J. DIFFERENTIAL GEOMETRY
84 (2010) 289-300

A CHARACTERIZATION OF THE STANDARD EMBEDDINGS OF $\mathbb{C}P^2$ AND Q^3

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Abstract

H. Hopf showed that the only constant mean curvature sphere $\mathbb{S}^2$ immersed in $\mathbb{R}^3$ is the round sphere. The Kähler framework is an adequate approach to generalize Hopf' s theorem to higher dimensions. When $\varphi : M \to \mathbb{R}^n$ is an isometric immersion from a Kähler manifold, the complexified second fundamental form α splits according to types. The $(1,1)$ part of the second fundamental form plays the role of the mean curvature for surfaces and will be called the pluri-mean curvature *pmc*. Therefore isometric immersions with parallel pluri-mean curvature (*ppmc* isometric immersions) generalize in a natural way the cmc immersions. It is a standard fact that $\mathbb{R}^8$ is the smallest space where $\mathbb{C}P^2$ can be embedded. The aim of this work is to generalize Hopf's theorem proving in particular that the only *ppmc* isometric immersion from $\mathbb{C}P^2$ into $\mathbb{R}^8$ is the standard immersion.

1. Introduction and statement of results

The smallest $\mathbb{R}^k$ into which $\mathbb{S}^2 = \mathbb{C}P^1$ may be embedded is $\mathbb{R}^3$. H. Hopf [**13**] showed that, up to congruence, the only constant mean curvature (*cmc*) isometric immersion from the sphere into $\mathbb{R}^3$ is the standard immersion. Affording higher dimensions in the domain manifold, an adequate setting is the class of Kähler manifolds. When M is a Kähler manifold and $\varphi : M \longrightarrow \mathbb{R}^n$ is an isometric immersion, the coupling of the second fundamental form α of φ with the complex structure J of M originates two operators. To describe these operators we denote respectively by T^cM, $T'M$ and $T''M$ the complexification of TM and the eigenbundles of J corresponding to the eigenvalues i and $-i$. We will denote π' and π'' respectively the orthogonal projections of T^cM onto $T'M$ and $T''M$. Accordingly, each $X \in T^cM$ is decomposed as $X = X' + X''$ where

$$X' = \pi'(X) = \tfrac{1}{2}(X - iJX), \quad X'' = \pi''(X) = \tfrac{1}{2}(X + iJX)$$

The authors wish to thank Fundação para a Ciência e Tecnologia, Portugal and CNPq and FAPEAM, Brasil, for support.
Received 03/07/2008.

(*type decomposition*). Then the complexification of α decomposes accordingly giving rise to the components

$$\begin{aligned}\alpha^{(1,1)}(X,Y) &= \alpha(X',Y'')+\alpha(X'',Y'),\\ \alpha^{(2,0)}(X,Y) &= \alpha(X',Y').\end{aligned}$$

H. Hopf discovered that the traceless part of the second fundamental form of an immersed surface with constant mean curvature ("*cmc*") is a holomorphic quadratic differential on the surface. This observation was the key to his well known theorem refered above. This holomorphic differential is nothing but the operator $\alpha^{(2,0)}$, and $\alpha^{(1,1)} = \langle\ ,\ \rangle H$ where $H = \frac{1}{2}\,\mathrm{trace}\,\alpha$ is the mean curvature vector. In higher dimensions, the mean curvature (trace of α) can be generalized to $\alpha^{(1,1)}$ which we call *pluri-mean curvature* (see [**3**] for a justification). For isometric immersions where this part of the second fundamental form is parallel (*parallel pluri-mean curvature, ppmc*), the other part $\alpha^{(2,0)}$ is again a (normal bundle valued) holomorphic quadratic differential. When $\alpha^{(1,1)}$ vanishes identically, the immersion is called $(1,1)$-geodesic or pluriminimal ([**6**], [**4**], [**5**]). When $\alpha^{(2,0)}$ vanishes identically, the immersion is called $(2,0)$-*geodesic*; such immersion are also *ppmc* and have been classified by Ferus [**12**]: they are the so called *standard embeddings* of Kähler symmetric spaces (cf. Section 5 and [**8**]).

Isometric immersions with parallel pluri-mean curvature share some geometric features of parallel mean curvature surfaces, namely the existence of a 1-parameter deformation through a smooth family of isometric *ppmc*-immersions which, up to a parallel isomorphism, have the same normal bundle ([**3**]). Just as in the case of immersions with parallel mean curvature, isometric *ppmc*-immersions can also be characterized by the pluriharmonicity of their Gauss maps ([**10**], [**3**]).

