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CLASSIFICATION OF COMPACT ANCIENT SOLUTIONS TO THE CURVE SHORTENING FLOW

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Abstract

We consider an embedded convex compact ancient solution Γ_t to the curve shortening flow in $\mathbb{R}^2$. We prove that Γ_t is either a family of contracting circles, which is a type I ancient solution, or a family of evolving Angenent ovals, which is of type II.

1. Introduction

We consider an ancient embedded solution $\Gamma_t \subset \mathbb{R}^2$ of the curve shortening flow

$$\frac{\partial \mathbf{X}}{\partial t} = -\kappa \, \mathbf{N} \tag{1.1}$$

which moves each point $\mathbf{X}$ on the curve Γ_t in the direction of the inner normal vector $\mathbf{N}$ to the curve at P by a speed which is equal to the curvature κ of the curve.

In [**4**] Gage and Hamilton proved that if Γ_0 is a convex curve embedded in $\mathbb{R}^2$, then equation (1.1) shrinks Γ_t to a point. In addition, the curve remains convex and becomes asymptotically circular close to its extinction time.

In [**5**] Grayson studied the evolution of non-convex embedded curves under (1.1). He proved that if Γ_0 is any embedded curve in $\mathbb{R}^2$, the solution Γ_t does not develop any singularities before it becomes strictly convex.

Let $\Gamma_t \subset \mathbb{R}^2$ be an embedded ancient solution to the curve shortening flow (1.1). If s is the arclength along the curve and $\mathbf{X} = (x, y)$, we can express (1.1) as a system

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial s^2}, \qquad \frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial s^2}.$$

The evolution for the curvature κ of Γ_t is given by

$$\kappa_t = \kappa_{ss} + \kappa^3, \tag{1.2}$$

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which is a strictly parabolic equation. Let θ be the angle between the tangent vector and the x-axis. For convex curves we can use the angle θ as a parameter. It has been computed in [**4**] that

$$\kappa_t = \kappa^2 \kappa_{\theta\theta} + \kappa^3. \tag{1.3}$$

It turns out that the evolution of the family Γ_t is completely described by the evolution (1.3) of the curvature κ. Gage and Hamilton observed that a positive 2π periodic function represents the curvature function of a simple closed strictly convex C^2 plane curve if and only if

$$\int_0^{2\pi} \frac{\cos\theta}{\kappa(\theta)}\, d\theta = \int_0^{2\pi} \frac{\sin\theta}{\kappa(\theta)}\, d\theta = 0. \tag{1.4}$$

We will assume from now on that Γ_t is an ancient solution of the curve shortening flow defined on $(-\infty, T)$. We will also assume that our extinction time $T = 0$.

It is natural to consider the *pressure* function

$$p := \kappa^2$$

which evolves by

$$p_t = p\, p_{\theta\theta} - \frac{1}{2}\, p_\theta^2 + 2\, p^2. \tag{1.5}$$

We say that an ancient solution to (1.3) is

- of type I, if it satisfies $\sup_{t\in(-\infty,-1]} \sup_{\Gamma_t} |t||p(x,t)| < \infty$;
- of type II, if $\sup_{t\in(-\infty,-1]} \sup_{\Gamma_t} |t||p(x,t)| = \infty$.

The ancient solution to (1.3) defined by

$$p(\theta, t) = \frac{1}{2(-t)}$$

corresponds to a family of *contracting circles*. This solution is of type I and at the same time falls in a category of contracting self-similar solutions (these are solutions of the flow whose shapes change homothetically during the evolution). We will show in the next section the existence of compact ancient solutions to (1.3) that are not self-similar. Since they have been discovered by Angenent, we will refer to them as to the *Angenent ovals*.

One very nice and important property of ancient solutions to the curve shortening flow is that $\kappa_t \geq 0$. This fact follows from Hamilton's Harnack estimate for convex curves [**7**]. By the strong maximum principle, $\kappa(\cdot, t) > 0$ for all $t < 0$. If we start at any time $t_0 \leq 0$, Hamilton proved that

$$\kappa_t + \frac{\kappa}{2(t - t_0)} - \frac{\kappa_s^2}{k} \geq 0. \tag{1.6}$$

Letting $t_0 \to -\infty$, we get

$$\kappa_t \geq 0. \tag{1.7}$$

In this note we provide the following classification of ancient convex solutions to the curve shottening flow.

