J. DIFFERENTIAL GEOMETRY
83 (2009) 637-661

MAXIMIZATION OF THE SECOND POSITIVE NEUMANN EIGENVALUE FOR PLANAR DOMAINS

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Abstract

We prove that the second positive Neumann eigenvalue of a bounded simply-connected planar domain of a given area does not exceed the first positive Neumann eigenvalue on a disk of half this area. The estimate is sharp and attained by a sequence of domains degenerating to a union of two identical disks. In particular, this result implies the Pólya conjecture for the second Neumann eigenvalue. The proof is based on a combination of analytic and topological arguments. As a by-product of our method we obtain an upper bound on the second eigenvalue for conformally round metrics on odd-dimensional spheres.

1. Introduction and main results

1.1. Neumann eigenvalues of planar domains. Let Ω be a bounded planar domain. The domain Ω is said to be *regular* if the spectrum of the Neumann boundary value problem on Ω is discrete. This is true, for instance, if Ω satisfies the cone condition, that is there are no outward pointing cusps (see [**NS**] for more refined conditions and a detailed discussion).

Let $0 = \mu_0 < \mu_1(\Omega) \le \mu_2(\Omega) \le \cdots \nearrow \infty$ be the Neumann eigenvalues of a regular domain Ω. According to a classical result of Szegő ([**Sz**], see also [**SY**, p. 137], [**Hen**, section 7.1]), for any regular simply-connected domain Ω

$$\mu_1(\Omega)\,\mathrm{Area}(\Omega) \le \mu_1(\mathbb{D})\pi \approx 3.39\,\pi, \tag{1.1.1}$$

where $\mathbb{D}$ is the unit disk, and $\mu_1(\mathbb{D})$ is the square of the first zero of the derivative $J_1'(x)$ of the first Bessel function of the first kind. The proof of Szegő's theorem relies on the Riemann mapping theorem and hence works only if Ω is simply-connected. However, inequality (1.1.1) holds without this assumption, as was later shown by Weinberger [**We**].

Received 10/16/2008.

The Pólya conjecture for Neumann eigenvalues [**Po1**] (see also [**SY**, p. 139]) states that for any regular bounded domain Ω

$$\mu_k(\Omega)\,\mathrm{Area}(\Omega) \le 4k\,\pi \tag{1.1.2}$$

for all $k \ge 1$. This inequality is true for all domains that tile the plane, e.g., for any triangle and any quadrilateral [**Po2**]. It follows from the two-term asymptotics for the eigenvalue counting function ([**Iv**], [**Me**]) that for any domain there exists a number K such that (1.1.2) holds for all $k > K$.

Inequality (1.1.1) implies that (1.1.2) is true for μ_1. The best one could show for $k \ge 2$ was $\mu_k \le 8\pi k$ ([**Kro**]). In the present paper we consider the case $k = 2$. Our main result is

Theorem 1.1.3. *Let Ω be a regular simply-connected planar domain. Then*

$$\mu_2(\Omega)\,\mathrm{Area}(\Omega) \le 2\,\mu_1(\mathbb{D})\,\pi \approx 6.78\,\pi, \tag{1.1.4}$$

with the equality attained in the limit by a family of domains degenerating to a disjoint union of two identical disks.

The second part of the theorem immediately follows from (1.1.4). Indeed, if Ω is a disjoint union of two identical disks then (1.1.4) is an equality. Joining the two disks by a passage of width ϵ we can construct a family of simply-connected domains such that the left-hand side in (1.1.4) converges to $2\mu_1(\mathbb{D})\pi$ as $\epsilon \to 0$.

Theorem 1.1.3 gives a positive answer to a question of Parnovski [**Par**], motivated by an analogous result proved in [**Na**] for the second eigenvalue on a sphere. Note that (1.1.4) immediately implies (1.1.2) for $k = 2$ for any regular simply-connected planar domain.

Remark 1.1.5. It would be interesting to check the bound (1.1.4) for non-simply connected domains. We believe it remains true in this case as well.

Remark 1.1.6. All estimates discussed in this section have analogues in the Dirichlet case. For example, (1.1.1) is the Neumann counterpart of the celebrated Faber-Krahn inequality ([**Fa**, **Kra1**], see also [**Hen**, section 3.2]), which states that among all bounded planar domains of a given area, the first Dirichlet eigenvalue is minimal on a disk. Similarly, Theorem 1.1.3 can be viewed as an analogue of the result due to Krahn and P. Szego ([**Kra2**], [**Hen**, Theorem 4.1.1]), who proved that among bounded planar domains of a given area, the second Dirichlet eigenvalue is minimized by the union of two identical disks.

1.2. Eigenvalue estimates on spheres. Let $(\mathbb{S}^n, g)$ be a sphere of dimension $n \ge 2$ with a Riemannian metric g. Let

$$0 < \lambda_1(\mathbb{S}^n, g) \le \lambda_2(\mathbb{S}^n, g) \le \cdots \nearrow \infty$$

be the eigenvalues of the Laplacian on $(\mathbb{S}^n, g)$. Hersch [**Her**] adapted the approach of Szegő to prove that $\lambda_1(\mathbb{S}^2, g)\,\mathrm{Area}(\mathbb{S}^2, g) \le 8\pi$ for any Riemannian metric g, with the equality attained on a sphere with the standard round metric g_0. In order to obtain a similar estimate in higher dimensions, one needs to restrict the Riemannian metrics to a fixed conformal class [**EI**]. Indeed, in dimension ≥ 3, if one only restricts the volume, λ_1 is unbounded [**CD**]. In particular, it was shown in [**EI**] (see also [**MW**]) that for any metric g in the class $[g_0]$ of conformally round metrics,

$$\lambda_1(\mathbb{S}^n, g)\,\mathrm{Vol}(\mathbb{S}^n, g)^{\frac{2}{n}} \le n\,\omega_n^{2/n}, \tag{1.2.1}$$

where

$$\omega_n = \frac{2\,\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}$$

is the volume of the unit round n-dimensional sphere. This result can be viewed as a generalization of Hersch's inequality, since all metrics on $\mathbb{S}^2$ are conformally equivalent to the round metric g_0.

A similar problem for higher eigenvalues is much more complicated. It was proved in [**CE**, Corollary 1] that

$$\lambda_k^c(\mathbb{S}^n, [g_0]) := \sup_{g\in[g_0]} \lambda_k(\mathbb{S}^n, g)\,\mathrm{Vol}(\mathbb{S}^n, g)^{\frac{2}{n}} \ge n\,(k\,\omega_n)^{2/n}, \tag{1.2.2}$$

The number $\lambda_k^c(\mathbb{S}^n, [g_0])$ is called the k-th conformal eigenvalue of $(\mathbb{S}^n, [g_0])$. It was shown in [**Na**] that for $k = 2$ and $n = 2$ inequality (1.2.2) becomes an equality, and the supremum is attained by a sequence of surfaces degenerating to a union of two identical round spheres. We conjecture that the same is true in all dimensions:

Conjecture 1.2.3. *The second conformal eigenvalue of* $(\mathbb{S}^n, [g_0])$ *equals*

$$\lambda_2^c(\mathbb{S}^n, [g_0]) = n\,(2\,\omega_n)^{2/n} \tag{1.2.4}$$

for all $n \ge 2$.

As a by-product of the method developed for the proof of Theorem 1.1.3, we prove an upper bound for $\lambda_2^c(\mathbb{S}^n, [g_0])$ when the dimension n is *odd* (this condition is explained in Remark 4.3.8). Our result is in good agreement with Conjecture 1.2.3.

Theorem 1.2.5. *Let* $n \in \mathbb{N}$ *be odd and let* $(\mathbb{S}^n, g)$ *be a* n*-dimensional sphere with a conformally round metric* $g \in [g_0]$. *Then*

$$\lambda_2(\mathbb{S}^n, g)\,\mathrm{Vol}(\mathbb{S}^n, g)^{\frac{2}{n}} < (n+1)\left(\frac{4\pi^{\frac{n+1}{2}}\Gamma(n)}{\Gamma(\frac{n}{2})\Gamma(n+\frac{1}{2})}\right)^{2/n} \tag{1.2.6}$$

Remark 1.2.7. Note that the Dirichlet energy is not conformally invariant in dimensions $n \geq 3$. Therefore, to prove Theorem 1.2.5 we have to work with the modified Rayleigh quotient (see [**Ber**, pp. 141–142], [**FN**]), which is strictly greater than the usual one unless the gradient of the test function has constant norm. This is the main reason why we do not believe the bound (1.2.6) is sharp. At the same time, the estimate (1.2.6) is just a little bit weaker than the conjectured bound (1.2.4): one can check numerically that the ratio of the constants at the right-hand sides of (1.2.6) and (1.2.4) is contained in the interval $(1, 1.04)$ for all n. Moreover, the difference between the two constants converges to 0 as the dimension $n \to \infty$, and hence (1.2.6) is "asymptotically sharp" as follows from (1.2.2).