The smallest $\mathbb{R}^k$ into which $\mathbb{C}P^2$ may be *ppmc*-immersed is $\mathbb{R}^8$. (The total Pontrjagin class of $M = \mathbb{C}P^2$ is $p(TM) = 1 + 3\xi^2$ where ξ is the standard generator of $H^2(M;\mathbb{Z})$ (cf. [**15**], p. 178). If $f : M \to \mathbb{R}^n$ is any immersion with normal bundle NM, then $TM \oplus NM$ is a trivial bundle. Thus $p(TM)p(NM) = 1$ whence $p(NM) = (1+3\xi^2)^{-1} = 1-3\xi^2$. Since $p(NM) = 1 + p_1(NM)$, we get

$$p_1(NM) = -3\xi^2. \qquad (a)$$

This excludes codimension one ($n = 5$) since the normal bundle of an oriented hyperplane is trivial. If the codimension is two ($n = 6$), the normal bundle is an oriented plane bundle, hence a complex line bundle. Let $\eta = c_1(NM) \in H^2(\mathbb{C}P^2;\mathbb{Z})$ be its first Chern class. Then by [**15**], p. 177 we have $1 - p_1(NM) = (1-\eta)(1+\eta) = 1-\eta^2$ and therefore

$$p_1(NM) = \eta^2. \qquad (b)$$

Comparing with (a) would yield $-3\xi^2 = \eta^2 \in H^4(\mathbb{C}P^2;\mathbb{Z}) \cong \mathbb{Z}$ which is impossible since -3 is not a square number. The same conclusion holds for codimension three ($n = 7$) provided that NM splits off a trivial subbundle. If the mean curvature vector is nowhere zero, it generates such a subbundle.)

The aim of this work is to generalize Hopf's theorem proving that the only *ppmc*-immersion from $\mathbb{C}P^2$ (with any Kähler metric) into $\mathbb{R}^8$ is the standard immersion. In fact we will prove more: We will show that any *ppmc* immersion with codimension ≤ 4 is a standard embedding. Besides the $\mathbb{S}^2$ and $\mathbb{C}P^2$, there is just one other case with codimension ≤ 4: The complex quadric $Q^3 \subset \mathbb{C}P^4$ which is the Grassmannian of oriented 2-planes in $\mathbb{R}^5$.

Theorem 1.1. *Let M be a compact Kählerian manifold with positive first Chern class and $\varphi : M \to \mathbb{R}^n$ a full indecomposable isometric ppmc immersion with codimension ≤ 4. Then either M is isometric to $\mathbb{S}^2$, $\mathbb{C}P^2$ or Q^3 (up to scaling), and φ is the standard embedding (up to congruence), or $\phi(M)$ is a minimal sphere in $\mathbb{S}^4$.*

Corollary 1.2. *Let $\varphi : \mathbb{C}P^2 \to \mathbb{R}^8$ be an immersion whose induced metric is Kähler. If φ is ppmc, then φ is the standard embedding of $\mathbb{C}P^2$ endowed with the Fubini-Study metric.*

Remark. The minimal spheres in $\mathbb{S}^4$ have been classified by R. Bryant [**2**].

2. Holomorphic differentials

Let M be a Kähler manifold and $\varphi : M \to \mathbb{R}^n$ an isometric immersion. Let $\alpha : S^2(TM) \to NM$ (where S^2 denotes the second symmetric power) be the second fundamental form (tacitly extended to the complexified bundles) with its components $\alpha^{(2,0)}, \alpha^{(1,1)}, \alpha^{(0,2)} = \overline{\alpha^{(2,0)}}$. Throughout the paper we assume that φ is *ppmc*, i.e. $\alpha^{(1,1)}$ is parallel with respect to the induced connections on TM and NM. In particular, the (unnormalized) mean curvature vector $H = \operatorname{trace} \alpha = \sum_i \alpha(E_i, \overline{E_i})$ (where $E_1, \dots, E_m$ is any unitary basis of $T'M$) is a parallel normal vector field.

Lemma 2.1. *The 4-form*

$$\beta : (A, B, C, D) \mapsto \langle \alpha(A, B), \alpha(C, D) \rangle \tag{1}$$

on $\otimes^4 T^c M$ – which is always symmetric in (A, B) and (C, D) – is symmetric in (B, C) iff $\langle R(B, C)A, D\rangle = 0$.

Proof. This is immediate from the Gauss equation

$$\langle \alpha(A, B), \alpha(C, D)\rangle - \langle \alpha(A, C), \alpha(B, D)\rangle = \langle R(B, C)D, A\rangle \qquad \textit{q.e.d.}$$

Lemma 2.2. *The form $\Lambda_4 = \langle \alpha^{(2,0)}, \alpha^{(2,0)}\rangle$ on $\otimes^4(T'M)$ is symmetric and holomorphic. Likewise, for any parallel normal vector field ξ, the symmetric 2-form $\Lambda_\xi = \langle \alpha^{(2,0)}, \xi\rangle$ on $\otimes^2(T'M)$ is holomorphic.*

Proof. Since M is Kähler, $R(B,C)D = 0$ if $B, C \in T'M$ and hence we see the symmetry of Λ_4 from the previous Lemma 2.1. For the holomorphicity we need two preparations:

(a) Let $z = (z_1, \dots, z_m)$ be a holomorphic chart and $Z_j = \partial/\partial z_j$ the corresponding holomorphic coordinate vector fields. Then

$$\nabla_{\overline{Z}_k} Z_j = \nabla_{Z_j} \overline{Z}_k \in T'M \cap T''M = 0. \tag{2}$$

