J. DIFFERENTIAL GEOMETRY
84 (2010) 489-523

CALABI FLOW AND PROJECTIVE EMBEDDINGS

Joel Fine,
appendix written by Kefeng Liu & Xiaonan Ma

Abstract

Let $X \subset \mathbb{CP}^N$ be a smooth subvariety. We study a flow, called balancing flow, on the space of projectively equivalent embeddings of X which attempts to deform the given embedding into a balanced one. If $L \to X$ is an ample line bundle, considering embeddings via $H^0(L^k)$ gives a sequence of balancing flows. We prove that, provided these flows are started at appropriate points, they converge to Calabi flow for as long as it exists. This result is the parabolic analogue of Donaldson's theorem relating balanced embeddings to metrics with constant scalar curvature [**12**]. In our proof we combine Donaldson's techniques with an asymptotic result of Liu and Ma [**17**] which, as we explain, describes the asymptotic behavior of the derivative of the map FS $\circ$ Hilb whose fixed points are balanced metrics.

1. Introduction

1.1. Overview of results. The idea of approximating Kähler metrics by projective embeddings goes back several years. The fundamental fact is that the projective metrics are dense in the space of all Kähler metrics. More precisely, let $L \to X^n$ be an ample line bundle over a complex manifold and let h be a Hermitian metric in L whose curvature defines a Kähler metric $\omega \in c_1(L)$. Together, h and ω determine an L^2-inner-product on the vector spaces $H^0(L^k)$. Using an L^2-orthonormal basis of sections for each $H^0(L^k)$ gives a sequence of embeddings $\iota_k \colon X \to \mathbb{CP}^{N_k}$ into larger and larger projective spaces and hence a sequence of projective metrics $\omega_k = \frac{1}{k}\iota_k^*\omega_{\mathrm{FS}}$ in the same cohomology class as ω.

Theorem 1 (Tian [**30**], Ruan [**28**]). *The metrics ω_k converge to ω in C^∞ as $k \to \infty$.*

(The sequence ω_k was considered by Yau in the case when ω is Kähler–Einstein [**35**]. Tian proved Theorem 1 with C^2-convergence; this was then improved to C^∞-convergence by Ruan.)

Received 1/13/2009.

From here it is natural to ask if any of the objects studied in Kähler geometry can be approximated by objects in projective geometry. An example of this phenomenon, due to Donaldson, is the strong relationship between balanced embeddings and Kähler metrics of constant scalar curvature, which is the central focus of this article.

Before stating Donaldson's result, we first recall the definition of a balanced embedding, originally due to Luo [**19**] and Zhang [**38**] (see also Bourguignon, Li, and Yau [**3**]). Let $\mu\colon \mathbb{CP}^N \to i\mathfrak{u}(N+1)$ be the Hermitian-matrix valued function, given in homogeneous unitary coordinates by $\mu = (\mu_{\alpha\beta})$ where

$$\mu_{\alpha\beta}[x_0 : \cdots : x_N] = \frac{x_\alpha \bar{x}_\beta}{\sum |x_\gamma|^2}.$$

Given a smooth subvariety $X \subset \mathbb{CP}^N$, we consider the integral of μ over X^n with respect to the Fubini–Study metric:

$$\bar{\mu} = \int_X \mu \, \frac{\omega_{\mathrm{FS}}^n}{n!}.$$

The subvariety is called *balanced* if $\bar{\mu}$ is a multiple of the identity.

Donaldson proved the following:

Theorem 2 (Donaldson [**12**]). *Suppose that for all large k there is a basis of $H^0(L^k)$ which gives a balanced embedding $\iota_k\colon X \to \mathbb{CP}^{N_k}$ and, moreover, that the metrics $\omega_k = \frac{1}{k}\iota_k^*\omega_{\mathrm{FS}}$ converge in C^∞ to a metric ω. Then ω has constant scalar curvature.*