Theorem 1.1. *Let $p(\theta, t) = \kappa^2(\theta, t)$ be an ancient solution to (1.5), definning a family of embedded closed convex curves in $\mathbb{R}^2$ that evolve by the curve shortening flow. Then,*

(i) either $p(\theta, t) = \frac{1}{(-2t)}$, which corresponds to contracting circles, or

(ii) $p(\theta, t) = \lambda(\frac{1}{1-e^{2\lambda t}} - \sin^2(\theta + \gamma))$, for two parameters $\lambda > 0$ and γ, which corresponds to the Angenent ovals.

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2. Proof of Theorem 1.1

We begin by showing the existence of the Angenent ovals (ancient compact solutions to the curve shortening flow that are not self-similar).

Proposition 2.1 (Angenent)**.** *There exist compact ancient solutions to (1.5) which contract to a point at time $t = 0$ and have the form*

$$p(\theta, t) = \lambda(\frac{1}{1 - e^{2\lambda t}} - \sin^2(\theta + \gamma)) \tag{2.1}$$

where $\lambda > 0$ and γ is a fixed angle.

Proof. We look for a solution to (1.5) in the form

$$p(\theta, t) = a(t) - b(t) \sin^2(\theta + \gamma).$$

Then

$$p_t = a'(t) - b'(t) \sin^2(\theta + \gamma),$$

while

$$p_\theta = -2\, b(t)\, \sin(\theta{+}\gamma)\, \cos(\theta{+}\gamma), \qquad p_{\theta\theta} = -2\, b(t)\, (\cos^2(\theta{+}\gamma){-}\sin^2(\theta{+}\gamma)).$$

If we plug everything in (1.5), we get

$$a'(t) - b'(t) \sin^2(\theta + \gamma) = -2a(t)\, b(t) + 2a^2(t).$$

Since this has to hold for every θ, we obtain that $b'(t) = 0$, which means $b(t) = \lambda$. If we choose θ at which $\sin(\theta + \gamma) = 0$, we have

$$a'(t) = -2\lambda a(t) + 2a(t)^2.$$

Solving this ODE yields to

$$a(t) = \frac{\lambda}{1 - e^{2\lambda(t+\mu)}}$$

for some constants $\lambda > 0$ and μ. Since $\lim_{t\to 0} p(\theta,t) = +\infty$, we must have $\mu = 0$; hence p is of the form (2.1). q.e.d.

We will prove the theorem by introducing a monotone functional along the flow. Denote by

$$\alpha(\theta,t) := p_\theta(\theta,t).$$

By using (1.5) it easily follows that

$$\alpha_t = p\,(\alpha_{\theta\theta} + 4\,\alpha). \tag{2.2}$$

We introduce the functional

$$I(\alpha) = \int_0^{2\pi} (\alpha_\theta^2 - 4\alpha^2)\, d\theta.$$

The following lemma shows the monotonicity of $I(\alpha)$ in time.

Lemma 2.2. *$I(\alpha(t))$ is decreasing along the flow (2.2). Moreover,*

$$\frac{d}{dt} I(\alpha(t)) = -2\int_0^{2\pi} \frac{\alpha_t^2}{p}\, d\theta.$$

Proof. We compute

$$\begin{aligned}
\frac{d}{dt} I(\alpha(t)) &= \int_0^{2\pi} (2\alpha_\theta \alpha_{\theta t} - 8\alpha\alpha_t)\, d\theta \\
&= -2\int_0^{2\pi} 2\alpha_{\theta\theta}\alpha_t\, d\theta - 8\int_0^{2\pi} \alpha\alpha_t\, d\theta \\
&= \int_0^{2\pi} \frac{2(\alpha_t - 4\alpha p)\alpha_t}{p}\, d\theta - 8\int_0^{2\pi} \alpha\alpha_t\, d\theta \\
&= -2\int_0^{2\pi} \frac{\alpha_t^2}{p}\, d\theta.
\end{aligned}$$

q.e.d.