(b) The Codazzi equations show for all $\bar{A} \in T''M$ and $B, C \in T'M$

$$(\nabla_{\bar{A}}\alpha)(B,C) = (\nabla_B \alpha)(\bar{A}, C) = 0 \tag{3}$$

since $\alpha^{(1,1)}$ is parallel. Thus derivatives of α vanish as soon the arguments are of mixed type. Hence $\nabla_{\bar{A}}\Lambda_\xi = \langle \nabla_{\bar{A}}\alpha^{(2,0)}, \xi\rangle = 0$ and similarly $\nabla_{\bar{A}}\Lambda_4 = 0$.

Now the partial derivatives with respect to $\overline{z}_k$ are:

$$\begin{array}{rcccl} \frac{\partial}{\partial \bar{z}_k}\Lambda_\xi(Z_a, Z_b) & = & \Lambda_\xi(\nabla_{\overline{Z}_k} Z_a, Z_b) + \Lambda_\xi(Z_a, \nabla_{\overline{Z}_k} Z_b) & = & 0, \\ \frac{\partial}{\partial \bar{z}_k}\Lambda_4(Z_a, Z_b, Z_c, Z_d) & = & \Lambda_4(\nabla_{\overline{Z}_k} Z_a, Z_b, Z_c, Z_d) + \dots & = & 0 \end{array}$$

which shows that these forms are holomorphic. q.e.d.

Now let us assume that M is compact with positive first Chern class. Then M allows a Kähler metric with positive Ricci curvature, cf. [**1**], (11.16), p.322. A Bochner type argument allows the conclusion that there are no nonzero holomorphic differentials on M (see [**14**]), in particular:

Corollary 2.3. *Let M be a compact Kähler manifold with positive first Chern class and $\varphi : M \to \mathbb{R}^n$ an isometric ppmc immersion. Then the forms $\Lambda_4 = \langle \alpha^{(2,0)}, \alpha^{(2,0)}\rangle$ and $\Lambda_\xi = \langle \alpha^{(2,0)}, \xi\rangle$ for every parallel normal field ξ vanish on all of M.*

A *ppmc* immersion φ will be called *half isotropic* if the last assertion is true, i.e. if $\langle \alpha^{(2,0)}, \alpha^{(2,0)}\rangle = 0$ and $\langle \alpha^{(2,0)}, \xi\rangle = 0$ for every parallel normal field $\xi \in N^o$ where

$$N^o = \{\alpha(A', B'');\ A, B \in TM\}. \tag{4}$$

We have seen that positive first Chern class implies half isotropic.

3. Indecomposability

Let M be a Kähler manifold. An isometric immersion $\varphi : M \to \mathbb{R}^n$ is *decomposable* if M is a Riemannian product of Kähler submanifolds, $M = M_1 \times M_2$, and there are isometric immersions $\varphi_i : M_i \to \mathbb{R}^{n_i}$ with $n = n_1 + n_2$ such that $\varphi = \varphi_1 \times \varphi_2$.

Lemma 3.1. *Let M be Kähler and $\varphi : M \to \mathbb{R}^n$ an isometric ppmc-immersion which is decomposable. Then both factors are ppmc.*

Proof. The type decomposition of the second fundamental form α is inherited to its components α_1 and α_2, and since the projections onto M_i are parallel, the components $\alpha_i^{(1,1)}$ of $\alpha^{(1,1)}$ are also parallel. q.e.d.

Passing to the components if necessary, we may assume from now on that our *ppmc* immersion $\varphi : M \to \mathbb{R}^n$ is *indecomposable.* Moreover we will always assume that φ is *full*, i.e. $\varphi(M)$ is not contained in a proper affine subspace of $\mathbb{R}^n$. We put

$$N^1 = \left\{\alpha^{(2,0)}(A,B) + \alpha^{(0,2)}(A,B);\ A,B \in TM\right\} \tag{5}$$

which is a subbundle of NM on an open subset $M_o \subset M$.

Lemma 3.2. *Let ξ be any parallel normal vector field with $\xi \perp N^1M$. Then the corresponding Weingarten operator $A_\xi \in \mathrm{Hom}(TM,TM)$ is parallel and commutes with J.*

Proof. Let $A_\xi^{(1,1)}$ be the $(1,1)$-Weingarten map of ξ,

$$\langle A_\xi^{(1,1)}(X), Y\rangle := \langle \alpha^{(1,1)}(X,Y), \xi\rangle.$$

Since $\alpha^{(1,1)}(JX,JY) = \alpha^{(1,1)}(X,Y)$, we have

$$\langle J^{-1}A_\xi^{(1,1)}(JX), Y\rangle = \langle A_\xi^{(1,1)}(JX), JY\rangle = \langle A_\xi^{(1,1)}(X), Y\rangle,$$

thus $J^{-1}A_\xi^{(1,1)}J = A_\xi^{(1,1)}$, so $A_\xi^{(1,1)}$ commutes with J. Since both $\alpha^{(1,1)}$ and ξ are parallel, so is $A_\xi^{(1,1)}$. But the (2,0) and (0,2) components of α are perpendicular to ξ, thus $\langle A_\xi(X), Y\rangle = \langle \alpha(X,Y), \xi\rangle = \langle \alpha^{(1,1)}(X,Y), \xi\rangle = \langle A_\xi^{(1,1)}X, Y\rangle$ whence $A_\xi = A_\xi^{(1,1)}$. q.e.d.

