Roman Domination

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Abstract

In his article “Defend the Roman Empire!” (1999), Ian Stewart discussed a strategy of Emperor Constantine for defending the Roman Empire. Motivated by this article, Cockayne et al. (2004) introduced the notion of Roman domination in graphs.

Let $G = (V, E)$ be a graph. A Roman dominating function of $G$ is a function $f : V \rightarrow \{0, 1, 2\}$ such that every vertex $v$ for which $f(v) = 0$ has a neighbor $u$ with $f(u) = 2$. The weight of a Roman dominating function $f$ is $w(f) = \sum_{v \in V} f(v)$. The Roman domination number of a graph $G$, denoted by $\gamma_R(G)$, is the minimum weight of all possible Roman dominating functions.

This paper introduces a quantity $R(xy)$ for each pair of non-adjacent vertices $\{x, y\}$ in $G$, called the Roman dominating index of $\{x, y\}$, which is defined by $R(xy) = \gamma_R(G) - \gamma_R(G + xy)$. We prove that $0 \leq R(xy) \leq 1$ and give a necessary and sufficient condition on $\{x, y\}$ for which $R(xy) = 1$.

This paper also introduces the Roman-critical graph. We call $G = (V, E)$ Roman-critical if $\gamma_R(G - e) > \gamma_R(G), \forall e \in E$. It is proved that a Roman-critical graph can only be a star graph whose order is not equal to 2, or the union of such graphs.

In addition, this paper shows that for each connected graph $G$ of order $n \geq 3$, $2 \leq \gamma_R(G) \leq \left\lfloor \frac{4n}{5} \right\rfloor$. A family of graphs for which the respective equality holds is also provided.

Finally, this paper finds the lower bound of the Roman domination number for 3-regular graphs and the exact value of the Roman domination number for $C_n \times P_2$ and 3-regular circulant graphs.
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Note: Additional information, a separate document, offers solutions to problems mentioned in section 3.
1. History and Motivation

About 1700 years ago, the Roman Empire was under attack, and Emperor Constantine had to decide where to station his four field army units to protect eight regions. His trick was to place the army units so that every region was either secured by its own army (one or two units) or was securable by a neighbor with two army units, one of which can be sent to the undefended region directly if a conflict breaks out.

Constantine chose to place two army units in Rome and two at his new capital, Constantinople. This meant only Britain could not be reached in one step. As it happened, Constantine’s successors lost control of Britain. The causes were surely more complex than anything that can be explained by this simple model. Nevertheless, Stewart (1) is right in arguing that if Constantine had been a better mathematician, the Roman Empire might have lasted a little longer than it did.

Indeed, there are six ways to improve on Constantine’s deployment. These results were obtained through a form of zero-one integer programming by ReVelle and Rosing (2).

Besides placing of Roman army units, the same sort of mathematics can also be used for optimizing the location of the declining number of British Fleets at the end of the 19th century or American Military Units during the Cold War (2). In addition to army placement, the same sort of mathematics is also useful when people want to know the best place in town to put a new hospital, fire station, or fast-food restaurant. Many times such optimization problems can be modeled by Roman domination or its variants.
2. Definitions and existing results

2.1. General definitions in graph theory

The following are some basic definitions in graph theory, many of which are adopted from Introduction to Graph Theory: H3 Mathematics (4).

A graph $G$ consists of a non-empty finite set $V(G)$ of vertices together with a finite set $E(G)$ (possibly empty) of edges such that:

1. each edge joins two distinct vertices in $V(G)$ and
2. any two distinct vertices in $V(G)$ are joined by at most one edge.

The number of vertices in $G$, denoted by $v(G)$, is called the order of $G$.

Let $u, v$ be any two vertices in $G$. They are said to be adjacent if they are joined by an edge, say, $e$ in $G$. We also write $e = uv$ or $e = vu$ (the ordering of $u$ and $v$ in the expression is immaterial), and we say that

1. $u$ is a neighbor of $v$ and vice versa,
2. the edge $e$ is incident with the vertex $u$ (and $v$) and
3. $u$ and $v$ are the two ends of $e$.

The set of all neighbors of $v$ in $G$ is denoted by $N(v)$; that is,

$$N(v) = \{x | x \text{ is a neighbor of } v\}.$$

The degree of $v$ in $G$, denoted by $d(v)$, is defined as the number of edges incident with $v$. The vertex $v$ is called an end-vertex if $d(v) = 1$.

A path in a graph $G$ is an alternating sequence of vertices and edges beginning and ending at vertices:

$$v_0e_0v_1e_1v_2 \ldots v_{k-1}e_{k-1}v_k$$

where $k \geq 1$, $e_i$ is incident with $v_i$ and $v_{i+1}$, for each $i = 0, 1, \ldots, k - 1$, and the vertices $v_i$’s and edges $e_i$’s need to be distinct. The length of the path above is defined as $k$, which is the number of occurrences of edges in the sequence.

A graph $G$ is said to be connected if every two vertices in $G$ are joined by a path, and disconnected if it is not connected.
The distance from \( u \) to \( v \), denoted by \( d(u, v) \), is defined as the smallest length of all \( u - v \) paths in \( G \). (Note that \( d(v) \) denotes the degree of \( v \) in \( G \).)

Let \( P_n \) denote a path of \( n \) vertices, \( P_n = v_1v_2 \ldots v_n \), and \( C_n \) a cycle of \( n \) vertices, \( C_n = v_1v_2 \ldots v_nv_1 \).

Notice that we have two definitions for path. What a ‘path’ really means should be clear from the context when it is mentioned.

A graph \( H \) is called a subgraph of graph \( G \) if \( V(H) \subseteq V(G) \) and \( E(H) \subseteq E(G) \). A subgraph \( H \) of a graph \( G \) is said to be spanning if \( V(H) = V(G) \).

A bipartite graph is a set of graph vertices decomposed into two disjoint sets such that no two graph vertices within the same set are adjacent.

### 2.2. Roman domination defined

Let \( G = (V, E) \) be a graph. A Roman dominating function is a function \( f : V \to \{0, 1, 2\} \) such that every vertex \( v \) for which \( f(v) = 0 \) has a neighbor \( u \) with \( f(u) = 2 \).

The weight of a Roman dominating function \( f \) is \( w(f) = \sum_{v \in V} f(v) \). This corresponds to the total number of army units required under a specific deployment scheme.

We are interested in finding Roman dominating function(s) of minimum weight for a particular graph. It makes sense in the army placement context, because we want to minimize the number of army units needed to secure a particular set of given regions.

A Roman dominating function of minimum weight among all the possible Roman dominating functions is called a \( \gamma_R \)-function. The Roman domination number of a graph \( G \), denoted by \( \gamma_R(G) \), is the weight of a \( \gamma_R \)-function – the minimum weight of all possible Roman dominating functions.
2.3. Existing results

The decision problem corresponding to computing $\gamma_R(G)$ is NP-complete. (3)

We use $[x]$ to denote the smallest integer larger than or equal to $x$ while $\lfloor x \rfloor$ to denote the largest integer smaller than or equal to $x$.

The following result was proved by Dreyer (3).

Proposition 1: For path $P_n$ and cycle $C_n$ of order $n$,

$$\gamma_R(P_n) = \gamma_R(C_n) = \left\lceil \frac{2n}{3} \right\rceil.$$ 

The next observation follows readily from the definition.

Proposition 2: If $H$ is a spanning subgraph of a graph $G$, then $\gamma_R(H) \geq \gamma_R(G)$.