Remark 1.2.8. It was conjectured in [**Na**] that if $n = 2$ then (1.2.2) is an equality for all $k \geq 1$, with the maximizer given by the union of k identical round spheres. One could view this as an analogue of the Pólya conjecture (1.1.2) for the sphere. Note that a similar "naive" guess about the maximizer of the k-th Neumann eigenvalue of a planar domain is false: a union of k equal disks can not maximize μ_k for all $k \geq 1$, because, as one can easily check, this would contradict Weyl's law. For the same reason, (1.2.2) can not be an equality for all $k \geq 1$ in dimensions $n \geq 5$.

1.3. Plan of the paper. The paper is organized as follows. In sections 2.1–2.5 we develop the "folding and rearrangement" technique based on the ideas of [**Na**] and apply it to planar domains. The topological argument used in the proof of Theorem 1.1.3 is presented in section 2.6. In section 2.7 we complete the proof of the main theorem using some facts about the subharmonic functions. In sections 3.1 and 3.2 we prove the auxiliary lemmas used in the proof of Theorem 1.1.3. In section 4.1 we present a somewhat stronger version of the classical Hersch's lemma ([**Her**]). In sections 4.2 and 4.3 we adapt the approach developed in sections 2.1-2.7 for the case of the sphere. In section 4.4 we use the modified Rayleigh quotient to complete the proof of Theorem 1.2.5.

Acknowledgments. We are very grateful to L. Parnovski for a stimulating question that has lead us to Theorem 1.1.3, and to M. Levitin for many useful discussions on this project. We would also like to thank B. Colbois and L. Polterovich for helpful remarks.

2. Proof of Theorem 1.1.3

2.1. Standard eigenfunctions for μ_1 on the disk. Let

$$\mathbb{D} = \left\{ z \in \mathbb{C} \,\middle|\, |z| < 1 \right\}$$

be the open unit disk. Let J_1 be the first Bessel function of the first kind, and let $\zeta \approx 1.84$ be the smallest positive zero of its derivative J_1'.

Set

$$f(r) = J_1(\zeta r).$$

Given $R \geq 0$ and $s = (R\cos\alpha, R\sin\alpha) \in \mathbb{R}^2$, define $X_s : \mathbb{D} \to \mathbb{R}$ by

$$X_s(z) = f(|z|)\frac{z \cdot s}{|z|} = Rf(r)\cos(\theta - \alpha), \tag{2.1.1}$$

where $r = |z|$, $\theta = \arg z$, and $z \cdot s$ denotes the scalar product in $\mathbb{R}^2$. The functions X_s are the Neumann eigenfunctions corresponding to the double eigenvalue

$$\mu_1(\mathbb{D}) = \mu_2(\mathbb{D}) = \zeta^2 \approx 3.39.$$

The functions X_{e_1} and X_{e_2} form a basis for this space of eigenfunctions (where the vectors $\{e_1, e_2\}$ form the standard basis of $\mathbb{R}^2$).

2.2. Renormalization of measure. We say that a conformal transformation T of the disk *renormalizes* a measure $d\nu$ if for each $s \in \mathbb{R}^2$,

$$\int_{\mathbb{D}} X_s \circ T \, d\nu = 0. \tag{2.2.1}$$

Finite signed measures on $\mathbb{D}$ can be seen as elements of the dual of the space $C(\overline{\mathbb{D}})$ of continuous functions. As such, the norm of a measure $d\nu$ is

$$\|d\nu\| = \sup_{f \in C(\overline{\mathbb{D}}), |f| \leq 1} \left| \int_{\mathbb{D}} f \, d\nu \right|. \tag{2.2.2}$$

The following result is an analogue of Hersch's lemma (see [**Her**], [**SY**]).

Lemma 2.2.3. *For any finite measure $d\nu$ on $\mathbb{D}$ there exists a point $\xi \in \mathbb{D}$ such that $d\nu$ is renormalized by the automorphism $d_\xi : \mathbb{D} \to \mathbb{D}$ defined by*

$$d_\xi(z) = \frac{z + \xi}{\bar{\xi} z + 1}.$$

Proof. Set $M = \int_{\mathbb{D}} d\nu$ and define the continuous map $C : \mathbb{D} \to \mathbb{D}$ by

$$\begin{aligned} C(\xi) &= \frac{1}{M\, f(1)} \int_{\mathbb{D}} (X_{e_1}, X_{e_2}) \; (d_\xi)_* d\nu \\ &= \frac{1}{M\, f(1)} \int_{\mathbb{D}} (X_{e_1} \circ d_\xi, X_{e_2} \circ d_\xi) \, d\nu. \end{aligned}$$

Let $e^{i\theta} \in S^1 = \partial\mathbb{D}$. For any $z \in \mathbb{D}$,

$$\lim_{\xi \to e^{i\theta}} d_\xi(z) = e^{i\theta}.$$

This means that the map C can be continuously extended to the closure $\overline{\mathbb{D}}$ by $C = \mathrm{id}$ on $\partial\mathbb{D}$. By the same topological argument as in Hersch's lemma (and as in the proof of the Brouwer fixed point theorem), a continuous map $C : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ such that $C(\xi) = \xi$ for $\xi \in \partial\mathbb{D}$ must be onto. Hence, there exists some $\xi \in \mathbb{D}$ such that $C(\xi) = 0 \in \mathbb{D}$. q.e.d.

Lemma 2.2.4. *For any finite measure $d\nu$ the renormalizing point ξ is unique.*

Proof. First, let us show that if the measure $d\nu$ is already renormalized then $\xi = 0$. Suppose that $\mathbb{D} \ni \eta \neq 0$ renormalizes $d\nu$. Without loss of generality assume that η is real and positive (if not, apply a rotation). Setting $s = 1$, by Lemma 3.1.1 we get that $X_s(d_\eta(z)) > X_s(z)$ for all $z \in \mathbb{D}$ and hence

$$\int_{\mathbb{D}} X_s \circ d_\eta \, d\nu > \int_{\mathbb{D}} X_s \, d\nu = 0,$$

which contradicts the hypothesis that η renormalizes $d\nu$.

Now let $d\nu$ be an arbitrary finite measure which is renormalized by $\xi \in \mathbb{D}$. Assume $\eta \in \mathbb{D}$ also renormalizes $d\nu$. Let us show that $\eta = \xi$. Taking into account that $d_{-\xi} \circ d_\xi = d_0 = \mathrm{id}$, we can write

$$(d_\eta)_* d\nu = \left(d_\eta \circ d_{-\xi}\right)_* \left(d_\xi\right)_* d\nu.$$

A straightforward computation shows that

$$d_\eta \circ d_{-\xi} = \frac{1 - \eta\bar{\xi}}{1 - \bar{\eta}\xi} d_\alpha,$$

where $\alpha = d_{-\xi}(\eta)$ and $\left|\frac{1-\eta\bar{\xi}}{1-\bar{\eta}\xi}\right| = 1$. This implies that $d\alpha$ renormalizes $\left(d_\xi\right)_* d\nu$ which is already renormalized. Hence, as we have shown above, $\alpha = d_{-\xi}(\eta) = 0$, and therefore $\xi = \eta$. q.e.d.

Given a finite measure, we write $\Gamma(d\nu) \in \mathbb{D}$ for its unique renormalizing point $\xi \in \mathbb{D}$.

Corollary 2.2.5. *The renormalizing point $\Gamma(d\nu) \in \mathbb{D}$ depends continuously on the measure $d\nu$.*

Proof. Let $(d\nu_n)$ be a sequence of measures converging to the measure $d\nu$ in the norm (2.2.2). Without loss of generality suppose that $d\nu$ is renormalized. Let $\xi_n \in \mathbb{D} \subset \overline{\mathbb{D}}$ be the unique element such that d_{ξ_n} renormalizes $d\nu_n$. Let (ξ_{n_k}) be a convergent subsequence, say to $\xi \in \overline{\mathbb{D}}$. Now, by definition of ξ_n there holds

$$0 = \lim_{k\to\infty} \left| \int_{\mathbb{D}} X_s \, (d_{\xi_{n_k}})_* d\nu_{n_k} \right| = \left| \int_{\mathbb{D}} X_s \, (d_\xi)_* d\nu \right|,$$

and hence d_ξ renormalizes $d\nu$. Since we assumed that $d\nu$ is normalized, by uniqueness we get $\xi = 0$. Therefore, 0 is the unique accumulation point of the set $\xi_n \in \mathbb{D}$ and hence by compactness we get $\xi_n \to 0$. This completes the proof of the lemma. q.e.d.

Corollary 2.2.5 will be used in the proof of Lemma 2.5.3, see section 3.2.