Proposition 3.3. *Let $\varphi : M \to \mathbb{R}^n$ be indecomposable, full, ppmc and half isotropic. Then $\varphi(M)$ is minimal in a round sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, and any parallel normal field in N^o is a multiple of the position vector.*

Proof. Let $\xi \in N^oM$ be any parallel normal field. By half isotropy, $\xi \perp N^1$. From Corollary 2.3 we have $\Lambda_\xi = \langle \xi, \alpha^{2,0}\rangle = 0$. Hence by the previous lemma the Weingarten operator A_ξ is parallel and its eigenspaces form parallel J-invariant distributions $E_1, \dots, E_r$. The parallelity of ξ also implies that $R^N(A,B)\xi = 0$ for any A,B and then the Ricci equation shows that A_ξ commutes with any other Weingarten operator A_η. Therefore $\alpha(E_i,E_j) = 0$ for $i \neq j$, and by Moore's theorem [**16**], φ is decomposable unless $r = 1$. Hence $A_\xi = \lambda_\xi I$ for some $\lambda_\xi \in \mathbb{R}$.

In particular all this holds for the mean curvature vector $\xi = H$. By compactness, $\lambda_H \neq 0$. Thus H is umbilic and $\varphi(M)$ is contained as a minimal submanifold in a sphere of radius $1/|\lambda|$.

But for any parallel normal field $\xi \perp H$ in N^o we have $\operatorname{trace} A_\xi = \langle H, \xi\rangle = 0$ and hence $\lambda_\xi = 0$. So ξ is a constant vector since $\partial_X\xi = -A_\xi(X) + \nabla_X^N\xi = 0$ for any tangent vector X. Moreover $\varphi(M) \subset \xi^\perp$. By the fullness assumption this shows $\xi = 0$. q.e.d.

4. The Riccati equation

Next we shall consider the distribution $\Delta = \ker \alpha^{(2,0)}$. As we shall see, this is an autoparallel distribution on M. We need some properties of such distributions.

Let M be a Riemannian manifold and $\Delta \subset TM$ an auto-parallel distribution, i.e. $\nabla_\Delta \Delta \subset \Delta$. Denoting $\Gamma = \Delta^\perp$, we also have $\nabla_\Delta \Gamma \subset \Gamma$, since $\langle \nabla_\Delta \Gamma, \Delta \rangle = -\langle \Gamma, \nabla_\Delta \Delta \rangle \subset \langle \Gamma, \Delta \rangle = 0$.

Proposition 4.1. *For any vector field $T \in \Delta$ we consider the tensor $C_T \in \mathrm{Hom}(\Gamma, \Gamma)$,*

$$C_T X = -(\nabla_X T)^\Gamma$$

for all $X \in \Gamma$. Then we have for all $S, T \in \Delta$:

$$\nabla_S C_T = C_T C_S + C_{\nabla_S T} + R(\ ,S)T. \tag{6}$$

Proof. For any $X \in \Gamma$ we have

$$(\nabla_S C_T)X = \nabla_S (C_T X) - C_T(\nabla_S X), \tag{7}$$

where

$$\begin{aligned}
\nabla_S(C_T X) &= -\nabla_S (\nabla_X T)^\Gamma \\
&= -\left(\nabla_S \nabla_X T\right)^\Gamma \\
&= \left(-R(S,X)T - \nabla_X \nabla_S T - \nabla_{[S,X]} T\right)^\Gamma, && (8)\\
-C_T(\nabla_S X) &= -C_T((\nabla_S X)^\Gamma) \\
&= -C_T((\nabla_X S)^\Gamma) - C_T([S,X]^\Gamma) \\
&= C_T C_S X + (\nabla_{[S,X]^\Gamma} T)^\Gamma. && (9)
\end{aligned}$$

Let $L = [S, X]$. Then

$$(\nabla_L T)^\Gamma = (\nabla_{L^\Gamma} T + \nabla_{L^\Delta} T)^\Gamma = (\nabla_{L^\Gamma} T)^\Gamma$$

since $\nabla_\Delta \Delta \subset \Delta \perp \Gamma$. Hence the last terms of (8) and (9) cancel each other. Moreover, $\nabla_S T \in \Delta$ and

$$-(\nabla_X \nabla_S T)^\Gamma = C_{\nabla_S T} X.$$

Further note that $\langle R(\Delta, \Gamma)\Delta, \Delta \rangle = 0$ since Δ is totally geodesic, so the curvature term $R(S, X)T$ in (8) is automatically in Γ. Thus inserting (8) and (9) into (7) proves (6). q.e.d.