Sections 3 and 4 offer a comprehensive study on how adding or deleting an edge will affect the Roman domination number of a graph and how from the change in Roman domination number we can deduce about some properties of the graph and its Roman dominating function.

3. Roman dominating index

In practice, armies, utility operators, etc are concerned about where to build a new road, a pipeline and others so as to reduce the size of the army or reap the most economic benefits. As such, I would like to introduce a new concept called the Roman dominating index. It will be useful in simplifying some Roman domination problems like the one shown in Section 3.2.

Let $G$ be a graph and $x, y$ two non-adjacent vertices in $G$. The Roman dominating index of $\{x, y\}$, denoted by $R(x, y)$, is defined by $R(xy) = \gamma_R(G) - \gamma_R(G + xy)$.

As $G$ is a spanning subgraph of $(G + xy)$, by Proposition 2, $R(xy) \geq 0$. In what follows, we shall show that this quantity is always bounded above by 1.
3.1. The bounds of the Roman dominating index

**Proposition 3:** Let $G$ be a graph. For any pair of non-adjacent vertices $\{x, y\}$ in $G$, $0 \leq R(xy) \leq 1$.

**Proof:** We need only to prove that $R(xy) \leq 1$. Let $G' = G + xy$ and $f'$ be a $\gamma_R$-function of $G'$. There are two cases to consider.

**Case 1:** $\{f'(x), f'(y)\} = \{0, 2\}$.

Without loss of generality, assume that $f'(x) = 0$ and $f'(y) = 2$, and define $f : V \rightarrow \{0, 1, 2\}$:

$$f(v) = \begin{cases} f'(v), & v \neq x, \\ 1, & v = x. \end{cases}$$

Then $f$ is a Roman dominating function of $G$ as removing edge $xy$ only raises the possibility that vertex $x$ may be unprotected, if $f'$ for $G'$ is to be used for $G$. Simply adding one more army to this vertex $x$ will resolve the issue—in this way, all vertices are again protected, with an increase of one in Roman domination number.

Clearly, $w(f) = \gamma_R(G') + 1$. Thus $\gamma_R(G) \leq w(f) = \gamma_R(G') + 1$. It follows that $R(xy) = \gamma_R(G) - \gamma_R(G') \leq 1$.

**Case 2:** The negation of case 1.

Neither of the two vertices $x$ and $y$ is protected by the other. Thus, the existence of the edge $xy$ in $G'$ does not help protect either vertex $x$ or $y$. Hence when edge $xy$ is removed from $G'$ to get $G$, $f'$ is still a Roman dominating function for $G$. Hence $\gamma_R(G) \leq w(f') = \gamma_R(G')$. So $R(xy) = \gamma_R(G) - \gamma_R(G') \leq 0$. Hence $R(xy) = 0$.

Summing up the aforementioned two cases of discussion on the upper bound, we have $R(xy) \leq 1$. ■

**Remark:** Both the lower and upper bounds are reachable.

To show the lower bound is reachable, the Roman dominating index of an edge which joins the two ends of a path of order three $P_3$ together to form a cycle $C_3$ is 0.
To show the upper bound is achievable, the Roman dominating index of any edge that joins two non-neighboring vertices in cycle \( C_4 \) is 1.

**Proposition 4:** Let \( \{x, y\} \) be a pair of non-adjacent vertices in a graph \( G \). Then \( R(xy) = 1 \) if and only if there exists a \( \gamma_R \)-function \( f \) of \( G \) such that \( \{f(x), f(y)\} = \{1, 2\} \).

**Proof:**

**Sufficiency:** We may assume that \( f(x) = 1 \) and \( f(y) = 2 \) for \( G \). Define \( f' \) on \( G' \) as follows:

\[
f'(v) = \begin{cases} f(v), v \neq x, \\ 0, v = x. \end{cases}
\]

\( f' \) is a Roman dominating function as \( x \) is protected by \( y \) in \( G' \).

Now that \( w(f') = \gamma_R(G) - 1 \), we have \( \gamma_R(G') \leq w(f') = \gamma_R(G) - 1 \). So \( R(xy) = \gamma_R(G) - \gamma_R(G') \geq 1 \). As \( R(xy) \leq 1 \), we have \( R(xy) = 1 \).

**Necessity:** Assume \( R(xy) = 1 \). As shown in the proof for Proposition 3, there exists a \( \gamma_R \)-function \( f' \) of \( G' \) such that \( \{f'(x), f'(y)\} = \{0, 2\} \).

Assume \( f'(x) = 0, f'(y) = 2 \), then we have a Roman dominating function \( f \) for \( G \) as defined by

\[
f(v) = \begin{cases} f'(v), v \neq x, \\ 1, v = x. \end{cases}
\]

Note that \( w(f) = \gamma_R(G') + 1 \). Thus \( w(f) = \gamma_R(G') + 1 = \gamma_R(G) - R(xy) + 1 = \gamma_R(G) \). By definition, \( f \) is a \( \gamma_R \)-function for \( G \), with \( f(x) = 1, f(y) = 2 \). □
3.2. An application of Proposition 3

**Problem 1:** Given a path $P_n$ of order $n \geq 3$, are there pairs of non-adjacent vertices $x, y$ in $P_n$ such that $R(xy) = 1$? If yes, which pairs?

**Solution:** Let $x, y$ be two vertices in $P_n$ as shown in Figure 2.

Dreyer (3) showed that if $n \equiv 0 (\mod 3)$, no vertices in $P_n$ are mapped to 1 in any $\gamma_{R}$-function. According to Proposition 4, $R(xy) = 0$.

Dreyer (3) showed that if $n \equiv 1 (\mod 3)$ where $n \neq 1$, there exists for any $\gamma_{R}$-function $f$ of $P_n$ a vertex mapped to 1 and vertices mapped to 2. Thus max $[R(xy)] = 1$. To find the exact vertices to connect to obtain this maximum value, we just need to find the possible value-1 and value-2 vertices in $f$. Without loss of generality, let $f(x) = 1$ and $f(y) = 2$. As the positions of value-1 and value-2 vertices in $f$ follow a simple pattern, it is easy to show that the Roman dominating index of 1 can be achieved if and only if $i \equiv 0, j \equiv 1, k \equiv 1 (\mod 3)$.

Similarly, if $n \equiv 2 (\mod 3)$ ($n \neq 2$), max $[R(xy)] = 1$. Let $f$ be a $\gamma_{R}$-function of $P_n$ and without loss of generality assume $f(x) = 1$ and $f(y) = 2$. The necessary and sufficient condition for which $R(xy) = 1$ is

- $i \equiv 0, j \equiv 1, k \equiv 2 (\mod 3)$, or
- $i \equiv 1, j \equiv 1, k \equiv 1 (\mod 3)$, or
- $i \equiv 0, j \equiv 2, k \equiv 1 (\mod 3)$.

**Remark 1:** By similar arguments, we can determine the condition to achieve a Roman dominating index of 1 for some other classes of graphs.

For cycle $C_n$ of order $n$, if $n \equiv 0 (\mod 3)$, $R(xy) = 0$. If $n \equiv 1$ or $2 (\mod 3)$, max $[R(xy)] = 1$.

For two disjoint paths/cycles or a path and a cycle, where both orders of the two
components are not multiples of three, by joining the two disjoint components, we can have a Roman dominating edge with $R(xy) = 1$.

The exact positions of the vertices to connect can be determined as before by finding possible value-1 and value-2 vertices in a $\gamma_R$-function of the graph.