An easy computation shows that $I(\alpha(t)) \equiv 0$ on both the circles and the Angenent ovals, which motivates the following:

Proposition 2.3. *For any ancient convex solution to (1.5), we have*

$$I(\alpha(t)) \equiv 0, \qquad \textit{for all } t \in (-\infty, 0).$$

The proof of the proposition will be given in two steps. In the first step we will show that $\liminf_{t\to-\infty} I(\alpha(t)) \le 0$, and in the second step we will prove that $\lim_{t\to 0} I(\alpha(t)) = 0$. The monotonicity of $I(\alpha(t))$ shown in Lemma 2.2 will then readily imply that $I(\alpha(t)) \equiv 0$, for all $t < 0$.

Lemma 2.4. *We have*

$$\liminf_{t\to-\infty} I(\alpha(t)) \le 0.$$

Proof. We follow a simplified proof which was suggested to us by the referee. On an ancient solution we have $k_t \geq 0$, which gives $p_t \geq 0$. Hence, $p(\cdot, t) \leq C < \infty$, for all $t < -1 < 0$.

If we differentiate (1.5) in θ, we get

$$(p_\theta)_t = p(p_\theta)_{\theta\theta} + 4pp_\theta, \tag{2.3}$$

which implies

$$\frac{1}{2p}\,(p_\theta^2)_t = p_\theta(p_\theta)_{\theta\theta} + 4p_\theta^2$$

and therefore

$$\left(\frac{p_\theta^2}{2p}\right)_t = \frac{(p_\theta^2)_t}{2p} - \frac{p_\theta^2\;p_t}{2p^2} \leq p_\theta(p_\theta)_{\theta\theta} + 4p_\theta^2$$

where we used that $p_t \geq 0$. Integrating the above inequality gives

$$\frac{d}{dt}\int_0^{2\pi} \frac{p_\theta^2}{2p}\,d\theta \leq \int_0^{2\pi} (\,p_\theta(p_\theta)_{\theta\theta} + 4p_\theta^2\,)\,d\theta,$$

and after integration by parts, we get

$$\frac{d}{dt}\int_0^{2\pi} \frac{p_\theta^2}{2p}\,d\theta = \int_0^{2\pi} (-p_{\theta\theta}^2 + 4p_\theta^2)\,d\theta = -I(\alpha(t)). \tag{2.4}$$

On the other hand, from the inequality

$$p\,p_{\theta\theta} - \frac{1}{2}\,p_\theta^2 + 2\,p^2 = p_t \geq 0,$$

dividing by p and integrating, we obtain

$$\int_0^{2\pi} -\frac{1}{2}\frac{p_\theta^2}{p} + 2\,p\,d\theta \geq 0$$

or

$$\int_0^{2\pi} \frac{p_\theta^2}{2p}\,d\theta \leq 2\int_0^{2\pi} p\,d\theta \leq C$$

since p is bounded for $t < -1 < 0$. Combining this with (2.4) implies that

$$\liminf_{t\to-\infty} I(\alpha(t)) \leq 0,$$

finishing the proof of the lemma. q.e.d.

We will next analyze behavior of the functional $I(\alpha(t))$ as $t \to 0$. For this we will use the following result shown by Gage and Hamilton in [**4**]. Consider the evolution of the normalized curvature $\tilde\kappa$, where the normalization is chosen so that the related convex curve encloses an area π. The rescaled curvature $\tilde\kappa$ is defined by

$$\tilde\kappa(\theta,\tau) = k(\theta,t)\sqrt{-2t}, \qquad \text{with } \tau = -\frac{1}{2}\log(-t). \tag{2.5}$$

The evolution equation for $\tilde\kappa$ is

$$\tilde\kappa_t = \tilde\kappa^2\tilde\kappa_{\theta\theta} + \tilde\kappa^3 - \tilde\kappa. \tag{2.6}$$

Theorem 2.5 (Gage, Hamilton)**.** *If Γ_0 is a closed convex curve embedded in the plane $\mathbb{R}^2$, the curve shortening flow shrinks Γ_t to a point in a circular manner. Moreover, the curvature and all its derivatives of the rescaled curve shortening flow converge exponentially to* 1 *and* 0*, respectively, with the rate $e^{-2\eta\tau}$, where τ is the new time variable introduced above and η is any constant in* $(0,1)$*.*

We will now prove:

