates

Let M be a finite dimensional manifold, μ be a probability measure on M with a smooth, strictly positive density in each coordinate chart. For $r > 0$, let $\|f\|_r := \left(\int_M |f|^r\,d\mu\right)^{1/r}$ denote the $L^r(\mu)$ – norm of $f : M \to \mathbb{C}$.

Let X_t be a time dependent vector field and let S_t denote its flow, i.e. $S_t(m)$ solves,

$$\frac{d}{dt}S_t(m) = X_t \circ S_t(m) \text{ with } S_0(m) = m \text{ for all } m \in M. \tag{2.1}$$

We will assume that X_t is forward complete, i.e. $S_t(m)$ exists for all $t \geq 0$ and $m \in M$. Define

$$\mu_t = (S_t)_* \mu = \mu \circ S_t^{-1}.$$

Since μ_t also has a strictly positive density in each coordinate chart the Radon-Nikodym derivative

$$J_t = d\mu_t/d\mu$$

exists for all $t \geq 0$. Our goal of this section is to prove Theorem 2.5 below which gives an upper bound on $\|J_t\|_q$ for $q \in (1,\infty)$. This result is a slight extension of part of Theorem 2.14 in Galaz-Fontes, Gross, and Sontz [**28**] to the setting of time dependent vector fields, X_t. For the readers convenience we will sketch the method introduced in [**28**, Theorem 2.14]. In what follows, $0\ln 0$ is to always be interpreted to be 0.

Lemma 2.1. *Suppose that $(t,m) \in (0,T) \times M \to h_t(m) \in [0,\infty)$ is a smooth bounded function and $r : (0,T) \to (1,\infty)$ is a C^1 – function. Then*

$$\begin{aligned}\frac{d}{dt}\ln\|h_t\|_{r(t)} =& \frac{\dot r(t)}{r(t)}\int_M \frac{h_t^{r(t)}}{\|h_t\|_{r(t)}^{r(t)}}\left(\ln\frac{h_t}{\|h_t\|_{r(t)}}\right)\,d\mu \\ &+ \frac{1}{r(t)}\int_M \frac{\frac{d}{ds}|_{s=t}h_s^{r(t)}}{\|h_t\|_{r(t)}^{r(t)}}d\mu.\end{aligned}\tag{2.2}$$

Proof. For the reader's convenience we will give a formal derivation of this identity and refer the reader to Gross [**33**, Lemma 1.1] for the technical details. For $r > 0$ and any bounded measurable function, $g : M \to \mathbb{R}$, a straight forward calculation shows

$$\frac{d}{dr}\ln\|g\|_r = \frac{1}{r}\int_M \frac{|g|^r}{\|g\|_r^r}\left(\ln\frac{|g|}{\|g\|_r}\right)\,d\mu.$$

If we further assume that $r > 1$ and $v : M \to \mathbb{R}$ is another bounded measurable function, then

$$\begin{aligned}\partial_v \ln\|g\|_r &= \partial_v\left[\frac{1}{r}\ln\left(\int_M |g|^r\,d\mu\right)\right] = \frac{1}{r}\frac{\int_M \partial_v |g|^r\,d\mu}{\int_M |g|^r\,d\mu} \\ &= \frac{1}{r}\int_M \frac{\partial_v |g|^r}{\|g\|_r^r}d\mu = \int_M \frac{|g|^{r-1}\operatorname{sgn}(g)}{\|g\|_r^r}v\,d\mu.\end{aligned}$$

These two identities along with the chain rule,

$$\frac{d}{dt}\ln\|h_t\|_{r(t)} = \frac{d}{ds}|_{s=t}\left[\ln\|h_t\|_{r(s)} + \ln\|h_s\|_{r(t)}\right],$$

easily give Eq. (2.2). q.e.d.

Lemma 2.2. *Let $W \in L^1(\mu)$ and $f \geq 0$ be a bounded measurable function. Then, for all $s > 0$,*

$$\int_M Wf d\mu \leq s\int_M f\ln\frac{f}{\mu(f)}d\mu + s\mathcal{B}(W/s)\int_M f d\mu, \tag{2.3}$$

where

$$\mathcal{B}(W) := \ln\left(\mu\left(e^W\right)\right) = \ln\left(\int_M e^W d\mu\right).$$

Proof. Recall that Young's inequality states, $xy \leq e^x + y\ln y - y$ for $x \in \mathbb{R}$ and $y \geq 0$, where $0\ln 0 := 0$. Applying Young's inequality with $x = W$ and $y = f$ and then integrating the result gives

$$\int_M Wf d\mu \leq \int_M e^W d\mu + \int_M [f\ln f - f]\, d\mu.$$

Replacing f by λf with $\lambda > 0$ in this inequality then shows

$$\begin{aligned}\int_M Wf d\mu &\leq \lambda^{-1}\left[\int_M e^W d\mu + \int_M [\lambda f\ln(\lambda f) - \lambda f]\, d\mu\right]\\ &= \lambda^{-1}\int_M e^W d\mu + \ln\lambda\int_M f d\mu + \int_M [f\ln f - f]\, d\mu.\end{aligned}$$

The minimizer of the right side of this inequality occurs at $\lambda = \left(\int_M e^W d\mu\right)\cdot\left(\int_M f d\mu\right)^{-1}$ and using this value for λ gives

$$\int_M Wf d\mu \leq \int_M f\ln\frac{f}{\mu(f)}d\mu + \mathcal{B}(W)\int_M f d\mu. \tag{2.4}$$

(The proof of Eq. (2.4) was predicated on the assumption that $\mathcal{B}(W) < \infty$ but clearly Eq. (2.4) remains valid when $\mathcal{B}(W) = \infty$.) The estimate in Eq. (2.3) follows directly from this by replacing W by W/s. q.e.d.

Definition 2.3. The *μ–divergence* of a smooth vector field, X, on M is the function $W = W_X^\mu$ defined by

$$\int_M X\varphi d\mu = \int_M \varphi W d\mu, \text{ for all } \varphi \in C_c^1(M).$$

Proposition 2.4. *Let X_t and S_t be as in Eq. (2.1), $W_t := W_{X_t}$ be the μ–divergence of X_t, $h \in C^1(M,[0,\infty))$, $h_t := h\circ S_t^{-1}$, and $r \in C^1((0,\tau),(1,\infty))$. Then for any $s > 0$ we have*

$$\frac{d}{dt}\ln\|h_t\|_{r(t)} \geq \left(\frac{\dot{r}}{r} - s\right)\int_M \frac{h_t^r}{\|h_t\|_r^r}\left(\ln\frac{h_t}{\|h_t\|_r}\right)\, d\mu - \frac{s}{r}\mathcal{B}\left(s^{-1}W_t\right). \tag{2.5}$$

Proof. Differentiating the identity $S_t \circ S_t^{-1}(m) = m$ and making use of the flow Eq. (2.1) implies

$$X_t(m) + (S_t)_* \frac{d}{dt} S_t^{-1}(m) = 0.$$

Therefore,

$$\frac{d}{dt} S_t^{-1}(m) = -\left(S_t^{-1}\right)_* X_t(m)$$

or equivalently,

$$\frac{d}{dt} f\left(S_t^{-1}(m)\right) = -X_t\left(f \circ S_t^{-1}\right)(m) \text{ for all } f \in C^1(M).$$

Using this identity along with Eq. (2.2) shows

$$\frac{d}{dt} \ln \|h_t\|_{r(t)} = \frac{\dot{r}}{r} \int_M \frac{h_t^r}{\|h_t\|_r^r} \left(\ln \frac{h_t}{\|h_t\|_r}\right) d\mu - \frac{1}{r} \int_M \frac{X_t h_t^r}{\|h_t\|_r^r} d\mu, \tag{2.6}$$

where $r = r(t)$ and $\dot{r} = \dot{r}(t)$. Combining this identity with the definition of W_t and the estimate in Eq. (2.3) with $W = W_t$ and $f = \frac{h_t^r}{\|h_t\|_r^r}$ then implies,

$$\begin{aligned}\frac{d}{dt} \ln \|h_t\|_{r(t)} =& \frac{\dot{r}}{r} \int_M \frac{h_t^r}{\|h_t\|_r^r} \left(\ln \frac{h_t}{\|h_t\|_r}\right) d\mu - \frac{1}{r} \int_M W_t \frac{h_t^r}{\|h_t\|_r^r} d\mu \\ \geq& \frac{\dot{r}}{r} \int_M \frac{h_t^r}{\|h_t\|_r^r} \left(\ln \frac{h_t}{\|h_t\|_r}\right) d\mu \\ & - \frac{s}{r}\left[\int_M \frac{h_t^r}{\|h_t\|_r^r} \ln \frac{h_t^r}{\|h_t\|_r^r} d\mu + \mathcal{B}(W_t/s)\right]\end{aligned}$$

which is the same as Eq. (2.5). q.e.d.

The following theorem is the extension of Galaz-Fontes, Gross, and Sontz [**28**, Theorem 2.14] from time–independent vector fields to time–dependent vector fields. These results generalize the fundamental results of Cruzerio [**12**] – also see [**5**, **6**, **13**, **18**, **50**, **51**] for other related results.

Theorem 2.5 (Jacobian Estimate)**.** *Let $q > 1$ and $r \in C([0,\tau],[1,\infty)) \cap C^1((0,\tau),(1,\infty))$ such that $r(0) = 1$, $r(\tau) = q$ and $\dot{r}(t) > 0$ for $0 < t < \tau$, then*

$$\|J_\tau\|_{q'} \leq e^{\Lambda(r)}, \tag{2.7}$$

where $q' := q/(q-1)$ is the conjugate exponent to q and

$$\Lambda(r) = \Lambda_X(r) := \int_0^\tau \frac{\dot{r}(t)}{r^2(t)} \mathcal{B}\left(\frac{r(t)}{\dot{r}(t)} W_t\right) dt. \tag{2.8}$$

Proof. Taking $s = \dot{r}/r$ in Eq. (2.5) gives

$$\frac{d}{dt} \ln \|h_t\|_{r(t)} \geq -\frac{\dot{r}}{r^2} \mathcal{B}\left(\frac{r}{\dot{r}} W_t\right)$$

which integrates to

$$\left\|h \circ S_{\tau}^{-1}\right\|_{q} = \|h_{\tau}\|_{q} \geq \|h\|_{1} \exp\left(-\int_{0}^{\tau} \frac{\dot{r}(t)}{r^{2}(t)} \mathcal{B}\left(\frac{r(t)}{\dot{r}(t)} W_{t}\right) dt\right).$$

Replacing h by $h \circ S_{\tau}$ in this inequality implies

$$\int_{M} h J_{\tau} d\mu = \|h \circ S_{\tau}\|_{1} \leq \|h\|_{q} e^{\Lambda(r)}. \tag{2.9}$$

Let $L^{q}(\mu)^{+}$ denote the almost everywhere non-negative functions in $L^{q}(\mu)$. Since Eq. (2.9) is valid for all $h \in C^{1}(M, [0, \infty))$ and the latter functions are dense in $L^{q}(\mu)^{+}$ (see the proof of Lemma 2.8 in [**28**]), it follow that Eq. (2.9) is valid for all $h \in L^{q}(\mu)^{+}$. Equation (2.7) now follows by the converse to Hölder's inequality. Indeed, let $K \subset M$ be a compact set and take $h = J_{\tau}^{q'-1} 1_{K} = J_{\tau}^{1/(q-1)} 1_{K}$ in Eq. (2.9) to find

$$\int_{M} J_{\tau}^{q'} 1_{K} d\mu \leq \left\|J_{\tau}^{1/(q-1)} 1_{K}\right\|_{q} e^{\Lambda(r)} = \left(\int_{M} J_{\tau}^{q'} 1_{K} d\mu\right)^{1/q} e^{\Lambda(r)}.$$

This inequality is equivalent to

$$\|J_{\tau} 1_{K}\|_{q'} = \left(\int_{M} J_{\tau}^{q'} 1_{K} d\mu\right)^{1-1/q} \leq e^{\Lambda(r)}.$$

Now replacing K by K_{n} with K_{n} compact and $K_{n} \uparrow M$ and passing to the limit as $n \to \infty$ in the previous inequality gives the estimate in Eq. (2.7). q.e.d.

3. Proof of Theorem 1.5

In this section we will give a proof of Theorem 1.5 assuming that Theorem 1.3 holds. We will use the following notation in the proofs.

Notation 3.1. Let $c(t)$ be defined by $c(0) = 1$ and

$$c(t) = \frac{t}{e^{t} - 1} \quad \text{for all} \quad t \neq 0. \tag{3.1}$$

Proof. (Proof of Theorem 1.5.) In order to abbreviate the notation, let $c := c(kT)/T$. Let $g \in C^{1}([0,1], G)$ be such that $g(0) = e \in G$ and $g(1) = y \in G$ and define $A_{t} := L_{g(t)*}^{-1} \dot{g}(t) \in \mathfrak{g}$. If we now let $X_{t} := \tilde{A}_{t} \in \Gamma(TG)$, then the flow, S_{t}, of X_{t} satisfies, $S_{t}(x) = xg(t)$. Indeed, because X_{t} is left invariant,

$$\frac{d}{dt} xg(t) = L_{x*} \dot{g}(t) = L_{x*} L_{g(t)*} A_{t} = L_{xg(t)*} A_{t} = X_{t}(xg(t)).$$

In order to apply the Jacobian estimate in Theorem 2.5, let $d\mu(x) = d\nu_T(x) := p_T(x)\,dx$ and observe that

$$\int_G h(S_1(x))\,d\mu(x) = \int_G h(xy)\,d\mu(x) = \int_G h(xy)\,p_T(x)\,dx$$
$$= \int_G h(x)\,p_T\left(xy^{-1}\right)dx = \int_G h(x)\frac{p_T\left(xy^{-1}\right)}{p_T(x)}d\mu(x)$$

from which it follows that

$$J_1(x) := \frac{d(S_1)_*\mu}{d\mu}(x) = \frac{p_T\left(xy^{-1}\right)}{p_T(x)}. \tag{3.2}$$

Moreover, if $W_t = W_{X_t}^{\nu_T}$ is the $\mu = \nu_T$ – divergence of X_t, by Theorem 1.3,

$$\mathcal{B}(\lambda W_t) = \ln\left(\int_G e^{\lambda W_t}d\mu\right) \le \frac{c(kT)}{T}\lambda^2|A_t|_{\mathfrak{g}}^2. \tag{3.3}$$

Hence it follows from Theorem 2.5 that

$$\left[\int_G\left(\frac{p_T\left(xy^{-1}\right)}{p_T(x)}\right)^{q'}p_T(x)\,dx\right]^{1/q'} = \|J_1\|_{q'} \le e^{\Lambda(r)}, \tag{3.4}$$

where

$$\Lambda(r) = \int_0^1\frac{\dot{r}(t)}{r^2(t)}\mathcal{B}\left(\frac{r(t)}{\dot{r}(t)}W_t\right)dt$$
$$\le c\int_0^1\frac{\dot{r}(t)}{r^2(t)}\frac{r^2(t)}{\dot{r}^2(t)}|A_t|_{\mathfrak{g}}^2\,dt = c\int_0^1\frac{|A_t|_{\mathfrak{g}}^2}{\dot{r}(t)}dt,$$

and $r \in C([0,1],[1,\infty)) \cap C^1((0,1),(1,\infty))$ such that $r(0) = 1$, $r(1) = q$ and $\dot{r}(t) > 0$ for $0 < t < 1$. We now want to choose $r(t)$ so as to minimize $\Lambda(r)$ subject to the constraints $\dot{r}(t) > 0$, $r(0) = 1$ and $r(1) = q$. To see how to choose r, let us differentiate $\Lambda(r)$ in a direction v such that $v(0) = 0 = v(1)$ and then require

$$0 \overset{\text{set}}{=} (\partial_v\Lambda)(r) = -\frac{c}{2}\int_0^1\frac{|A_t|_{\mathfrak{g}}^2}{\dot{r}^2(t)}\dot{v}(t)\,dt = -\frac{c}{2}\int_0^1 v(t)\frac{d}{dt}\left(\frac{|A_t|_{\mathfrak{g}}^2}{\dot{r}^2(t)}\right)dt.$$

Since $v(t)$ is arbitrary, we should require $\frac{|A_t|_{\mathfrak{g}}^2}{\dot{r}^2(t)} = \kappa^{-2}$, where $\kappa > 0$ is a constant, i.e. $\dot{r}(t) = \kappa|A_t|_{\mathfrak{g}}$. Hence we take

$$r(t) = 1 + \kappa\int_0^t|A_\tau|_{\mathfrak{g}}\,d\tau,$$

where

$$\kappa := (q-1)\left(\int_0^1|A_\tau|_{\mathfrak{g}}\,d\tau\right)^{-1}$$

has been chosen so that $r(1) = q$. With this choice of r,

$$\Lambda(r) := \frac{c}{2}\int_0^1 \frac{|A_t|_{\mathfrak{g}}^2}{\kappa |A_t|_{\mathfrak{g}}} dt = \frac{c}{2\kappa}\int_0^1 |A_t|_{\mathfrak{g}}\, dt = \frac{c}{2(q-1)}\left(\int_0^1 |A_t|_{\mathfrak{g}}\, dt\right)^2$$

and using this value for $\Lambda(r)$ in Eq. (3.4) along with the identity, $(q-1)^{-1} = q' - 1$ implies

$$\begin{aligned}\left(\int_G \left[\frac{p_T(xy^{-1})}{p_T(x)}\right]^{q'} p_T(x)\, dx\right)^{1/q'} &= \|J_1\|_{q'} \\ &\le \exp\left(\frac{c(q'-1)}{2}\left(\int_0^1 |A_t|_{\mathfrak{g}}\, dt\right)^2\right).\end{aligned}$$

Upon noting that $q' := q(q-1)^{-1}$ ranges over $(1,\infty)$ as q ranges over $(1,\infty)$, the proof of Theorem 1.5 is complete. q.e.d.