**Remark 2:** Without using Proposition 3 and 4, it may be much more tedious to solve Problem 1 as in some previous (successful) attempts by the author. For details, refer to Additional information.

### 3.3. Discussion on adding successive new edges to a path

**Problem 2:** Given a path $P_n$ of order $n \geq 3$, a positive integer $m$ with $m \leq n$, and a vertex $v$ not in $P_n$, how do we add $m$ new edges to join $v$ and $m$ vertices in $P_n$ so that the resulting graph $G$ has the largest $\gamma_R(G)$? What is the value of this largest $\gamma_R(G)$? What about the smallest one?

**Detailed solution** is available in Additional information.

**Result:**

**Largest:**

If $m \leq \left\lfloor \frac{n+1}{3} \right\rfloor + 1$, then $\gamma_R(G) = \left\lfloor \frac{2n+2}{3} \right\rfloor$, and $f(v) = \begin{cases} 1, & \text{if } n \equiv 0 \text{ or } 1 \pmod{3}, \\ 0, & \text{if } n \equiv 2 \pmod{3}. \end{cases}$

If $m \geq \left\lfloor \frac{n+1}{3} \right\rfloor + 1$, then $\gamma_R(G) = n - m + 2$, and $f(v) = 2$.

**Smallest:**

If $m \leq 3$, then $\gamma_R(G) = \left\lfloor \frac{2n}{3} \right\rfloor$, and $f(v) = 0$.

If $m \geq 3$, then $\gamma_R(G) = \left\lfloor \frac{2}{3}(n - m) \right\rfloor + 2$, and $f(v) = 2$. ■
Remark 1: This problem can model the transition from a segmented, line-like distribution system of gas/water/heat, to a centralized, star-like one.

Remark 2: Following the trend of adding an edge between two disjoint graph in sections 3.2 and adding successive edges in section 3.3, a direction for further research is to combine these two cases and look into the effect of adding successive edges between two disjoint graphs.

4. Roman-critical graphs

Let $G = (V, E)$ be a graph. We call $G$ Roman-critical if $\gamma_R(G - xy) > \gamma_R(G)$ for all $xy \in E$.

The star graph $S_n$ of order $n$ is a tree on $n$ vertices with one vertex having degree $(n - 1)$ and the other $(n - 1)$ having vertex degree 1. Note that $S_1$ is a single vertex. A galaxy is a union of star graphs.

Now we will characterize Roman-critical graphs.

**Lemma:** Let $G = (V, E)$ be a graph, $\{x, y\}$ a pair of adjacent vertices in $G$, and $f$ a $\gamma_R$-function of $G$ defined on $V$. If $\gamma_R(G - xy) = \gamma_R(G) + 1$, then $\{f(x), f(y)\} = \{0, 2\}$.

**Proof:**

We shall prove the contrapositive of the given proposition, i.e. If $\{f(x), f(y)\} \neq \{0, 2\}$, then $\gamma_R(G - xy) = \gamma_R(G)$.

Checking the cases where $\{f(x), f(y)\} \neq \{0, 2\}$, we find that $f$ will still be a Roman dominating function for $(G - xy)$.

Thus, $\omega(f) \geq \gamma_R(G - xy) \geq \gamma_R(G) = \omega(f)$. So $\gamma_R(G - e) = \gamma_R(G)$.
**Proposition 5:** $G$ is a Roman-critical graph if and only if $G$ is a galaxy without $S_2$ as a component.

**Proof:**

**Sufficiency:** Let $f$ be the $\gamma_R$-function of $G$. $f(v) = \begin{cases} 2, & d(v) \geq 2, \\ 0, & d(v) = 1. \end{cases}$

Let $d(x) = 1$. Then any $\gamma_R$-function $f'$ of $(G - xy)$ must have

$$f'(v) = \begin{cases} 1, & v = x, \\ f(v), & v \neq x. \end{cases}$$

Note that $f(x) = 0$. Thus $\gamma_R(G - xy) - \gamma_R(G) = 1$. This implies that $G$ is Roman-critical.

**Necessity:** By lemma, $G$ is Roman-critical implies that $\{f(x), f(y)\} = \{0, 2\}$ for all $\gamma_R$-functions $f$ of $G$ and for all pairs of adjacent vertices $x, y$. We may assume that $f(x) = 0, f(y) = 2$.

Assume that $d(x) > 1$. Let any neighbor of $x$ other than $y$ be $z$. Since $x$ does not need $z$’s protection (if any) and $z$ is not protected by $x$, $\gamma_R(G - xz) = \gamma_R(G)$. Thus, $G$ is not Roman-critical, a contradiction.

Assume now that $d(y) = 1$. Let $f'$ be a function from $V(G - xy)$ to $\{0, 1, 2\}$ as follows:

$$f'(v) = \begin{cases} 1, & v = x, y, \\ f(v), & v \neq x or y. \end{cases}$$

Clearly, $f'$ is a Roman dominating function for $(G - xy)$. Note that

$$\gamma_R(G - xy) \leq w(f') = w(f) = \gamma_R(G).$$

Hence $G$ is not Roman-critical.

Thus $d(x) = 1$ and $d(y) \geq 2$. Hence, $G$ is a galaxy without $S_2$ as a component. $\blacksquare$
5. Bound of Roman domination number

The **diameter** of a graph $G$, denoted by $D(G)$, is defined as

$$D(G) = \max\{d(u,v) | u, v \text{ are in } V\}.$$  

Note that $[a + b] \geq [a] + [b]$ for any real number $a$ and $b$. We now establish the following result.

**Proposition 6:** For any tree $T$ of order $n \geq 3$, $2 \leq \gamma_R(T) \leq \left\lfloor \frac{4n}{5} \right\rfloor$.

**Proof:** The lower bound is trivial as no matter how large the order is, a star always has a Roman domination number of 2.

I will prove the upper bound by **mathematical induction** on the diameter of the tree, $D(T)$.

**Base cases:** If $D(T) = 2, 3, \text{ or } 4$, $\gamma_R(T) \leq \left\lfloor \frac{4n}{5} \right\rfloor$.

**Case 1:** $D(T) = 2$. Obviously $\gamma_R(T) = 2 \leq \left\lfloor \frac{4n}{5} \right\rfloor$.

**Case 2:** $D(T) = 3$. Find the path $v_0e_0v_1e_1v_2e_2v_3$ which maximizes $d(v_2)$.

If $d(v_1) > 2$ and $d(v_2) > 2$, we can remove $e_1$ and thus get two isolated trees $T_1$ and $T_2$ of diameter 2. $\gamma_R(T) \leq \gamma_R(T_1) + \gamma_R(T_2) \leq \left\lfloor \frac{4n}{5} \right\rfloor$.

Otherwise, let $f(v_2) = 2$ and $f(v_0) = 1 \Rightarrow \gamma_R(T) = 3 \leq \left\lfloor \frac{4n}{5} \right\rfloor$.

**Case 3:** $D(T) = 4$. Find a path $v_0e_0v_1e_1v_2e_2v_3e_3v_4$ which maximizes $d(v_3)$.

If $d(v_3) > 2$, we can remove it together with all its neighboring end vertices as a tree of diameter 2. Repeat this until the tree decreases in diameter to some cases previously discussed, or becomes a tree $T'$ where $d(v_1) = d(v_3) = 2$ and $d(v_2) \geq 2$. The former is handled by previously discussed trees of diameter 2 or 3. As to the latter, construct a Roman dominating function $f$ such that $f(v_2) = 2$ and $f(v) = 1$ for all end-vertices $v$ in $T'$. Thus we have $\gamma_R(T') \leq \frac{2 + d(v_2)}{2d(v_2) + 1} n' \leq \left\lfloor \frac{4n'}{5} \right\rfloor$, where $n'$ is the order of $T'$. 