Lemma 2.6. *We have*

$$\lim_{t\to 0} I(\alpha(t)) = 0.$$

Proof. To prove the lemma we will analyze the normalized flow (2.6) since due to Theorem 2.5 we have good decay estimates on $\tilde\kappa - 1$ and the derivatives of $\tilde\kappa$. Notice that

$$I(\alpha(t)) = e^{4\tau}\int_0^{2\pi}(\tilde\alpha_\theta^2 - 4\tilde\alpha^2)\,d\theta,$$

with $\tilde\alpha$ denoting the corresponding rescaled $\alpha = p_\theta$. By Theorem 2.5 we have

$$|\tilde\kappa - 1| \le C(\eta)\,e^{-2\eta\tau}, \qquad |\frac{\partial^m\tilde\kappa}{\partial\theta^m}| \le C_m(\eta)\,e^{-2\eta\tau}, \qquad \forall\eta\in(0,1). \tag{2.7}$$

These estimates imply the bounds $|\tilde\alpha| \le Ce^{-2\eta\tau}$ and $|\tilde\alpha_\theta| \le Ce^{-2\eta\tau}$ and show that $I(\alpha(t)) = O(1)$. To conclude the lemma we will need to show that $I(\alpha(t)) = o(1)$. We will do so by analyzing the linearization of (2.6) around $\tilde\kappa = 1$. For this reason we set $w := \tilde k - 1$. It is easy to see that

$$w_\tau = w_{\theta\theta} + 2\,w + w\,w_{\theta\theta}(w+2) + w^2\,(w+3),$$

which we can rewrite as

$$w_\tau = \mathcal{L}(w) + \mathcal{R}(w),$$

where

$$\mathcal{L}(w) = w_{\theta\theta} + 2w \qquad \text{and} \qquad \mathcal{R}(w) = w\,w_{\theta\theta}(w+2) + w^2\,(w+3).$$

Note that $\mathcal{R}(w)$ is an error term that is quadratic in w and its derivatives. Hence, by (2.7) we have $|\mathcal{R}(w(\theta,\tau))| \le Ce^{-4\eta\tau}$. The spectrum for $\mathcal{L}$ on an interval $[0,2\pi]$ is given by

$$\lambda_l = 2 - l^2, \qquad l \ge 0,$$

with corresponding eigenvectors $f_l(\theta) = \cos(l\theta)$ and $g_l(\theta) = \sin(l\theta)$. The semigroup representation formula for w gives

$$w(\theta,s) = e^{s\mathcal{L}}w(\theta,0) + \int_0^s e^{(s-\tau)\mathcal{L}}\,\mathcal{R}(w(\theta,\tau))\,d\tau. \tag{2.8}$$

Since the system of functions $\{\frac{\cos(l\theta)}{\sqrt\pi}\}_{l\ge0}$ and $\{\frac{\sin(l\theta)}{\sqrt\pi}\}_{l\ge0}$ is an orthonormal basis (of the space of continuous and periodic functions with period

2π) with respect to the inner product given by $(f, g) = \int_0^{2\pi} f(\theta)\, g(\theta)\, d\theta$, we have

(2.9)
$$w(\theta, 0) = \sum_{l \geq 0} (\alpha_l\, f_l + \beta_l\, g_l), \qquad \mathcal{R}(w(\theta, \tau)) = \sum_{l \geq 0} (\alpha_l(\tau)\, f_l + \beta_l(\tau)\, g_l).$$

We have

$$e^{s\mathcal{L}} w(\theta, 0) = \alpha_0\, e^{2s} + \sum_{l=1,2} [\alpha_l\, \cos l\theta + \beta_l\, \sin l\theta]\, e^{\lambda_l s} + o(e^{-2s}). \tag{2.10}$$

Also, setting

$$A_l(s) = \int_0^s \alpha_l(\tau) e^{-\lambda_l \tau}\, d\tau \quad \text{and} \quad B_l(s) = \int_0^s \beta_l(\tau) e^{-\lambda_l \tau}\, d\tau,$$

we may write

$$\mathcal{R}(w(\theta, \tau)) = A_0(s)\, e^{2s} + \sum_{l \geq 1} [\, A_l(s)\, \cos l\theta + B_l(s)\, \sin l\theta]\, e^{\lambda_l s} \tag{2.11}$$

where $\lambda_l < -2$, for $l \geq 3$.