4. Properties of the Hodge – de Rham semigroups

This section gathers a number of technical functional analytic results needed to establish the representation formula in Theorem 5.4 below. Let (M,g) be a complete Riemannian manifold, dV denote the volume measure on M associated to g, ∇ denote the Levi-Civita covariant derivative, $\Lambda^k = \Lambda^k(T^*M)$, $\Lambda = \oplus_{k=0}^{\dim M}\Lambda^k$, $\Omega^k(M)$ $(\Omega_c^k(M))$ denote the space of (compactly supported) smooth k – forms over M, and $\Omega(M) = \oplus_{k=0}^{\dim M}\Omega^k(M)$ be the space of all smooth forms over M. If α and β are measurable k – forms, let

$$\langle \alpha, \beta\rangle_m := \sum_{j_1,\dots,j_k=1}^{d} \alpha(e_{j_1},\dots,e_{j_k})\,\beta(e_{j_1},\dots,e_{j_k}),$$

where $\{e_j\}_{j=1}^d$ is any orthonormal frame for T_mM. When $m \to \langle\alpha,\beta\rangle_m$ is integrable, let

$$(\alpha,\beta) := \int_M \langle\alpha,\beta\rangle\, dV$$

and let $L^2(\Lambda^k)$ denote the measurable k – forms, α, such that $(\alpha,\alpha) < \infty$. Further let

$$L^2(\Lambda) := \oplus_{k=0}^{\dim M} L^2\left(\Lambda^k\right).$$

Two measurable k – forms, α and β, are take to be equivalent if $\alpha = \beta$ a.e.

Let $d : \Omega(M) \to \Omega(M)$ be the differential operator taking k – forms to $k+1$ – forms, δ be the formal L^2 – adjoint of $-d$,

$$\Delta := -(\delta d + d\delta) = -(d+\delta)^2$$

be the Hodge-de Rham Laplacian on $\Omega(M)$, and $\square$ be the Bochner (i.e. flat) Laplacian on $\Omega(M)$. More precisely if α is a k – form, $\delta\alpha$ is the $k-1$ form defined by

$$(\delta\alpha)_m := \sum_{j=1}^{d} \left(\nabla_{e_j}\alpha\right)(e_j, -) \tag{4.1}$$

and

$$(\square\alpha)_m := \sum_{j=1}^{d} \nabla^2_{e_j\otimes e_j}\alpha := \sum_{j=1}^{d} \left(\nabla^2_{E_j}\alpha - \nabla_{\nabla_{E_j}E_j}\alpha\right)_m,$$

where $\{E_j\}_{j=1}^{\dim M}$ is an local orthonormal frame for TM defined in a neighborhood of m. The next two theorems summarize the properties about these operators that will be needed in this paper.

Theorem 4.1. *The operators, $d_k := d|_{\Omega_c^k(M)} : \Omega_c^k(M) \to \Omega_c^{k+1}(M)$ for $k = 0, 1, 2\ldots, \dim M - 1$ are $L^2(\Lambda^k)$ – closable with closure denoted by $\bar{d}_k$. Let us now further assume that (M, g) is complete. Then:*

1) *Each of the operators, $\Delta_k := \Delta|_{\Omega_c^k(M)}$ for $k = 0, 1, 2\ldots, \dim M$ thought of as unbounded operators on $L^2(\Lambda^k)$, are essentially self-adjoint operators. Let $\bar{\Delta}_k$ denote the (self-adjoint) closure of Δ_k.*
2) *Each operator, $\bar{\Delta}_k$, is non-negative. Let $e^{t\bar{\Delta}_k}$ denotes the contraction semi-group on $L^2(\Lambda^k)$ associated to $\bar{\Delta}_k$.*
3) *For $k \in \{0, 1, \ldots, \dim M - 1\}$ and $t > 0$, $\bar{d}_k e^{t\bar{\Delta}_k} = e^{t\bar{\Delta}_{k+1}}\bar{d}_k$ on the domain of $\bar{d}_k$.*
4) *$\delta e^{t\bar{\Delta}_k}\omega = e^{t\bar{\Delta}_{k-1}}\delta\omega$ for all $\omega \in \Omega_c^k(M)$ with $k = 1, 2, \ldots, \dim M$.*

Proof. Let $\delta_k := \delta|_{\Omega_c^k(M)} : \Omega_c^k(M) \to \Omega_c^{k-1}(M)$. As $-\delta_{k+1} \subset d_k^*$, d_k^* is densely defined and hence d_k is closable. For items 1. and 2., see Gaffney [**26**], Roelcke [**53**], Chernoff [**11**], [**68**], and Strichartz [**60**]. Item 3. is a simple application of Theorem A.2 of Appendix A below. In applying this theorem, take $W = L^2(\Lambda^{k-1})$, $X = L^2(\Lambda^k)$, $Y = L^2(\Lambda^{k+1})$ and $Z = L^2(\Lambda^{k+2})$ with $A = \bar{d}_{k-1}$, $B = \bar{d}_k$, and $C := \bar{d}_{k+1}$. By convention $\Omega^{-1}(M) = \{0\} = \Omega^{\dim M+1}(M)$ and $d_{-1} = 0 = d_{\dim M}$. With these assignments, the self-adjoint operators, L and S, in Theorem A.2 become

$$L = \bar{d}_{k-1}d_{k-1}^* + d_k^*\bar{d}_k \text{ and } S = \bar{d}_k d_k^* + d_{k+1}^*\bar{d}_{k+1}. \tag{4.2}$$

As $\Delta_k|_{\Omega_c^k(M)} \subset -L$ and $-L$ is self-adjoint (see Theorem A.1 below), it follows that $\bar{\Delta}_k = -L$ and similarly, $\bar{\Delta}_{k+1} = -S$. For item 4., let $\omega \in \Omega_c^k(M)$ and $\varphi \in \Omega_c^{k-1}(M)$. Then

$$\begin{aligned}\left(\delta e^{t\bar{\Delta}_k}\omega, \varphi\right) &= -\left(e^{t\bar{\Delta}_k}\omega, d\varphi\right) = -\left(\omega, e^{t\bar{\Delta}_k}\bar{d}\varphi\right) = -\left(\omega, \bar{d}e^{t\bar{\Delta}_{k-1}}\varphi\right)\\ &= \left(\delta\omega, e^{t\bar{\Delta}_{k-1}}\varphi\right) = \left(e^{t\bar{\Delta}_{k-1}}\delta\omega, \varphi\right).\end{aligned}$$

q.e.d.

Remark 4.2. With a little more work it is possible to show that $\bar{d}_k = -\delta_{k+1}^*$ and that $\bar{\delta}_k e^{t\bar{\Delta}_k} = e^{t\bar{\Delta}_{k-1}}\bar{\delta}_k$ on the domain of $\bar{\delta}_k$. We will omit the proof of these results as they are not needed for this paper.

We are primarily concerned with zero and one forms. A key ingredient in the sequel is the Bochner identity,

$$\Delta\alpha = \Box\alpha - \alpha \circ \mathrm{Ric} \text{ for all } \alpha \in \Omega^1(M). \tag{4.3}$$

Assumption 1. *For the rest of this paper we will assume that* (M, g) *is a complete Riemannian manifold such that* $\mathrm{Ric} \geq k$ *for some* $k \in \mathbb{R}$, *i.e.* $\mathrm{Ric}_m \geq kI_{T_mM}$ *for all* $m \in M$.

Theorem 4.3 (Semi-group domination)**.** *Suppose that* (M, g) *is a complete Riemannian manifold such that* $\mathrm{Ric} \geq k$ *for some* $k \in \mathbb{R}$. *Then for all* $f \in L^2(\Lambda^0)$ *and* $\alpha \in L^2(\Lambda^1)$,

$$\left|e^{t\bar{\Delta}_0} f\right| \leq e^{t\bar{\Delta}_0}|f| \leq \|f\|_\infty \text{ a.e.} \tag{4.4}$$

and

$$\left|e^{t\bar{\Delta}_1}\alpha\right| \leq e^{-kt}e^{t\bar{\Delta}_0}|\alpha| \leq e^{-kt}\|\alpha\|_\infty \text{ a.e.}, \tag{4.5}$$

where $\|f\|_\infty$ *and* $\|\alpha\|_\infty$ *denote the essential supremums of the functions,* $|f|$ *and* $m \to |\alpha_m|$ *respectively.*

Proof. The inequality in Eq. (4.4) is an immediate consequence Eqs. (1.2), (1.1) and the positivity of the heat kernel, $p_t(x, y)$. This inequality may also be proved using the semi-group domination ideas that will be used below to prove Eq. (4.5). The proof of Eq. (4.5) will be an application of the results in Simon [**57**, **58**] and Hess, Schrader, and Uhlenbrock [**35**] along with a Kato [**40**] type inequality. The general Kato inequality we need is given in Theorem B.2 of Appendix B. We apply Theorem B.2 with $E = \Lambda^1(T^*M)$ to conclude,

$$(\Box\alpha, \varphi \operatorname{sgn}_e(\alpha)) \leq (|\alpha|, \Delta\varphi) \tag{4.6}$$

for all $\alpha \in \Omega_c^1(M)$ and $\varphi \in C^\infty(M)_+ := C^\infty(M \to [0, \infty))$. In Eq. (4.6),

$$\operatorname{sgn}_e(\alpha) := 1_{\alpha \neq 0}\frac{\alpha}{|\alpha|} + 1_{\alpha = 0}e,$$

where e is any measurable section of E such that $\langle\Box\alpha, e\rangle = 0$ on M. This inequality and the Bochner identity in Eq. (4.3) shows

$$\begin{aligned}(\Delta_1\alpha, \varphi \operatorname{sgn}_e(\alpha)) &= (\Box\alpha, \varphi \operatorname{sgn}_e(\alpha)) - (\alpha \circ \mathrm{Ric}, \varphi \operatorname{sgn}_e(\alpha)) \\ &\leq (|\alpha|, \Delta\varphi) - (\alpha \circ \mathrm{Ric}, \varphi \operatorname{sgn}_e(\alpha)).\end{aligned} \tag{4.7}$$

To evaluate the last term, let Y be the vector field on M such that $\alpha = \langle Y, \cdot \rangle$. Then $\alpha \circ \mathrm{Ric} = \langle \mathrm{Ric}\, Y, \cdot \rangle$ and

$$\begin{aligned}\langle \alpha \circ \mathrm{Ric}, \mathrm{sgn}_e(\alpha) \rangle &= 1_{\alpha \neq 0} \frac{1}{|\alpha|} \langle \alpha \circ \mathrm{Ric}, \alpha \rangle = 1_{\alpha \neq 0} \frac{1}{|\alpha|} \langle \mathrm{Ric}\, Y, Y \rangle \\ &\geq k 1_{\alpha \neq 0} \frac{1}{|\alpha|} \langle Y, Y \rangle = k 1_{\alpha \neq 0} \frac{1}{|\alpha|} |\alpha|^2 = k |\alpha|.\end{aligned}$$

Therefore,

$$(\alpha \circ \mathrm{Ric}, \varphi\, \mathrm{sgn}_e(\alpha)) = \int_M \langle \alpha \circ \mathrm{Ric}, \mathrm{sgn}_e(\alpha) \rangle \varphi dV \geq k (|\alpha|, \varphi)$$

which combined with Eq. (4.7) implies

$$(\Delta_1 \alpha, \varphi\, \mathrm{sgn}_e(\alpha)) \leq (|\alpha|, \Delta\varphi) - k (|\alpha|, \varphi) \tag{4.8}$$

or equivalently,

$$(H_0 \alpha, \varphi\, \mathrm{sgn}_e(\alpha)) \geq (|\alpha|, -\Delta\varphi),$$

where $H_0 := -(\Delta + k)|_{\Omega_c^1(M)}$. In particular, if $g \in C_c^\infty(M)_+$, $\lambda > 0$, $\varphi = \left(-\bar{\Delta}_0 + \lambda\right)^{-1} g$, and $\alpha_1 \in \Omega_c^1(M)$ and we define $\alpha_2 := \varphi\, \mathrm{sgn}_e(\alpha_1) \in L^2(\Lambda^1)$, then $(\alpha_1, \alpha_2)_{L^2(\Lambda_1)} = (|\alpha_1|, |\alpha_2|)_{L^2(\Lambda_0)}$, $|\alpha_2| = \varphi$, and

$$(H_0 \alpha_1, \alpha_2)_{L^2(\Lambda_1)} \geq \left(|\alpha_1|, -\bar{\Delta}_0 \varphi\right)_{L^2(\Lambda_0)}.$$

Hence we have verified the hypothesis of Proposition 2.14 and Theorem 2.15 in [**35**] and as a consequence,

$$\left| e^{-t\bar{H}_0} \alpha \right| \leq e^{-t\left(-\bar{\Delta}_0\right)} |\alpha| \text{ a.e. for all } \alpha \in L^2\left(\Lambda^1\right). \tag{4.9}$$

As $\bar{H}_0 = -\bar{\Delta}_1 - k$ and hence, $e^{-t\bar{H}_0} = e^{t\bar{\Delta}_1} e^{tk}$, Eq. (4.9) is equivalent to the first inequality in Eq. (4.5). q.e.d.

5. A path integral derivative formula

5.1. Brownian motion and the divergence formula. We start with a filtered probability space, $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}\right)$, satisfying the usual hypothesis. For each $x \in M$, let $\{\Sigma_t^x : t < \zeta(x)\}$ be an M – valued Brownian motion on $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}\right)$, starting from x, with possibly finite lifetime $\zeta(x)$. Recall Σ_t^x is said to be an M–valued Brownian motion provided it is a Markov diffusion process starting at x with transition semigroup determined by the heat kernel, $p_t(\cdot, \cdot)$. Because of our standing assumption, $\mathrm{Ric} \geq k$, it is well–known that $\int_M p_t(x, y) dy = 1$ for all $x \in M$ and consequently that $\zeta(x) = \infty$, see [**3**, **27**, **68**, **16**, **42**, **29**, **30**, **32**] and the books [**63**, Theorem 8.62], [**37**, Chapter 4.] and [**14**, Theorem 5.2.6]. For our purposes it will be convenient to construct Σ_t^x as a solution to a stochastic differential equation which we will describe shortly.

Notation 5.1. Given two isometric isomorphic real finite–dimensional inner product spaces, V and W, let $O(V,W)$ denote the set of linear isometries from V to W.

Let $//_t(\sigma)$ denote parallel translation along a curve σ in TM and all associated bundles. We also introduce the horizontal vector fields on the orthogonal frame bundle over M as

$$B_v(u) = \frac{d}{dt}|_0 //_t(\sigma)\, u \text{ for } v \in \mathbb{R}^d \text{ and } u \in O\left(\mathbb{R}^d, T_x M\right),$$

where $\sigma(t)$ is a curve in M such that $\dot{\sigma}(0) = uv$.

Notation 5.2. Given a semi-martingale, Y_t, we will denote its Itô differential by dY_t and its Fisk-Stratonovich differential by $\circ dY_t$.

Let b_t denote a $\mathbb{R}^d$ – valued Brownian motion, $x \in M$, and $u_0 \in O\left(\mathbb{R}^d, T_x M\right)$, then Σ_t^x may be defined as the unique solution to the stochastic differential equation,

$$\begin{aligned} \circ d\Sigma_t^x &= u_t \circ db_t \text{ with } \Sigma_0^x = x, \\ \circ du_t &= B_{\circ db_t}(u_t) \text{ with } u_0. \end{aligned}$$

The *stochastic parallel translation along* Σ_t^x up to time t is taken to be, $//_t := u_t u_0^{-1} \in O\left(T_x M, T_{\Sigma_t^x} M\right)$. Suppose that $f(t,m)$ $(\alpha(t,m))$ is a smooth time dependent function (one form), then the Itô differentials of $f(t,\Sigma_t^x)$ and $\alpha(t,\Sigma_t^x)//_t$ are

(5.1)
$$d\left[f(t,\Sigma_t^x)\right] = \left(\frac{\partial}{\partial t} f(t,\Sigma_t^x) + \frac{1}{2}\Delta_0 f(t,\Sigma_t^x)\right) dt + \langle \operatorname{grad} f(t,\cdot), //_t db_t\rangle$$

and

(5.2)
$$d\left[\alpha(t,\Sigma_t^x)//_t\right] = \left(\frac{\partial}{\partial t}\alpha(t,\Sigma_t^x) + \frac{1}{2}\Box\alpha(t,\Sigma_t^x)\right) dt + \left[\nabla_{//_t db_t}\alpha(t,\cdot)\right]//_t.$$

See (for example) [**24**, **46**, **63**, **37**, **20**] for more on the general background used in this section.

5.2. The divergence formula.

Let Q_t denote the $\operatorname{End}(T_x M)$– valued process satisfying the ordinary differential equation,

$$\frac{d}{dt}Q_t = -\frac{1}{2}\operatorname{Ric}^{//_t} Q_t \quad \text{with } Q_0 = id_{T_x M}, \tag{5.3}$$

where

$$\operatorname{Ric}^{//_t} := //_t^{-1}\operatorname{Ric}_{\Sigma_t^x} //_t. \tag{5.4}$$

Lemma 5.3. *If* Ric $\geq k$ *for some* $k \in \mathbb{R}$ *and* $\|\cdot\|_{op}$ *denotes the operator norm on* $T_x M$, *then*

$$\|Q_t\|_{op} \leq e^{-kt/2}. \tag{5.5}$$

Similarly if $\mathrm{Ric} \le K$ *for some* $K \in \mathbb{R}$, *then*

$$\left\|Q_t^{-1}\right\|_{op} \le e^{Kt/2}. \tag{5.6}$$

Proof. For any $v \in T_xM$, we have

$$\frac{d}{dt}|Q_t v|^2 = \left\langle -\mathrm{Ric}^{//_t} Q_t v, Q_t v\right\rangle \le -k\,|Q_t v|^2$$

from which Eq. (5.5) easily follows. To prove Eq. (5.6), let $R_t := \left(Q_t^{-1}\right)^*$ and observe that

$$\frac{d}{dt}R_t = -\left(Q_t^{-1}\dot{Q}_t Q_t^{-1}\right)^* = \frac{1}{2}\left(Q_t^{-1}\mathrm{Ric}^{//_t} Q_t Q_t^{-1}\right)^* = \frac{1}{2}\mathrm{Ric}^{//_t} R_t.$$

Hence reasoning as above we may conclude that

$$\left\|Q_t^{-1}\right\|_{op} = \left\|\left(Q_t^{-1}\right)^*\right\|_{op} = \|R_t\|_{op} \le e^{Kt/2}.$$

q.e.d.