Theorem 3 (Donaldson [**12**]). *Suppose that* $\mathrm{Aut}(X, L)$ *is discrete and that the class $c_1(L)$ contains a metric ω of constant scalar curvature. Then for all large k there is a basis of $H^0(L^k)$ giving a balanced embedding $\iota_k\colon X \to \mathbb{CP}^{N_k}$ and, moreover, the metrics $\omega_k = \frac{1}{k}\iota_k^*\omega_{\mathrm{FS}}$ converge in C^∞ to ω.*

The goal of this article is to prove the parabolic analogue of Donaldson's Theorems. In [**4**] Calabi introduced a parabolic flow, *Calabi flow*, which one might hope deforms a given Kähler metric toward a constant scalar curvature one. The flow is

$$\frac{\partial \omega}{\partial t} = i\bar{\partial}\partial \, S\left(\omega(t)\right),$$

where S denotes scalar curvature.

As we will explain, the projective analogue of this is *balancing flow*. Given an embedding $\iota\colon X \to \mathbb{CP}^N$, let $\bar{\mu}_0$ denote the trace-free part of $\bar{\mu}$, so that $\bar{\mu}_0 = 0$ if and only if the embedding is balanced. The Hermitian matrix $\bar{\mu}_0$ defines a vector field on $\mathbb{CP}^N$ and consequently an infinitesimal deformation of the embedding ι. This defines balancing flow:

$$\frac{\mathrm{d}\iota}{\mathrm{d}t} = -\bar{\mu}_0(\iota).$$

It is not hard to show that the flow exists for all time (see §1.2).

Our results concern the asymptotics of a certain sequence of balancing flows. Let h be a Hermitian metric in L whose curvature gives a Kähler form $\omega \in c_1(L)$. As in the description of Theorem 1, let ι_k be the embedding defined via a basis of $H^0(L^k)$ which is orthonormal with respect to the $\mathrm{L}^2(h, \omega)$ inner-product and let $\omega_k = \frac{1}{k}\iota_k^*\omega_{\mathrm{FS}}$ be the sequence of projective approximations to ω. For each k, we run a sped-up version of the balancing flow, so that $\iota_k(t)$ solves

$$\frac{\mathrm{d}\iota_k}{\mathrm{d}t} = -2\pi k^2 \bar{\mu}_0(\iota_k), \quad \iota_k(0) = \iota_k. \tag{1}$$

We study the sequence $\omega_k(t) = \frac{1}{k}\iota_k(t)^*\omega_{\mathrm{FS}}$ of metric flows, proving the parabolic analogue of Donaldson's Theorems 2 and 3:

Theorem 4. *Suppose that for each $t \in [0, T]$ the metric $\omega_k(t)$ converges in C^∞ to a metric $\omega(t)$ and, moreover, that this convergence is C^1 in t. Then the limit $\omega(t)$ is a solution to Calabi flow starting at ω.*

Theorem 5. *Suppose that the Calabi flow $\omega(t)$ starting at ω exists for $t \in [0, T]$. Then for each t, the metric $\omega_k(t)$ converges in C^∞ to $\omega(t)$. Moreover, this convergence is C^1 in t.*

1.2. A moment map interpretation. The following picture will not be used directly in our proofs, but it gave the original motivation for this work, so it is perhaps worth mentioning briefly here. First, we recall the standard moment map set-up, which consists of a group K acting by Kähler isometries on a Kähler manifold Z, along with an equivariant moment map $m \colon Z \to \mathfrak{g}^*$. The action extends to the complexified group $G = K^{\mathbb{C}}$ giving a holomorphic, but no longer isometric, action. The problem one is interested in is finding a zero of m in a given G-orbit. By K-equivariance, this becomes a question about the behavior of a certain function, called the *Kempf–Ness function*, $F \colon G/K \to \mathbb{R}$ on the symmetric space G/K. This function is geodesically convex and its derivative is essentially m. Hence there is a zero of the moment map in the orbit if and only if F attains its minimum. A natural way to search for such a minimum is to consider the downward gradient flow of F.