We have

$$|\sum_{l \geq 3} [\, A_l(s)\, \cos l\theta + B_l(s)\, \sin l\theta]\, e^{\lambda_l s}\, | = o(e^{-2s}), \quad \text{as } s \to \infty. \tag{2.12}$$

Indeed, since $|\mathcal{R}(w(\theta, \tau))| \leq C e^{-4\eta\tau}$, for $\eta \in (0, 1)$, from the Fourier representation (2.9) for $\mathcal{R}(w(\cdot, t))$, we get the bounds

$$|\alpha_l(\tau)| \leq C e^{-4\eta\tau}, \qquad |\beta_l(\tau)| \leq C e^{-4\eta\tau}, \quad \text{for all } l \geq 1. \tag{2.13}$$

Note that for $l \geq 3$ the above implies

$$e^{\lambda_l s}\, |A_l(s)\, \cos l\,\theta + B_l(s)\, \sin l\,\theta| \leq C\, e^{\lambda_l s} \int_0^s e^{(-\lambda_l - 4\eta)\tau}\, d\tau \leq \frac{o(e^{-2s})}{l^2 - 2 - 4\eta}$$

since $\lambda_l < -2$ and $-4\eta < -2$ for $\eta \in (\frac{1}{2}, 1)$. This finishes the proof of (2.12), since the series $\sum_{l \geq 3} \frac{1}{l^2 - 2 - 4\eta}$ converges.

Combining (2.8), (2.10), (2.11), and (2.12), we conclude that

$$\begin{aligned} w(\theta, s) = {} & \bar{A}_0(s)\, e^{2s} + [\, \bar{A}_1(s)\, \cos\theta + \bar{B}_1(s)\, \sin\theta\,]\, e^{s} \\ & + [\, \bar{A}_2(s)\, \cos 2\theta + \bar{B}_2(s)\, \sin 2\theta\,]\, e^{-2s} + o(e^{-2s}), \end{aligned} \tag{2.14}$$

with $\bar{A}_l(s) = \alpha_l + A_l(s)$ and $\bar{B}_l(s) = \beta_l + B_l(s)$. We will next estimate the first two terms in the above expansion. By definition we have $\tilde{\kappa} = 1 + w$, where $w = O(e^{-2\eta s})$, with $\eta \in (0, 1)$, by (2.7). Hence,

$$\frac{1}{\tilde{\kappa}} = 1 - w + O(w^2). \tag{2.15}$$

By (1.4), multiplying (2.15) by $\cos\theta$ or $\sin\theta$, integrating it over $\theta \in [0, 2\pi]$, and using (2.14) and (2.7), we obtain

$$\bar{A}_1(s)\, e^{s} = o(e^{-2s}) \quad \text{and} \quad \bar{B}_1(s)\, e^{s} = o(e^{-2s}).$$

The above discussion implies that

$$\tilde\kappa(\theta,s) = 1 + \bar A_0(s)\, e^{2s} + [\bar A_2(s)\cos(2\theta) + \bar B_2(s)\sin(2\theta)]e^{-2s} + o(e^{-2s}). \tag{2.16}$$

Therefore, the pressure $\tilde p := \tilde\kappa^2$ satisfies

$$\begin{aligned}\tilde p(\theta,s) &= (1+\bar A_0(s)\, e^{2s})^2 + 2\,[\,\bar A_2(s)\cos 2\theta + \bar B_2(s)\sin 2\theta\,]\, e^{-2s}\\ &\quad + 2\bar A_0(s)\,[\,\bar A_2(s)\,\cos 2\theta + \bar B_2(s)\,\sin 2\theta\,] + \bar A_0(s)\, o(1) + o(e^{-2s}).\end{aligned} \tag{2.17}$$

By the decay of w and (2.14) we have $|\bar A_0(s) e^{2s}| \le C\, e^{-2\eta s}$, which implies the bound $\bar A_0(s) = O(e^{-2s(1+\eta)})$. Similarly, we can see that $\bar A_2(s)$ and $\bar B_2(s)$ are of the order $O(e^{2(1-\eta)s})$, which shows that

$$\bar A_0(s)\,[\,\bar A_2(s)\cos 2\theta + \bar B_2(s)\sin 2\theta\,] = o(e^{-2s}).$$

Differentiating (2.17) in θ and using the above estimates, we finally conclude that

$$\tilde\alpha(\theta,s) = \tilde p_\theta(\theta,s) = 4\,[\,\bar B_2(s)\cos 2\theta - \bar A_2(s)\sin 2\theta]\, e^{-2s} + o(e^{-2s}),$$

which easily yields to

$$I(\alpha(t)) = e^{4\tau}\int_0^{2\pi} (\tilde\alpha_\theta^2 - 4\tilde\alpha^2)\, d\theta = o(1)$$

since $\int_0^{2\pi}(\sin^2 2\theta - \cos^2 2\theta)\, d\theta = 0$ and $\int_0^{2\pi} \sin 2\theta \cos 2\theta\, d\theta = 0$, finishing the proof of the Lemma. q.e.d.