When M is compact, the following result is Theorem 5.10 of Driver and Thalmaier [**23**].

Theorem 5.4 (A divergence formula)**.** *Assume the Ricci curvature,* Ric, *on* M *satisfies,* $k \le \mathrm{Ric} \le K$ *for some* $-\infty < k \le K < \infty$. *Let* $T > 0$ *and* $\tilde{\ell}$ *be a* C^1 *– adapted real-valued process such that* $\tilde{\ell}_0 = 0$, $\tilde{\ell}_T = 1$, *and*

$$\int_0^T \left|\frac{d}{d\tau}\tilde{\ell}_\tau\right| d\tau \le C, \tag{5.7}$$

where $C < \infty$ *is a non-random constant. Then for every* C^2 *– vector field,* Y, *on* M *with compact support the following identity holds*

$$\mathbb{E}\left[\nabla \cdot Y\left(\Sigma_T^x\right)\right] = \mathbb{E}\left[\left\langle Y(\Sigma_T^x), //_T Q_T \int_0^T \tilde{\ell}_t' Q_t^{-1} db_t\right\rangle\right], \tag{5.8}$$

where $\nabla \cdot Y$ *is the divergence of* Y *and* $\tilde{\ell}_t' := \frac{d}{dt}\tilde{\ell}_t$.

Proof. The proof will consist of adding some technical details to the proof of Theorem 5.10 in [**23**]. Suppose a is a smooth one form on M with compact support,

$$a_t := e^{(T-t)\bar{\Delta}_1/2}a, \tag{5.9}$$

$\tilde{\ell}_\tau$ is an adapted continuously differentiable real–valued process, and ℓ_0 is a fixed vector in T_xM. Then as shown in [**23**, Theorem 3.4] (and repeated below in Lemma C.1 for the readers convenience) the process,

$$Z_t := \left(a_t\left(\Sigma_t^x\right) \circ //_t\right) Q_t \left[\int_0^t Q_\tau^{-1}\left(\frac{d}{d\tau}\tilde{\ell}_\tau\right) db_\tau + \ell_0\right] - \left(\delta a_t\right)\left(\Sigma_t^x\right)\tilde{\ell}_t \tag{5.10}$$

is a local martingale. From Theorems 4.1 and 4.3 we have

$$|a_t| \leq e^{-(T-t)k/2} \|a\|_\infty \leq e^{T|k|/2} \|a\|_\infty$$

and

$$|\delta a_t| = \left| e^{(T-t)\bar{\Delta}_0/2} \delta a \right| \leq \|\delta a\|_\infty .$$

Making use of these estimates along with Lemma 5.3 and Eq. (5.7) shows that Z_t is a bounded local martingale and hence, by a localization argument, a martingale. In particular, it follows that $t \to \mathbb{E} Z_t$ is constant for $0 \leq t \leq T$ and hence

$$\begin{aligned} \left(e^{T\Delta/2} a\right) (\Sigma_0^x) \ell_0 - \delta \left(e^{T\Delta/2} a\right) (\Sigma_0^x) \tilde{\ell}_0 = Z_0 = \mathbb{E} Z_T \\ = \mathbb{E} \left[\left(a (\Sigma_T^x) \circ //_T\right) Q_T \left(\int_0^T Q_\tau^{-1} \left(\frac{d}{d\tau} \tilde{\ell}_\tau \right) db_\tau + \ell_0 \right) \right] \\ - \mathbb{E} \left[\delta a (\Sigma_T^x) \tilde{\ell}_T \right] . \end{aligned}$$

If we now suppose that $\ell_0 = 0$, $\tilde{\ell}_0 = 0$, and $\tilde{\ell}_T = 1$, the above formula reduces to

$$0 = \mathbb{E} \left[\left(a (\Sigma_T^x) \circ //_T\right) Q_T \int_0^T Q_\tau^{-1} \left(\frac{d}{d\tau} \tilde{\ell}_\tau \right) db_\tau - \delta a (\Sigma_T^x) \right] .$$

This identity is equivalent to the identity in Eq. (5.8) as is seen by taking $a(x) v := \langle Y(x), v \rangle$ for all $x \in M$ and $v \in T_x M$ and recalling that

$$\delta a = \sum_{i=1}^{d} i_{e_i} \nabla_{e_i} \langle Y, \cdot \rangle = \sum_{i=1}^{d} i_{e_i} \langle \nabla_{e_i} Y, \cdot \rangle = \nabla \cdot Y.$$

q.e.d.

Example 5.5. *Taking $\tilde{\ell}_t = t/T$ in Eq. (5.8) shows*

$$\mathbb{E} \left[\nabla \cdot Y (\Sigma_T^x) \right] = \frac{1}{T} \mathbb{E} \left[\left\langle Y(\Sigma_T^x), //_T Q_T \int_0^T Q_t^{-1} db_t \right\rangle \right] . \tag{5.11}$$

6. Exponential integrability of W_A^T

In this section and for the remainder of the paper we will again go back to the setting where $M = G$ is a connected uni-modular Lie group equipped with a left–invariant Riemannian metric as described in the introduction. We are now going to use Theorem 5.4 to estimate $W_A := W_A^T$ in Definition 1.7. In order to do this we will use Eq. (5.8) to find a useful path integral expression for W_A, see Theorem 6.4 below.

For $A, B \in \mathfrak{g}$, let $D_A B := \nabla_A \tilde{B} \in \mathfrak{g}$ where ∇ is the Levi-Civita covariant derivative on TG. Observe that $\nabla_{\tilde{A}} \tilde{B}$ is a left invariant vector field and $\left(\nabla_{\tilde{A}} \tilde{B}\right)(e) = \nabla_A \tilde{B} = D_A B$. Hence we have the identity, $\nabla_{\tilde{A}} \tilde{B} = \widetilde{D_A B}$.

Lemma 6.1. *Suppose that* $\{A_i\}_{i=1}^{\dim \mathfrak{g}}$ *is an orthonormal basis for* $\mathfrak{g}$ *and* G *is uni-modular. Then*

1) $\sum_{i=1}^{\dim \mathfrak{g}} D_{A_i} A_i = 0$ *or equivalently* $\sum_{i=1}^{\dim \mathfrak{g}} \nabla_{\tilde{A}_i} \tilde{A}_i = 0$.
2) *The divergence of* $\tilde{B}$, $\nabla \cdot \tilde{B}$, *is zero for all* $B \in \mathfrak{g}$.
3) $\Delta_0 = \sum_{i=1}^{\dim \mathfrak{g}} \tilde{A}_i^2$ *is the Laplace Beltrami operator on* G.

Proof. 1) The formula for $D_A B$ is

$$D_A B = \frac{1}{2}\left(ad_A B - ad_A^* B - ad_B^* A\right)$$

and hence $D_A A = -ad_A^* A$ and for any $B \in \mathfrak{g}$ we find

$$\begin{aligned}\left(\sum_{i=1}^{\dim \mathfrak{g}} D_{A_i} A_i, B\right)_{\mathfrak{g}} &= -\sum_{i=1}^{\dim \mathfrak{g}} \left(A_i, ad_{A_i} B\right)_{\mathfrak{g}} \\ &= -\sum_{i=1}^{\dim \mathfrak{g}} \left(A_i, ad_B A_i\right)_{\mathfrak{g}} = -\operatorname{tr}\left(ad_B\right).\end{aligned}$$

Since G is uni-modular, $\det\left(Ad_{e^{tB}}\right) = 0$ for all t and therefore $\operatorname{tr}\left(ad_B\right) = 0$.

2) The following simple computation shows $\nabla \cdot \tilde{B} = 0$

$$\begin{aligned}\nabla \cdot \tilde{B} = \sum_{i=1}^{\dim \mathfrak{g}} \left(\nabla_{\tilde{A}_i} \tilde{B}, \tilde{A}_i\right)_{TG} &= \sum_{i=1}^{\dim \mathfrak{g}} \left(D_{A_i} B, A_i\right)_{\mathfrak{g}} \\ = -\sum_{i=1}^{\dim \mathfrak{g}} \left(B, D_{A_i} A_i\right)_{\mathfrak{g}} = 0.\end{aligned}$$

3) Observe that $\left\{\tilde{A}_i\right\}_{i=1}^{\dim \mathfrak{g}}$ is a globally defined orthonormal frame for TG and that

$$\Delta_0 = \sum_{i=1}^{\dim \mathfrak{g}} \left[\tilde{A}_i^2 - \nabla_{\tilde{A}_i} \tilde{A}_i\right] = \sum_{i=1}^{\dim \mathfrak{g}} \tilde{A}_i^2.$$

q.e.d.

In Theorem 6.4 below, we will specialize Theorem 5.4 in order to find a probabilistic representation for W_A of Definition 1.7. This representation will then be used to estimate $\int_G e^{W_A} d\nu_T$ for all $A \in \mathfrak{g}$. Let $\{\Sigma_t\}_{t \geq 0}$ be a Brownian motion on G such that $\Sigma_0 = e$, b_t be the $\mathfrak{g}$ – valued Brownian motion defined by,

$$b_t := \int_0^t //_\tau (\Sigma)^{-1} \circ d\Sigma_\tau,$$

and β_t be the $\mathfrak{g}$ – valued semi-martingale defined by

$$\beta_t := \int_0^t \theta\left(\circ d\Sigma_\tau\right) = \int_0^t L_{\Sigma_\tau^{-1}*} \circ d\Sigma_\tau,$$

where $\theta\left(v_g\right) := L_{g^{-1}*} v_g$ for all $v_g \in T_g G$. As a reflection of the fact that $\sum_{i=1}^{\dim \mathfrak{g}} \tilde{A}_i^2$ is the Laplace–Beltrami operator, β_t is another $\mathfrak{g}$–valued Brownian motion. This will also be evident from the following proposition.

Proposition 6.2. *Fix $T > 0$ and let $U_t \in O\left(\mathfrak{g}\right)$ be the unique solution to the stochastic differential equation*

$$dU_t + D_{\circ d\beta_t} U_t = 0 \text{ with } U_0 = I. \tag{6.1}$$

Further define $Y_t := U_t Q_t$, and $V_t := Y_T Y_t^{-1}$. Then

$$//_t := L_{\Sigma_t *} U_t \tag{6.2}$$

and

$$\int_0^t U_\tau^{-1} \circ d\beta_\tau = \int_0^t U_\tau^{-1} d\beta_\tau = \int_0^t //_\tau^{-1} \circ d\Sigma_\tau = b_t. \tag{6.3}$$

Proof. The fact that $//_t := L_{\Sigma_t *} U_t$ is explained in [**19**, Theorem 6.6] and hence

$$b_t = \int_0^t U_t^{-1} L_{\Sigma_t *}^{-1} \circ d\Sigma_\tau = \int_0^t U_t^{-1} \theta\left(\circ d\Sigma_\tau\right) = \int_0^t U_\tau^{-1} \circ d\beta_\tau,$$

i.e. $d\beta_t = U_t \circ db_t$. Letting $\{A_i\}_{i=1}^{\dim \mathfrak{g}}$ be an orthonormal basis for $\mathfrak{g}$, it follows from Lemma 6.1 and the fact that $\{U_t A_i\}_{i=1}^{\dim \mathfrak{g}}$ is also an orthonormal basis for $\mathfrak{g}$ that

$$\begin{aligned} dU_t db_t &= -\frac{1}{2} D_{d\beta_t} U_t db_t = -\frac{1}{2} D_{U_t db_t} U_t db_t \\ &= -\frac{1}{2} \sum_{i=1}^{\dim \mathfrak{g}} D_{U_t A_i} U_t A_i \, dt = 0. \end{aligned}$$

This allows us to conclude that $d\beta_t = U_t \circ db_t = U_t db_t$ which completes the proof of the proposition. q.e.d.

Proposition 6.3. *Let $Y_t := U_t Q_t$ and for fixed $T > 0$ let $V_t := Y_T Y_t^{-1}$ and $\mathcal{G}_t := \overline{\sigma\left(\beta_\tau - \beta_s : t \le s, \tau \le T\right)}$ – the completion of the σ – algebra generated by $\{\beta_\tau - \beta_s : t \le s, \tau \le T\}$. Then*

1) *V_t is $\mathcal{G}_t$ – measurable, and*
2) *V_t is the unique solution to the backwards stochastic differential equation,*

$$dV_t = V_t \left(D_{\circ d\beta_t} + \frac{1}{2} \operatorname{Ric}_e \, dt \right) \text{ with } V_T = I.$$

Proof. Because $L_{\Sigma_t *}$ is an isometry of G, it follows that

$$\mathrm{Ric}^{//_t} = //_t^{-1} \mathrm{Ric}_{\Sigma_t} //_t = U_t^{-1} L_{\Sigma_t *}^{-1} \mathrm{Ric}_{\Sigma_t} L_{\Sigma_t *} U_t = U_t^{-1} \mathrm{Ric}_e U_t. \tag{6.4}$$

Using this identity and the definition of Y_t we find, $Y_0 = Id$ and

$$\begin{aligned} dY_t &= -D_{\circ d\beta_t} U_t Q_t - \frac{1}{2} U_t \mathrm{Ric}^{//_t} Q_t dt \\ &= -D_{\circ d\beta_t} Y_t - \frac{1}{2} U_t \mathrm{Ric}^{//_t} U_t^{-1} Y_t dt \qquad (6.5) \\ &= -D_{\circ d\beta_t} Y_t - \frac{1}{2} \mathrm{Ric}_e \, Y_t dt. \qquad (6.6) \end{aligned}$$

Since $dY_t^{-1} = -Y_t^{-1} (\circ dY_t) Y_t^{-1}$, it follows that Y_t^{-1} satisfies,

$$dY_t^{-1} = Y_t^{-1} D_{\circ d\beta_t} + \frac{1}{2} Y_t^{-1} \mathrm{Ric}_e \, dt \text{ with } Y_0^{-1} = Id. \tag{6.7}$$

For $T \geq t \geq 0$, let $Y_{T,t}$ solve,

$$d_T Y_{T,t} = -D_{\circ d\beta_T} Y_{T,t} - \frac{1}{2} \mathrm{Ric}_e \, Y_{T,t} dT \text{ with } Y_{t,t} = Id,$$

and observe that $Y_{T,t}$ is $\overline{\sigma (\beta_\tau - \beta_s : t \leq s, \tau \leq T)}$ – measurable. By the uniqueness of solutions to linear stochastic differential equations we may conclude

$$Y_T = Y_{T,t} Y_t \text{ a.s. for all } 0 \leq t \leq T$$

and hence it follows that $V_t = Y_T Y_t^{-1} \overset{\text{a.s.}}{=} Y_{T,t}$ is also $\overline{\sigma (\beta_\tau - \beta_s : t \leq s, \tau \leq T)}$ – measurable. Moreover we have,

$$\begin{aligned} dV_t &= Y_T d\left(Y_t^{-1}\right) = -Y_T Y_t^{-1} (\circ dY_t) Y_t^{-1} \\ &= -V_t \left(-D_{\circ d\beta_t} - \frac{1}{2} \mathrm{Ric}_e \, dt \right) \\ &= V_t \left(D_{\circ d\beta_t} + \frac{1}{2} \mathrm{Ric}_e \, dt \right) \text{ with } V_T = Id. \end{aligned}$$

See [**19**, Section 4.1] for more on the backwards stochastic integral interpretation of this equation. q.e.d.

Theorem 6.4. *If $A \in \mathfrak{g}$ and $\ell \in C^1([0,T], \mathbb{R})$ with $\ell(0) = 0$ and $\ell(T) = 1$, then*

$$W_A(x) = \mathbb{E}\left[\left(A, \int_0^T \dot{\ell}(\tau) V_\tau d\overleftarrow{\beta}_\tau \right) \middle| \Sigma_T = x \right], \tag{6.8}$$

where $\int_0^T \dot{\ell}(\tau) V_\tau d\overleftarrow{\beta}_\tau$ is a backwards Itô integral and V_t satisfies the (backwards) stochastic differential equation,

$$dV_t = \frac{1}{2} V_t \mathrm{Ric}_e \, dt + V_t D_{\circ d\beta_t} \text{ with } V_T = Id.$$

Proof. Let $f \in C_c^\infty(G)$ and

$$Y(x) := f(x)\tilde{A}(x) = f(x)L_{x*}A.$$

As shown in Lemma 6.1, $\nabla \cdot \tilde{A} = 0$ from which it follows that

$$\nabla \cdot Y = \left(\operatorname{grad} f, \tilde{A}\right)_{TG} = \tilde{A}f.$$

Therefore an application of Theorem 5.4 (with $\tilde{\ell}_t$ now being denoted by $\ell(t)$) shows,

$$\begin{aligned}\mathbb{E}\left[\left(\tilde{A}f\right)(\Sigma_T)\right] &= \mathbb{E}\left[f(\Sigma_T)\left\langle \tilde{A}(\Sigma_T), //_T Q_T \int_0^T \dot{\ell}(\tau) Q_\tau^{-1} db_\tau\right\rangle\right] \\ &= \mathbb{E}\left[f(\Sigma_T)\left\langle A, L_{\Sigma_T^{-1}*}//_T Q_T \int_0^T \dot{\ell}(\tau) Q_\tau^{-1} db_\tau\right\rangle\right]. \qquad (6.9)\end{aligned}$$

From Eq. (6.3)

$$\begin{aligned}&\left\langle A, L_{\Sigma_T^{-1}*}//_T Q_T \int_0^T \dot{\ell}(\tau) Q_\tau^{-1} db_\tau\right\rangle \\ &\qquad = \left\langle A, U_T Q_T \int_0^T \dot{\ell}(\tau) Q_\tau^{-1} U_\tau^{-1} d\beta_\tau\right\rangle \\ &\qquad = \left\langle A, Y_T \int_0^T \dot{\ell}(\tau) Y_\tau^{-1} d\beta_\tau\right\rangle \qquad (6.10) \\ &\qquad = \left\langle A, \int_0^T \dot{\ell}(\tau) V_\tau d\beta_\tau\right\rangle. \qquad (6.11)\end{aligned}$$

Moreover, we may write the last expression as a backwards Itô integral, since

$$dV_\tau d\beta_\tau = V_\tau D_{d\beta_\tau} d\beta_\tau = V_\tau \sum_{A \in ONB(\mathfrak{g})} D_A A \cdot dt = 0,$$

wherein we have used Lemma 6.1 again for the last equality. Hence we now have

$$\left\langle A, L_{\Sigma_T^{-1}*}//_T Q_T \int_0^T \dot{\ell}(\tau) Q_\tau^{-1} db_\tau\right\rangle = \left\langle A, \int_0^T \dot{\ell}(\tau) V_\tau d\overleftarrow{\beta}_\tau\right\rangle.$$

These computations may be justified by the same methods introduced in [**19**]. This completes the proof because,

$$\begin{aligned}\mathbb{E}\left[W_A(\Sigma_T) f(\Sigma_T)\right] &= \mathbb{E}\left[\left(\tilde{A}f\right)(\Sigma_T)\right] \\ &= \mathbb{E}\left[f(\Sigma_T)\left\langle A, \int_0^T \dot{\ell}(\tau) V_\tau d\overleftarrow{\beta}_\tau\right\rangle\right]\end{aligned}$$

for all $f \in C_c^\infty(G)$. q.e.d.