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**Inductive hypothesis:** If $\gamma_R(T) \leq \left\lfloor \frac{4n}{5} \right\rfloor$ for any tree $T$ where $k - 3 \leq D(T) \leq k - 1$, then for any tree $T$ of $D(T) = k$, $\gamma_R(T) \leq \left\lfloor \frac{4n}{5} \right\rfloor$. To show this:

1. Given a tree $T$ where $D(T) = k$, find its longest path $v_0e_0v_1e_1v_2 \ldots v_{k-3}e_{k-3}v_{k-2}e_{k-2}v_{k-1}e_{k-1}v_k$.

2. Remove edge $e_{k-3}$. Since there is only one path linking a vertex to another in any tree, removing an edge means that these two vertices are no longer linked by edges or vertices. Thus two disjoint trees $T_{b1}$ and $T_1$ result. $T_{b1}$ contains path $v_{k-2}e_{k-2}v_{k-1}e_{k-1}v_k$. $d(v_{k-2}, v_k) = 2$ implies that $D(T_{b1}) \geq 2$. Since we chose the longest path in $T$, $D(T_{b1}) \leq 4$. Thus $2 \leq D(T_{b1}) \leq 4$ and $T_{b1}$ falls in base cases aforementioned. Let $v(T_{b1})$ denote the order of $T_{b1}$. We have $\gamma_R(T_{b1}) \leq \left\lfloor \frac{4v(T_{b1})}{5} \right\rfloor$.

$T_1$ contains path $v_0e_0v_1e_1v_2 \ldots v_{k-4}e_{k-4}v_{k-3}$. $d(v_0, v_{k-3}) = k - 3$ implies that $D(T_1) \geq k - 3$. In addition, $D(T_1) \leq D(T) = k$. Thus $k - 3 \leq D(T_1) \leq k$.

If $D(T_1) = k$, do note that there are fewer paths of length $k$ in $T_1$ than in $T$ as path $v_0e_0v_1e_1v_2 \ldots v_{k-3}e_{k-3}v_{k-2}e_{k-2}v_{k-1}e_{k-1}v_k$ and possibly others no longer exist in $T_1$.

3. If $D(T_1) = k$, repeat steps 1 and 2. At the $i^{th}$ repetition of steps 1 and 2, we divide $T_{i-1}$ into $T_{bi}$ and $T_i$. As the number of path of length $k$ is finite and this number decreases each time we apply step 1 and 2, we are certain that after $m$ repeats $D(T_m)$ will for the first time be smaller than $k$. So we have $k - 3 \leq D(T_m) \leq k - 1$.

$$\gamma_R(T_m) \leq \left\lfloor \frac{4v(T_m)}{5} \right\rfloor$$

by the inductive hypothesis. Thus, for $T$ whose $D(T) = k$ we have

$$\gamma_R(T) \leq \gamma_R\left(\sum_{i=1}^{m} T_{bi} + T_m\right) = \sum_{i=1}^{m} \gamma_R(T_{bi}) + \gamma_R(T_m) \leq \sum_{i=1}^{m} \left\lfloor \frac{4v(T_{bi})}{5} \right\rfloor + \left\lfloor \frac{4v(T_m)}{5} \right\rfloor$$

$$\leq \left\lfloor \frac{4}{5} \sum_{i=1}^{m} v(T_{bi}) + \frac{4}{5} v(T_m) \right\rfloor = \left\lfloor \frac{4v(T)}{5} \right\rfloor$$

$\blacksquare$
**Remark 1:** This bound is achievable by constructing trees of the following structures.

![Figure 4](image-url)

**Remark 2:** Given a tree $T$ of order $n \geq 3$, $\gamma_R(T) = \frac{4n}{5}$ if and only if $T$ has a structure like the right most ones shown in Figure 4.

**Proof:** Sufficiency is shown directly by Proposition 6. For necessity, we need a closer examination of the proof for Proposition 6. We find that given $2 \leq D(T) \leq 4$, only $\gamma_R(P_5) = \frac{4n}{5}$. Only when $T_{bi} = P_5$ for all $1 \leq i \leq m$ will we have $\gamma_R(T) = \frac{4n}{5}$. $\blacksquare$

**Corollary:** For any connected graph $G$ of order $n \geq 3$, $2 \leq \gamma_R(G) \leq \left\lceil \frac{4n}{5} \right\rceil$.

**Proof:** Proof for lower bound is trivial while the one for upper bound follows immediately from Proposition 2 and 6. $\blacksquare$

### 6. Roman domination in 3-regular graphs

A graph is called a **3-regular graph** if the degree of all vertices are 3. The order of any 3-regular graph can only be even.

For a graph $G = (V, E)$, let $f$ be a function from $V$ to the set $\{0, 1, 2\}$, and let $(V_0, V_1, V_2)$ be the ordered partition of $V$ induced by $f$, where $V_i = \{v \in V | f(v) = i\}$ for $i = 0, 1, 2$. There is a one-to-one correspondence between the functions $f: V \to \{0, 1, 2\}$ and the ordered partitions $(V_0, V_1, V_2)$ of $V$. Thus, we will write $f = (V_0, V_1, V_2)$.

For any vertex $v \in V$, the **open neighborhood** of $v$ is the set $N(v) = \{u \in V | uv \in E\}$ and the **closed neighborhood** is the set $N[v] = N(v) \cup \{v\}$.
Proposition 7: If $G$ is a 3-regular graph of order $n$, then
\[ \gamma_R(G) \geq \begin{cases} \frac{n}{2}, & n \equiv 0 \pmod{4}, \\ \frac{n}{2} + 1, & n \equiv 2 \pmod{4}. \end{cases} \]

Proof: Because in a 3-regular graph every vertex has a degree of three, a vertex in $V_2$ can protect its closed neighborhood, namely itself and its three neighbors. The most efficient protection occurs when there is no intersection between the closed neighborhood of any vertex in $V_2$.

Thus, when $n = 4m$, $\gamma_R(G) \geq \frac{4m}{3+1} \times 2 = 2m = \frac{n}{2}$.

When $n = 4m + 2$, the two extra vertices will increase the Roman domination number by 2.

Thus, when $n = 4m + 2$, $\gamma_R(G) \geq \frac{4m}{3+1} \times 2 + 2 = 2m + 2 = \frac{n}{2} + 1$. $\blacksquare$

Remark: the lower bounds are achievable as shown in Proposition 8.

6.1. Roman domination in $C_{\frac{n}{2}} \times P_2$

For graph $G$ and $H$, the Cartesian product of $G$ and $H$, denoted by $G \times H$, is the graph with vertex set \{$(u,v)|u \in V(G), v \in V(H)$\}. Two vertices $(u_1,v_1)$ and $(u_2,v_2)$ in $G \times H$ are adjacent if and only if one of the following is true: $u_1 = u_2$ and $v_1$ is adjacent to $v_2$ in $H$; or $v_1 = v_2$ and $u_1$ is adjacent to $u_2$ in $G$.