We will now complete the proof of Theorem 1.1.

Proof of Theorem 1.1. By Proposition 2.3 we have

$$I(\alpha(t)) \equiv 0, \qquad \text{for all } t<0.$$

Lemma 2.2 implies that $\alpha_t \equiv 0$, that is,

$$p\,(\alpha_{\theta\theta} + 4\alpha) = 0,$$

which means (since $p>0$) that

$$\alpha_{\theta\theta} + 4\alpha = 0$$

and therefore

$$\alpha(\theta,t) = a_0(t)\cos 2(\theta+\gamma(t)) + b_0(t)\sin 2(\theta+\gamma(t))$$

for some functions in time $a_0(t)$, $b_0(t)$, and $\gamma(t)$. Since $\alpha = p_\theta$, by integrating in θ we obtain

$$p(\theta,t) = a(t)\,\sin 2(\theta+\gamma(t)) + b(t)\,\cos 2(\theta+\gamma(t)) + c(t) \tag{2.18}$$

for $a(t) = \frac{a_0(t)}{2}$, $b(t) = -\frac{b_0(t)}{2}$, and another function in time $c(t)$. If we plug $p(\theta, t)$ back to equation (1.5), we find that a, b, and c satisfy the ODEs

$$a'(t) - 2\, b(t)\, \gamma'(t) = 0, \qquad b'(t) + 2\, a(t)\, \gamma'(t) = 0, \tag{2.19}$$

and

$$c'(t) = 2\, c(t)^2 - 2\, (a(t)^2 + b(t)^2). \tag{2.20}$$

Equations (2.19) imply that

$$\frac{d}{dt}\, (a^2(t) + b^2(t)) = 0.$$

Hence we can set

$$a^2(t) + b^2(t) = \lambda^2$$

for a parameter $\lambda \geq 0$. We distinguish between two cases.

Case 1. *We have* $\lambda = 0$. By (2.18) the pressure $p(\theta, t)$ becomes $p(\theta, t) = c(t)$, which satisfies

$$c'(t) = 2\, c^2(t).$$

Using that the $\lim_{t \to 0} p(\theta, t) = +\infty$, we get

$$p(\theta, t) = c(t) = \frac{1}{(-2t)},$$

which corresponds to contracting circles by the curve shortening flow.

Case 2. *We have* $\lambda > 0$. If we express

$$a(t) = \lambda\, \sin(2h(t)), \qquad b(t) = \lambda\, \cos(2h(t))$$

for some function $h(t)$ and plug them back to the equations (2.19), we find

$$2\, \lambda\, \sin(2h(t))\, (\gamma'(t) - h'(t)) = 0, \qquad 2\, \lambda\, \cos(2h(t))\, (h'(t) - \gamma'(t)) = 0,$$

which implies that $\gamma(t) = h(t) - \gamma$, for a parameter γ. We conclude from (2.18) that

$$p(\theta, t) = \lambda\, \cos 2(\theta + \gamma) + c(t). \tag{2.21}$$

We will now compute $c(t)$. It follows from (2.20) that $c(t)$ satisfies the ODE

$$c'(t) = 2\, (c(t)^2 - \lambda^2).$$

Solving this ODE and using that $\lim_{t \to 0} p(\theta, t) = +\infty$ (which implies the $\lim_{t \to 0} c(t) = +\infty$), we obtain

$$c(t) = -\lambda\, \coth 2\lambda t.$$

Combining the above, we conclude the desired formula

$$p(\theta, t) = \lambda \left(\frac{1}{1 - e^{2\lambda t}} - \sin^2(\theta + \gamma) \right)$$

for two parameters $\lambda > 0$ and γ.

This concludes the proof of the Theorem. q.e.d.

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