2.3. Variational characterization of μ_2. It follows from the Riemann mapping theorem and Lemma 2.2.3 that for any simply-connected domain Ω there exists a conformal equivalence $\phi : \mathbb{D} \to \Omega$, such that the pullback measure

$$d\mu(z) = \phi^*(dz) = |\phi'(z)|^2 \, dz$$

satisfies for any $s \in S^1$

$$\int_{\mathbb{D}} X_s(z) \, d\mu(z) = 0. \tag{2.3.1}$$

Using a rotation if necessary, we may also assume that

$$\int_{\mathbb{D}} X_{e_1}^2(z) \, d\mu(z) \geq \int_{\mathbb{D}} X_s^2(z) \, d\mu(z) \tag{2.3.2}$$

for any $s \in S^1$. The proof of Theorem 1.1.3 is based on the following variational characterization of $\mu_2(\Omega)$:

$$\mu_2(\Omega) = \inf_E \sup_{0 \neq u \in E} \frac{\int_{\mathbb{D}} |\nabla u|^2 \, dz}{\int_{\mathbb{D}} u^2 \, d\mu} \tag{2.3.3}$$

where E varies among all two-dimensional subspaces of the Sobolev space $H^1(\mathbb{D})$ that are orthogonal to constants, that is for each $f \in E$, $\int_{\mathbb{D}} f \, d\mu = 0$. Note that the Dirichlet energy is conformally invariant in two dimensions, and hence the numerator in (2.3.3) can be written using the standard Euclidean gradient and the Lebesgue measure.

2.4. Folding of hyperbolic caps. It is well-known that the group of automorphisms of the disk coincides with the isometry group of the Poincaré disk model of the hyperbolic plane [**Bea**, section 7.4]. Therefore, for any $\xi \in \mathbb{D}$, the automorphism

$$d_\xi(z) = \frac{z + \xi}{\bar{\xi} z + 1}$$

is an isometry. Note that we have $d_0 = \mathrm{id}$ and $d_\xi(0) = \xi$ for any ξ.

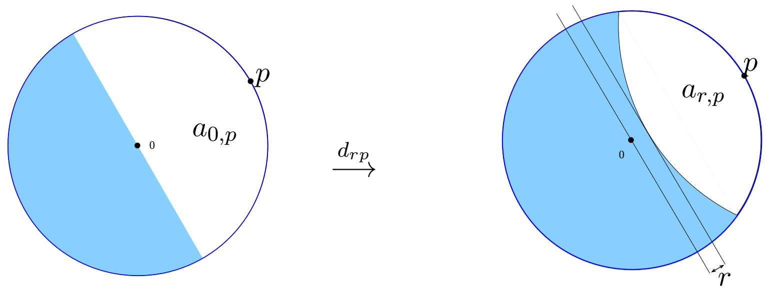
Let γ be a geodesic in the Poincaré disk model, that is a diameter or the intersection of the disk with a circle which is orthogonal to $\partial\mathbb{D}$. Each connected component of $\mathbb{D} \setminus \gamma$ is called a *hyperbolic cap* on $\mathbb{D}$. The space of hyperbolic caps is parametrized as follows. Given $(r, p) \in (-1, 1) \times S^1$ let

$$a_{r,p} = d_{rp}(a_{0,p}),$$

where

$$a_{0,p} = \{x \in \mathbb{D} \; : \; x \cdot p > 0\}$$

is the half-disk such that p is the center of its boundary half-circle. The limit $r \to 1$ corresponds to a cap degenerating to a point on the boundary $\partial\mathbb{D}$ (that is, $a \to p$), while the limit $r \to -1$ corresponds to degeneration to the full disk $\mathbb{D}$ (that is, $a \to \mathbb{D}$). Given $p \in \mathbb{D}$, we define the automorphism $R_p(z) = -p^2 \bar{z}$. It is the reflection with respect to



the line going through 0 and orthogonal to the segment joining 0 and p. For each cap $a_{r,p}$, let us define a conformal automorphism

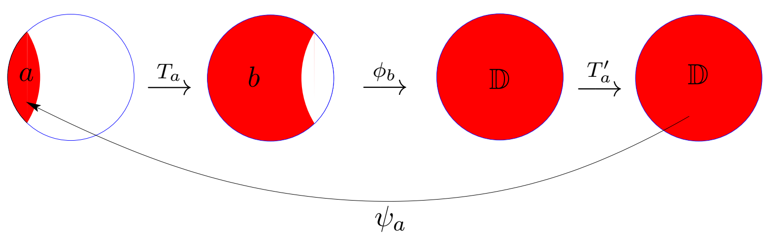
$$\tau_a = d_{rp} \circ R_p \circ d_{-rp}.$$

One can check that this is the reflection with respect to the hyperbolic geodesic $\partial a_{r,p}$. In particular, $\tau_a(a) = \mathbb{D} \setminus \overline{a}$ and τ_a is the identity on ∂a.

2.5. Folding and rearrangement of measure. Given a measure $d\mu$ on $\mathbb{D}$ and a hyperbolic cap $a \subset \mathbb{D}$, the *folded measure* $d\mu_a$ is defined by

$$d\mu_a = \begin{cases} d\mu + \tau_a^* d\mu & \text{on } a, \\ 0 & \text{on } \mathbb{D} \setminus \overline{a}. \end{cases}$$

Clearly, the measure $d\mu_a$ depends continuously in the norm (2.2.2) on the cap $a \subset \mathbb{D}$. For each cap $a \in \mathbb{D}$ let us construct the following conformal equivalence $\psi_a : \mathbb{D} \to a$. First, observe that it follows from



the proof of the Riemann mapping theorem [**Ta**, p.342] that there exists a family $\phi_a : a \to \mathbb{D}$ of conformal equivalences depending continuously on the cap a such that $\lim_{a \to \mathbb{D}} \phi_a = \text{id}$ pointwise. Let $\xi(a) = \Gamma(d\mu_a)$ be the normalizing point for the measure $d\mu_a$ and set $T_a = d_{\xi(a)}$. The measure $(T_a)_* d\mu_a$ is supported in the cap $b = T_a(a)$. Pushing this measure to the full disk using $\phi_b : b \to \mathbb{D}$ leads to the measure

$$(\phi_b \circ T_a)_* d\mu_a.$$

Let $\eta(a) = \Gamma((\phi_b \circ T)_* d\mu_a)$ and set

$$T'_a := d_{\eta(a)} : \mathbb{D} \to \mathbb{D}.$$

The conformal equivalence $\psi_a : \mathbb{D} \to a$ is defined by

$$\psi_a = \left(T'_a \circ \phi_b \circ T_a\right)^{-1}.$$

The pull-back by ψ_a of the folded measure is

$$d\nu_a = \psi_a^* d\mu_a. \tag{2.5.1}$$

It is clear from the above construction that $d\nu_a$ is a normalized measure on the whole disk. We call $d\nu_a$ the *rearranged measure*. It also follows from the construction that the conformal transformations $\psi_a : \mathbb{D} \to a$ depend continuously on a and

$$\lim_{a \to \mathbb{D}} \psi_a = \mathrm{id} : \mathbb{D} \to \mathbb{D} \tag{2.5.2}$$

in the sense of the pointwise convergence. We will make use of the following important property of the rearranged measure.

Lemma 2.5.3. *If a sequence of hyperbolic caps $a \in \mathbb{D}$ degenerates to a point $p \in \partial\mathbb{D}$, the limiting rearranged measure is a "flip-flop" of the original measure $d\mu$:*

$$\lim_{a \to p} d\nu_a = R_p^* d\mu. \tag{F}$$

We call (F) the flip-flop property. The proof of Lemma 2.5.3 will be presented at the end of the paper.

2.6. Maximizing directions. Given a finite measure $d\nu$ on $\mathbb{D}$, consider the function $V : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$V(s) = \int_{\mathbb{D}} X_s^2 \, d\nu.$$

This function is a quadratic form since the mapping $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$(s, t) \mapsto \int_{\mathbb{D}} X_s X_t \, d\nu$$

is symmetric and bilinear (the latter easily follows from (2.1.1)). In particular, $V(s) = V(-s)$ for any s.

Let $\mathbb{R}P^1 = S^1/\mathbb{Z}_2$ be the projective line. We denote by $[s] \in \mathbb{R}P^1$ the element of the projective line corresponding to the pair of points $\pm s \in S^1$. We say that $[s] \in \mathbb{R}P^1$ is a *maximizing direction* for the measure $d\nu$ if $V(s) \geq V(t)$ for any $t \in S^1$. The measure $d\nu$ is called *simple* if there is a unique maximizing direction. Otherwise, the measure $d\nu$ is said to be *multiple*. We have the following

Lemma 2.6.1. *A measure $d\nu$ is multiple if and only if $V(s)$ does not depend on $s \in S^1$.*

Proof. Since $V(s)$ is a symmetric quadratic form, it can be diagonalized. This means that there exists an orthonormal basis (v_1, v_2) of $\mathbb{R}^2$, such that for any $s = \alpha v_1 + \beta v_2 \in \mathbb{D}$ we have $V(s) = M\alpha^2 + m\beta^2$. for some numbers $0 < m \leq M$. It is clear now that the measure $d\nu$ is multiple if and only if $M = m$, and therefore $V(s)$ takes the same value for all $s \in S^1$. q.e.d.