Proposition 8: If $G = C_{\frac{n}{2}} \times P_2$, then
\[ \gamma_R(G) = \begin{cases} \frac{n}{2}, & n \equiv 0 \pmod{8}, \\ \frac{n}{2} + 1, & n \equiv 2, 4, 6 \pmod{8}. \end{cases} \]

Proof:

$C_{\frac{n}{2}} \times P_2$ is a 3-regular graph.
Let $V\left(C_n^2\right) = \{0,1,2, \ldots, \frac{n}{2} - 1\}$ and $uv \in E(C_n^2)$ if and only if $u - v \equiv -1$ or $1 \pmod{\frac{n}{2}}$. Let $V(P_2) = \{1,2\}$.

**Case 1:** $n = 8m \ (m \geq 1)$. Let

$$
V_2 = \{(4k,1) | 0 \leq k \leq m - 1\} \cup \{(4l+2,2) | 0 \leq l \leq m - 1\},
$$

$$
V_1 = \emptyset,
$$

$$
V_0 = V - V_2.
$$

$f = (V_0,V_1,V_2)$ is a Roman dominating function for $G$ with weight $4m$. By Proposition 7, $\gamma_R(G) \geq 4m$. So $\gamma_R(G) = 4m = \frac{n}{2}$.

**Case 2:** $n = 8m + 2 \ (m \geq 1)$. Let

$$
V_2 = \{(4k,1) | 0 \leq k \leq m\} \cup \{(4l+2,2) | 0 \leq l \leq m - 1\},
$$

$$
V_1 = \emptyset,
$$

$$
V_0 = V - V_2.
$$

$f = (V_0,V_1,V_2)$ is a Roman dominating function for $G$ with weight $(4m+2)$. By Proposition 7, $\gamma_R(G) \geq 4m + 2$. So $\gamma_R(G) = 4m + 2 = \frac{n}{2} + 1$.

**Case 3:** $n = 8m + 4 \ (m \geq 1)$. By Proposition 7, $\gamma_R(G) \geq 4m + 2$. We will show by a coloring method that the equality cannot be reached. Since $G$ is a bipartite graph. Let its partite sets be $X$ and $Y$. Color each vertex in $X$ black and each vertex in $Y$ white. Each vertex in one set is adjacent to exactly $3$ vertices in the other set.

Assume for contradiction that $\gamma_R(G) = 4m + 2$ can be reached. Thus, $2|V_2| + |V_1| = 4m + 2$. Since a vertex in $V_2$ can protect at most $4$ vertices, $4|V_2| + |V_1| \geq 8m + 4$. Eliminate $|V_2|$ from the previous two expressions, we have $|V_1| \leq 0$. Hence, we have $|V_2| = 2m + 1, V_1 = \emptyset, V_0 = V - V_2$, and there must be no intersection between the closed neighborhoods of any two vertices in $V_2$.

In $V_2$, we assume that there exist $s$ black vertices and $t$ white vertices. Then $V_2$ protects $(s + 3t)$ black vertices and $(3s + t)$ white vertices. Since $V_2$ should protect all vertices of $G$ without overlapping, $(s + 3t)$ is the number of black vertices in $G$ and $(s + 3t)$ is the number of white vertices in $G$. Hence both $(s + 3t)$ and $(3s + t)$ are even.
But \( s + t = |V_2| = 2m + 1 \) is odd. Thus \( s + 3t = (s + t) + 2t \) and \( 3s + t = (s + t) + 2s \) are both odd, a contradiction. Thus \( \gamma_R(G) = 4m + 2 \) cannot be reached.

We will show \( \gamma_R(G) = 4m + 3 \). Let
\[
\begin{align*}
V_2 &= \{(4k, 1)|0 \leq k \leq m\} \cup \{(2 + 4l, 2)|0 \leq l \leq m - 1\}, \\
V_1 &= \{(4m + 1, 2)\}, \\
V_0 &= V - V_1 - V_2.
\end{align*}
\]

\( f = (V_0, V_1, V_2) \) is a Roman dominating function for \( G \) with weight \( (4m + 3) \). So \( \gamma_R(G) = 4m + 3 = \frac{n}{2} + 1 \).

**Case 4:** \( n = 8m + 6 \) (\( m \geq 0 \)). Let
\[
\begin{align*}
V_2 &= \{(4k, 1), (4k + 2, 2)|0 \leq k \leq m\}, \\
V_1 &= \emptyset, \\
V_0 &= V - V_2.
\end{align*}
\]

\( f = (V_0, V_1, V_2) \) is a Roman dominating function for \( G \) with weight \( (4m + 4) \).

By Proposition 7, \( \gamma_R(G) \geq 4m + 4 \). So \( \gamma_R(G) = 4m + 4 = \frac{n}{2} + 1 \). \( \blacksquare \)

### 6.2. Roman domination in 3-regular circulant graph

A **circulant graph** \( C_n(a_1, a_2, \ldots, a_k) \) with \( n \) vertices 0,1,2,...,\( n-1 \) refers to a simple graph whose vertex \( i \) is adjacent to \( i \pm a_1, i \pm a_2, \ldots, i \pm a_k \) (take the remainder \( r \) mod \( n \), \( 0 \leq r \leq n - 1 \)), where \( a_1, a_2, \ldots, a_k \) are positive integers and \( 0 < a_i < \frac{n+1}{2}, a_i \neq a_j \) (\( i \neq j \), \( i, j = 1,2, \ldots, k \)).

The necessary and sufficient condition for a circulant graph \( C_n(a_1, a_2, \ldots, a_k) \) to be connected is that the greatest common divisor of \( (n, a_1, a_2, \ldots, a_k) \), denoted by \( \gcd(n, a_1, a_2, \ldots, a_k) \) is 1. A 3-regular circulant graph must be \( C_n(a, \frac{n}{2}) \), where \( n \) is an even number larger than 2, \( 1 \leq a \leq \frac{n}{2} - 1 \). If \( G = C_n(a, \frac{n}{2}) \) is a connected 3-regular circulant graph, then \( \gcd(n, a) = 1 \) or 2. In addition, if \( \gcd(n, a) = 1 \), then \( C_n(a, \frac{n}{2}) \) is isomorphic to \( C_n(1, \frac{n}{2}) \). If \( \gcd(n, a) = 2 \), then \( \frac{n}{2} \) must be odd and \( C_n(a, \frac{n}{2}) \) is isomorphic to \( C_2 \times P_2 \).
Proposition 9: If \( G = C_n \langle a, \frac{n}{2} \rangle \) is a 3-regular circulant graph and \( \gcd(n, a) = 1 \), then
\[
\gamma_R(G) = \begin{cases} 
\frac{n}{2}, & \text{if } n \equiv 4 \pmod{8}, \\
\frac{n}{2} + 1, & \text{if } n \equiv 0, 2, 6 \pmod{8}.
\end{cases}
\]

Proof: \( G \cong C_n \langle 1, \frac{n}{2} \rangle \).

Case 1: \( n = 8m + 4 \) (\( m \geq 0 \)). Let
\[
V_2 = \{4k|k = 0, 1, \ldots, 2m\}, \\
V_1 = \emptyset, \\
V_0 = V - V_2.
\]
Odd number vertices are obviously neighbors of vertices in \( V_2 \). Even number vertices are either in \( V_2 \) or equal to \( 2 + 4l \) (\( l = 0, 1, \ldots, m - 1 \)). The vertices \( 2 + 4l \) are also neighbors of vertices in \( V_2 \) as
\[
(2 + 4l) + \frac{n}{2} = (2 + 4l) + (4m + 2) = 4(l + m + 1) \equiv 4k(\pmod{n}).
\]
Thus, \( f = (V_0, V_1, V_2) \) is a Roman dominating function for \( G \).