In [**12**], Donaldson explains how this is relevant in our situation. On the one hand, scalar curvature can be interpreted as a moment map, an observation due to Donaldson [**10**] and Fujiki [**16**]. In this case the symmetric space is the space $\mathcal{H}$ of positively curved Hermitian metrics in L or, equivalently once a reference metric in $c_1(L)$ is chosen, the space of Kähler potentials (see the work of Donaldson [**11**], Mabuchi [**21**], and Semmes [**29**] for a description of this symmetric space structure). Finding a zero of the moment map corresponds to finding a constant scalar curvature metric in the given Kähler class. In this context, the Kempf–Ness function is Mabuchi's K-energy and the gradient flow is Calabi flow.

On the other hand, in [**12**] Donaldson showed how to fit balanced metrics into a finite dimensional moment-map picture. (This picture has been subsequently studied in [**27**, **34**].) If $L \to X$ is very ample, then every basis of $H^0(L)$ defines an embedding $X \subset \mathbb{CP}^N$. We consider the space $Z \cong \mathrm{GL}(N+1)$ of all bases. There is a Kähler structure on Z, whose definition involves the fact that each point gives an embedding of X, and $\mathrm{U}(N+1)$ acts isometrically with a moment map which is essentially $-i\bar{\mu}_0$. So finding a zero of the moment map corresponds to finding a balanced embedding. This time, the Kempf–Ness function is a normalized version of what is called the "F^0-functional" by some authors (it is the function denoted $\tilde{Z}$ in [**14**]). The gradient flow on the Bergman space $\mathcal{B} = \mathrm{GL}\,/\,\mathrm{U}$ is balancing flow. One immediate consequence of this is that balancing flow exists for all time.

Taking successively higher powers L^k of L gives a sequence of moment-map problems on successively larger Bergman spaces $\mathcal{B}_k$, each of which lives inside $\mathcal{H}$. Put loosely, Donaldson's Theorems 2 and 3 say that the zeros of the finite-dimensional moment maps in $\mathcal{B}_k$ converge in $\mathcal{H}$ to a zero of the infinite-dimensional moment map. Theorems 4 and 5 say that provided we choose the finite-dimensional gradient flows to start at a appropriate points then they converge to the infinite-dimensional gradient flow.

In fact, the only aspect of this picture that we use directly in the proofs is that balancing flow is distance decreasing on $\mathcal{B}$, which follows from the fact that it is the downward gradient flow of a geodesically convex function. This was discovered prior to the moment-map interpretation by Paul [**24**] and Zhang [**38**]. We remark in passing that Calabi and Chen [**5**] have proved that Calabi flow on $\mathcal{H}$ is distance decreasing using the symmetric space metric of Donaldson–Mabuchi–Semmes. This is strongly suggested by the standard moment-map picture, but doesn't follow directly because $\mathcal{H}$ is infinite-dimensional.

1.3. Additional Context. Calabi suggested in [**4**] that, when one exists, a constant scalar curvature Kähler metric should be considered a "canonical" representative of a Kähler class. Since this suggestion, such metrics have been the focus of much work. Additional motivation is provided by the conjectural equivalence between the existence of a Kähler metric of constant scalar curvature representing $c_1(L)$ and the stability, in a certain sense, of the underlying polarisation $L \to X$. This began with a suggestion of Yau [**36**] which was refined by Tian [**31**, **32**] and Donaldson [**13**].

Calabi flow, meanwhile, has received less attention. This is no doubt due to the fact that, as it is a fourth-order fully nonlinear parabolic PDE, there are few standard analytic techniques which apply directly. A start is made in the foundational article by Chen and He [**8**] which includes a proof of short-time existence and also shows that when a constant scalar curvature metric ω exists and the Calabi flow starts sufficiently close to ω then the flow exists for all time and converges to ω. There are also some long time existence results in which a priori existence of a constant scalar curvature metric is replaced by a "small energy" assumption. For example, Tosatti and Weinkove [**33**] show that, assuming $c_1(X) = 0$, if the Calabi flow starts at a metric with sufficiently small Calabi energy, the flow exists for all time and converges to a constant scalar curvature metric. Chen and He [**7**] have proved a similar result for Fano toric surfaces.