Note that by (2.3.2), $[e_1]$ is a maximizing direction for the measure $d\mu$.

Proposition 2.6.2. *If the measure $d\mu$ is simple, then there exists cap $a \subset \mathbb{D}$ such that the rearranged measure $d\nu_a$ is multiple.*

The proof of this proposition is based on a topological argument, somewhat more subtle than the one used in the proof of Lemma 2.2.3. This is a proof by contradiction. We assume the measure $d\mu$ as well as the measures $d\nu_a$ to be simple. Given a cap $a \subset \mathbb{D}$, let $[s(a)] \in \mathbb{R}P^1$ be the unique maximizing direction for $d\nu_a$. Since $d\nu_a$ depends continuously on a and X_s depends continuously on s, it follows that the map $a \mapsto [s(a)]$ is continuous. Let us understand the behavior of the maximizing directions as the cap a degenerates to the full disk and to a point.

Lemma 2.6.3. *Assume the measures $d\mu$ as well as each $d\nu_a$ to be simple. Then*

$$\lim_{a\to\mathbb{D}}[s(a)] = [e_1] \tag{2.6.4}$$

$$\lim_{a\to e^{i\theta}}[s(a)] = [e^{2i\theta}]. \tag{2.6.5}$$

Proof. First, note that formula (2.6.4) immediately follows from (2.5.2) and (2.3.2). Let us prove (2.6.5). Set $p = e^{i\theta}$. Lemma 2.5.3 implies

$$\lim_{a\to p}\int_{\mathbb{D}} X_s^2\,d\nu_a = \int_{\mathbb{D}} X_s^2\,R_p^*d\mu = \int_{\mathbb{D}} X_s^2\circ R_p\,d\mu = \int_{\mathbb{D}} X_{R_p s}^2\,d\mu. \tag{2.6.6}$$

Since $[e_1]$ is the unique maximizing direction for $\mathbb{D}$, the right hand side of (2.6.6) is maximal for $R_p s = \pm e_1$. Applying R_p on both sides we get $s = \pm e^{2i\theta}$ and hence $[s] = [e^{2i\theta}]$. q.e.d.

Proof of Proposition 2.6.2. Suppose that for each cap $a \subset \mathbb{D}$ the measure $d\nu_a$ is simple. Recall that the space of caps is identified with $(-1,1)\times S^1$. Define $h : (-1,1)\times S^1 \to \mathbb{R}P^1$ by $h(r,p) = [s(a_{r,p})]$. It follows from Lemma 2.6.3) that h extends to a continuous map on $[-1,1]\times S^1$ such that

$$h(-1,e^{i\theta}) = [e_1], h(1,e^{i\theta}) = [e^{2i\theta}].$$

This means that h is a homotopy between a trivial loop and a non-contractible loop on $\mathbb{R}P^1$. This is a contradiction. q.e.d.

2.7. Test functions. Assume that $d\mu$ is simple. By Proposition 2.6.2 and Lemma 2.6.1 there exists a cap $a \subset \mathbb{D}$ such that

$$\int_{\mathbb{D}} X_s^2\,d\nu_a(z)$$

does not depend on the choice of $s \in S^1$. Let $a^* = \mathbb{D}\setminus \overline{a}$.

Definition 2.7.1. Given a function $u : a \to \mathbb{R}$, the *lift* of u, $\tilde{u} : \mathbb{D} \to \mathbb{R}$ is given by

$$\tilde{u}(z) = \begin{cases} u(z) & \text{if } z \in a, \\ u(\tau_a z) & \text{if } z \in a^*. \end{cases}$$

Given $u : a \to \mathbb{R}$ we have

$$\int_a u \, d\mu_a = \int_a u \, d\mu + \int_{a^*} u \circ \tau_a \, d\mu = \int_{\mathbb{D}} \tilde{u} \, d\mu.$$

For every $s \in \mathbb{R}^2$, set

$$u_a^s = X_s \circ \psi_a^{-1} : a \to \mathbb{R}.$$

We will use the two-dimensional space

$$E = \left\{ \tilde{u}_a^s \,\middle|\, s \in \mathbb{R}^2 \right\}$$

of test functions in the variational characterization (2.3.3) of μ_2. Note that since τ is the identity map on the geodesic $\gamma = \partial a \cap \mathbb{D}$, the functions $\tilde{u}_a^s$ can be extended continuously to γ, and hence $\tilde{u}_a^s \in H^1(\mathbb{D})$.

Proposition 2.7.2. *For each* $s \in \mathbb{R}^2$

$$\frac{\int_{\mathbb{D}} |\nabla \tilde{u}_a^s|^2 \, dz}{\int_{\mathbb{D}} (\tilde{u}_a^s)^2 \, d\mu} \leq 2\mu_1(\mathbb{D}). \tag{2.7.3}$$

We split the proof of Proposition 2.7.2 in two parts.

Lemma 2.7.4. *For any hyperbolic cap* $a \subset \mathbb{D}$,

$$\int_{\mathbb{D}} |\nabla \tilde{u}_a^s|^2 \, dz = \left(2\pi \int_{r=0}^1 f^2(r) r \, dr \right) \mu_1(\mathbb{D}).$$

Lemma 2.7.5.

$$\int_{\mathbb{D}} (\tilde{u}_a^s)^2 \, d\mu \geq \pi \left(\int_{r=0}^1 f^2(r) r \, dr \right). \tag{2.7.6}$$

Proof of Lemma 2.7.4. It follows from the definition of the lift that

$$\int_{\mathbb{D}} |\nabla \tilde{u}_a^s|^2 \, dz = \int_a |\nabla u_a^s|^2 \, dz + \int_{a^*} |\nabla (u_a^s \circ \tau_a)|^2 \, dz.$$

By conformal invariance of the Dirichlet energy, the two terms on the right hand side are equal, so that

$$\begin{aligned} \int_{\mathbb{D}} |\nabla \tilde{u}_a^s|^2 \, dz &= 2 \int_a |\nabla u_a^s|^2 \, dz = 2 \int_a |\nabla (X_s \circ \psi_a^{-1})|^2 \, dz \\ &= 2 \int_{\mathbb{D}} |\nabla X_s|^2 \, dz \qquad \longleftarrow \text{ (by conformal invariance)} \end{aligned}$$

(2.7.7)

$$= 2\mu_1(\mathbb{D}) \int_{\mathbb{D}} X_s^2 \, dz \qquad \longleftarrow \text{ (since } X_s \text{ is the first eigenfunction on a disk)}$$

It follows from (2.1.1) that given two orthogonal directions $s, t \in S^1$ we have

$$\int_{\mathbb{D}} (X_s^2 + X_t^2)\, dz = \int_{\mathbb{D}} f^2(|z|)\, dz.$$

Therefore, by symmetry we get

$$\int_{\mathbb{D}} X_s^2\, dz = \frac{1}{2} \int_{\mathbb{D}} f^2(|z|)\, dz = \pi \int_{r=0}^{1} f^2(r) r\, dr.$$

This completes the proof of the lemma. q.e.d.

To prove Lemma 2.7.5 we use the following result.

Lemma 2.7.8. *The rearranged measure $d\nu_a$ on $\mathbb{D}$ can be represented as $d\nu_a = \delta(z)dz$, where $\delta : \mathbb{D} \to \mathbb{R}$ is a subharmonic function.*

Proof. Indeed, $d\nu_a = \psi_a^* d\mu_a$, where the measure $d\mu_a$ on the cap a is obtained as the sum of measures $d\mu$ and $\tau_a^* d\mu$. Both measures $d\mu$ and $\tau_a^* d\mu$ correspond to flat Riemannian metrics on a, because $d\mu$ is the pullback of the Euclidean measure dz on the domain Ω by the conformal map $\phi : \mathbb{D} \to \Omega$ (see section 2.3). Since the maps ψ_a and τ_a are also conformal, one has $\psi_a^* d\mu = \alpha(z)dz$ and $\psi_a^*(\tau_a^* d\mu) = \beta(z)dz$ for some *subharmonic* functions $\alpha(z)$, $\beta(z)$. Indeed, the metrics corresponding to these measures are flat (they are pullbacks by ψ_a of flat metrics on a that we mentioned above), and the well-known formula for the Gaussian curvature in isothermal coordinates yields $\Delta \log \alpha(z) = \Delta \log \beta(z) = 0$ (cf. [**BR**, p. 663]). Therefore, $\alpha(z)$ and $\beta(z)$ are subharmonic as exponentials of harmonic functions [**Le**, p. 45]. Finally, $d\nu_a = \delta(z)dz$, where $\delta(z) = \alpha(z) + \beta(z)$ is subharmonic as a sum of subharmonic functions. This completes the proof of the lemma. q.e.d.