Our next goal is to bound $\int_G e^{W_A} d\nu_T$ for all $A \in \mathfrak{g}$. In order to do this it will be necessary to estimate the size of the process V_t.

Lemma 6.5. *Suppose $k \in \mathbb{R}$ is chosen so that $\mathrm{Ric} \geq kI$, then*

$$|V_t^* A|^2 \leq |A|^2 e^{-k(T-t)} \text{ for all } A \in \mathfrak{g}. \tag{6.12}$$

Proof. Since

$$dV_t = \frac{1}{2} V_t \operatorname{Ric}_e dt + V_t D_{\circ d\beta_t},$$

we have

$$dV_t^* = \frac{1}{2} \operatorname{Ric}_e V_t^* dt - D_{\circ d\beta_t} V_t^*$$

wherein we have used the fact that $D_A : \mathfrak{g} \to \mathfrak{g}$ is antisymmetric. In particular it now follows that

$$\begin{aligned} d\,|V_t^* A|^2 &= 2\,(\circ dV_t^* A, V_t^* A) = 2\left(\frac{1}{2} \operatorname{Ric}_e V_t^* A dt - D_{\circ d\beta_t} V_t^* A, V_t^* A\right) \\ &= (\operatorname{Ric}_e V_t^* A, V_t^* A)\, dt \geq k\,|V_t^* A|^2\, dt \text{ with } |V_T^* A|^2 = |A|^2 . \end{aligned}$$

We may write this inequality as

$$\frac{d}{dt} \ln |V_t^* A|^2 \geq k \text{ with } |V_T^* A|^2 = |A|^2$$

which upon integration gives,

$$\ln |A|^2 - \ln |V_t^* A|^2 = \ln |V_T^* A|^2 - \ln |V_t^* A|^2 \geq k\,(T-t)\,.$$

Hence $|A|^2 / |V_t^* A|^2 \geq e^{k(T-t)}$ which is equivalent to Eq. (6.12).

q.e.d.

Lemma 6.6. *Let $k \in \mathbb{R}$ and $T > 0$, then*

$$\inf \left\{ \int_0^T \dot{\ell}^2(\tau)\, e^{-k(T-\tau)} d\tau \right\} \leq \frac{k}{e^{kT} - 1}, \tag{6.13}$$

where the infimum is taken over all $\ell \in C^1([0,T], \mathbb{R})$ such that $\ell(0) = 0$ and $\ell(T) = 1$.

Proof. By a simple calculus of variation argument, $\ell \in C^1([0,T], \mathbb{R})$ with $\ell(0) = 0$ and $\ell(T) = 1$ is a critical point for the function,

$$K(\ell) := \int_0^T \dot{\ell}^2(\tau)\, e^{-k(T-\tau)} d\tau, \tag{6.14}$$

iff $\dot{\ell}(\tau)\, e^{k\tau}$ is constant in τ. This constraint and the boundary conditions imply that K has a unique critical point at

$$\ell_c(\tau) = \frac{e^{-k\tau} - 1}{e^{-kT} - 1}.$$

Plugging this value of ℓ_c into K then shows $K(\ell_c) = k\left(1 - e^{-kT}\right)^{-1}$ from which Eq. (6.13) follows. q.e.d.

6.1. Proof of Theorems 1.3 and 1.4. With the above results as preparation, we are now in position to complete the proofs of Theorem 1.3 and 1.4.

Proof. Proof of Theorem 1.3. Let $\ell \in C^1([0,T], \mathbb{R})$ such that $\ell(0) = 0$ and $\ell(T) = 1$. From Theorem 6.4, Lemma 6.5, Jensen's inequality for conditional expectations, and a standard martingale argument (see the proof of Lemma 7.6 and especially Eq. 7.17 in [**18**]) we have

$$\begin{aligned}
\int_G e^{W_A} d\nu_T &= \mathbb{E}\left[\exp\left(\mathbb{E}\left[\left\langle A, \int_0^T \dot{\ell}(\tau) V_\tau d\overleftarrow{\beta}_\tau\right\rangle \middle| \sigma(\Sigma_T)\right]\right)\right] \\
&\leq \mathbb{E}\left[\mathbb{E}\left[\exp\left(\left\langle A, \int_0^T \dot{\ell}(\tau) V_\tau d\overleftarrow{\beta}_\tau\right\rangle\right) \middle| \sigma(\Sigma_T)\right]\right] \\
&= \mathbb{E}\left[\exp\left(\left\langle A, \int_0^T \dot{\ell}(\tau) V_\tau d\overleftarrow{\beta}_\tau\right\rangle\right)\right] \\
&\leq \exp\left(\frac{1}{2}\left\| \int_0^T \dot{\ell}^2(\tau) \left|V_\tau^* A\right|^2 d\tau \right\|_{L^\infty(P)}\right) \\
&\leq \exp\left(\frac{|A|^2}{2} \int_0^T \dot{\ell}^2(\tau) e^{-k(T-\tau)} d\tau\right),
\end{aligned}$$

where P is the underlying probability measure. Since ℓ was arbitrary, it follows from Lemma 6.6 that,

$$\begin{aligned}
\int_G e^{W_A} d\nu_T &\leq \inf_\ell \exp\left(\frac{1}{2}\int_0^T \dot{\ell}^2(\tau) |A|^2 e^{-k(T-\tau)} d\tau\right) \\
&\leq \exp\left(\frac{1}{2}\frac{k}{e^{kT}-1}|A|^2\right) = \exp\left(\frac{1}{2T} c(kT) |A|^2\right).
\end{aligned}$$

q.e.d.

Proof. (Proof of Theorem 1.4.) From Theorem 6.4, Lemma 6.5, Jensen's inequality for conditional expectations, and Burkholder-Davis-Gundy inequality (see for example [**61**, Corollary 6.3.1a on p.344], [**49**, Appendix A.2], or [**48**, p. 212] and [**39**, Theorem 17.7] for the real case), there exists $C_q < \infty$ such that

$$\begin{aligned}
\int_G |W_A|^q \, d\nu_T &= \mathbb{E}\left[\left|\mathbb{E}\left[\left\langle A, \int_0^T \dot{\ell}(\tau) V_\tau d\overleftarrow{\beta}_\tau \right\rangle \middle| \sigma(\Sigma_T)\right]\right|^q\right] \\
&\leq \mathbb{E}\left[\mathbb{E}\left[\left|\left\langle A, \int_0^T \dot{\ell}(\tau) V_\tau d\overleftarrow{\beta}_\tau \right\rangle\right|^q \middle| \sigma(\Sigma_T)\right]\right] \\
&= \mathbb{E}\left[\left|\left\langle A, \int_0^T \dot{\ell}(\tau) V_\tau d\overleftarrow{\beta}_\tau \right\rangle\right|^q\right] \\
&= \mathbb{E}\left[\left|\int_0^T \dot{\ell}(\tau) \left\langle V_\tau^* A, d\overleftarrow{\beta}_\tau \right\rangle\right|^q\right] \\
&\leq C_q^q \mathbb{E}\left[\left|\int_0^T \dot{\ell}^2(\tau) |V_\tau^* A|^2 \, d\tau\right|^{q/2}\right] \\
&\leq C_q^q \left(|A|^2 \int_0^T \dot{\ell}^2(\tau) e^{-k(T-\tau)} d\tau\right)^{q/2}.
\end{aligned}$$

Using Lemma 6.6, we may optimize this last estimate over the admissible ℓ to find,

$$\int_G |W_A|^q \, d\nu_T \leq C_q^q \left(|A|^2 \frac{k}{e^{kT}-1}\right)^{q/2} = C_q^q \left(|A|^2 \frac{c(kT)}{T}\right)^{q/2}$$

which is equivalent to Eq. (1.4). q.e.d.

7. Applications

Lemma 7.1. *Suppose that* $T > 0$, $q > 1$, *and* $f \in L^q(\nu_T) \cap C^2(G)$ *such that* $\Delta f \in L^q(\nu_T)$. *Then* $f, \Delta f \in L^q(\nu_t)$ *for* $0 < t \leq T$ *and*

(7.1)
$$\frac{\partial}{\partial t} \int_G p_t(x,y) f(y) \, dy = \frac{1}{2} \int_G p_t(x,y) \Delta f(y) \, dy \quad \text{for all} \quad 0 < t < T.$$

Proof. Since the Ricci curvature is left translation invariant, it is bounded on G. Applying the Li – Yau Harnack inequality (see Eq. (D.6 below), we have for any $\gamma > 1/2$ that there exists $K = K(\gamma, T) < \infty$ such that

$$p_t(x) \leq K \left(\frac{T}{t}\right)^{d\gamma} p_T(x) \ \forall \ (x,t) \in G \times (0,T]. \tag{7.2}$$

In particular it follows that

$$\|f\|_{L^q(\nu_t)} \leq K \left(\frac{T}{t}\right)^{d\gamma/q} \|f\|_{L^q(\nu_T)} \ \forall \ 0 < t \leq T. \tag{7.3}$$

Using $q'-1=(q-1)^{-1}$ and Eq. (1.6), it follows that

$$\begin{aligned}\int_G p_t(y,x)\,|f(x)|\,dx &= \int_G \frac{p_t(y,x)}{p_t(x)}\,|f(x)|\,d\nu_t(x)\\ &\le \left\|\frac{p_t(y,\cdot)}{p_t(\cdot)}\right\|_{L^{q'}(\nu_t)}\cdot\|f\|_{L^q(\nu_t)}\\ &\le \|f\|_{L^q(\nu_t)}\exp\left(\frac{c(kt)(q'-1)}{2t}|y|^2\right)\\ &\le \|f\|_{L^q(\nu_t)}\exp\left(\frac{c(kt)}{2t(q-1)}|y|^2\right). \qquad (7.4)\end{aligned}$$

Therefore the integrals in Eq. (7.1) are well defined. Moreover,

$$\begin{aligned}\int_G p_t(y,x)\,f(x)\,dx &= \int_G p_t\left(y^{-1}x\right)f(x)\,dx = \int_G p_t(x)\,f(yx)\,dx\\ &= \int_G f\circ L_y(x)\,p_t(x)\,dx\end{aligned}$$

and for any $r\in(1,q)$,

$$\begin{aligned}\|f\circ L_y\|^r_{L^r(\nu_t)} &= \int_G |f(yx)|^r\,p_t(x)\,dx = \int_G |f(x)|^r\,p_t\left(y^{-1}x\right)dx\\ &= \int_G |f(x)|^r\,\frac{p_t\left(y^{-1}x\right)}{p_t(x)}d\nu_t(x)\\ &\le \|f\|_{L^q(\nu_t)}\exp\left(\frac{c(kt)\,q\,(q-r)^{-1}}{2t}|y|^2\right)\end{aligned}$$

wherein we have used Hölder's inequality and Eq. (1.9) for the last inequality. From these remarks and the fact that $\Delta(f\circ L_y)=(\Delta f)\circ L_y$, it suffices to prove Eq. (7.1) in the special case when $y=e$. From Eq. (7.2) and the dominated convergence theorem, the function

$$F(t)=\int_G f(x)\,d\nu_t(x)\quad\text{for all}\quad t\in(0,T]$$

is continuous. Our goal now is to show F is differentiable and that $\dot F(t)=\frac{1}{2}\int_G \Delta f(x)\,d\nu_t(x)$ for all $0<t<T$. To prove this suppose that $h\in C_c^\infty(G)$ and consider,

$$F_h(t):=\int_G f(x)\,h(x)\,p_t(x)\,dx.$$

To simplify notation in the computation below, let $\{A_i\}_{i=1}^{\dim\mathfrak{g}}$ be an orthonormal basis for $\mathfrak{g}$, $\nabla f=\left(\tilde A_i f\right)_{i=1}^{\dim\mathfrak{g}}$, and $\nabla\cdot U=\sum\tilde A_i U_i$, where $U=(U_i)_{i=1}^{\dim\mathfrak{g}}$ with $U_i\in C^\infty(G)$. Using $\frac{\partial}{\partial t}p_t(x)=\frac{1}{2}\Delta p_t(x)$, and a few integration by parts we find

$$\begin{aligned}
\dot{F}_h(t) &= \frac{1}{2}\int_G f(x)\,h(x)\,\Delta p_t(x)\,dx \\
&= \frac{1}{2}\int_G \Delta(fh)\ p_t\,dV = \frac{1}{2}\int_G (f\Delta h + 2\nabla f\cdot\nabla h + h\Delta f)\ p_t\,dV \\
&= \frac{1}{2}\int_G (f\Delta h + h\Delta f)\ p_t\,dV - \int_G f\ \nabla\cdot[\nabla h\ p_t]\ dV \\
&= \frac{1}{2}\int_G (f\Delta h + h\Delta f)\ p_t\,dV - \int_G f\,[\Delta h\ p_t + \nabla h\cdot\nabla p_t]\ dV \\
&= -\frac{1}{2}\int_G f\Delta h\ d\nu_t - \int_G f\ \nabla h\cdot\frac{\nabla p_t}{p_t}d\nu_t + \frac{1}{2}\int_G h\Delta f\ d\nu_t. \qquad (7.5)
\end{aligned}$$

Therefore,

$$\dot{F}_h(t) - \frac{1}{2}\int_G \Delta f\ d\nu_t = -\frac{1}{2}R_h(t) - S_h(t) + \frac{1}{2}U_h(t),$$

where, making use of Eqs. (7.3) and (1.4), we have

$$\begin{aligned}
|R_h(t)| &\le \int_G |f|\,|\Delta h|\ d\nu_t \le \|f\|_{L^q(\nu_t)}\,\|\Delta h\|_{L^{q'}(\nu_t)} \\
&\le K^2\left(\frac{T}{t}\right)^{d\gamma}\|f\|_{L^q(\nu_T)}\,\|\Delta h\|_{L^{q'}(\nu_T)}, \qquad (7.6)
\end{aligned}$$

$$\begin{aligned}
|S_h(t)| &= \sum_i \int_G |f|\left|\tilde{A}_i h\right|\left|W^t_{A_i}\right| d\nu_t \\
&\le \sum_i \left\|f\cdot\tilde{A}_i h\right\|_{L^q(\nu_t)}\left\|W^t_{A_i}\right\|_{L^{q'}(\nu_t)} \\
&\le C_q\sqrt{\frac{c(kt)}{t}}K\left(\frac{T}{t}\right)^{d\gamma/q}\sum_i\left\|f\cdot\tilde{A}_i h\right\|_{L^q(\nu_T)}. \qquad (7.7)
\end{aligned}$$

and

$$\begin{aligned}
|U_h(t)| &\le \int_G |\Delta f|\,|h-1|\ d\nu_t \le \|\Delta f\|_{L^q(\nu_t)}\,\|1-h\|_{L^{q'}(\nu_t)} \\
&\le K^2\left(\frac{T}{t}\right)^{d\gamma}\|\Delta f\|_{L^q(\nu_T)}\,\|1-h\|_{L^{q'}(\nu_T)}. \qquad (7.8)
\end{aligned}$$

From [**22**, Lemma 3.6], we may choose $\{h_n\}_{n=1}^\infty \subset C_c^\infty(G,[0,1])$ such that $h_n(x) = 1$ whenever $|x| \le n$ and $\sup_n \sup_{x\in G}\left|\left(\tilde{A}_{i_1}\ldots\tilde{A}_{i_k}h_n\right)(x)\right| < \infty$ for all $i_1,\ldots,i_k \in \{1,2,\ldots,\dim\mathfrak{g}\}$ and $k\in\mathbb{N}$. It then follows from Eqs. (7.3), (7.5), (7.6), (7.7), and (7.8) and the dominated convergence theorem that, as $n\to\infty$,

$$\left|\dot{F}_{h_n}(t) - \frac{1}{2}\int_G \Delta f\ d\nu_t\right| \le \frac{1}{2}|R_{h_n}(t)| + |S_{h_n}(t)| + \frac{1}{2}|U_{h_n}(t)| \to 0$$

uniformly on compact subsets of $(0,T)$. Moreover, by the dominated convergence theorem, $F_{h_n}(t) \to F(t)$ as $n \to \infty$ and therefore we may conclude that $\dot{F}(t) = \frac{1}{2}\int_G \Delta f \, d\nu_t$ for $t \in (0,T)$. q.e.d.