Corollary 4.2. *If (M, J) is Kähler and $\Delta \subset TM$ autoparallel with $J\Delta = \Delta$ and $C_T J = J C_T$ (i.e. C_T is $\mathbb{C}$-linear), then $R(\ ,T)T$ is $\mathbb{C}$-linear on Γ.*

Proof. This is immediate from (6) for $S = T$ and the parallelity of J. q.e.d.

5. The kernel of $\alpha^{(2,0)}$

Now let M be a Kähler manifold and $\varphi : M \to \mathbb{R}^n$ an isometric *ppmc* immersion. Let us consider

$$\Delta = \ker \alpha^{(2,0)} = \left\{X \in TM;\ \alpha(X', Y') = 0\ \forall_{Y \in TM}\right\} \tag{10}$$

which is of maximal dimension on an open subset $M_o \subset M$ and hence a distribution on M_o. We denote Δ' the projection of Δ to $T'M$. Clearly Δ is J-invariant.

When $\Delta = TM$, i.e. $\alpha^{(2,0)} = 0$, the immersion is called $(2,0)$-*geodesic*. In this case, the Codazzi equations immediately show $\nabla\alpha = 0$. Such immersions (so called *extrinsic symmetric spaces*) have been classified by D. Ferus [**12**]. The (2,0)-geodesic ones are the *standard embeddings* of the Kähler symmetric spaces, defined as follows. A *Kähler symmetric space* is a Kähler manifold M which is also a symmetric space such that all point reflections are holomorphic. If M is compact without local euclidean factor, the almost complex structure J_p at any point $p \in M$ defines an element of its transvection Lie algebra $\mathfrak{g}$; this map $p \mapsto J_p$: $M \to \mathfrak{g}$ is the standard embedding (cf. [**8**]). E.g. for $M = \mathbb{S}^2$, the transvection Lie algebra is $\mathfrak{g} = \mathfrak{so}(3) \cong \mathbb{R}^3$ and for $M = \mathbb{C}P^2$ we have $\mathfrak{g} = \mathfrak{su}(3) \cong \mathbb{R}^8$.

Lemma 5.1. *For all $S, T \in \Delta$ and $A \in TM$ we have:*

$$\nabla_{A''}T' \in \Delta', \tag{11}$$

$$\nabla_S T \in \Delta. \tag{12}$$

Proof. The Codazzi equations give for all $B \in TM$:

$$(\nabla_{A''}\alpha)(T', B') = (\nabla_{T'}\alpha)(A'', B') = 0$$

since $\alpha^{(1,1)}$ is parallel. Hence $\alpha(\nabla_{A''}T', B') = -\alpha(T', \nabla_{A''}B') = 0$ since $T \in \Delta$. This proves (11). For (12) we have to show

$$\alpha(\nabla_S T', B') = 0$$

for all $S, T \in \Delta$ and $B \in TM$. We split $S = S' + S''$. Since $\alpha(\nabla_{S''}T', B') = 0$ by (11), it remains to show $\alpha(\nabla_{S'}T', B') = 0$. But $(\nabla_{S'}\alpha)(T', B') = (\nabla_{B'}\alpha)(T', S') = \nabla_{B'}\left(\alpha(T', S')\right) - \alpha(\nabla_{B'}T', S') - \alpha(T', \nabla_{B'}S') = 0$. Thus

$$\alpha(\nabla_{S'}T', B') = -\alpha(T', \nabla_{S'}B') = 0.$$

q.e.d.

Corollary 5.2. *Δ is autoparallel and hence integrable, and the leaves are totally geodesic Kähler submanifolds which are (2,0)-geodesic in the ambient euclidean space.*

Proof. Δ is autoparallel by (12), hence integrable with totally geodesic leaves, and since Δ is J-invariant, the leaves are Kähler submanifolds of M. Moreover they are (2,0)-geodesic since $\alpha^{(2,0)} = 0$ on Δ.

q.e.d.

Now let $\Gamma = \Delta^\perp$. Consider the tensor field $C : \Delta \to \mathrm{Hom}(\Gamma, \Gamma)$ defined by

$$C_T(X) = -(\nabla_X T)^\Gamma \tag{13}$$

for $T \in \Delta$ and $X \in \Gamma$.

Lemma 5.3. *C_T commutes with J.*

Proof. By (11) we have $(\nabla_{X''}T')^\Gamma = 0 = (\nabla_{X'}T'')^\Gamma$. Extending the Γ-projection complex linearly and using the splitting $X = X' + X''$ and $T = T' + T''$ we have

$$(\nabla_X T)^\Gamma = (\nabla_{X'}T')^\Gamma + (\nabla_{X''}T'')^\Gamma$$

and consequently

$$(\nabla_{JX} T)^\Gamma = i(\nabla_{X'}T')^\Gamma - i(\nabla_{X''}T'')^\Gamma = J(\nabla_X T)^\Gamma.$$

Now the claim follows from the definition of C_T, see (13). q.e.d.