Balanced metrics have been written about several times since their introduction. Independently, Luo [**19**] and Zhang [**38**] have proved that an embedding can be balanced if and only if it is stable in the sense of GIT, giving the projective analogue of the conjectural relationship between constant scalar curvature metrics and stability mentioned above. The restriction on $\mathrm{Aut}(X, L)$ in Donaldson's Theorem 3 has been relaxed by Mabuchi [**22**, **23**], while the picture in [**12**] has been related to the Deligne pairing by Phong and Sturm in [**26**, **27**], an approach which also leads to a sharpening of some estimates.

1.4. Overview of proofs.

1.4.1. Bergman asymptotics. The key technical result which underpins Theorems 1, 2, and 3 concerns the *Bergman function*. Given a Kähler metric $\omega \in c_1(L)$, let h be a Hermitian metric in L with curvature $2\pi i\omega$. Let s_α be a basis of $H^0(L^k)$ which is orthornomal with respect to the L^2-inner-product determined by h and ω. The Bergman function $\rho_k(\omega)\colon X \to \mathbb{R}$ is defined by

$$\rho_k(\omega) = \sum_\alpha |s_\alpha|^2.$$

where $|\cdot|$ denotes the pointwise norm on sections using h.

The central result concerns the asymptotics of ρ_k and is due to the work of Catlin [**6**], Lu [**18**], Tian [**30**], and Zelditch [**37**]. We state it as it appears in [**12**] (see proposition 6 there and the discussion afterward).

Theorem 6.

1) *For fixed ω there is an asymptotic expansion as $k \to \infty$,*

$$\rho_k(\omega) = A_0(\omega)k^n + A_1(\omega)k^{n-1} + \cdots,$$

where $n = \dim X$ and where the $A_i(\omega)$ are smooth functions on X which are polynomials in the curvature of ω and its covariant derivatives.

2) *In particular,*

$$A_0(\omega) = 1, \quad A_1(\omega) = \frac{1}{2\pi}S(\omega).$$

3) *The expansion holds in C^∞ in that for any $r, M > 0$,*

$$\left\| \rho_k(\omega) - \sum_{i=0}^{M} A_i(\omega) k^{n-i} \right\|_{C^r(X)} \leq K_{r,M,\omega} k^{n-M-1}$$

for some constants $K_{r,M,\omega}$. Moreover, the expansion is uniform in that for any r and M there is an integer s such that if ω runs over a set of metrics which are bounded in C^s, and with ω bounded below, the constants $K_{r,M,\omega}$ are bounded by some $K_{r,M}$ independent of ω.

This expansion essentially proves Theorem 1, since ρ_k relates the original metric ω to the projective approximation ω_k:

$$\omega = \omega_k + \frac{i}{2k} \bar{\partial}\partial \log \rho_k.$$

Tian's Theorem follows from the fact that the leading term in the expansion of ρ_k is constant.

To see the link with balanced embeddings, note that the embedding ι_k is balanced if and only if ρ_k is constant. The fact that the second term in the expansion of ρ_k is the scalar curvature of ω is at the root of the relationship between the asymptotics of balanced embeddings and constant scalar curvature metrics described by Donaldson's Theorems 2 and 3.

The asymptotics of the Bergman kernel will also be critical in the proofs of Theorems 4 and 5, but of equal importance is another asymptotic result, due to Liu and Ma [**17**]. In fact, for our application a slight strengthening of this result is desirable. Profs. Liu and Ma were kind enough to provide a proof of this improvement and this appears in the appendix to this article. Liu and Ma's theorem concerns a sequence of integral operators Q_k, introduced by Donaldson in [**15**] and defined as follows. Let $B_k(p, q)$ denote the Bergman kernel of L^k; in other words, if s^* denotes the section of $(\bar{L}^k)^*$ which is metric-dual to a section s of L^k, then B_k is the section of $L^k \otimes (\bar{L}^k)^* \to X \times X$ given by

$$B_k(p, q) = \sum_\alpha s_\alpha(p) \otimes s_\alpha^*(q)$$

(where s_α is an L^2-orthonormal basis of holomorphic sections as before). Now define a sequence of functions $K_k \colon X \times X \to \mathbb{R}$ by

$$K_k(p, q) = \frac{1}{k^n} |B_k(p, q)|^2 = \frac{1}{k^n} \sum_{\alpha,\beta} (s_\alpha, s_\beta)(p)(s_\beta, s_\alpha)(q)$$