7.1. The proof of Proposition 1.8.

Proof. Now suppose, as in Proposition 1.8, $T > 0$, $q > 1$, and $f \in L^q(\nu_T)$ such that $\Delta f = 0$. As in the proof of Lemma 7.1, we may reduce the proof to the case where $y = e$. Let $F(t) := \int_G f d\nu_t$. By Lemma 7.1 and the mean value theorem, $F(T) = F(t)$ for all $t \in (0,T)$ and in particular, $F(T) = \lim_{t\downarrow 0} F(t)$. We are going to finish the proof by showing $\lim_{t\downarrow 0} F(t) = f(e)$. To do this, let $h \in C_c^\infty(G,[0,1])$ be chosen so that $h(x) = 1$ if $|x| \leq 1$. Then

$$F(t) = \int_G f(x)\, h(x)\, p_t(x)\, dx + r(t),$$

where

$$\begin{aligned} |r(t)| &\leq \int_G |f(x)|\,|1-h(x)|\, p_t(x)\, dx \leq \int_{|x|\geq 1} |f(x)|\, p_t(x)\, dx \\ &= \int_{|x|\geq 1} |f(x)| \frac{p_t(x)}{p_T(x)} d\nu_T(x) \leq \sup_{|x|\geq 1} \frac{p_t(x)}{p_T(x)} \|f\|_{L^1(\nu_T)}. \end{aligned} \tag{7.9}$$

Since $\lim_{t\downarrow 0}\int_G f(x)\, h(x)\, p_t(x)\, dx = f(e)\, h(e) = f(e)$, it suffices to show $\lim_{t\downarrow 0}|r(t)| = 0$. To estimate $r(t)$ we will make use of some crude upper and lower bounds on the heat kernel, $p_t(x)$, for example see [**65**, Theorem V.4.4 or Theorem IX.1.2.] for more precise bounds. According to either of these theorems, there exists a constant $c > 0$ such that

$$\begin{aligned} \frac{p_t(x)}{p_T(x)} \leq & \frac{ct^{-d/2}\exp\left(-c|x|^2/t\right)}{c^{-1}T^{-d/2}\exp\left(-c^{-1}|x|^2/T\right)} \\ &= c^2\left(\frac{T}{t}\right)^{d/2}\exp\left(\left(\frac{1}{cT}-\frac{c}{t}\right)|x|^2\right). \end{aligned}$$

From this estimate it follows that $\lim_{t\downarrow 0}\sup_{|x|\geq 1}(p_t(x)/p_T(x)) = 0$ which combined with Eq. (7.9) shows $\lim_{t\downarrow 0}|r(t)| = 0$. q.e.d.

7.2. Applications to infinite–dimensional groups. For this section, suppose that G is a topological group, $\mathcal{B}$ is the Borel σ − algebra over G, and G_0 is a dense subgroup of G which is endowed with the structure of an infinite–dimensional Hilbert Lie group. Further assume that $\mathfrak{g}_0 := \mathrm{Lie}(G_0) = T_e G_0$ is equipped with a Hilbertian inner product, $\langle\cdot,\cdot\rangle_{\mathfrak{g}_0}$. We will also assume that $(G,\mathcal{B})$ is also equipped with a probability measure, ν, to be thought of as the "heat kernel" measure at some time $T > 0$ associated to the given inner product on $\mathfrak{g}_0$. We will now give two theorems which guarantee that ν is quasi-invariant under both left and right translations by elements of G_0. The two cases considered are where G can be thought of as either a projective or inductive limit of finite–dimensional Lie groups.

Theorem 7.2 (Projective Limits)**.** *Suppose that $T > 0$, A is a directed set, $\{G_\alpha\}_{\alpha\in A}$ is a collection of finite dimensional uni-modular Lie groups, and $\{\pi_\alpha : G \to G_\alpha\}_{\alpha\in A}$ is a collection of continuous group homomorphisms satisfying the following properties.*

1) *$\mathcal{B}$ is equal to the σ – algebra generated by the projections, $\{\pi_\alpha\}_{\alpha\in A}$.*
2) *$\pi_\alpha|_{G_0} : G_0 \to G_\alpha$ is a smooth surjection. Let $d\pi_\alpha : \mathfrak{g}_0 \to \mathfrak{g}_\alpha$ be the differential of π_α at e.*
3) *$\nu_\alpha := (\pi_\alpha)_* \nu = \nu \circ \pi_\alpha^{-1}$ is the time T heat kernel measure on G_α determined by the unique inner product. $(\cdot,\cdot)_\alpha$ on $\mathfrak{g}_\alpha$ which makes*
$$d\pi_\alpha|_{\mathrm{Nul}(\pi_\alpha)} : \mathrm{Nul}\,(\pi_\alpha)^\perp \to \mathfrak{g}_\alpha$$
an isometric isomorphism of inner product spaces.
4) *There exists $k \in \mathbb{R}$ such that $\mathrm{Ric}_\alpha \geq k g_\alpha$ for all $\alpha \in A$, where Ric_α is the Ricci tensor on G_α equipped with the left invariant metric determined by $\langle\cdot,\cdot\rangle_\alpha$.*

Under these assumptions, to each $h \in G_0$, $\nu \circ R_h^{-1}$ is absolutely continuous relative to ν. Moreover, if $J_h := d\left(\nu \circ R_h^{-1}\right)/d\nu$ is the Radon-Nikodym derivative of $\nu \circ R_h^{-1}$ with respect to ν and $1 \leq q < \infty$, then

$$\|J_h\|_{L^q(\nu)} \leq \exp\left(\frac{c\,(kT)\,(q-1)}{2T} d^2_{G_0}(e,h)\right), \tag{7.10}$$

where d_{G_0} is the Riemannian distance function on G_0.

Proof. Since the estimate in Eq. (7.10) holds for $q = 1$, we may assume without loss of generality that $1 < q < \infty$. Let $\mathbb{H}$ denote the linear space of bounded measurable functions of the form $f = u \circ \pi_\alpha$, where $\alpha \in A$ and $u : G_\alpha \to \mathbb{R}$ is a bounded measurable function on G_α. Because of assumption 1., $\mathbb{H}$ is dense in $L^q(G,\nu)$. (An easy proof may be given using a functional form of the monotone class theorem, see for example [**38**, Theorem A.1 on p. 309].) By Theorem 1.5 in the form of Eq. (1.7),

$$J_\alpha(x) := \frac{\nu_\alpha\left(dx \cdot \pi_\alpha\left(h^{-1}\right)\right)}{\nu_\alpha(dx)} \text{ for } x \in G_\alpha,$$

satisfies

$$\|J_\alpha\|_{L^q(G_\alpha,\nu_\alpha)} \leq \exp\left(\frac{c\,(kT)\,(q-1)}{2T} d^2_{G_\alpha}(e,\pi_\alpha(h))\right) \text{ for all } 1 < q < \infty.$$

Using this result and assumption 3, if $f = u \circ \pi_\alpha \in \mathbb{H}$, then

$$\begin{aligned}
\int_G |f(xh)|\, d\nu(x) &= \int_G |u \circ \pi_\alpha(xh)|\, d\nu(x) \\
&= \int_G |u(\pi_\alpha(x)\pi_\alpha(h))|\, d\nu(x) \\
&= \int_{G_\alpha} |u(y \cdot \pi_\alpha(h))|\, d\nu_\alpha(y) \\
&= \int_{G_\alpha} |u(y)|\, J_\alpha(y)\, d\nu_\alpha(y).
\end{aligned}$$

An application of Hölder's inequality then implies,

$$\begin{aligned}
\int_G |f(xh)|\, d\nu(x) &\leq \|u\|_{L^q(G_\alpha,\nu_\alpha)} \cdot \|J_\alpha\|_{L^{q'}(G_\alpha,\nu_\alpha)} \\
&\leq \|f\|_{L^q(G,\nu)} \exp\left(\frac{c(kT)(q'-1)}{2T} d^2_{G_\alpha}(e, \pi_\alpha(h))\right). \qquad (7.11)
\end{aligned}$$

Now suppose that $k \in C^1([0,1], G_0)$ such that $k(0) = e$ and $k(1) = h$. Then the length of $t \to \pi_\alpha(k(t)) \in G_\alpha$ is given by

$$\ell_{G_\alpha}(\pi_\alpha \circ k) = \int_0^1 \left| L_{\pi_\alpha(k(t))^{-1}*}\pi_\alpha\left(\dot{k}(t)\right)\right|_{\mathfrak{g}_\alpha} dt.$$

Since

$$\begin{aligned}
L_{\pi_\alpha(k(t))^{-1}*}\pi_\alpha\left(\dot{k}(t)\right) &= \frac{d}{ds}|_0 \pi_\alpha(k(t))^{-1}\pi_\alpha(k(t+s)) \\
&= \frac{d}{ds}|_0 \pi_\alpha\left(k(t)^{-1}k(t+s)\right) = d\pi_\alpha\left(L_{k(t)^{-1}*}\dot{k}(t)\right)
\end{aligned}$$

and

$$\left| L_{\pi_\alpha(k(t))^{-1}*}\pi_\alpha\left(\dot{k}(t)\right)\right|_{\mathfrak{g}_\alpha} = \left| d\pi_\alpha\left(L_{k(t)^{-1}*}\dot{k}(t)\right)\right|_{\mathfrak{g}_\alpha} \leq \left| L_{k(t)^{-1}*}\dot{k}(t)\right|_{\mathfrak{g}_0},$$

it follows that

$$d_{G_\alpha}(e, \pi_\alpha(h)) \leq \ell_{G_\alpha}(\pi_\alpha \circ k) \leq \int_0^1 \left| L_{k(t)^{-1}*}\dot{k}(t)\right|_{\mathfrak{g}_0} dt = \ell_{G_0}(k).$$

Taking the infimum over all such k implies

$$d_{G_\alpha}(e, \pi_\alpha(h)) \leq d_{G_0}(e, h).$$

Combining this inequality with Eq. (7.11) gives the estimate,

$$\int_G |f(xh)|\, d\nu(x) \leq \|f\|_{L^q(G,\nu)} \exp\left(\frac{c(kT)(q'-1)}{2T} d^2_{G_0}(e, h)\right). \qquad (7.12)$$

The afore mentioned density of $\mathbb{H}$ in $L^q(G, \nu)$ along with Eq. (7.12) shows the linear functional $\varphi : \mathbb{H} \to \mathbb{R}$, defined by

$$\varphi_h(f) := \int_G f(xh)\, d\nu(x),$$

extends uniquely to a continuous linear functional, $\bar{\varphi}_h$, on $L^q(G,\nu)$ satisfying

$$|\bar{\varphi}_h(f)| \le \|f\|_{L^q(G,\nu)} \exp\left(\frac{c(kT)(q'-1)}{2T} d_{G_0}^2(e,h)\right) \ \forall\ f \in L^q(G,\nu).$$

Since $L^q(G,\nu)^* \cong L^{q'}(G,\nu)$, there exists $J_h \in L^{q'}(G,\nu)$ such that

$$\|J_h\|_{L^{q'}(G,\nu)} \le \exp\left(\frac{c(kT)(q'-1)}{2T} d_{G_0}^2(e,h)\right)$$

and

$$\bar{\varphi}_h(f) = \int_G f(x) J_h(x)\, d\nu(x) \quad \text{for all} \quad f \in L^q(G,\nu).$$

Restricting this formula to $\mathbb{H}$ shows,

$$\begin{aligned} \int_G f(x)\, \nu\left(dxh^{-1}\right) &= \int_G f(xh)\, d\nu(x) = \bar{\varphi}_h(f) \\ &= \int_G f(x) J_h(x)\, d\nu(x) \quad \text{for all} \quad f \in \mathbb{H}. \end{aligned} \tag{7.13}$$

Another monotone class argument (again use [**38**, Theorem A.1 on p. 309]) shows that Eq. (7.13) remains valid for all bounded measurable functions, $f : G \to \mathbb{R}$. Therefore, we have shown that $J_h := d\nu \circ R_h^{-1}/d\nu$ exists and satisfies the bound in Eq. (7.10). q.e.d.

We now turn to the inductive limit quasi-invariance theorem. The following result is an abstraction of the quasi-invariance result in [**18**]. For related results of this type see, Fang [**25**] and Airault and Malliavin [**1**].

Theorem 7.3 (Inductive Limits)**.** *Again, let $T > 0$, $G_0 \subset G$, and $(G, \mathcal{B}, \nu)$ be as described at the start of this section. Further assume there exists, $\{G_\alpha\}_{\alpha\in A}$, where A is a directed set and for each $\alpha \in A$, G_α is a finite dimensional uni-modular Lie subgroup of G_0 such that $G_\alpha \subset G_\beta$ if $\alpha < \beta$. Let $i_\alpha : G_\alpha \to G_0$ denote the smooth injection map. The following properties are assumed to hold.*

1) *$\cup_{\alpha\in A} G_\alpha$ is a dense subgroup of G_0.*
2) *For all $f \in BC(G,\mathbb{R})$ (the bounded continuous maps from G to $\mathbb{R}$),*
$$\int_G f d\nu = \lim_{\alpha\to\infty} \int_{G_\alpha} (f \circ i_\alpha)\, d\nu_\alpha,$$
where ν_α is the time, T, heat kernel measure on G_α associated to inner product, $(\cdot,\cdot)_{\mathfrak{g}_\alpha}$, defined to be the restriction of $(\cdot,\cdot)_{\mathfrak{g}_0}$ to $\mathfrak{g}_\alpha \times \mathfrak{g}_\alpha$.
3) *There exists $k \in \mathbb{R}$ such that $\mathrm{Ric}_\alpha \ge k g_\alpha$ for all $\alpha \in A$, where Ric_α and g_α are the left invariant Ricci and the metric tensors on G_α induced by $(\cdot,\cdot)_{\mathfrak{g}_\alpha}$.*

4) *For each $\alpha \in A$, there exits a smooth section, $s_\alpha : G_0 \to G_\alpha$ (i.e. $s_\alpha \circ i_\alpha = id_{G_\alpha}$) satisfying the following property. Given $\alpha_0 \in A$, and $k \in C^1([0,1], G_0)$ with $k(0) = e$, there exists an increasing sequence, $\{\alpha_n\}_{n=1}^{\infty} \subset A$ (i.e. $\alpha_0 < \alpha_1 < \alpha_2 < \dots$), such that*

$$\ell_{G_0}(k(\cdot)) = \lim_{n\to\infty} \ell_{G_{\alpha_n}}(s_{\alpha_n} \circ k). \tag{7.14}$$

*(We do **not** assume that $s_\alpha : G_0 \to G_\alpha$ is a homomorphism.)*

Under these assumptions, to each $h \in G_0$, $\nu \circ R_h^{-1}$ is absolutely continuous relative to ν and the Moreover, the Radon-Nikodym derivative, $J_h := d(\nu \circ R_h^{-1})/d\nu$, again satisfies the bounds in Eq. (7.10).

Proof. As in the proof of Theorem 7.2 it suffices to assume $q \in (1, \infty)$ throughout the proof. Let $\alpha_0 \in A$, $h \in G_{\alpha_0}$, and $\alpha_0 < \alpha_1 < \alpha_2 < \dots < \alpha_n < \dots$ be as in item 4. above. By Theorem 1.5 in the form of Eq. (1.7), the Radon-Nikodym derivative, $J_{\alpha_n}(x)$, of $\nu_{\alpha_n}\left(dx \cdot s_{\alpha_n}(h)^{-1}\right) = \nu_{\alpha_n}(dx \cdot h^{-1})$ relative to $\nu_{\alpha_n}(dx)$ satisfies the estimate,

$$\begin{aligned}\|J_{\alpha_n}\|_{L^{q'}(G_{\alpha_n},\nu_{\alpha_n})} &\leq \exp\left(\frac{c(kT)(q'-1)}{2T} d^2_{G_{\alpha_n}}(e, h^{-1})\right) \\ &= \exp\left(\frac{c(kT)(q'-1)}{2T} d^2_{G_{\alpha_n}}(e, h)\right) \\ &\leq \exp\left(\frac{c(kT)(q'-1)}{2T} \ell^2_{G_{\alpha_n}}(s_{\alpha_n} \circ \sigma)\right),\end{aligned}$$

where σ is any path in $C^1([0,1], G_0)$ such that $\sigma(0) = e$ and $\sigma(1) = h$. Assuming the $f \in BC(G)$, by the definition of J_{α_n} and Hölder's inequality,

$$\begin{aligned}\int_{G_{\alpha_n}} |f(x \cdot h)|\, d\nu_{\alpha_n}(x) &= \int_{G_{\alpha_n}} J_{\alpha_n}(x) |f(x)|\, d\nu_{\alpha_n}(x) \\ &\leq \|f\|_{L^q(G_{\alpha_n},\nu_{\alpha_n})} \cdot \exp\left(\frac{c(kT)(q'-1)}{2T} \ell^2_{G_{\alpha_n}}(s_{\alpha_n} \circ \sigma)\right).\end{aligned}$$

Using the assumptions in items 2. and 4. of the theorem, we may pass to the limit $(n \to \infty)$ in this inequality to find,

$$\int_G |f(x \cdot h)|\, d\nu(x) \leq \|f\|_{L^q(G,\nu)} \cdot \exp\left(\frac{c(kT)(q'-1)}{2T} \ell^2_{G_0}(\sigma)\right). \tag{7.15}$$

Optimizing this inequality over $\sigma \in C^1([0,1], G_0)$ joining e to h gives

$$\int_G |f(x \cdot h)|\, d\nu(x) \leq \|f\|_{L^q(G,\nu)} \cdot \exp\left(\frac{c(kT)(q'-1)}{2T} d^2_{G_0}(e, h)\right). \tag{7.16}$$

Up to now we have verified Eq. (7.16) for any $h \in \cup_{\alpha\in A} G_\alpha$. As the latter set is dense in G_0, the dominated convergence theorem along with the continuity of $d^2_{G_0}(e, h)$ in h allows us to conclude that the estimate in

Eq. (7.16) is valid for all $h \in G_0$. Since $BC(G,\mathbb{R})$ is dense in $L^q(G,\nu)$ (again use [**38**, Theorem A.1 on p. 309]) and because of Eq. (7.16), the linear functional, $\varphi_h : BC(G) \to \mathbb{R}$ defined by

$$\varphi_h(f) = \int_G f(xh)\, d\nu(x), \tag{7.17}$$

has a unique extension to an element, $\bar{\varphi}_h$, of $L^q(G,\nu)^*$ satisfying

$$|\bar{\varphi}_h(f)| \le \|f\|_{L^q(G,\nu)} \cdot \exp\left(\frac{c(kT)(q'-1)}{2T} d_{G_0}^2(e,h)\right) \quad \forall\ f \in L^q(G,\nu). \tag{7.18}$$

As in the latter part of the proof of Theorem 7.2, the estimate in Eq. (7.18) implies the existence of a function, $J_h \in L^{q'}(G,\nu)$, such that

$$\bar{\varphi}_h(f) = \int_G f(x)\, J_h(x)\, d\nu(x) \tag{7.19}$$

and

$$\|J_h\|_{L^{q'}(G,\nu)} \le \exp\left(\frac{c(kT)(q'-1)}{2T} d_{G_0}^2(e,h)\right).$$

Furthermore, from Eqs. (7.17) and (7.19) it follows that

$$\int_G f(xh)\, d\nu(x) = \int_G f(x)\, J_h(x)\, d\nu(x) \text{ for all } f \in BC(G). \tag{7.20}$$

Another monotone class argument [**38**, Theorem A.1 on p. 309] then shows Eq. (7.20) is valid for all bounded measurable functions, $f : G \to \mathbb{R}$. Hence $\nu\left(dxh^{-1}\right) = J_h(x)\,\nu(dx)$ and $J_h(x)$ satisfies the estimate in Eq. (7.10). q.e.d.