6. Small codimension

Let $N^1 = N' + N'' \subset NM$ where N' is spanned by the values of $\alpha^{(2,0)}$ and $N'' = \overline{N'}$ by the values of $\alpha^{(0,2)}$; these are subbundles on an open subset $M_o \subset M$. By Corollary 2.3, N' and N'' are isotropic, $\langle N', N' \rangle = 0$. Denoting by $(\ , \ )$ the hermitean inner product, $(X, Y) = \langle X, \overline{Y} \rangle$, we have $(N', N'') = \langle N', N' \rangle = 0$ and thus

$$\dim N^1 = 2 \dim N'.$$

Moreover note that $N^1 \perp H \neq 0$, hence $N \supset \mathbb{R}H \oplus N^1$ and therefore

$$\operatorname{codim} \varphi(M) \geq 2 \dim N' + 1. \tag{14}$$

Definition. A *ppmc* immersion φ is said to be *isotropic* if

$$\langle \alpha^{(2,0)}, \alpha^{(2,0)} \rangle = 0 = \langle \alpha^{(2,0)}, \alpha^{(1,1)} \rangle$$

i.e. the values of $\alpha^{(2,0)}, \alpha^{(0,2)}, \alpha^{(1,1)}$ span subbundles which are mutually perpendicular with respect to the hermitian inner product.

Lemma 6.1. *Let M be Kähler and $\varphi : M \to \mathbb{R}^n$ an isometric ppmc immersion of codimension ≤ 4. If $\operatorname{codim} \Delta \geq 2$, then φ is isotropic.*

Proof. We have seen above that $\langle \alpha^{(2,0)}, \alpha^{(2,0)} \rangle = \Lambda_4 = 0$. We need to show $\langle \alpha^{(2,0)}, \alpha^{(1,1)} \rangle = 0$, i.e.

$$\langle \alpha(X', Y'), \alpha(Z', W'') \rangle = 0$$

for all $X, Y, Z, W \in TM_o$. Let us fix Z and W. The values of $\alpha^{(2,0)}$ lie in N' which is complex one-dimensional on M_o since $N' \neq 0$ and $2\dim N' + 1 \leq 4$ by (14). Thus the subspace of all U with $\alpha(U', Z') = 0$ has dimension $n-1$. By Lemma 2.1 we have for any such U

$$\langle \alpha(U', V'), \alpha(Z', W'') \rangle = \langle \alpha(U', Z'), \alpha(V', W'') \rangle = 0$$

Since $\dim \Delta \leq n-2$, we may choose our U outside Δ, and hence we find some V such that $\alpha(U', V') \neq 0$. But since $\dim N' = 1$ we may replace the particular element $\alpha(U', V')$ by an arbitrary $\alpha(X', Y') \in N'$ and obtain $\langle \alpha(X', Y'), \alpha(Z', W'') \rangle = 0$. q.e.d.

Lemma 6.2. (cf. [**11**]) *Let M be a compact Kähler manifold with $c_1(M) > 0$ and $\varphi : M \to \mathbb{R}^n$ an isometric ppmc immersion with codimension ≤ 4. If* $\operatorname{codim} \Delta = 1$*, then either $\varphi(M)$ is $(2,0)$-geodesic or φ is decomposable into a product of two ppmc immersions one of which is $(2,0)$-geodesic.*

Proof. By Theorem 4.1, the tensor field $C_T : \Gamma \to \Gamma$ corresponding to Δ (see (13)) satisfies the Riccati equation (6). Since Γ is complex one-dimensional (with the complex structure defined by J) and C_T is complex linear by Lemma 5.3, it is a complex multiple of the identity, $C_T = \lambda I$. Let γ be a geodesic on a maximal leaf of Δ and denote by T its velocity field. Then $C_{T(t)} = \lambda(t) I$ where the complex function $\lambda(t)$ satisfies the Riccati type equation

$$\lambda' = \lambda^2 + r \tag{15}$$

where $R(\ ,T)T = r(t)I$ with $r(t) = \left\langle R^M(Y,T)T, Y \right\rangle_{\gamma(t)}$. We will see in the subsequent Lemma 6.3 that $r \geq 0$. It is well known that 0 is the only real solution of (15) which is defined on the whole real line (any other solution has a pole). Therefore C_T has no real eigenvalues. But if λ is complex, $\lambda = \mu + i\eta$, we replace T by the vector $\tilde{T} = \mu T - \eta JT = \bar{\lambda} T$ and get $C_{\tilde{T}} = \bar{\lambda} C_T = \bar{\lambda}\lambda I = (\mu^2 + \eta^2) I$ at the initial point $t = 0$. Extending $\tilde{T}$ to the tangent vector field along a geodesic $\tilde{\gamma}$, we obtain $C_{\tilde{T}} = \tilde{\lambda} I$ with $\tilde{\lambda}(0) \in \mathbb{R}$. Then $\tilde{\lambda}(t)$ is a real solution of (15) and as before we conclude $\tilde{\lambda} = 0$ which implies $\lambda = 0$. We conclude that $C_T = 0$ for all $T \in \Delta$ which shows that Δ is not only autoparallel, but even fully parallel, and then the same holds for $\Gamma = \Delta^{\perp}$. Hence M_o is locally a product of two nontrivial Kähler manifolds M_1 and M_2.