So \( \gamma_R(G) \leq 4m + 2 \). However, by Proposition 7, \( \gamma_R(G) \geq 4m + 2 \). So
\[
\gamma_R(G) = 4m + 2 = \frac{n}{2}.
\]

Case 2: \( n = 8m + 6 \) (\( m \geq 0 \)). Let
\[
V_2 = \{4k|k = 0, 1, \ldots, m\} \cup \{\frac{n}{2} + 2 + 4l|l = 0, 1, \ldots, m\}, \\
V_1 = \emptyset, \\
V_0 = V - V_2.
\]
Similar to case 1, we can check that \( f = (V_0, V_1, V_2) \) is a Roman dominating function for \( G \). By Proposition 7, \( \gamma_R(G) \geq 4m + 4 \). So \( \gamma_R(G) = 4m + 4 = \frac{n}{2} + 1 \).

Case 3: \( n = 8m \) (\( m \geq 1 \)). By Proposition 7, \( \gamma_R(G) \geq 4m \). We will show by contradiction that the equality cannot be reached. Suppose \( \gamma_R(G) = 4m \). Then \( |V_2| = 2m, V_1 = \emptyset, V_0 = V - V_2 \). Because \( G \) is a 3-regular graph and there are \( 8m \) vertices in \( G \), there should be no intersection between the closed neighborhoods of any two vertices in \( V_2 \). Without loss of generality, let \( 0 \in V_2 \). Thus \( 1, 2 \in V_0 \). If
3 ∈ V₂, consider \( \frac{n}{2} + 1 \) in \( V(G) \). This vertex can only be protected by vertex \\
\( \frac{n}{2}, \frac{n}{2} + 1 \) or \( \frac{n}{2} + 2 \). Thus, one of these three vertices belongs to \( V₂ \). But its closed \\
neighborhood will intersect with the closed neighborhood of 0 or 3, a 

contradiction. Hence, \( 3 ∉ V₂ \). Then, \( 4 ∈ V₂ \); otherwise 2 and 3 have to be 
covered by vertices \( \frac{n}{2} + 2 \) and \( \frac{n}{2} + 3 \), but their closed neighborhoods intersect.

Similarly, 5, 6 and 7 do not belong to \( V₂ \) but 8 does. Generally, \( 4k - 1, 4k - 2 \) and \( 4k - 3 \) do not belong to \( V₂ \) but \( 4k \) does. Because \( \frac{n}{2} = 4m \), \( \frac{n}{2} \) belongs to 
\( V₂ \); but now it intersects with the closed neighborhood of vertex 0, a contradiction. 

Hence it is proved that \( γ_R(G) > 4m \).

On the other hand, let 

\[
V₂ = \{4k|k = 0, 1, \ldots, m - 1\} \cup \{\frac{n}{2} + 2 + 4l|l = 0, 1, \ldots, m - 1\}, \\
V₁ = \{4m - 1\}, \\
V₀ = V - V₂ - V₁.
\]

\( f = (V₀, V₁, V₂) \) is a Roman dominating function for \( G \). Thus, \( γ_R(G) \leq 4m + 1 \) 
and hence \( γ_R(G) = 4m + 1 = \frac{n}{2} + 1 \).

**Case 4:** \( n = 8m + 2 \) \((m ≥ 1)\). Let 

\[
V₂ = \{4k|k = 0, 1, \ldots, m\} \cup \{\frac{n}{2} + 2 + 4l|l = 0, 1, \ldots, m - 1\}, \\
V₁ = \emptyset, \\
V₀ = V - V₂.
\]

Similar to case 1, we can check that \( f = (V₀, V₁, V₂) \) is a Roman dominating 
function for \( G \) with weight \( (4m + 2) \). By Proposition 7, \( γ_R(G) ≥ 4m + 2 \). So 
\( γ_R(G) = 4m + 2 = \frac{n}{2} + 1 \).

**Proposition 10:** If \( G = Cₙ\left(a, \frac{n}{2}\right) \) is a 3-regular circulant graph, \((n, a) = 2\), then 
\( γ_R(G) = \frac{n}{2} + 1 \).

**Proof:** The result follows from Proposition 8.
7. Areas for future research

One limitation of Roman domination is that when there are too many regions in the neighborhood of a region with two armies, when multiple attacks are launched simultaneously on the neighborhood, the two armies may not be enough to defend them. Thus we devise a new kind of Roman domination called $k$-Roman domination.

Let $G = (V, E)$ be a graph, $f : V \to \{0, 1, 2, \ldots, k\}$ and $V_i = \{v \in V | f(v) = i\} (i = 0, 1, 2, \ldots, k)$. If $V_0 \subseteq N[\bigcup V_i] (i \geq 1)$ and for all $v \in V_i (i \geq 1)$, $|N(v) \cap V_0| \leq i$, then we call $f$ a $k$-Roman dominating function for $G$.

In addition, since some roads between regions may be one-way only, we can apply Roman domination to directed graphs. We may redefine Roman dominating function as $f : V \to \{0, 1, 2\}$ such that every vertex $v$ for which $f(v) = 0$ has a neighbor $u$ with $f(u) = 2$ and there exists an arc $uv$.

Bibliography


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Additional Information for Roman Domination

Wang Shizhi

Supervisor: Prof. Koh Khee Meng

Mentor: Mr. Dennis Yeo
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1. Alternative solution for the problem in Section 3.2

Note: The author discovered the following alternative solution before proving Propositions 3 and 4, the two propositions which enable the shorter solution presented in Section 3.2 on the page 10 of the Report.

Problem 1: Given a path $P_n$ of order $n \geq 3$, are there pairs of non-adjacent vertices $x, y$ in $P_n$ such that $R(xy) = 1$? If yes, which pairs?

The alternative solution:

We denote the path of order $n$ by $P_n$, and the new graph formed by adding an extra edge by $P_n'$.

Case 1: Neither vertices $x$ nor $y$ is assigned 2 under $f'_{\gamma_R}$, $\gamma_R$-function for $P_n'$.

It is obvious that graph $G$ and $G'$ will have the same Roman domination number and thus the Roman dominating index of edge $xy$ will always be zero. According to Proposition 1,

$$\gamma_R(P_n') = \left\lceil \frac{2n}{3} \right\rceil, f(x) \neq 2 \text{ and } f(y) \neq 2$$

Case 2: One of vertices $x$ and $y$ is assigned 2 by function $f'_{\gamma_R}$.

Without loss of generality, let $f'_{\gamma_R}(x) = 0$ and $f'_{\gamma_R}(y) = 2$. As shown in Figure 1.1, $i$ is the number of vertices on the left of $x$, $j$ between $x$ and $y$ (not inclusive) and $k$ on the right of $y$.