Corollary 7.4. *Under the hypothesis of either Theorem* 7.2 *or* 7.3*, the heat kernel measure, ν, is quasi-invariant under left translations by elements of $h \in G_0$. Moreover, the Radon-Nikodym derivative, $J_h^l := d\left(\nu \circ L_h^{-1}\right)/d\nu$ satisfies the same bound as $d\left(\nu \circ R_h^{-1}\right)/d\nu$ which is given in Eq.* (7.10)*.*

Proof. Since the heat kernel measures $\{\nu_\alpha\}_{\alpha \in A}$ on the Lie groups, $\{G_\alpha\}_{\alpha \in A}$, are invariant under inversion, $x \to x^{-1}$, it follows that ν also inherits this property. Hence if $f : G \to \mathbb{R}$ is a bounded measurable function, then

$$\begin{aligned}\int_G f(hx)\, d\nu(x) &= \int_G f\left(hx^{-1}\right) d\nu(x) = \int_G f\left(\left(xh^{-1}\right)^{-1}\right) d\nu(x) \\ &= \int_G f\left(x^{-1}\right) J_{h^{-1}}(x)\, d\nu(x) = \int_G f(x)\, J_{h^{-1}}\left(x^{-1}\right) d\nu(x),\end{aligned}$$

from which it follows that $J_h^l(x) = J_{h^{-1}}\left(x^{-1}\right)$ for ν – a.e. x. Therefore,

$$\left\|J_h^l\right\|_{L^q(\nu)} = \|J_{h^{-1}}\|_{L^q(\nu)} \le \exp\left(\frac{c(kT)(q-1)}{2T} d_{G_0}^2\left(e,h^{-1}\right)\right)$$

which completes the proof since $d^2_{G_0}\left(e,h^{-1}\right) = d^2_{G_0}\left(h,e\right) = d^2_{G_0}\left(e,h\right).$
q.e.d.

See Driver [**18**] for an explicit application of the projective limit Theorem 7.2 in the setting of loop groups and see Driver and Gordina [**21**] for an application of the inductive limit Theorem 7.3 to an infinite dimensional Heisenberg group setting.

Appendix A. A commutator theorem

In this section we will develop the abstract functional analytic results which were used in the proofs of Theorems 4.1 and 4.3. Results similar to the next theorem may be found in Brüning and Lesch [**7**], Xue-Mei Li [**44**, **45**] and in Bueler [**8**].

Theorem A.1. *Let W, X, and Y be Hilbert spaces and $A : W \to X$ and $B : X \to Y$ be densely defined closed (unbounded) operators such that* $\mathrm{Ran}(A) \subset \mathrm{Nul}(B)$. *Let $Q : X \to W \oplus Y$ be the unbounded linear operator defined by: $\mathcal{D}(Q) = \mathcal{D}(A^*) \cap \mathcal{D}(B)$ and for $x \in \mathcal{D}(Q)$, $Qx := (A^*x, Bx)$. Let us also define $R : W \oplus Y \to X$ by $\mathcal{D}(R) = \mathcal{D}(A) \oplus \mathcal{D}(B^*)$ and $R(w,y) := Aw + B^*y$. Then*

1) $\mathrm{Ran}(A)$ and $\mathrm{Ran}(B^*)$ are orthogonal.
2) R is closed.
3) $Q = R^*$ is a closed densely defined operator.
4) The operator, $L := AA^* + B^*B$, on X is densely defined, non-negative, and self adjoint operator. Moreover, $L := Q^*Q$.

Proof. We will denote all of the inner products on these Hilbert spaces by $\langle \cdot, \cdot \rangle$. Let $w \in \mathcal{D}(A)$ and $y \in \mathcal{D}(B^*)$. Since $\mathrm{Ran}(A) \subset \mathrm{Nul}(B)$, $0 = \langle BAw, y \rangle = \langle Aw, B^*y \rangle$ which proves item 1. For item 2., suppose that $(w_n, y_n) \in \mathcal{D}(R)$ are such that there exists $(w,y) \in W \oplus Y$ and $x \in X$ such that

$$(w_n, y_n) \to (w,y) \text{ as } n \to \infty \quad \text{and}$$
$$R(w_n, y_n) \to x \text{ as } n \to \infty.$$

We must show that $w \in \mathcal{D}(A)$, $y \in \mathcal{D}(B^*)$ and that $x = Aw + B^*y$. We are given that $Aw_n + B^*y_n \to x$ as $n \to \infty$. But by the first item and the Cauchy criteria, this implies that both $\lim_{n\to\infty} Aw_n$ and $\lim_{n\to\infty} B^*y_n$ exist. Because both A and B^* are closed, this implies that $w \in \mathcal{D}(A)$, $y \in \mathcal{D}(B^*)$ and that

$$Aw + B^*y = \lim_{n\to\infty} Aw_n + \lim_{n\to\infty} B^*y_n = \lim_{n\to\infty}\left(Aw_n + B^*y_n\right).$$

Hence we have proved item 2. To check item 3, note that as R is closed, it follows that R^* is densely defined. Therefore we need only show that $R^* = Q$. For this, let us recall that $x \in \mathcal{D}(R^*)$ and $R^*x =$

(w, y) iff $\langle (w, y), (w', y') \rangle = \langle x, R(w', y') \rangle$ for all $(w', y') \in \mathcal{D}(R)$. This is equivalent to the following statements:

- $\langle w, w' \rangle + \langle y, y' \rangle = \langle x, Aw' + B^* y' \rangle$ for all $w' \in \mathcal{D}(A)$ and $y' \in \mathcal{D}(B^*)$.
- $\langle w, w' \rangle = \langle x, Aw' \rangle$ and $\langle y, y' \rangle = \langle x, B^* y' \rangle$ for all $w' \in \mathcal{D}(A)$ and $y' \in \mathcal{D}(B^*)$.
- $x \in \mathcal{D}(A^*)$, $x \in \mathcal{D}(B^{**}) = \mathcal{D}(B)$, $A^* x = w$ and $Bx = B^{**} x = y$.
- $x \in \mathcal{D}(Q)$ and $Qx = (w, y)$.

Thus we have proved item 3. of the theorem. Item 4. By a Theorem of Von-Neumann, [**52**, Theorem X.25], $Q^* Q$ is a non-negative densely defined self adjoint operator on X. So it suffices to show that $Q^* Q = AA^* + B^* B$. By items 2. and 3., $Q^* = R^{**} = R$. Therefore, $Q^* Q = RQ$. Now the following are equivalent:

- $x \in \mathcal{D}(RQ)$ and $RQx = x'$.
- $x \in \mathcal{D}(A^*) \cap \mathcal{D}(B)$, $Qx := (A^* x, Bx) \in \mathcal{D}(R)$, and $R(A^* x, Bx) = x'$.
- $x \in \mathcal{D}(A^*) \cap \mathcal{D}(B)$, $A^* x \in \mathcal{D}(A)$, $Bx \in \mathcal{D}(B^*)$ and $AA^* x + B^* Bx = x'$.
- $x \in \mathcal{D}(AA^*) \cap \mathcal{D}(B^* B)$ and $AA^* x + B^* Bx = x'$.
- $x \in \mathcal{D}(AA^* + B^* B)$ and $AA^* x + B^* Bx = x'$.

This shows $Q^* Q = AA^* + B^* B$ and thus completes the proof.

q.e.d.

Theorem A.2 (Commutator Theorem)**.** *Let W, X, Y, and Z be Hilbert spaces and $A : W \to X$, $B : X \to Y$, and $C : Y \to Z$ be densely defined closed (unbounded) operators such that* $\mathrm{Ran}(A) \subset \mathrm{Nul}(B)$ *and* $\mathrm{Ran}(B) \subset \mathrm{Nul}(C)$. *Let $L := AA^* + B^* B$ and $S := BB^* + C^* C$. Then $Be^{-tL} x = e^{-tS} Bx$ for all $x \in \mathcal{D}(B)$ and any $t \geq 0$.*

Proof. Let $\lambda > 0$. Observe that $BL = BB^* B$ on $\mathcal{D}(BL) = \mathcal{D}(AA^*) \cap \mathcal{D}(BB^* B)$ and that $SB = BB^* B$ on $\mathcal{D}(SB) = \mathcal{D}(BB^* B)$. In particular we have shown

$$SB = BB^* B = BL \text{ on } \mathcal{D}(BL) = \mathcal{D}(AA^*) \cap \mathcal{D}(BB^* B) \tag{A.1}$$

and hence,

$$(1 + \lambda S) B = B (1 + \lambda L) \text{ on } \mathcal{D}(BL). \tag{A.2}$$

If $x \in D(B)$ and $x' := (1 + \lambda L)^{-1} x$, then $x' \in D(L) \subset D(B)$ and

$$Lx' = (1 + \lambda L) x' - \lambda x' = x - \lambda x' \in D(B).$$

Therefore $x' \in D(BL)$ and so by Eq. (A.2) applied to $x' = (1 + \lambda L)^{-1} x$ we discover that,

$$(1 + \lambda S) B (1 + \lambda L)^{-1} x = B (1 + \lambda L) (1 + \lambda L)^{-1} x = Bx.$$

Applying $(1 + \lambda S)^{-1}$ to both sides of this equation shows

$$B(1 + \lambda L)^{-1} = (1 + \lambda S)^{-1} B \text{ on } D(B). \tag{A.3}$$

Multiplying Eq. (A.3) on the right by $(1+\lambda L)^{-1}$ gives

$$B(1+\lambda L)^{-2} = (1+\lambda S)^{-1} B\,(1+\lambda L)^{-1} = (1+\lambda S)^{-2} B \text{ on } D\,(B)\,,$$

wherein we have used Eq. (A.3) again in the second equality. Continuing this way inductively allows us to conclude.

$$B(1+\lambda L)^{-n} = (1+\lambda S)^{-n} B \text{ on } D\,(B) \text{ for all } n \in \mathbb{N}. \tag{A.4}$$

To complete the proof the theorem recall $e^{-tL} = s - \lim_{n\to\infty} \left(1+\frac{t}{n}L\right)^{-n}$ and that $e^{-tS} = s - \lim_{n\to\infty} \left(1+\frac{t}{n}S\right)^{-n}$. Hence, taking $\lambda = t/n$ in Eq. (A.4) and then passing to the limit allows us to conclude

$$\lim_{n\to\infty} B\left(1+\frac{t}{n}L\right)^{-n} x = \lim_{n\to\infty}\left(1+\frac{t}{n}S\right)^{-n} Bx = e^{-tS}Bx$$

for all $x \in D\,(B)$.

Since B is closed, it follows that, for all $x \in D\,(B)$, that

$$e^{-tL}x = \lim_{n\to\infty}\left(1+\frac{t}{n}L\right)^{-n} x \in D\,(B)$$

and

$$Be^{-tL}x = \lim_{n\to\infty} B\left(1+\frac{t}{n}L\right)^{-n} x = e^{-tS}Bx.$$

q.e.d.

Appendix B. A Kato type inequality

Let E be a real Euclidean vector bundle over a Riemannian manifold, M, $\Gamma^\infty(E)$ $(\Gamma_c^\infty(E))$ be the smooth (compactly supported) sections of E, and $\mathcal{H} := L^2(E)$ be the space of square integrable sections of E. Further assume that E is equipped with a metric compatible connection, ∇^E, and that $\Box = \Box^E$ is the associated Bochner Laplacian on $\Gamma^\infty(E)$. To be more explicit, if $\{e_i\}_{i=1}^{\operatorname{rank}(E)}$ is a local orthonormal frame, then

$$\Box f = \operatorname{tr}\left(\nabla^{T^*M\otimes E}\nabla^E f\right) = \sum_i \left(\nabla^E_{e_i}\nabla^E_{e_i} f - \nabla^E_{\nabla^{TM}_{e_i}e_i} f\right).$$

To simplify notation in the computations below, we will drop the superscripts, E and TM from the symbols since they can be deduced from the context.

Notation B.1. Given a measurable section, $e : M \to E$, and $f \in \mathcal{H}$, let

$$\operatorname{sgn}_e(f) := 1_{f\neq 0}\frac{f}{|f|} + 1_{f=0}\,e = \begin{cases} \frac{f}{|f|} & \text{if } f \neq 0 \\ e & \text{if } f = 0 \end{cases}.$$

With this notation we have the polar decomposition, $f = |f| \operatorname{sgn}_e(f)$, which is valid no matter what the choice of e.

Theorem B.2 (Kato's Inequality)**.** *Let* $\varepsilon > 0$, $f \in \Gamma^\infty(E)$, $|f|_\varepsilon := \sqrt{|f|^2 + \varepsilon^2}$, *and* $\hat{f}_\varepsilon := f/|f|_\varepsilon$. *Then*

$$d|f|_\varepsilon = \left\langle \hat{f}_\varepsilon, \nabla f \right\rangle \text{ and}$$

$$\Delta_0 |f|_\varepsilon = \frac{1}{|f|_\varepsilon} \sum_i \left(|\nabla_{e_i} f|^2 - \left| \left\langle \hat{f}_\varepsilon, \nabla_{e_i} f \right\rangle \right|^2 \right) + \left\langle \hat{f}_\varepsilon, \square f \right\rangle \tag{B.1}$$

$$\geq \left\langle \hat{f}_\varepsilon, \square f \right\rangle. \tag{B.2}$$

Moreover if $\varphi \in C^\infty(M)_+$ *and* $f \in C_c^\infty(E)$, *then*

$$(\square f, \varphi \operatorname{sgn}_e(f)) \leq (|f|, \Delta_0 \varphi), \tag{B.3}$$

where e *is any measurable section of* E *such that* $\langle \square f(x), e(x) \rangle_x = 0$ *and* $|e(x)|_x = 1$ *on the set where* $f = 0$.

Proof. This theorem is mostly a straightforward computation. (See [**35**], where a local coordinate version of this calculation is done.) We start by computing the gradient of $|f|_\varepsilon$ as

$$d|f|_\varepsilon = \frac{1}{2\sqrt{|f|^2 + \varepsilon^2}} d|f|^2 = \frac{1}{\sqrt{|f|^2 + \varepsilon^2}} \langle f, \nabla_\cdot f \rangle.$$

With this in hand we have the following formula for the Hessian of $|f|_\varepsilon$;

$$\begin{aligned} \nabla d|f|_\varepsilon = &- \left(|f|^2 + \varepsilon^2\right)^{-3/2} \langle f, \nabla_\cdot f \rangle^2 \\ &+ \frac{1}{\sqrt{|f|^2 + \varepsilon^2}} \left(\langle \nabla_\cdot f, \nabla_\cdot f \rangle + \left\langle f, \nabla^2_{(\cdot,\cdot)} f \right\rangle \right). \end{aligned}$$

Taking the trace of this result gives

$$\begin{aligned} \Delta_0 |f|_\varepsilon = &- \left(|f|^2 + \varepsilon^2\right)^{-3/2} \sum_i |\langle f, \nabla_{e_i} f \rangle|^2 \\ &+ \frac{1}{\sqrt{|f|^2 + \varepsilon^2}} \left(\sum_i |\nabla_{e_i} f|^2 + \langle f, \square f \rangle \right) \end{aligned}$$

which is equivalent to Eq. (B.1). Equation (B.2) follows by the Cauchy-Schwarz inequality which implies

$$|\nabla_{e_i} f|^2 - \left| \left\langle \hat{f}_\varepsilon, \nabla_{e_i} f \right\rangle \right|^2 \geq |\nabla_{e_i} f|^2 - \left| \hat{f}_\varepsilon \right|^2 \cdot |\nabla_{e_i} f|^2 \geq 0.$$

If we now assume that $f \in \Gamma_c^\infty(E)$ and $\varphi \in C^\infty(M, [0, \infty))$, then multiplying Eq. (B.2) by φ and integrating gives,

$$\int_M \left\langle \square f, \frac{f}{|f|_\varepsilon} \right\rangle \varphi dV \leq \int_M \Delta_0 |f|_\varepsilon \varphi dV = \int_M |f|_\varepsilon \Delta_0 \varphi dV \tag{B.4}$$

where we have done two integrations by parts to get the last equality. Letting $\varepsilon \downarrow 0$ in Eq. (B.4) then implies

$$\int_M \langle \Box f, \operatorname{sgn}_0(f) \rangle \varphi dV \le \int_M |f| \, \Delta_0 \varphi dV \tag{B.5}$$

which is to say

$$\langle \Box f, \operatorname{sgn}_0(f) \rangle \le \Delta_0 |f| \ \text{ (in the distributional sense).} \tag{B.6}$$

If we now choose e to be a measurable section of E such that $|e| = 1$ and $\langle \Box f, e \rangle = 0$, then $\langle \Box f, \operatorname{sgn}_0(f) \rangle = \langle \Box f, \operatorname{sgn}_e(f) \rangle$ and we may rewrite Eqs. (B.5) and (B.6) as,

$$\int_M \langle \Box f, \operatorname{sgn}_e(f) \rangle \varphi dV \le \int_M |f| \, \Delta_0 \varphi dV$$

and

$$\langle \Box f, \operatorname{sgn}_e(f) \rangle \le \Delta_0 |f| \ \text{ (in the distributional sense).}$$

These last two equations are equivalent to Eq. (B.3). q.e.d.

Appendix C. A local martingale

In this appendix we will continue to use the notation in Section 5.1 unless otherwise stated.