To prove that $\varphi_{|M_o}$ is a product of immersions we first notice that $\alpha(S', Y') = 0$ for all $S \in \Delta$ and $Y \in \Gamma$. Using Lemma 2.1 and the vanishing of curvature tensor components with mixed Δ and Γ entries, we get

$$0 = \langle \alpha(S', Y'), \alpha(S'', Y'') \rangle = \langle \alpha(S', Y''), \alpha(S'', Y') \rangle.$$

This shows $\alpha(S', Y'') = 0$ and henceforth $\alpha(S, Y) = 0$ whenever $S \in \Delta$ and $Y \in \Delta^{\perp}$. Then $\varphi_{|M_o}$ splits as a product of immersions [**16**]. An analyticity argument allows the conclusion that M is globally a product of two Riemann surfaces M_1 and M_2 and φ is a product of two *ppmc* immersions, φ_1 and φ_2 where one of the factors (the integral leaves of Δ) is $(2,0)$-geodesic. q.e.d.

Lemma 6.3. *For all $T \in \Delta$ and $Y \in \Gamma = \Delta^\perp$ we have*

$$\langle R(Y,T)T, Y\rangle \geq 0 \tag{16}$$

Proof. We consider the complex multilinear extension of the curvature tensor and claim that, whenever $T, S \in \Delta$ and $Y \in \Gamma$,

$$R(Y'', T')S' \in \Delta', \quad R(Y', T'')S'' \in \Delta''. \tag{17}$$

To prove this claim we remember from (11) that $\nabla_{Z''}T' \in \Delta'$ (respectively $\nabla_{Z'}T'' \in \Delta$) whenever T is a section of Δ and $Z \in TM$. Using this and the fact that Δ is an auto-parallel distribution, we know that $\nabla_{T'}\nabla_{Y''}S'$, $\nabla_{Y''}\nabla_{T'}S'$ and $\nabla_{[T',Y'']}S'$ are in Δ', hence $R(Y'',T')S' \in \Delta'$. This proves (17).

We also recall that on any Kähler manifold we have $R(Y',T') = 0 = R(Y'',T'')$. Thus

$$\begin{aligned} R(Y,T) &= R(Y',T'') + R(Y'',T'), \\ \langle R(A,B)Y,T\rangle &= \langle R(A,B)Y',T''\rangle + \langle R(A,B)Y'',T'\rangle \end{aligned}$$

for arbitrary A, B. Since $T''M$ is isotropic ("Isotropic" means that the inner product vanishes: $\langle X+iJX, Y+iJY\rangle = \langle X,Y\rangle - \langle JX,JY\rangle + i\,(\langle X,JY\rangle + \langle JX,Y\rangle) = 0$ for all $X \in TM$), we conclude from (17), the Gauss equation and $\alpha(T',Y') = 0$:

$$\begin{aligned} \langle R(Y,T)T,Y\rangle &= \langle R(Y'',T')T'',Y'\rangle + \langle R(Y',T'')T',Y''\rangle \\ &= 2\langle \alpha(T',T''), \alpha(Y'',Y')\rangle. \end{aligned}$$

Again from Gauss equation (Lemma 2.1) we obtain

$$\langle \alpha(T',T''), \alpha(Y'',Y')\rangle = \langle \alpha(T',Y''), \alpha(T'',Y')\rangle.$$

Thus

$$\langle R(Y,T)T,Y\rangle = 2\langle \alpha(T',Y''), \alpha(T'',Y')\rangle \geq 0.$$

q.e.d.

7. The isotropic case

Recall that a *ppmc* immersion is *isotropic* if $\langle \alpha^{(2,0)}, \alpha^{(2.0)}\rangle = 0$ and $N^o \perp N^1$. Clearly "isotropic" is stronger than "half isotropic". A general study of this case has been done in [**7**], but in the present situation of low codimension we can do better.

Proposition 7.1. *Let $\varphi : M \to \mathbb{R}^n$ be full indecomposable isotropic ppmc with codimension ≤ 4. Then either $\varphi(M)$ is an isotropic minimal surface ("superminimal surface") in $\mathbb{S}^4$ or M is isometric to $\mathbb{S}^2$ or $\mathbb{C}P^2$*

or Q^3 (up to scaling) and φ is the standard embedding $\mathbb{S}^2 \hookrightarrow \mathbb{R}^3 = \mathfrak{so}(3)$ or $\mathbb{C}P^2 \hookrightarrow \mathbb{R}^8 = \mathfrak{su}(3)$ or $Q^3 \hookrightarrow \mathbb{R}^{10} = \mathfrak{so}(5)$.

Proof. We have to show that $\alpha^{(2,0)} = 0$; then φ is a standard embedding of a Kähler symmetric space with codimension ≤ 4 and we are done.