Numbers assigned to vertices $x$ and $y$ are already fixed (0 and 2 respectively). In addition, vertex $y$ can protect its three neighbors in $G'$. The remaining is to find the Roman domination number for three paths, of order $i, (j - 1)$, and $(k - 1)$, which can be easily done using Formula 1,

$$\gamma_R(P_n') = \left\lceil \frac{2i}{3} \right\rceil + \left\lceil \frac{2(j - 1)}{3} \right\rceil + \left\lceil \frac{2(k - 1)}{3} \right\rceil + 2, f(x) = 2 \text{ or } f(y) = 2$$

$$n = i + j + k + 2$$
Combining the two cases, we have
\[ γ_{R}(P_n') = \min \left[ \frac{2n}{3}, \frac{2i}{3} + \frac{2(j - 1)}{3} + \frac{2(k - 1)}{3} + 2 \right], \]
where \( n = i + j + k + 2, \)
and \( \min[a, b] = \) the smaller value of \( a \) and \( b \) (if \( a = b, \min[a, b] = a = b \))
\[ ∴ R(x y) = \left[ \frac{2n}{3} \right] - \min \left[ \frac{2n}{3}, \frac{2i}{3} + \frac{2(j - 1)}{3} + \frac{2(k - 1)}{3} + 2 \right] \]

The following result can be checked:

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<th>( j )(mod3)</th>
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Thus the conclusion follows that, for $P_n$:

If $n \equiv 0 \pmod{3}$, $\max[R(xy)] = 0$.

If $n \equiv 1 \pmod{3}$, $\max[R(xy)] = 1$. Refer to case 5 for which two vertices to connect.

If $n \equiv 2 \pmod{3}$, $\max[R(xy)] = 1$. Refer to cases 6, 8 and 14 for which two vertices to connect. ■

**Remark:** Similar problems on other classes of graph as mentioned in Remark 1 of Section 3.2 can also be solved in the same way whereby graphs of unknown Roman domination number is transformed to some classes of graphs of known Roman domination number such as a path.
2. Detailed discussion on adding successive new edges

Note: This section corresponds to Section 3.3 on page 11 of the Report. For brevity, only results are presented in Section 3.3; for completeness, the detailed derivations are presented below.

Problem 2: Given a path $P_n$ of order $n \geq 3$, a positive integer $m$ with $m \leq n$, and a vertex $v$ not in $P_n$, how do we add $m$ new edges to join $v$ and $m$ vertices in $P_n$ so that the resulting graph $G$ has the largest $\gamma_R(G)$? What is the value of this largest $\gamma_R(G)$? What about the smallest one?

![Figure 1: Adding successive new edges in detail](image)

Solution:

Largest $\gamma_R(G)$:

As shown in Figure 1, let the new vertex be $v$ and the vertices on the path $v_1, v_2, ..., v_n$.

We compare minimum weight under the two cases below to find the largest Roman domination number.

Case 1: $v$ is mapped to 2.

Sub-case 1.1: $m \geq \left\lfloor \frac{n}{3} \right\rfloor$, where $\lfloor x \rfloor$ is floor function.

By connecting $v_{3i}, 1 \leq i \leq \left\lfloor \frac{n}{3} \right\rfloor$ to $v$, we have $n_i \leq 2$, for $0 \leq i \leq m$. By
mapping to 1 any vertex on path that is not connected to \( v \), and \( v \) to 2, we have

\[
\text{minimum weight} = n - m + 2.
\]

**Sub-case 1.2:** \( m < \left\lceil \frac{n}{3} \right\rceil \).

We connect the \( m \) edges to vertices \( v_{3i}, 1 \leq i \leq m \).

This maximizes weight as, for any integer \( x \) and \( y \),

\[
y_R(x + y - 2) + y_R(2) = \left\lceil \frac{2}{3} (x + y - 2) \right\rceil + \left\lceil \frac{2}{3} \times 2 \right\rceil \\
\geq \left\lceil \frac{2}{3} x \right\rceil + \left\lceil \frac{2}{3} y \right\rceil = y_R(x) + y_R(y).
\]

By mapping \( v_{3i-2} \) and \( v_{3i-1} \) to 1 and applying Proposition 1 to the \((n - 3m)\) consecutive vertices, we have,

\[
\text{minimum weight} = 2m + \left\lceil \frac{2}{3} (n - 3m) \right\rceil + 2 = \left\lceil \frac{2}{3} n \right\rceil + 2.
\]

**Case 2:** \( v \) is not mapped to 2.

**Sub-case 2.1:** \( n \equiv 0 \pmod{3} \). We must map vertices \( v_{3i-1} \) \((1 \leq i \leq \frac{n}{3})\) to 2 while others on the path to 0.

When \( m \leq \frac{2n}{3} \), we connect the new edges to vertices others than \( v_{3i-1} \). Then, \( v \) must be mapped to 1. Thus,

\[
\text{minimum weight} = \frac{2n}{3} + 1, m \leq \frac{2n}{3}.
\]

When \( m > \frac{2n}{3} \), \( v \) will be connected to some \( v_{3i-1} \). We map \( v \) to 0. Thus,

\[
\text{minimum weight} = \frac{2n}{3}, m > \frac{2n}{3}.
\]

In summary,

\[
\text{minimum weight} = \begin{cases} 
\frac{2n}{3} + 1, & m \leq \frac{2n}{3} \\
\frac{2n}{3}, & m > \frac{2n}{3}.
\end{cases}
\]
Sub-case 2.2: \( n \equiv 1 \pmod{3} \).

If \( f(v_n) = 1 \), vertices \( v_{3i-1} \) \((1 \leq i \leq \frac{n-1}{3})\) will be mapped to 2 while others on the path to 0. If \( f(v_1) = 1 \), vertices \( v_{3i} \) will be mapped to 2 while others on the path to 0. Only vertices \( v_{3i-2} \) \((1 \leq i \leq \frac{n+2}{3})\) will never be mapped to 2.

Thus when \( m \leq \frac{n+2}{3} \), we connect the new edges to vertices \( v_{3i-2} \). Then, \( v \) must be mapped to 1. Thus,

\[
\text{minimum weight} = \left\lfloor \frac{2n}{3} \right\rfloor + 1 = \frac{2n + 4}{3}, m \leq \frac{n + 2}{3}.
\]

When \( m > \frac{n+2}{3} \), \( v \) will be connected to some \( v_{3i-1} \) or \( v_{3i} \) \((1 \leq i \leq \frac{n-1}{3})\).

Then, \( v \) will be mapped to 0. Thus,

\[
\text{minimum weight} = \left\lfloor \frac{2n}{3} \right\rfloor = \frac{2n + 1}{3}, m > \frac{n + 2}{3}.
\]

In summary,

\[
\text{minimum weight} = \begin{cases} \frac{2n + 4}{3}, & m \leq \frac{n + 2}{3} \\ \frac{2n + 1}{3}, & m > \frac{n + 2}{3} \end{cases}
\]

Sub-case 2.3: \( n \equiv 2 \pmod{3} \).

If \( f(v_{n-1}) = f(v_n) = 1 \), vertices \( v_{3i-1} \) \((1 \leq i \leq \frac{n-2}{3})\) will be mapped to 2 while others on the path to 0. If \( f(v_1) = f(v_n) = 1 \), vertices \( v_{3i} \) will be mapped to 2 while others on the path to 0. If \( f(v_1) = f(v_2) = 1 \), vertices \( v_{3i+1} \) will be mapped to 2 while others on the path to 0.

\( v_1 \) can also be mapped to 2: \( f(v_1) = 2, f(v_2) = 0, f(v_{3i+1}) = 2 \), all other vertices being mapped to 0. By symmetry, \( v_n \) can also be mapped to 2.

Thus,

\[
\text{minimum weight} = \left\lfloor \frac{2n}{3} \right\rfloor = \frac{2n + 2}{3}
\]

Now we have obtained the minimum weight possible under the two cases, we
can compare them and determine which one offers the smaller value. We will have three cases.