Lemma C.1 (Local martingale lemma)**.** *Let $\tilde{\ell}_t \in \mathbb{R}$ be an adapted continuously differentiable real valued process, $\ell_0 \in T_x M$,*

$$\ell_t = Q_t \left[\int_0^t Q_\tau^{-1} \left(\frac{d}{d\tau} \tilde{\ell}_\tau \right) db_\tau + \ell_0 \right], \tag{C.1}$$

$a \in \Omega_c^1(M)$, and

$$Z_t := (a_t(\Sigma_t) \circ //_t) \, \ell_t - \delta a_t(\Sigma_t) \, \tilde{\ell}_t, \tag{C.2}$$

be as in Eq. (5.10). Then Z_t is a local martingale whose Itô differential is given by

$$\begin{aligned} dZ_t = \left(\nabla_{//_t db_t} a_t \right)(\Sigma_t) \circ //_t \ell_t + (a_t(\Sigma_t) \circ //_t) \left(\frac{d}{dt} \tilde{\ell}_t \right) db_t \\ - \left(\nabla_{//_t db_t} a_t \right)(\Sigma_t) \, \tilde{\ell}_t. \end{aligned} \tag{C.3}$$

Proof. The proof of this lemma is purely a computation. For the sake of the reader's understanding we will give a slightly inefficient proof designed to motivate the form of Z_t in Eq. (5.10). Let a_t be as in Eq. (5.9) and then set

$$N_t := a_t(\Sigma_t) \circ //_t.$$

Then by Itô's lemma in Eq. (5.2), Theorem 4.1, and Bochner identity in Eq. (4.3), we find

$$\begin{aligned} dN_t &= \left(\nabla_{//_t db_t} a_t\right)(\Sigma_t) \circ //_t + \frac{1}{2}\left((\square - \Delta)\, a_t(\Sigma_t)\right) \circ //_t dt \\ &= \left(\nabla_{//_t db_t} a_t\right)(\Sigma_t) \circ //_t + \frac{1}{2}\left[a_t(\Sigma_t) \circ \mathrm{Ric} \circ //_t\right] dt. \end{aligned} \tag{C.4}$$

Also by Itô's lemma in Eq. (5.1) and item 4. of Theorem 4.1,

$$\begin{aligned} d\left[\delta a_t(\Sigma_t)\right] &= d\left[\left(e^{(T-t)\bar{\Delta}_0/2}\delta a\right)(\Sigma_t)\right] \\ &= \left(\nabla_{//_t db_t}\left[e^{(T-t)\bar{\Delta}_0/2}\delta a\right]\right)(\Sigma_t) = \left(\nabla_{//_t db_t}\left[\delta a_t\right]\right)(\Sigma_t). \end{aligned} \tag{C.5}$$

Now suppose $\ell_t \in T_xM$ and $\tilde{\ell}_t \in \mathbb{R}$ are arbitrary continuous Brownian semi-martingales such that

$$d\ell_t = \alpha_t\, db_t + \beta_t\, dt \text{ and } d\tilde{\ell}_t = \tilde{\alpha}_t\, db_t + \tilde{\beta}_t\, dt$$

with α_t, β_t, $\tilde{\alpha}_t$, and $\tilde{\beta}_t$ being continuous adapted processes with values in $\mathrm{End}(T_xM)$, T_xM, T_xM^*, and $\mathbb{R}$ respectively and let

$$Z_t = N_t\ell_t - (\delta a_t)(\Sigma_t)\,\tilde{\ell}_t. \tag{C.6}$$

Making use of Eqs. (C.4) and (C.5), the Itô differential of Z in Eq. (C.6) is,

$$\begin{aligned} dZ_t &= \left(\nabla_{//_t db_t} a_t\right)(\Sigma_t) \circ //_t\ell_t + \frac{1}{2}\left[a_t(\Sigma_t) \circ \mathrm{Ric} \circ //_t\ell_t\right] dt \\ &\quad + \left(a_t(\Sigma_t) \circ //_t\right)\left[\alpha_t db_t + \beta_t\, dt\right] + \left(\nabla_{//_t e_i} a_t\right)(\Sigma_t) \circ //_t\alpha_t e_i dt \\ &\quad - \left(\nabla_{//_t db_t}\left[\delta a_t\right]\right)(\Sigma_t)\,\tilde{\ell}_t - \delta a_t(\Sigma_t)\left[\tilde{\alpha}_t db_t + \tilde{\beta}_t dt\right] \\ &\quad - \left(\nabla_{//_t e_i}\left[\delta a_t\right]\right)\tilde{\alpha}_t e_i dt \\ &= \left(\nabla_{//_t db_t} a_t\right)(\Sigma_t) \circ //_t\ell_t + \left(a_t(\Sigma_t) \circ //_t\right)\alpha_t db_t \\ &\quad - \left(\nabla_{//_t db_t}\left[\delta a_t\right]\right)(\Sigma_t)\,\tilde{\ell}_t - \delta a_t(\Sigma_t)\,\tilde{\alpha}_t db_t \\ &\quad + \begin{pmatrix} \frac{1}{2}\left[a_t(\Sigma_t) \circ \mathrm{Ric} \circ //_t\ell_t\right] + \left(a_t(\Sigma_t) \circ //_t\right)\beta_t \\ + \left(\nabla_{//_t e_i} a_t\right)(\Sigma_t) \circ //_t\alpha_t e_i - \delta a_t(\Sigma_t)\,\tilde{\beta}_t - \left(\nabla_{//_t e_i}\left[\delta a_t\right]\right)\tilde{\alpha}_t e_i \end{pmatrix} dt. \end{aligned} \tag{C.7}$$

Our goal is to choose α_t, β_t, $\tilde{\alpha}_t$, and $\tilde{\beta}_t$ in such as way that Z_t is a local martingale. To do this we need to make the term in the parenthesis in Eq. (C.7) vanish. Grouping the terms according to the number of derivatives on a_t, the term in parenthesis in Eq. (C.7) will vanish provided

$$\frac{1}{2}\left[a_t\left(\Sigma_t\right)\circ \operatorname{Ric}\circ //_t\ell_t\right]+\left(a_t\left(\Sigma_t\right)\circ //_t\right)\beta_t=0,$$
$$\left(\nabla_{//_te_i}a_t\right)\left(\Sigma_t\right)\circ //_t\alpha_te_i-\delta a_t\left(\Sigma_t\right)\tilde{\beta}_t=0,$$
$$\text{and}\qquad \left(\nabla_{//_te_i}\left[\delta a_t\right]\right)\tilde{\alpha}_te_i=0.$$

Moreover because of Eq. (4.1), these equations may be satisfied by choosing $\tilde{\alpha}\equiv 0$ (so that $\tilde{\ell}_t$ is differentiable and $\frac{d\tilde{\ell}_t}{dt}=\tilde{\beta}_t$),

$$\beta_t=-\frac{1}{2}//_t^{-1}\operatorname{Ric}//_t\ell_t=:-\frac{1}{2}\operatorname{Ric}^{//_t}\ell_t,$$

and

$$\alpha_t=\tilde{\beta}_tI_{T_xM}=\frac{d\tilde{\ell}_t}{dt}I_{T_xM}.$$

Thus we have shown,

$$Z_t:=\left(a_t\left(\Sigma_t\right)\circ //_t\right)\ell_t-\delta a_t\left(\Sigma_t\right)\tilde{\ell}_t,$$

is a local martingale provided $\tilde{\ell}_t$ is an adapted C^1 – process and ℓ solves

$$d\ell_t=\frac{d\tilde{\ell}_t}{dt}db_t-\frac{1}{2}\operatorname{Ric}^{//_t}\ell_t\,dt. \tag{C.8}$$

To solve this equation for ℓ_t, let Q_t solve the ODE in Eq. (5.3) and write $\ell_t=Q_tk_t$ where $k_t:=Q_t^{-1}\ell_t$. Plugging this expression for ℓ_t into Eq. (C.8) using,

$$d\ell_t=-\frac{1}{2}\operatorname{Ric}^{//_t}Q_tk_tdt+Q_tdk_t,$$

implies,

$$-\frac{1}{2}\operatorname{Ric}^{//_t}Q_tk_tdt+Q_tdk_t=\frac{d\tilde{\ell}_t}{dt}db_t-\frac{1}{2}\operatorname{Ric}^{//_t}Q_tk_t\,dt$$

from which we learn, $dk_t=Q_t^{-1}\frac{d\tilde{\ell}_t}{dt}db_t$. Integrating this equation and multiplying the result on the left by Q_t gives Eq. (C.1). Equation (C.3) now follows from Eq. (C.7) with $\tilde{\alpha}=0$ and $\alpha_t=\frac{d\tilde{\ell}_t}{dt}I_{T_xM}$. q.e.d.

Appendix D. Wang's dimension free Harnack inequality

Suppose that $p_T(\cdot,\cdot)>0$ is the heat kernel at time $T>0$ on a complete connected Riemannian manifold (M) and for measurable $f: M\to[0,\infty)$, let

$$\left(P_Tf\right)(x):=\int_M p_T(x,y)\,f(y)\,dV(y).$$

Hence if $f\in L^2(V)$, then $P_Tf=e^{T\bar{\Delta}_0/2}f$. The following lemma reflects the fact that $(L^q)^*$ and $L^{q'}$ are isometrically isomorphic Banach spaces for $1<q<\infty$ and $q'=q/(q-1)$ – the conjugate exponent to q.

Lemma D.1. *Let* $x, y \in M$, $T > 0$, $q \in (1, \infty)$, *and* $C \in (0, \infty]$. *Then*

$$[(P_T f)(x)]^q \le C^q (P_T f^q)(y) \text{ for all } f \ge 0 \tag{D.1}$$

if and only if

$$\left(\int_M \left[\frac{p_T(x,z)}{p_T(y,z)} \right]^{q'} p_T(y,z)\, dV(z) \right)^{1/q'} \le C. \tag{D.2}$$

Proof. Since

$$(P_T f)(x) = \int_M \frac{p_T(x,z)}{p_T(y,z)} f(z)\, p_T(y,z)\, dV(z),$$

if $d\mu(z) := p_T(y,z)\, dV(z)$ and $g(x) := \frac{p_T(x,\cdot)}{p_T(y,\cdot)}$, then

$$(P_T f)(x) = \int_M f(x)\, g(x)\, d\mu(x). \tag{D.3}$$

Since $g \ge 0$ and $L^q(\mu)^*$ is isomorphic to $L^{q'}(\mu)^*$ under the pairing in Eq. (D.3), it follows that

$$\|g\|_{L^{q'}(\mu)} = \sup_{f \ge 0} \frac{\int_M f(x)\, g(x)\, d\mu(x)}{\|f\|_{L^q(\mu)}} = \sup_{f \ge 0} \frac{(P_T f)(x)}{[(P_T f^q)(y)]^{1/q}}.$$

The last equation may be written more explicitly as,

$$\left(\int_M \left[\frac{p_T(x,z)}{p_T(y,z)} \right]^{q'} p_T(y,z)\, dV(z) \right)^{1/q'} = \sup_{f \ge 0} \frac{(P_T f)(x)}{[(P_T f^q)(y)]^{1/q}},$$

and from this equation the lemma easily follows. q.e.d.

The following theorem appears in [**66**, **67**] – see also [**2**].

Theorem D.2 (Wang's Harnack inequality)**.** *Suppose that* M *is a complete connected Riemannian manifold such that* $\mathrm{Ric} \ge kI$ *for some* $k \in \mathbb{R}$. *Then for all* $q > 1$, $f \ge 0$, $T > 0$, *and* $x, y \in M$, *we have*

$$(P_T f)^q(y) \le (P_T f^q)(z) \exp\left(q' \frac{k}{e^{kT} - 1} d^2(y,z) \right), \tag{D.4}$$

where $q' = q/(q-1)$ *is the conjugate exponent to* q.

In applying Wang's results the reader should use $k = -K$, $V \equiv 0$, and replace T by $T/2$ since Wang's generator is Δ rather than $\Delta/2$.

Corollary D.3. *Let* (M, g) *be a complete Riemannian manifold such that* $\mathrm{Ric} \ge kI$ *for some* $k \in \mathbb{R}$. *Then for every* $y, z \in M$ *and* $q \in [1, \infty)$,

$$\left(\int_M \left[\frac{p_T(y,x)}{p_T(z,x)} \right]^q p_T(z,x)\, dV(x) \right)^{1/q} \le \exp\left(\frac{c(kT)(q-1)}{2T} d^2(y,z) \right) \tag{D.5}$$

where $c(\cdot)$ is defined as in Eq. (3.1), $p_t(x,y)$ is the heat kernel on M and $d(y,z)$ is the Riemannian distance from x to y for $x,y \in M$.

Proof. From Lemma D.1 and Theorem D.2 with

$$C = \exp\left(\frac{q'}{q}\frac{k}{e^{kT}-1}d^2(y,z)\right) = \exp\left(\frac{1}{q-1}\frac{k}{e^{kT}-1}d^2(y,z)\right),$$

it follows that it follows that

$$\left(\int_M \left[\frac{p_T(x,z)}{p_T(y,z)}\right]^{q'} p_T(y,z)\,dV(z)\right)^{1/q'} \leq \exp\left(\frac{1}{q-1}\frac{k}{e^{kT}-1}d^2(y,z)\right).$$

Using $q-1 = (q'-1)^{-1}$ and then interchanging the roles of q and q' gives Eq. (D.5). q.e.d.

For comparison sake, recall that the classical Li - Yau Harnack inequality (see Li and Yau [**43**] and Davies [**14**, Theorem 5.3.5]) states if $\alpha > 1$, $s > 0$, and $\mathrm{Ric} \geq -K$ for some $K \geq 0$, then

$$\frac{p_t(y,x)}{p_{t+s}(z,x)} \leq \left(\frac{t+s}{t}\right)^{d\alpha/2} \exp\left(\frac{\alpha d^2(y,z)}{2s} + \frac{d\cdot\alpha K s}{8(\alpha-1)}\right), \tag{D.6}$$

for all $x,y,z \in M^d$ and $t > 0$. However when $s = 0$, Eq. (D.6) gives no information on $p_t(y,x)/p_t(z,x)$ when $y \neq z$.

Remark D.4. Since our heat equation is determined by $\Delta_0/2$ rather than Δ_0, the reader should replace t and s by $t/2$ and $s/2$ when applying the results in [**43**, **14**].

Appendix E. Consequences of Hamilton's estimates

Let $T \in (0,\infty)$, M $(d = \dim(M))$ be a complete Riemannian manifold with $\mathrm{Ric} \geq -KI$ for some $K \geq 0$, and let $V(x,r) := \mathrm{Vol}(B(x,r))$ be the volume of the ball, $B(x,r)$, centered at $x \in M$ with radius $r > 0$. Suppose, for $0 \leq t \leq t_1$, that $u(t,x)$ is a positive solution to the heat equation, $\frac{\partial}{\partial t}u = \frac{1}{2}\Delta u$. The Hamilton type gradient bounds [**34**, **59**, **41**] state that if

$$m := \sup\{u(t,x) : 0 \leq t \leq t_1,\ x \in M\},$$

then

$$t|\nabla \log(u(t,x))|^2 \leq 2(1+Kt)\log(m/u(t,x)) \text{ for all } (t,x) \in [0,t_1]\times M. \tag{E.1}$$

The standard heat kernel bounds (see for example Theorems 5.6.4, 5.6.6, and 5.4.12 in Sallof-Coste [**55**] and for more detailed bounds see [**43**, **14**, **54**, **15**, **31**]) state that there exist constants, $c = c(K,d,T)$ and

$C = C(K, d, T)$, such that,

$$\frac{c}{V\left(x, \sqrt{t/2}\right)} \exp\left(-C\frac{d^2(x,y)}{t}\right) \le p(t,x,y) \le \frac{C}{V\left(x, \sqrt{t/2}\right)} \exp\left(-c\frac{d^2(x,y)}{t}\right), \tag{E.2}$$

for all $x, y \in M$ and $t \in (0, T]$.

Let $s \in (0, T]$, $o \in M$, $t_1 = s/2$ and $u(t,x) = p_{s/2+t}(o,x)$. Combining Eqs. (E.1) and (E.2) then shows,

$$t|\nabla_x \log p_{s/2+t}(o,x))|^2 \le 2(1+Kt)\log\left(\frac{C}{c}\frac{V\left(0, \sqrt{s/4+t/2}\right)}{V\left(o, \sqrt{s/4}\right)} \exp\left(C\frac{d^2(o,y)}{s/2+t}\right)\right). \tag{E.3}$$

Taking $t = s/2$ in Eq. (E.3) and then replacing s by t in the resulting inequality implies,

$$\frac{t}{2}|\nabla_x \log p_t(o,x))|^2 \le 2\left(1 + K\frac{t}{2}\right)\log\left(\frac{C}{c}\frac{V\left(0, \sqrt{t/2}\right)}{V\left(o, \sqrt{t/4}\right)} \exp\left(C\frac{d^2(o,y)}{t}\right)\right). \tag{E.4}$$

Using the volume estimate (see [**10**] and [**55**, Theorem 5.6.4]),

$$\frac{V(x,\sigma)}{V(x,s)} \le \left(\frac{\sigma}{s}\right)^d \exp\left(\sqrt{(d-1)K}\sigma\right) \;\; \forall\, x \in M \text{ and } 0 \le s < \sigma,$$

it follows that

$$\frac{V(x,\sqrt{t/2})}{V(x,\sqrt{t/4})} \le 2^{d/2}\exp\left(\sqrt{(d-1)Kt/2}\right) \le 2^{d/2}\exp\left(\sqrt{(d-1)KT/2}\right). \tag{E.5}$$

Combining Eqs. (E.4) and (E.5) then allows us to conclude that there exist constants, c_1 and c_2 depending on T, K, and d such that

$$|\nabla_x \log p_t(o,x))| \le \left(\frac{c_1}{\sqrt{t}} + c_2\frac{d(o,x)}{t}\right) \;\; \text{for all} \;\; t \in (0,T] \text{ and } o, x \in M. \tag{E.6}$$

For this estimate in the compact case with its relations to stochastic analysis, see [**17**, **47**, **62**, **64**, **36**].

Proposition E.1. *Continuing the notation and assumptions used above, there exist constants, $C_1(d,K)$ and $C_2(d,K,t)$ such that,*

$$\int_M \exp\left(\lambda\left|\nabla_x \log p_t(o,x))\right|\right) p_t(o,x)\,dx \le C(d,K,t)\exp\left(C(d,K)\lambda^2/t\right) \tag{E.7}$$

for all $o \in M$ and $t \in (0,T]$.