Proof of Lemma 2.7.5*.* Set

$$G(r) = \int_{B(0,r)} \delta(z)\, dz = \int_0^r \int_0^{2\pi} \delta(\rho\, e^{i\phi}) \rho\, d\rho\, d\phi.$$

By Lemma 2.7.8 the function δ is subharmonic. The function

$$W(\rho) = \int_0^{2\pi} \delta(\rho\, e^{i\phi}) d\phi$$

is 2π times the average of δ over the circle of radius ρ, hence it is monotone non-decreasing in ρ ([**Le**, p. 46]). Therefore, since $r \leq 1$, we get as in [**SY**, p.138] that

$$\text{(2.7.9)} \quad G(r) = \int_0^r W(\rho)\rho\, d\rho = r^2 \int_0^1 W(r\,\rho)\rho\, d\rho \leq r^2 \int_0^1 W(\rho)\, \rho\, d\rho = r^2 G(1) = \pi r^2.$$

Now, because $\tilde{u}_a^s$ is the lift of $u_a^s = X_s \circ \psi_a$, we have

$$\int_{\mathbb{D}} (\tilde{u}_a^s)^2 \, d\mu = \int_a (u_a^s)^2 \, d\mu_a = \int_{\mathbb{D}} X_s^2 \, d\nu_a.$$

Moreover since $V_a(s)$ doesn't depend on $s \in S^1$,

$$\begin{aligned} V_a(s) = \int_{\mathbb{D}} X_s^2 \, d\nu_a &= \frac{1}{2} \int_{\mathbb{D}} \left(X_{e_1}^2 + X_{e_2}^2 \right) \, d\nu_a \\ &= \frac{1}{2} \int_{\mathbb{D}} f^2(|z|) \, \delta(z) \, dz = \frac{1}{2} \int_{r=0}^{1} f^2(r) G'(r) \, dr. \end{aligned} \tag{2.7.10}$$

Integrating by parts and taking into account that $G(r) \leq \pi r^2$ due to (2.7.9), we get

$$\begin{aligned} \int_{r=0}^{1} f^2(r) G'(r) \, dr = f^2(1) G(1) - \int_0^1 \frac{d}{dr} \big(f^2(r) \big) G(r) \, dr \ \geq \\ f^2(1) G(1) - \pi \int_0^1 \frac{d}{dr} \big(f^2(r) \big) r^2 \, dr = 2\pi \int_0^1 f^2(r) r \, dr. \end{aligned} \tag{2.7.11}$$

This completes the proof of Lemma 2.7.5 and Proposition 2.7.2.

q.e.d.

Remark 2.7.12. The proof of Lemma 2.7.5 is quite similar to the proof of (1.1.1), see [**Sz**, p. 348] and [**SY**, p. 138]. Our approach is somewhat more direct since it explicitly uses the subharmonic properties of the measure.

Proof of Theorem 1.1.3. Assume that $d\mu$ is simple. Then (1.1.4) immediately follows from Proposition 2.7.2 and the variational characterization (2.3.3) of μ_2.

Suppose now that $d\mu$ is multiple. In fact, the proof is simpler in this case. Indeed, it follows from Lemma 2.6.1, that any direction $[s] \in \mathbb{R}P^1$ is maximizing for $d\mu$ so that we can use the space

$$E = \left\{ X_s \,\middle|\, s \in \mathbb{R}^2 \right\}$$

of test functions in the variational characterization (2.3.3) of μ_2. Inspecting the proof of Proposition 2.7.2 we notice that the factor 2 disappears in (2.7.7) and hence in (2.7.3) as well. Therefore, in this case we get using (2.3.3) that $\mu_2(\Omega) \leq \mu_1(\mathbb{D})$. This completes the proof of the theorem. q.e.d.

Remark 2.7.13. When $d\mu$ is multiple, we get a stronger estimate

$$\mu_2(\Omega) \leq \mu_1(\mathbb{D}).$$

To illustrate this case, consider $\Omega = \mathbb{D}$. Then indeed $\mu_2(\mathbb{D}) = \mu_1(\mathbb{D})$.

3. Proofs of auxiliary lemmas

3.1. Uniqueness of the renormalizing point. The following lemma is used in the proof Lemma 2.2.4.

Lemma 3.1.1. *Let $r \in (0,1)$ and $s = 1$. Then $X_s(d_r(z)) > X_s(z)$ for all $z \in \mathbb{D}$.*

Proof. We have $X_s(z) = f(|z|)\cos\theta_1$ and $X_s(d_r(z)) = f(|d_r(z)|)\cos\theta_2$, where $\theta_1 = \arg z$ and $\theta_2 = \arg d_r(z)$. We need to show

$$f(|d_r(z)|)\cos\theta_2 > f(|z|)\cos\theta_1 \tag{3.1.2}$$

for all $z \in \mathbb{D}$. Note that the function f is monotone increasing, positive on the interval $(0,1]$, and $f(0) = 0$. Set $z = a + ib$. It is easy to check that for $|z| = 0$ the inequality in question is satisfied and therefore in the sequel we assume that $a^2 + b^2 > 0$.

Let us compare $|z|$ and $|d_r(z)|$. We note that $|z| = |\bar{z}|$. Since

$$|d_r(z)| = \frac{|z+r|}{|rz+1|},$$

we need to compare $|z+r|$ and $|r|z|^2 + \bar{z}|$. This boils down to comparing $(a+r)^2 + b^2$ and $((r(a^2+b^2)+a)^2 + b^2$, or, equivalently, $(a+r)^2$ and $((r(a^2+b^2)+a)^2$. Note that $a^2 + b^2 < 1$ since $z \in \mathbb{D}$. We have three cases:

(i) $a \geq 0$. Then $|d_r(z)| > |z|$.
(ii) $a < 0$ and $a + r \leq 0$. Then $|d_r(z)| < |z|$.
(iii) $a < 0$ and $a + r > 0$.

Let us now study the arguments θ_1 and θ_2.

We have:

$$d_r(z) = \frac{z+r}{rz+1} = \frac{(a+r)+ib}{(ar+1)+ibr} = \frac{(a+r)(ar+1) + b^2r + ib(1-r^2)}{(ar+1)^2 + b^2r^2}.$$

Taking into account that $ar + 1 > 0$, we obtain from this formula that in case (iii) $\cos\theta_2 > 0$. On the other hand, $\cos\theta_1 < 0$ in this case, and therefore the inequality (3.1.2) is satisfied since $f > 0$.

Consider now case (i). Using the formula above we get that

$$\tan\theta_2 = \frac{b(1-r^2)}{(a+r)(ar+1) + b^2r}.$$

If $a = 0$ then (3.1.2) is true since $\cos\theta_1 = 0$ and one may easily check that $\cos\theta_2 > 0$. So let us assume that $a \neq 0$. Then $\tan\theta_1 = b/a$. Note that the tangent is a monotone increasing function. If $b = 0$ then $\theta_1 = \theta_2 = 0$ and (3.1.2) is satisfied since $|d_r(z)| > |z|$. If $b \neq 0$, dividing by b and taking into account that $a > 0$, $r > 0$ we easily get:

$$\frac{1}{a} > \frac{1-r^2}{(a+r)(ar+1) + b^2r}.$$

Therefore, if $b > 0$ we get that $\tan\theta_1 > \tan\theta_2$ implying $0 < \theta_2 < \theta_1 < \pi/2$, and if $b < 0$ we get that $\tan\theta_1 < \tan\theta_2$ implying that $3\pi/2 < \theta_1 < \theta_2 < 2\pi$. At the same time, in the first case the cosine is monotonically decreasing, and in the second case the cosine is monotonely increasing. Therefore, for any $b \neq 0$ we get $0 < \cos\theta_1 < \cos\theta_2$, which implies (3.1.2).

Finally, consider the case (ii). If $(a + r)(ar + 1) + b^2 r \geq 0$ then we immediately get (3.1.2) since in this case $\cos\theta_2 \geq 0$ and $\cos\theta_1 < 0$. So let us assume $(a + r)(ar + 1) + b^2 r < 0$. If $b = 0$ then $\theta_1 = \theta_2 = \pi$, hence $\cos\theta_1 = \cos\theta_2 = -1$ and (3.1.2) is satisfied because $|d_r(z)| < |z|$. If $b \neq 0$, as in case (ii) we compare $\tan\theta_1$ and $\tan\theta_2$. We claim that again

$$\frac{1}{a} > \frac{1 - r^2}{(a + r)(ar + 1) + b^2 r}.$$

Since by our hypothesis the denominators in both cases are negative, it is equivalent to $a - ar^2 < a^2 r + ar^2 + a + r + b^2 r$. After obvious transformations we see that this reduces to $a^2 + 2ar + 1 + b^2 = (a + r)^2 + (1 - r^2) + b^2 > 0$ which is true.