These functions are the kernels for a sequence of integral operators acting on $C^\infty(X)$ defined by

$$(Q_k f)(p) = \int_X K_k(p, q) f(q) \frac{\omega^n(q)}{n!}.$$

Liu and Ma's theorem (following a suggestion of Donaldson [**15**]) relates the asymptotics of the operators Q_k to the heat kernel $\exp(-s\Delta)$ of (X, ω).

Theorem 7 (Liu and Ma [**17**], see also the appendix)**.** *For any choice of positive integer r, there exists a constant C such that for all sufficiently large integers k and any $f \in C^\infty(X)$,*

$$\left\| \left(\frac{\Delta}{k}\right)^r \left\{ Q_k(f) - \exp\left(-\frac{\Delta}{4\pi k}\right) f \right\} \right\|_{L^2} \leq \frac{C}{k} \|f\|_{L^2},$$

where the norms are taken with respect to ω. Moreover, the estimate is uniform in the sense that there is an integer s such that the constant C can be chosen independently of ω provided ω varies over a set of metrics which is bounded in C^s and with ω bounded below.

Liu and Ma's original article [**17**] deals with the cases $r = 0$ and $r = 1$; the remaining cases are considered in the appendix to this article, which also proves the following C^m estimate:

Theorem 8 (See appendix)**.** *For any choice of positive integer r, there exists a constant C such that for all sufficiently large k and for any $f \in C^\infty(X)$,*

$$\|Q_k(f) - f\|_{C^m} \leq \frac{C}{k} \|f\|_{C^m},$$

where the norms are taken with respect to ω. Moreover, the estimate is uniform in the sense that there is an integer s such that the constant C can be chosen independently of ω provided ω varies over a set of metrics which is bounded in C^s and with ω bounded below.

Just as the Bergman function ρ_k appears when comparing a Kähler metric ω to its algebraic approximations ω_k, the operators Q_k appear when one relates infinitesimal deformations of the metric ω to the corresponding deformations of the approximations ω_k. Since the Calabi flow deforms ω, it is clear that this will be of interest to us.

To see how the operators Q_k arise, let $h(t) = e^{\phi(t)} h$ denote a path of positively curved Hermitian metrics in L, giving a path of Kähler forms $\omega(t) \in c_1(L)$. The infinitesimal change in the L^2-inner-product on $H^0(L^k)$ corresponds to the Hermitian matrix A whose elements are

$$A_{\alpha\beta} = \int_X \left(k\dot{\phi} + \Delta\dot{\phi}\right) (s_\alpha, s_\beta) \frac{\omega^n}{n!}.$$

The term $k\dot{\phi}$ is due to the change in the fibrewise metric, while $\Delta\dot{\phi}$ is due to the change in volume form. The infinitesimal change in ω_k

corresponding to A is given by the potential

$$\begin{aligned}\frac{1}{k}\operatorname{tr}(A\mu) &= \int_X \left(\dot\phi + k^{-1}\Delta\dot\phi\right)(p)\frac{(s_\alpha,s_\beta)(p)(s_\beta,s_\alpha)(q)}{\rho_k(q)}\,\frac{\omega^n(p)}{n!},\\ &= \int_X \left(\dot\phi + k^{-1}\Delta\dot\phi\right)(p)\frac{k^n}{\rho_k(q)}K_k(p,q)\,\frac{\omega^n(p)}{n!},\\ &= \left(Q_k(\dot\phi) + k^{-1}Q_k(\Delta\dot\phi)\right)\left(1+O(k^{-1})\right).\end{aligned}$$

(The fact that the potential is $\operatorname{tr}(A\mu)$ is essentially a restatement of the fact that $-i\mu$ is a moment map for the action of $\mathrm{U}(N+1)$ on $\mathbb{CP}^N$, while the factor of k^{-1} is due to the rescaling needed to remain in a fixed Kähler class.) It follows from Liu and Ma's results that $\frac{1}{k}\operatorname{tr}(A\mu)\to\dot\phi$ in C^∞ and hence that the convergence of algebraic approximations $\omega_k(t)$ to $\omega(t)$ is also C^1 in the t direction.