Proof. Let $v(r) := \mathrm{Vol}(B(o,r))$, $\kappa := \sqrt{K/(d-1)}$, $\gamma := (d-1)\kappa = \sqrt{K(d-1)}$, and ω_{d-1} be the volume of the standard $d-1$ sphere. Using Bishop's comparison theorem (see [**9**, **56**]) which states,

$$dv(r) \le \omega_{d-1}\left(\frac{\sinh \kappa r}{\kappa}\right)^{d-1} dr \le \left(\frac{\omega_{d-1}}{2\kappa}\right)^{d-1} e^{\kappa(d-1)r}dr, \tag{E.8}$$

along with the estimates in Eqs. (E.2) and (E.6), we have

$$\int_M \exp\left(\lambda\left|\nabla_x \log p_t(o,x))\right|\right) p_t(o,x)\,dx$$
$$\le Ct^{-d/2}\int_0^\infty \exp\left(\lambda\left(\frac{c_1}{\sqrt{t}}+c_2\frac{r}{t}\right)\right)\exp\left(-\frac{C}{2t}r^2\right)dv(r)$$
$$\le C\left(\frac{\omega_{d-1}}{2\kappa}\right)^{d-1} t^{-d/2}\int_0^\infty \exp\left(\lambda\left(\frac{c_1}{\sqrt{t}}+c_2\frac{r}{t}\right)\right)\exp\left(-\frac{C}{2t}r^2\right)e^{\gamma r}dr \tag{E.9}$$
$$= C(d,K,T)t^{-d/2}\exp\left(\lambda\frac{c_1}{\sqrt{t}}\right)\int_0^\infty \exp\left(\left(\gamma+\lambda\frac{c_2}{t}\right)r\right)\exp\left(-\frac{C}{2t}r^2\right)dr. \tag{E.10}$$

Equation (E.7) follows easily from Eq. (E.10) and the following two estimates

$$c_1\frac{\lambda}{\sqrt{t}} \le \frac{1}{2}\left(c_1^2+\frac{\lambda^2}{2t}\right)$$

and

$$\int_0^\infty \exp\left(\left(\gamma+\lambda\frac{c_2}{t}\right)r\right)\exp\left(-\frac{C}{2t}r^2\right)dr$$
$$\le \int_{-\infty}^\infty \exp\left(\left(\gamma+\lambda\frac{c_2}{t}\right)r\right)\exp\left(-\frac{C}{2t}r^2\right)dr$$
$$= \sqrt{2\pi t/C}\exp\left(\frac{t}{2C}\left(\gamma+\lambda\frac{c_2}{t}\right)^2\right). \tag{E.11}$$

q.e.d.

Remark E.2. When $M=\mathbb{R}^d$, using Laplace asymptotics, one may show;

$$\lim_{d\to\infty} e^{-\frac{\lambda}{\sqrt{t}}\sqrt{d-1}}\int_{\mathbb{R}^d}\exp\left(\lambda\left|\nabla_x \log p_t(o,x))\right|\right)p_t(o,x)\,dx = e^{\lambda^2/4t}\ \forall\, t,\lambda>0.$$

In particular, this implies that we can not take both $C(d,0,t)$ and $C(d,0)$ in Eq. (E.7) to be independent of the dimension, $d = \dim(M)$.

References

[1] H. Airault & P. Malliavin, *Quasi-invariance of Brownian measures on the group of circle homeomorphisms and infinite-dimensional Riemannian geometry*, J. Funct. Anal. **241** (2006), no. 1, 99–142. MR 2264248, Zbl 1111.58032.

[2] M. Arnaudon, A. Thalmaier, & F.-Y. Wang, *Harnack inequality and heat kernel estimates on manifolds with curvature unbounded below*, Bull. Sci. Math. **130** (2006), no. 3, 223–233. MR 2215664, Zbl 1089.58024.

[3] R. Azencott, *Behavior of diffusion semi-groups at infinity*, Bull. Soc. Math. France **102** (1974), 193–240. MR 0356254, Zbl 0293.60071.

[4] V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform*, Comm. Pure Appl. Math. **14** (1961), 187–214. MR 0157250, Zbl 0107.09102.

[5] D. Bell, *A quasi-invariance theorem for measures on Banach spaces*, Trans. Amer. Math. Soc. **290** (1985), no. 2, 851–855. MR 792833, Zbl 0577.46045.

[6] V. Bogachev & E. Mayer-Wolf, *Absolutely continuous flows generated by Sobolev class vector fields in finite and infinite dimensions*, J. Funct. Anal. **167** (1999), no. 1, 1–68. MR 1710649, Zbl 0956.60079.

[7] J. Brüning & M. Lesch, *Hilbert complexes*, J. Funct. Anal. **108** (1992), no. 1, 88–132. MR 1174159, Zbl 0826.46065

[8] E. L. Bueler, *The heat kernel weighted Hodge Laplacian on noncompact manifolds*, Trans. Amer. Math. Soc. **351** (1999), no. 2, 683–713. MR 1443866, Zbl 0920.58002.

[9] J. Cheeger & D. G. Ebin, *Comparison theorems in Riemannian geometry*, North-Holland Publishing Co., Amsterdam, 1975, North-Holland Mathematical Library, Vol. 9. MR 0458335, Zbl 0309.53035.

[10] J. Cheeger, M. Gromov, & M. Taylor, *Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds*, J. Differential Geom. **17** (1982), no. 1, 15–53. MR 658471, Zbl 0493.53035.

[11] P. R. Chernoff, *Essential self-adjointness of powers of generators of hyperbolic equations*, J. Functional Analysis **12** (1973), 401–414. MR 0369890, Zbl 0263.35066.

[12] A. B. Cruzeiro, *équations différentielles ordinaires: non explosion et mesures quasi-invariantes*, J. Funct. Anal. **54** (1983), no. 2, 193–205. MR 724704, Zbl 0523.28020.

[13] A. B. Cruzeiro, *équations différentielles sur l'espace de Wiener et formules de Cameron-Martin non-linéaires*, J. Funct. Anal. **54** (1983), no. 2, 206–227. MR 724705, Zbl 0524.47028.

[14] E. B. Davies, *Heat kernels and spectral theory*, Cambridge Tracts in Mathematics, vol. 92, Cambridge University Press, Cambridge, 1989. MR 990239, Zbl 0699.35006.

[15] E. B. Davies, *The state of the art for heat kernel bounds on negatively curved manifolds*, Bull. London Math. Soc. **25** (1993), no. 3, 289–292. MR 1209255, Zbl 0802.58053.

[16] J. Dodziuk, *Maximum principle for parabolic inequalities and the heat flow on open manifolds*, Indiana Univ. Math. J. **32** (1983), no. 5, 703–716. MR 711862, Zbl 0526.58047.

[17] B. K. Driver, *A Cameron-Martin type quasi-invariance theorem for pinned Brownian motion on a compact Riemannian manifold*, Trans. Amer. Math. Soc. **342** (1994), no. 1, 375–395. MR 1154540, Zbl 0792.60013.

[18] B. K. Driver, *Integration by parts and quasi-invariance for heat kernel measures on loop groups*, J. Funct. Anal. **149** (1997), no. 2, 470–547. MR 1472366, Zbl 0887.58062.

[19] B. K. Driver, *Integration by parts for heat kernel measures revisited*, J. Math. Pures Appl. (9) **76** (1997), no. 8, 703–737. MR 1470881, Zbl 0907.60052.

[20] B. K. Driver, *Curved Wiener space analysis*, Real and stochastic analysis, Trends Math., Birkhäuser Boston, Boston, MA, 2004, pp. 43–198. MR 2090752, Zbl 1085.60032.

[21] B. K. Driver & M. Gordina, *Heat kernel analysis on infinite-dimensional Heisenberg groups*, J. Funct. Anal. **255** (2008), no. 9, 2395–2461. MR 2473262, Zbl pre05505662.

[22] B. K. Driver & L. Gross, *Hilbert spaces of holomorphic functions on complex Lie groups*, New trends in stochastic analysis (Charingworth, 1994), World Sci. Publishing, River Edge, NJ, 1997, pp. 76–106. MR 1654507, Zbl 0944.60013.

[23] B. K. Driver & A. Thalmaier, *Heat equation derivative formulas for vector bundles*, J. Funct. Anal. **183** (2001), no. 1, 42–108. MR 1837533, Zbl 0983.58018.

[24] M. Émery, *Stochastic calculus in manifolds*, Universitext, Springer-Verlag, Berlin, 1989, With an appendix by P.-A. Meyer. MR 1030543, Zbl 0697.60060.

[25] S. Fang, *Integration by parts for heat measures over loop groups*, J. Math. Pures Appl. (9) **78** (1999), no. 9, 877–894. MR 1725745, Zbl 0997.58022.

[26] M. P. Gaffney, *The harmonic operator for exterior differential forms*, Proc. Nat. Acad. Sci. U. S. A. **37** (1951), 48–50. MR 0048138, Zbl 0042.10205.

[27] M. P. Gaffney, *The conservation property of the heat equation on Riemannian manifolds.*, Comm. Pure Appl. Math. **12** (1959), 1–11. MR 0102097, Zbl 0102.09202.

[28] F. Galaz-Fontes, L. Gross, & S. B. Sontz, *Reverse hypercontractivity over manifolds*, Ark. Mat. **39** (2001), no. 2, 283–309. MR 1861062, Zbl 1021.58020.

[29] A. A. Grigor′yan, *Stochastically complete manifolds*, Dokl. Akad. Nauk SSSR **290** (1986), no. 3, 534–537. MR 860324, Zbl 0632.58041.

[30] A. A. Grigor′yan, *Bounded solutions of the Schrödinger equation on noncompact Riemannian manifolds*, Trudy Sem. Petrovsk. **51** (1989), no. 14, 66–77, 265–266. MR 1001354, Zbl 0744.58070.

[31] A. Grigor′yan, *Heat kernel of a noncompact Riemannian manifold*, Stochastic analysis (Ithaca, NY, 1993), Proc. Sympos. Pure Math., vol. 57, Amer. Math. Soc., Providence, RI, 1995, pp. 239–263. MR 1335475, Zbl 0829.58041

[32] A. Grigor′yan, *Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds*, Bull. Amer. Math. Soc. (N.S.) **36** (1999), no. 2, 135–249. MR 1659871, Zbl 0927.58019.

[33] L. Gross, *Logarithmic Sobolev inequalities*, Amer. J. Math. **97** (1975), no. 4, 1061–1083. MR 0420249, Zbl 0318.46049.

[34] R. S. Hamilton, *A matrix Harnack estimate for the heat equation*, Comm. Anal. Geom. **1** (1993), no. 1, 113–126. MR 1230276, Zbl 0799.53048.

[35] H. Hess, R. Schrader, & D. A. Uhlenbrock, *Domination of semigroups and generalization of Kato's inequality*, Duke Math. J. **44** (1977), no. 4, 893–904. MR 0458243, Zbl 0379.47028.

[36] E. P. Hsu, *Estimates of derivatives of the heat kernel on a compact Riemannian manifold*, Proc. Amer. Math. Soc. **127** (1999), no. 12, 3739–3744. MR 1618694, Zbl 0948.58025.

[37] E. P. Hsu, *Stochastic analysis on manifolds*, Graduate Studies in Mathematics, vol. 38, American Mathematical Society, Providence, RI, 2002. MR 1882015, Zbl 0994.58019.

[38] S. Janson, *Gaussian Hilbert spaces*, Cambridge Tracts in Mathematics, vol. 129, Cambridge University Press, Cambridge, 1997. MR 1474726, Zbl 0887.60009.

[39] O. Kallenberg, *Foundations of modern probability*, second ed., Probability and its Applications (New York), Springer-Verlag, New York, 2002. MR 1876169, Zbl 0996.60001.

[40] T. Kato, *Schrödinger operators with singular potentials*, Proceedings of the International Symposium on Partial Differential Equations and the Geometry of Normed Linear Spaces (Jerusalem, 1972), vol. 13, 1972, pp. 135–148 (1973). MR 0333833, Zbl 0246.35025.

[41] B. L. Kotschwar, *Hamilton's gradient estimate for the heat kernel on complete manifolds*, Proc. Amer. Math. Soc. **135** (2007), no. 9, 3013–3019 (electronic). MR 2317980, Zbl 1127.58021

[42] P. Li, *Uniqueness of L^1 solutions for the Laplace equation and the heat equation on Riemannian manifolds*, J. Differential Geom. **20** (1984), no. 2, 447–457. MR 788288, Zbl 0561.53045.

[43] P. Li & S.-T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156** (1986), no. 3-4, 153–201. MR 834612, Zbl 0611.58045.

[44] X.-M. Li, *Stochastic differential equations on noncompact manifolds: moment stability and its topological consequences*, Probab. Theory Related Fields **100** (1994), no. 4, 417–428. MR 1305780, Zbl 0815.60049.

[45] X.-M. Li, *Strong p-completeness of stochastic differential equations and the existence of smooth flows on noncompact manifolds*, Probab. Theory Related Fields **100** (1994), no. 4, 485–511. MR 1305784, Zbl 0815.60050.

[46] P. Malliavin, *Stochastic analysis*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 313, Springer-Verlag, Berlin, 1997. MR 1450093, Zbl 0878.60001.

[47] P. Malliavin & D. W. Stroock, *Short time behavior of the heat kernel and its logarithmic derivatives*, J. Differential Geom. **44** (1996), no. 3, 550–570. MR 1431005, Zbl 0873.58063.

[48] M. Métivier, *Semimartingales*, de Gruyter Studies in Mathematics, vol. 2, Walter de Gruyter & Co., Berlin, 1982, A course on stochastic processes. MR 688144, Zbl 0503.60054.

[49] D. Nualart, *The Malliavin calculus and related topics*, Probability and its Applications (New York), Springer-Verlag, New York, 1995. MR 1344217, Zbl 0837.60050

[50] G. Peters, *Flows on the Wiener space generated by vector fields with low regularity*, C. R. Acad. Sci. Paris Sér. I Math. **320** (1995), no. 8, 1003–1008. MR 1328727, Zbl 0828.47040.

[51] G. Peters, *Anticipating flows on the Wiener space generated by vector fields of low regularity*, J. Funct. Anal. **142** (1996), no. 1, 129–192. MR 1419419, Zbl 0907.58005.

[52] M. Reed & B. Simon, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975. MR 0493420, Zbl 0308.47002.

[53] W. Roelcke, *Über den Laplace-Operator auf Riemannschen Mannigfaltigkeiten mit diskontinuierlichen Gruppen*, Math. Nachr. **21** (1960), 131–149. MR 0151927, Zbl 0197.36401.

[54] L. Saloff-Coste, *Uniformly elliptic operators on Riemannian manifolds*, J. Differential Geom. **36** (1992), no. 2, 417–450. MR 1180389, Zbl 0735.58032.

[55] L. Saloff-Coste, *Aspects of Sobolev-type inequalities*, London Mathematical Society Lecture Note Series, vol. 289, Cambridge University Press, Cambridge, 2002. MR 1872526, Zbl 0991.35002.

[56] R. Schoen & S.-T. Yau, *Lectures on differential geometry*, Conference Proceedings and Lecture Notes in Geometry and Topology, I, International Press, Cambridge, MA, 1994, Lecture notes prepared by Wei Yue Ding, Kung Ching Chang [Gong Qing Zhang], Jia Qing Zhong and Yi Chao Xu, Translated from the Chinese by Ding and S. Y. Cheng, Preface translated from the Chinese by Kaising Tso. MR 1333601, Zbl 0830.53001.

[57] B. Simon, *An abstract Kato's inequality for generators of positivity preserving semigroups*, Indiana Univ. Math. J. **26** (1977), no. 6, 1067–1073. MR 0461209, Zbl 0389.47021

[58] B. Simon, *Kato's inequality and the comparison of semigroups*, J. Funct. Anal. **32** (1979), no. 1, 97–101. MR 533221, Zbl 0413.47037.

[59] P. Souplet & Qi S. Zhang, *Sharp gradient estimate and Yau's Liouville theorem for the heat equation on noncompact manifolds*, Bull. London Math. Soc. **38** (2006), no. 6, 1045–1053. MR 2285258, Zbl 1109.58025.

[60] R. S. Strichartz, *Analysis of the Laplacian on the complete Riemannian manifold*, J. Funct. Anal. **52** (1983), no. 1, 48–79. MR 705991, Zbl 0515.58037.

[61] D. W. Stroock, *Probability theory, an analytic view*, Cambridge University Press, Cambridge, 1993. MR 1267569, Zbl 0960.60001.

[62] D. W. Stroock, *An estimate on the Hessian of the heat kernel*, Itô's stochastic calculus and probability theory, Springer, Tokyo, 1996, pp. 355–371. MR 1439536, Zbl 0868.58075.

[63] D. W. Stroock *An introduction to the analysis of paths on a Riemannian manifold*, Mathematical Surveys and Monographs, vol. 74, American Mathematical Society, Providence, RI, 2000. MR 1715265, Zbl 0942.58001.

[64] D. W. Stroock & J. Turetsky, *Upper bounds on derivatives of the logarithm of the heat kernel*, Comm. Anal. Geom. **6** (1998), no. 4, 669–685. MR 1664888, Zbl 0928.58031.

[65] N. Th. Varopoulos, L. Saloff-Coste, & T. Coulhon, *Analysis and geometry on groups*, Cambridge Tracts in Mathematics, vol. 100, Cambridge University Press, Cambridge, 1992. MR 1218884, Zbl 0813.22003.

[66] F.-Y. Wang, *Logarithmic Sobolev inequalities on noncompact Riemannian manifolds*, Probab. Theory Related Fields **109** (1997), no. 3, 417–424. MR 1481127, Zbl 0887.35012.

[67] F.-Y. Wang, *Equivalence of dimension-free Harnack inequality and curvature condition*, Integral Equations Operator Theory **48** (2004), no. 4, 547–552. MR 2047597, Zbl 1074.47020.

[68] S.-T. Yau, *On the heat kernel of a complete Riemannian manifold*, J. Math. Pures Appl. (9) **57** (1978), no. 2, 191–201. MR 505904, Zbl 0405.35025.

Department of Mathematics, 0112
University of California, San Diego
La Jolla, CA 92093-0112
E-mail address: driver@euclid.ucsd.edu

Department of Mathematics
University of Connecticut
Storrs, CT 06269
E-mail address: gordina@math.uconn.edu