Therefore, taking into account that tangent is monotone increasing, we get that if $b > 0$ then $\pi/2 < \theta_2 < \theta_1 < \pi$, and if $b < 0$ then $\pi < \theta_1 < \theta_2 < 3\pi/2$. This implies that in either case $\cos\theta_1 < \cos\theta_2 < 0$. Together with the inequality $|d_r(z)| < |z|$ this gives (3.1.2) in case (ii). This completes the proof of the lemma. q.e.d.

3.2. Proof of Lemma 2.5.3. Let $\mathcal{M}$ be the space of signed finite measures on $\mathbb{D}$ endowed with the norm (2.2.2). Recall that the map $\Gamma : \mathcal{M} \to \mathbb{D}$ is defined by $\Gamma(d\nu) = \xi$ in such a way that $d_\xi : \mathbb{D} \to \mathbb{D}$ renormalizes $d\nu$. It is continuous by Corollary 2.2.5. The key idea of the proof of the "flip-flop" lemma is to replace the folded measure $d\mu_a$ by

$$d\hat{\mu}_a := (\tau_a)_* d\mu.$$

It is clear that

$$||d\mu_a - d\hat{\mu}_a|| \to 0 \qquad (3.2.1)$$

in the norm (2.2.2) as a degenerates to a point $p \in \partial\mathbb{D}$. At the same time, the next lemma shows that the "flip-flop"property is true for *each* cap when the rearranged measure $d\nu_a$ is replaced by $(d_{\zeta_a})_* d\hat{\mu}_a$, where $\zeta_a = \Gamma(d\hat{\mu}_a)$.

Lemma 3.2.2. *Let $a = a_{r,p}$ be a hyperbolic cap. Then*

$$(d_{\zeta_a})_* d\hat{\mu}_a = (d_{\zeta_a})_* (\tau_a)_* d\mu = R_p^* d\mu.$$

Proof. Let us show that $\zeta_a = -\frac{2r}{r^2+1} p$. Recall that $\tau_a(z) = d_{rp} \circ R_p \circ d_{-rp}$. A simple explicit computation then leads to

$$d_{\zeta_a} \circ \tau_a = R_p.$$

This implies

$$\int_{\mathbb{D}} X_s \circ d_{\zeta_a}\, d\hat{\mu}_a = \int_{\mathbb{D}} X_s \circ d_{\zeta_a} \circ \tau_a \, d\mu$$
$$= \int_{\mathbb{D}} X_s \circ R_p \, d\mu = \int_{\mathbb{D}} X_{R_p s}\, d\mu = 0$$

which proves the claim. q.e.d.

Let $\eta_a := \Gamma((d_{\zeta_a})_* d\mu_a)$ be the renormalizing vector for the measure $(d_{\zeta_a})_* d\mu_a$.

Lemma 3.2.3. *As the cap a degenerates to a point $p \in \partial\mathbb{D}$, $\eta_a \to 0$.*

Proof. Since d_{ζ_a} is a diffeomorphism, $(d_{\zeta_a})_* : \mathcal{M} \to \mathcal{M}$ is an isometry so that

$$\begin{aligned}(d_{\zeta_a})_* d\mu_a &= (d_{\zeta_a})_*(d\mu_a - d\hat{\mu}_a) + (d_{\zeta_a})_* d\hat{\mu}_a \\ &= \underbrace{(d_{\zeta_a})_*(d\mu_a - d\hat{\mu}_a)}_{\to 0} + \underbrace{(d_{\zeta_a} \circ \tau_a)}_{R_p}{}_* d\mu \to (R_p)_* d\mu.\end{aligned}$$

Here we have used (3.2.1). Continuity of Γ leads to

$$0 = \Gamma((R_p)_* d\mu) = \lim_{a\to p} \Gamma((d_{\zeta_a})_* d\mu_a) = \lim_{a \to p} \eta_a.$$

Note that the first equality follows from (2.3.1) and the identity $X_s \circ R_p = X_{R_p s}$ that we used earlier. q.e.d.

Set

$$q(a) = \frac{\overline{\zeta_a}\eta_a + 1}{\zeta_a \overline{\eta_a} + 1}, \quad \xi(a) = d_{\zeta_a}(\eta_a) = \left(\frac{\eta_a + \zeta_a}{\overline{\zeta_a}\eta_a + 1}\right). \tag{3.2.4}$$

A direct computation (cf. the proof of Lemma 2.2.4) leads to

$$\tilde{T}_a(z) := d_{\eta_a} \circ d_{\zeta_a} = q(a) d_{\xi(a)}(z).$$

It follows from its definition that $\tilde{T}_a$ renormalizes $d\mu_a$. Hence, $\Gamma(d\mu_a) = \xi(a)$ and $d_{\xi(a)} = T_a$, where the transformation T_a was defined in section 2.5. We have

$$\begin{aligned}T_{a*} d\mu_a &= (\frac{1}{q(a)} d_{\eta_a})_* (d_{\zeta_a})_* d\mu_a \\ &= (\frac{1}{q(a)} d_{\eta_a})_* (d_{\zeta_a})_* \left(d\hat{\mu}_a + (d\mu_a - d\hat{\mu}_a)\right).\end{aligned}$$

Now, it follows from Lemma 3.2.3 that $\lim_{a\to p} q(a) = 1$ and $\lim_{a \to p} d_{\eta_a} = \mathrm{id}$, because $\eta_a \to 0$. Therefore, taking into account (3.2.1) we get

$$\lim_{a\to p} T_{a*} d\mu_a = \lim_{a \to p} (d_{\xi_a})_* d\hat{\mu}_a = R_p^* d\mu.$$

To complete the proof of Lemma 2.5.3 it remains to show that as the cap a degenerates to p, $||T_{a*} d\mu_a - d\nu_a|| \to 0$. By definition $d\nu_a = \psi_a^* d\mu$, where $\psi_a = (T'_a \circ \phi_b \circ T_a)^{-1}$ (see section 2.5). Let us show that $b = T_a(a) \to \mathbb{D}$ as $a \to p$. Indeed,

$$T_a = d_{\xi(a)} = d_{\zeta_a} \circ (d_{-\zeta_a} \circ d_{\xi(a)}) = R_p \circ \tau_a \circ (d_{-\zeta_a} \circ d_{\xi(a)}).$$

Since $\eta_a \to 0$ when $a \to p$, it follows from (3.2.4) that the composition $d_{-\zeta_a} \circ d_{\xi(a)}$ tends to identity. Therefore, the cap $T_a(a)$ gets closer to $\mathbb{D} \setminus R_p(a)$ when a goes to p and thus $\lim_{a\to p} T_a(a) = \mathbb{D}$. This implies $\lim_{a\to p} \phi_{T_a(a)} = \mathrm{id}$ and $\lim_{a\to p} T'_a = \mathrm{id}$, and hence $\lim_{a\to p} ||T_{a*}d\mu_a - d\nu_a|| = 0$. q.e.d.

4. Proof of Theorem 1.2.5

4.1. Hersch's lemma and uniqueness of the renormalizing map. The proof of Theorem 1.2.5 is quite similar to the proof of Theorem 1.1.3. We use the following notation

$$\mathbb{B}^{n+1} = \{x \in \mathbb{R}^{n+1}, |x| < 1\}$$
$$\mathbb{S}^n = \partial \mathbb{B}^{n+1}.$$

The standard round metric on $\mathbb{S}^n$ is g_0. Given a conformally round metric $g \in [g_0]$ we write dg for its induced measure. Given $s \in \mathbb{R}^{n+1}$, define $X_s : \mathbb{S}^n \to \mathbb{R}$ by

$$X_s(x) = (x, s).$$

Similarly to (2.3.1) and (2.3.2), we assume that for each $s \in \mathbb{S}^n$:

$$\int_{\mathbb{S}^n} X_s\, dg = 0. \tag{4.1.1}$$

$$\int_{\mathbb{S}^n} X_{e_1}^2\, dg \geq \int_{\mathbb{S}^n} X_s^2\, dg. \tag{4.1.2}$$

Given $p \in \mathbb{S}^n$, $R_p : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is the reflection with respect to the hyperplane going through 0 and orthogonal to the segment joining 0 and p, that is

$$R_p(x) = x - 2(p, x)p.$$

Given $\xi \in \mathbb{B}^{n+1}$. define $d_\xi : \overline{\mathbb{B}}^{n+1} \to \overline{\mathbb{B}}^{n+1}$ by

$$d_\xi(x) = \frac{(1 - |\xi|^2)x + (1 + 2(\xi, x) + |x|^2)\xi}{1 + 2(\xi, x) + |\xi|^2|x|^2}. \tag{4.1.3}$$

Note that $d_\xi(0) = \xi$ and $d_\xi \circ d_{-\xi} = \mathrm{id}$. The map d_ξ is a conformal (Möbius) transformation of $\mathbb{S}^n$ [**SY**, p. 142]. Indeed, one can check that for $\xi \neq 0$,

$$d_\xi = \gamma_\xi \circ R_{\frac{\xi}{|\xi|}}$$

where γ_ξ is the spherical inversion with center $\frac{\xi}{|\xi|^2}$ and radius $\frac{1-|\xi|^2}{|\xi|^2}$. Note that for $n = 1$, the map d_ξ coincides with the one introduced in Lemma 2.2.3, where complex notation was used for convenience.