Thus assume that $\alpha^{(2,0)}$ does not vanish identically. Then the subbundle $N^o \subset NM$ must have rank one; otherwise in view of (14), each fibre of N^o would have dimension two and we would have another parallel normal field perpendicular to H in N^o which is impossible by Proposition 3.3. Thus φ takes values in the sphere $\mathbb{S}^{n-1}$, and the restriction $\varphi_S : M \to \mathbb{S}^{n-1}$ is *pluriminimal* or $(1,1)$-*geodesic*, i.e. the second fundamental form α_S of φ_S has vanishing (1,1)-component. By the subsequent lemma, M is a surface. Thus $\varphi(M)$ is an isotropic minimal surface of $\mathbb{S}^{n-1}$ with $n \leq 6$. But such minimal surfaces do not exist in $\mathbb{S}^5$ which is not an inner symmetric space (cf. [**9**]), thus $\varphi(M) \subset \mathbb{S}^4$.

q.e.d.

Lemma 7.2. *([**5**], [**17**]) Let M be a compact Kähler manifold and $\varphi_S : M \to \mathbb{S}^{n-1}$ a pluriminimal immersion. Then M is a surface.*

Proof. Let $\dim M = 2m$. Composing φ_S with the embedding $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, we get a *ppmc* immersion $\varphi : M \to \mathbb{R}^n$. Taking, at each $x \in M$, an orthonormal basis E_i, JE_i, $1 \leq i \leq m$, and using Gauss equations we obtain that

$$\left\langle \alpha(E_i', E_i''), \alpha(E_j'', E_j') \right\rangle = \left\langle \alpha(E_i', E_j''), \alpha(E_i'', E_j') \right\rangle,$$

from whence $H = 0$ which cannot happen. In fact,

$$\alpha(E_i', E_i'') = \langle E_i', E_i'' \rangle\, H = \tfrac{1}{2}\, H$$

while $\alpha(E_i', E_j'') = \langle E_i', E_j'' \rangle\, H = 0$ for $i \neq j$. q.e.d.

Proof of the Main Theorem: The proof of Theorem 1.1 is obtained from Lemma 6.1, Lemma 6.2 and Proposition 7.1.

References

[1] A. Besse, *Einstein Manifolds*, A Series of Modern Surveys, Springer-Verlag, 1987.

[2] R. Bryant, *Conformal and minimal immersions of compact surfaces into the 4-sphere*, J. Diff. Geom. **17** (1982), 455–473.

[3] F.E. Burstall, J.H. Eschenburg, M.J. Ferreira & R. Tribuzy, *Kähler submanifolds with parallel pluri-minimal curvature*, Diff. Geom and Apl. **20** (2004), 47–66.

[4] M. Dajczer & D. Gromoll, *Real Kähler submanifolds and uniqueness of the Causs map*, J. Diff. Geom. **22** (1985), 13–25.

[5] M. Dajczer & L. Rodrigues, *Rigidity of real Kähler submanifolds*, Duke Math. J. **53** (1986), 211–220.

[6] J. Eells & L. Lemaire, *Another report on Harmonic Maps*, Bull. London Math. Soc. **86** vol.20 (1988).

[7] J.-H. Eschenburg, M.J. Ferreira & R. Tribuzy, *Isotropic ppmc immersions*, Diff. Geom. Appl. **25** (2007), 351–355.

[8] J.-H. Eschenburg & P. Quast, *Pluriharmonic maps into Kähler symmetric spaces and Sym's formula*, Math. Z. **264** (2010), 469–481.

[9] J.-H. Eschenburg & R. Tribuzy, *Associated families of pluriharmonic maps and isotropy*, manuscripta math. **95** (1998), 295–310.

[10] M. Ferreira & R. Tribuzy, *Kählerian submanifolds of* $\mathbb{R}^n$ *with pluriharmonic Gauss map*, Bull.. Soc. Math. Belg. **45** (1993), 183–197.

[11] M. Ferreira & R. Tribuzy, *On the Nullity of Isometric Immersions from Kähler Manifolds*, Rend. Sem. Mat. Univ. Pol. Torino **65** (2007), 345–352.

[12] D. Ferus, *Symmetric Submanifolds of Euclidean Space*, Math. Ann. **247** (1980), 81–93.

[13] H. Hopf, *Differential Geometry in the Large*, Lecture Notes in Mathematics, 1000, Springer-Verlag (1983).

[14] S. Kobayashi & H.H. Wu, *On Holomorphic Sections of Certain Hermitian Vector Bundles*, Math. Ann. **198** (1970), 1–4.

[15] J.W. Milnor & J.D. Stasheff, *Characteristic Classes*, Annals of Mathematics Studies **76**, Princeton 1974.

[16] J.D. Moore, *Isometric Immersions of Riemannian Products*, J. Diff. Geom. **5** (1971), 159–168.

[17] S. Udagawa, *Holomorphicity of certain stable harmonic maps and minimal immersions*, Proc. London Math. Soc. (3), **57** (1988), 577–598.

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