**Case 1:** When $n \equiv 0 \pmod{3}$,

$$\gamma_R(G) = \min\left\{ \begin{array}{ll}
\frac{2n}{3} + 2, m < \frac{n}{3} \\
\frac{2n}{3} + 1, m \leq \frac{2n}{3} \\
n - m + 2, m \geq \frac{n}{3}
\end{array} \right. \left\{ \begin{array}{ll}
m < \frac{n}{3} \\
m \leq \frac{2n}{3} \\
m \geq \frac{n}{3}
\right. \right. $$

When $0 \leq m < \frac{n}{3}$, $\gamma_R(G) = \frac{2n}{3} + 1$, $v$ mapped to 1.

When $\frac{n}{3} \leq m \leq \frac{2n}{3}$, we compare the values $(n - m + 2)$ and $(\frac{2n}{3} + 1)$. When $m \leq \frac{n}{3} + 1$, $n - m + 2 \geq \frac{2n}{3} + 1$; when $m \geq \frac{n}{3} + 1$, $n - m + 2 \leq \frac{2n}{3} + 1$.

Thus, when $\frac{n}{3} \leq m \leq \frac{n}{3} + 1$, $\gamma_R(G) = \frac{2n}{3} + 1$, $v$ mapped to 1; when $\frac{n}{3} + 1 \leq m \leq \frac{2n}{3}$, $\gamma_R(G) = n - m + 2$, $v$ mapped to 2.

When $\frac{2n}{3} + 2 \leq n$, we compare the values $(n - m + 2)$ and $\frac{2n}{3}$. When $m \leq \frac{n}{3} + 2$, $n - m + 2 \geq \frac{2n}{3}$; when $m \geq \frac{n}{3} + 2$, $n - m + 2 \leq \frac{2n}{3}$.

As $n \geq 3$, we have $m \geq \frac{2n}{3} + 1 \geq \frac{n}{3} + 2$. Thus, $\gamma_R(G) = n - m + 2$, $v$ mapped to 2.

To conclude for the case where $n \equiv 0 \pmod{3}$, when $0 \leq m \leq \frac{n}{3} + 1$, $\gamma_R(G) = \frac{2n}{3} + 1$, $v$ mapped to 1.

When $\frac{n}{3} + 1 \leq m \leq n$, $\gamma_R(G) = n - m + 2$, $v$ mapped to 2.

**Case 2:** When $n \equiv 1 \pmod{3}$,

$$\gamma_R(G) = \min\left\{ \begin{array}{ll}
\frac{2n + 1}{3} + 2, m < \frac{n - 1}{3} \\
\frac{2n + 4}{3} + 1, m \leq \frac{n + 2}{3} \\
n - m + 2, m \geq \frac{n - 1}{3}
\end{array} \right. \left\{ \begin{array}{ll}
m < \frac{n - 1}{3} \\
m \leq \frac{n + 2}{3} \\
m \geq \frac{n - 1}{3}
\right. \right. $$

When $0 \leq m < \frac{n - 1}{3}$, $\gamma_R(G) = \frac{2n + 4}{3}$, $v$ mapped to 1.

When $\frac{n - 1}{3} \leq m \leq \frac{n + 2}{3}$, we have only two possible integer values for $m$.

When $m = \frac{n - 1}{3}$, $n - m + 2 = \frac{2n}{3} + \frac{7}{3} > \frac{2n + 4}{3}$,
When \( m = \frac{n+2}{3}, \) \( n - m + 2 = \frac{2}{3} n + \frac{4}{3} = \frac{2n+4}{3}. \)

Thus when \( \frac{n-1}{3} \leq m \leq \frac{n+2}{3}, \) \( \gamma_R(G) = \frac{2n+4}{3}, \) \( v \) mapped to 1.

When \( \frac{n+2}{3} < m \leq n, \) we have \( n - m + 2 \leq \frac{2n+1}{3}, \) with equality holds only when \( m = \frac{n+5}{3}. \) Thus \( \gamma_R(G) = n - m + 2, \) \( v \) mapped to 2.

To conclude for the case where \( n \equiv 1 \pmod{3}, \)

When \( 0 \leq m \leq \frac{n+2}{3}, \) \( \gamma_R(G) = \frac{2n+4}{3}, \) \( v \) mapped to 1.

When \( \frac{n+2}{3} \leq m \leq n, \) \( \gamma_R(G) = n - m + 2, \) \( v \) mapped to 2.

**Case 3:** When \( n \equiv 2 \pmod{3} \) or in another word \( n = 3k_n + 2, \) Roman domination number for the resultant graph is

\[
\min \left\{ \frac{2n+2}{3} + 2, m < \frac{n-2}{3}, \frac{2n+2}{3} \right\}.
\]

When \( 0 \leq m < \frac{n-2}{3}, \) \( \gamma_R(G) = \frac{2n+2}{3}, \) \( v \) mapped to 0.

When \( \frac{n-2}{3} \leq m \leq \frac{n+4}{3}, n - m + 2 \geq \frac{2n+2}{3}, \) \( \gamma_R(G) = \frac{2n+2}{3}, \) \( v \) mapped to 0.

When \( \frac{n+4}{3} \leq m \leq n, n - m + 2 \leq \frac{2n+2}{3}, \) \( \gamma_R(G) = n - m + 2, \) \( v \) mapped to 2.

**Result:** We summarize the three cases:

If \( m \leq \left\lfloor \frac{n+1}{3} \right\rfloor + 1, \)

\[
\gamma_R(G) = \left\lfloor \frac{2n+2}{3} \right\rfloor, \quad f(v) = \begin{cases} 1, & \text{if } n \equiv 0 \text{ or } 1 \pmod{3} \\ 0, & \text{if } n \equiv 2 \pmod{3} \end{cases}.
\]

If \( m \geq \left\lceil \frac{n+1}{3} \right\rceil + 1, \)

\[
\gamma_R(G) = n - m + 2, \quad f(v) = 2.
\]
Smallest $\gamma_R(G)$:

To find $G$ with the smallest $\gamma_R(G)$, we have two cases.

**Case 1:** $v$ is mapped to 2. $G$ ought to have as many $n_i \equiv 0 \pmod{3}$ as possible. One simple way to do that is to connect $e_1$ to the first vertex on the path, $e_2$ the second vertex, $e_3$ the third vertex and so on.

$$\text{minimum weight} = \left\lceil \frac{2}{3} (n - m) \right\rceil + 2.$$

**Case 2:** $v$ is not mapped to 2. We can always connect $v$ to the path such that it is adjacent to a value-2 vertex. Thus,

$$\text{minimum weight} = \left\lfloor \frac{2n}{3} \right\rfloor.$$

Comparing the two cases, the minimum Roman domination number is

$$\min \left\lceil \frac{2}{3} (n - m) \right\rceil + 2, \left\lfloor \frac{2n}{3} \right\rfloor.$$

By property of ceiling function,

$$\left\lceil \frac{2}{3} (n - m) \right\rceil + 2 = \left\lceil \frac{2}{3} (n - m) + 2 \right\rceil = \left\lceil \frac{2}{3} n + (2 - \frac{2}{3} m) \right\rceil.$$

Thus,

If $m \leq 3$,

$$\gamma_R(G) = \left\lceil \frac{2n}{3} \right\rceil,$$

$$f(v) = 0;$$

If $m \geq 3$,

$$\gamma_R(G) = \left\lceil \frac{2}{3} (n - m) \right\rceil + 2,$$

$$f(v) = 2. \quad \blacksquare$$