Similarly to the disk case, the transformation d_ξ is said to *renormalize* a measure $d\nu$ on the sphere $\mathbb{S}^n$ if for each $s \in \mathbb{R}^{n+1}$,

$$\int_{\mathbb{S}^n} X_s \circ d_\xi \, d\nu = 0. \tag{4.1.4}$$

This condition is clearly equivalent to

$$\int_{\mathbb{S}^n} x_i \circ d_\xi \, d\nu = 0, \quad i = 1, 2, \ldots, n+1,$$

which means that the center of mass of the measure $(d_\xi)_* d\nu$ on $\mathbb{S}^n$ is at the origin. The following result is a combination of Hersch's lemma [**Her**] and a uniqueness result announced in [**Na**].

Proposition 4.1.5. *For any finite measure $d\nu$ on $\mathbb{S}^n$, there exists a unique point $\xi \in \mathbb{B}^{n+1}$ such that d_ξ renormalizes $d\nu$. Moreover, the dependence of the point $\xi \in \mathbb{B}^{n+1}$ on the measure $d\nu$ is continuous.*

Proof. The existence of ξ is precisely Hersch's lemma (see [**Her**], [**SY**, p. 144], [**LY**, p. 274]).

Let us prove uniqueness. First, let us show that if $d\nu$ is a renormalized measure then $\xi = 0$. It follows from (4.1.3) by a straightforward computation that if $\mathbb{B}^{n+1} \ni \xi \neq 0$ then $X_\xi(x) < X_\xi(d_\xi(x))$ for any $x \in \mathbb{S}^n$. Assume that d_ξ renormalizes $d\nu$ for some $\xi \neq 0$. Then

$$0 = \int_{\mathbb{S}^n} X_\xi \, d\nu < \int_{\mathbb{S}^n} X_\xi \circ d_\xi \, d\nu = 0,$$

and we get a contradiction.

Now, let $d\nu$ be an arbitrary finite measure and suppose that it is renormalized by d_ξ and d_η. Writing $d_\eta = d_\eta \circ d_{-\xi} \circ d_\xi$ we get

$$\int_{\mathbb{S}^n} X_s \circ d_\eta \circ d_{-\xi} \, d\tilde{\sigma} = 0 \tag{4.1.6}$$

where the measure $d\tilde{\sigma} = (d_\xi)_* d\sigma$ is renormalized. At the same time, it easy to check that $d_\eta \circ d_{-\xi} = R \circ d_{d_{-\xi}(\eta)}$, where R is an orthogonal transformation. Indeed, since $-d_{-\xi}(\eta) = d_\xi(-\eta)$ we have

$$d_\eta \circ d_{-\xi} \circ d_{d_\xi(-\eta)}(0) = d_\eta(-\eta) = 0,$$

and it is well known that any Möbius transformation of the unit ball preserving the origin is orthogonal [**Bea**, Theorem 3.4.1]. Since R preserves the center of mass at zero, it follows from (4.1.6) that $d_{d_{-\xi}(\eta)}$ renormalizes the measure $d\tilde{\sigma}$, which is already renormalized. Therefore, as we have shown above, $d_{-\xi}(\eta) = 0$ and hence $\xi = \eta$.

Similarly to Corollary 2.2.5, uniqueness of the renormalizing point implies that its dependence on the measure is continuous. q.e.d.

4.2. Spherical caps, folding and rearrangement. The set $\mathcal{C}$ of all spherical caps is parametrized as follows: given $p \in \mathbb{S}^n$ let

$$a_{0,p} = \{x \in \mathbb{S}^n \ : (x,p) > 0\}$$

be the half-sphere centered at p. Given $-1 < r < 1$, let

$$a_{r,p} = d_{rp}(a_{0,p}).$$

To every spherical cap $a \in \mathcal{C}$ we associate a *folded* measure:

$$d\mu_a = \begin{cases} dg + \tau_a^* dg & \text{on } a, \\ 0 & \text{on } a^*, \end{cases}$$

where $a^* = \mathbb{S}^n \setminus \overline{a} \in \mathcal{C}$ is the cap adjacent to a, and τ_a is the conformal reflection with respect to the boundary circle of a. That is, for $a = a_{r,p}$

$$\tau_a = d_{rp} \circ R_p \circ d_{-rp}.$$

Let $\xi(a) \in \mathbb{B}^{n+1}$ be the unique point such that $d_{\xi(a)}$ renormalizes $d\mu_a$. We obtain a *rearranged folded measure*

$$d\nu_a = (d_{\xi(a)})_* d\mu_a. \tag{4.2.1}$$

4.3. Maximizing directions. Given a finite measure $d\nu$ on $\mathbb{S}^n$, define

$$V(s) = \int_{\mathbb{S}^n} X_s^2 d\nu.$$

Let $\mathbb{R}P^n$ be the projective space and let $[s] \in \mathbb{R}P^n$ be the point corresponding to $\pm s \in \mathbb{S}^n$. We say that $[s] \in \mathbb{R}P^n$ is a *maximizing direction* for $d\nu$ if $V(s) \geq V(t)$ for all $t \in \mathbb{S}^n$. We say that the spherical cap is *simple* if the maximizing direction is unique. Otherwise, similarly to Lemma 2.6.1, there exists a two-dimensional subspace $W \subset \mathbb{R}^{n+1}$ such that any $s \in W \cap \mathbb{S}^n$ is a maximizing direction for $d\nu$. In particular for each $s, t \in W$, $V(s) = V(t)$. In this case the measure $d\nu$ is called *multiple*.

Proposition 4.3.1. *Let $g \in [g_0]$ be a conformally round metric on a sphere $\mathbb{S}^n$ of odd dimension. If the measure dg is simple then there exists a spherical cap such that the rearranged folded measure $d\nu_a$ is multiple.*

The proof of Proposition 4.3.1 is similar to the proof of Proposition 2.6.2. We assume the measures dg as well as each $d\nu_a$ to be simple. Given a cap $a \subset \mathbb{S}^n$ let $[s(a)] \in \mathbb{R}P^1$ be the unique maximizing direction for $d\nu_a$. The map $a \mapsto [s(a)]$ is continuous. The following spherical version of the "flip-flop" property is proved exactly as Lemma 2.5.3.

Lemma 4.3.2. *If a sequence of spherical caps $a \in \mathcal{C}$ degenerates to a point $p \in \mathbb{S}^n$, the limiting rearranged measure is a "flip-flop" of the original measure dg:*

$$\lim_{a \to p} d\nu_a = R_p^* dg. \tag{4.3.3}$$

Similarly to Lemma 2.6.3 we study the maximizing directions for degenerating caps.

Lemma 4.3.4. *Suppose the measures dg as well as each $d\nu_a$ are simple. Then*

$$\lim_{a\to\mathbb{S}^n}[s(a)] = [e_1] \tag{4.3.5}$$

$$\lim_{a\to p}[s(a)] = [R_p e_1]. \tag{4.3.6}$$

Proof of Proposition 4.3.1. By convention (4.1.2), $[e_1]$ is the unique maximizing direction for dg. Recall that the space of caps has been identified with $(-1,1)\times\mathbb{S}^n$. The continuous map

$$h : [-1,1]\times\mathbb{S}^n \to \mathbb{R}P^n$$

is defined by

$$h(r,p) = \begin{cases} [e_1] & \text{for } r=-1,\\ [s(a_{r,p})] & \text{for } -1<r<1,\\ [R_p e_1] & \text{for } r=1.\end{cases}$$

That is, h is an homotopy between a constant map and the map

$$\phi : \mathbb{S}^n \to \mathbb{R}P^n$$

defined by $\phi(p) = [R_p e_1]$. We will show that this is impossible when n is odd by computing its degree. The map ϕ lifts to the map $\psi : \mathbb{S}^n \to \mathbb{S}^n$ defined by

$$\psi(p) = -R_p e_1 = 2(e_1,p)p - e_1. \tag{4.3.7}$$

The two solutions of $\psi(p) = e_1$ are e_1 and $-e_1$. It is easy to check that since the dimension n is odd, both differentials

$$D_{e_1}\psi : T_{e_1}\mathbb{S}^n \to T_{e_1}\mathbb{S}^n$$
$$D_{-e_1}\psi : T_{-e_1}\mathbb{S}^n \to T_{e_1}\mathbb{S}^n$$

preserve the orientation. This implies $\deg(\psi) = 2$. Moreover, the quotient map $\pi : \mathbb{S}^n \to \mathbb{S}^n$ has degree 2 for n odd. It follows that

$$\begin{aligned}\deg(\phi) &= \deg(\pi\circ\psi)\\ &= \deg(\pi)\deg(\psi) = 4.\end{aligned}$$

Since the degree of a map is invariant under homotopy, this is a contradiction. q.e.d.

Remark 4.3.8. In even dimensions one of the differentials $D_{\pm e_1}$ preserves the orientation and the other reverses it. Therefore, $\deg(\psi) = 0$ and the proof of Proposition 4.3.1 does not work in this case. In dimension two the existence of a multiple cap was proved in [**Na**] using a more sophisticated topological argument.

4.4. Test functions and the modified Rayleigh quotient. Let g_0 be the standard round metric on the sphere $\mathbb{S}^n$, so that

$$\omega_n := \int_{\mathbb{S}^n} dg_0 = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}. \tag{4.4.1}$$

Let $g \in [g_0]$ be a conformally round Riemannian metric of volume one, that is $\int_{\mathbb{S}^n} dg = 1$. The Rayleigh quotient of a non-zero function $u \in H^1(\mathbb{S}^n)$ is

$$R(u) = \frac{\int_{\mathbb{S}^n} |\nabla_g u|_g^2 \, dg}{\int_{\mathbb{S}^n} u^2 \, dg}.$$

We use the following variational characterization of $\lambda_2(g)$:

$$\lambda_2(g) = \inf_E \sup_{0 \neq u \in E} R(u) \tag{4.4.2}$$

where E varies among all two-dimensional subspaces of the Sobolev space $H^1(\mathbb{S}^n)$ that are orthogonal to constants, in the sense that for each $f \in E$, $\int_{\mathbb{S}^n} f \, dg = 0$. Following [**FN**], we use a *modified Rayleigh quotient*:

$$R'(u) = \frac{\left(\int_{\mathbb{S}^n} |\nabla_g u|_g^n \, dg\right)^{2/n}}{\int_{\mathbb{S}^n} u^2 \, dg}.$$

It follows from Hölder inequality that $R(u) \leq R'(u)$ for each $0 \neq u \in H^1(\mathbb{S}^n)$. It is easy to check that $\int_{\mathbb{S}^n} |\nabla_g u|_g^n \, dg$ is conformally invariant for each dimension n so that we can rewrite the modified Rayleigh quotient as follows:

$$R'(u) = \frac{\left(\int_{\mathbb{S}^n} |\nabla u|^n \, dg_0\right)^{2/n}}{\int_{\mathbb{S}^n} u^2 \, dg}$$

where the gradient and it's norm are with respect to the round metric g_0.

Assume that dg is simple and let $a \subset \mathbb{S}^n$ be a spherical cap such that $d\nu_a$ is multiple. Let $W \subset \mathbb{R}^{n+1}$ be the corresponding two dimensional subspace of maximizing directions. Given a function $u : a \to \mathbb{R}$, the *lift* of u, $\tilde{u} : \mathbb{S}^n \to \mathbb{R}$ is defined exactly as in Definition 2.7.1.

Proposition 4.4.3. *Given $s \in W \subset \mathbb{R}^{n+1}$, the function $u_a^s = X_s \circ d_{\xi(a)} : a \to \mathbb{R}$ is such that*

$$R'(\tilde{u}_a^s) < (n+1) \left(4 \frac{\pi^{\frac{n+1}{2}} \Gamma(n)}{\Gamma(\frac{n}{2}) \Gamma(n + \frac{1}{2})} \right)^{2/n}.$$

Proof. The conformal invariance of the modified Dirichlet energy in the numerator of $R'(u)$ implies

$$\int_{\mathbb{S}^n} |\nabla_g \tilde{u}_a^s|_g^n \, dg = \int_a |\nabla_g u_a^s|_g^n \, dg + \int_{a^*} |\nabla_g (u_a^s \circ \tau_a)|_g^n \, dg \tag{4.4.4}$$
$$= 2\int_a |\nabla_g u_a^s|_g^n \, dg = 2\int_{d_{\xi(a)}(a)} |\nabla_g X_s|_g^n \, dg$$
$$< 2\int_{\mathbb{S}^n} |\nabla_{g_0} X_s|_{g_0}^n \, dg_0.$$

Here the second and third equalities follows from conformal invariance. To obtain the inequality at the end we again use the conformal invariance as well as the fact that $d_{\xi(a)}(a) \subsetneq \mathbb{S}^n$. To estimate the denominator in the modified Rayleigh quotient we first note that for any $x = (x_1, \ldots x_{n+1}) \in \mathbb{S}^n$,

$$\sum_{j=1}^{n+1} \tilde{u}_a^{e_j}(x)^2 = \sum_{j=1}^{n+1} x_j^2 = 1.$$

Therefore, given that $\int_{\mathbb{S}^n} dg = 1$ we obtain:

$$\sum_{j=1}^{n+1} \int_{\mathbb{S}^n} (\tilde{u}_a^{e_j})^2 \, dg = 1.$$

Now, since W is a subspace of maximizing directions for the measure $d\nu_a$ defined by (4.2.1), for each $s \in W$ we have

$$\int_{\mathbb{S}^n} (\tilde{u}_a^s)^2 \, dg \geq \frac{1}{n+1}. \tag{4.4.5}$$

Set

$$K_n := \int_{\mathbb{S}^n} |\nabla_{g_0} X_s|_{g_0}^n \, dg_0.$$

Combining (4.4.4) and (4.4.5) we get

$$R'(\tilde{u}_a^s) < (n+1)\,(2K_n)^{2/n}. \tag{4.4.6}$$

Proposition 4.4.3 then follows from the lemma below.

Lemma 4.4.7. *The constant K_n is given by*

$$K_n = \frac{2\pi^{\frac{n+1}{2}}\Gamma(n)}{\Gamma(\frac{n}{2})\Gamma(n+\frac{1}{2})}.$$

Proof. Recall that g_0 is the standard round metric on the unit sphere $\mathbb{S}^n$. If we consider $X_s(x) = (x, s)$ as a function on $\mathbb{R}^{n+1}$ then its gradient is just the constant vector s:

$$\mathrm{grad}_{\mathbb{R}^{n+1}} X_s = s.$$

This means that for any point $p \in \mathbb{S}^n$ the gradient of the function $X_s : \mathbb{S}^n \to \mathbb{R}$ at p is the projection of s on the tangent space $T_p\mathbb{S}^n$:

$$\nabla X_s(p) = s - (s,p)p.$$

Therefore, taking into account that $|s| = |p| = 1$, we get

$$|\nabla X_s(p)|^n = (|s-(s,p)p|^2)^{n/2} = (1-(s,p)^2)^{n/2},$$

and hence

$$K_n = \int_{\mathbb{S}^n} (1-(s,p)^2)^{n/2} dg_0.$$

Let θ be the angle between the vectors p and s. Making a change of variables we obtain

$$K_n = \omega_{n-1} \int_0^\pi (1-\cos^2\theta)^{n/2} (\sin\theta)^{n-1}\, d\theta = \omega_{n-1} \int_0^\pi \sin^{2n-1}\theta\, d\theta,$$

where ω_{n-1} is the volume of the standard round sphere $\mathbb{S}^{n-1}$ given by (4.4.1).

The calculation of a table integral [**GR**, 3.621(4)]

$$\int_0^\pi \sin^{2n-1}\theta\, d\theta = \frac{\sqrt{\pi}\,\Gamma(n)}{\Gamma(n+\frac{1}{2})}$$

completes the proofs of Lemma 4.4.7 and Proposition 4.4.3. q.e.d.

Proof of Theorem 1.2.5. If the measure dg is simple, then (1.2.6) follows from Proposition 4.4.3 and the variational principle (4.4.2). If dg is multiple, then, as in the proof of Theorem 1.1.3 at the end of section 2.7, one can work directly with this measure without any folding and rearrangement. Inspecting the proof of Proposition 4.4.3 we notice that the factor $2^{2/n}$ disappears in (4.4.4) and hence also in (4.4.6). Therefore, in this case we get an even better bound than (1.2.6). This completes the proof of Theorem 1.2.5. q.e.d.

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