We can describe this calculation in the notation of [**14**]. Recall that $\mathcal{H}$ denotes the space of positively curved Hermitian metrics in L, while $\mathcal{B}_k$ denotes the space of projective Hermitian metrics in L^k, i.e., those obtained by pulling back the Fubini–Study metric from $\mathcal{O}(1)\to\mathbb{CP}^{N_k}$ using embeddings via $H^0(L^k)$. Given $h\in\mathcal{H}$ in L, using an $L^2(h)$-orthonormal basis of $H^0(L^k)$ to embed X gives a projection $\mathrm{Hilb}_k\colon\mathcal{H}\to\mathcal{B}_k$; meanwhile, taking the k^{th} root gives an inclusion $\mathrm{FS}_k\colon\mathcal{B}_k\to\mathcal{H}$. Composing gives a map $\Phi_k=\mathrm{FS}_k\circ\mathrm{Hilb}_k\colon\mathcal{H}\to\mathcal{H}$ and this calculation shows that the derivative of Φ_k at a given point h satisfies $(\mathrm{d}\Phi_k)_h = Q_k+O(k^{-1})$. So, while the expansion of ρ_k tells us that $\Phi_k(h)/k^n\to h$, Liu–Ma's asymptotics give that $(\mathrm{d}\Phi_k)_h(\dot\phi)\to\dot\phi$.

1.4.2. Outline of arguments. Our overall approach follows the general scheme of Donaldson's proofs of Theorems 2 and 3. We begin in §2 by proving Theorem 4. Just as Donaldson's Theorem 2 is implied more or less directly by the uniformity of the asymptotic expansion of the Bergman kernel, so our result will follow easily from this combined with the uniformity in Liu and Ma's Theorem.

We then move on to the proof of Theorem 5. As with Theorem 3, this part requires substantially more effort. It is shown in §3 that the standard sequence of projective approximations to Calabi flow gives an $O(k^{-1})$ approximation to balancing flow. This is analogous to the fact that the standard sequence of projective approximations to a constant scalar curvature metric are themselves close to being balanced. Just as in the case of balanced metrics, however, $O(k^{-1})$ is not strong enough; for later arguments it becomes important to improve this to beat any power of k^{-1}.

Donaldson solved this problem by considering instead a perturbation of the constant scalar curvature metric ω:

$$\omega + i\bar\partial\partial\sum_{j=1}^m k^{-j}\eta_j$$

where the potentials η_j solve partial differential equations of the form $L\eta = f$ for a certain linear elliptic operator L. We apply the same idea to Calabi flow, using time-dependent potentials $\eta_j(t)$ which are required to solve parabolic equations $\dot{\eta} + L\eta = g$ associated to the same operator L.

After this perturbation we have, for each k, a path of projective metrics $\omega'_k(t)$ which on the one hand converge to Calabi flow and on the other hand are $O(k^{-m})$ from the balancing flow $\omega_k(t)$. Here we are working for each k in the Bergman space $\mathcal{B}_k = \mathrm{GL}(N_k+1)/\,\mathrm{U}(N_k+1)$ and $O(k^{-m})$ means with respect to the Riemannian distance function d_k of the symmetric space metric. The remainder of the proof is concerned with uniformly controlling the C^r norm on metric tensors by the Riemannian distance d_k. This involves two parts: first, some analytic estimates proved in §4 reduce the problem to controlling $\bar{\mu}$; second, some estimates in projective geometry proved in §5 show how to control $\bar{\mu}$ by d_k. §6 puts all the pieces together and completes the proof of Theorem 5.

Acknowledgements. The author was supported by an FNRS postdoctoral fellowship. I am very grateful to Gábor Székelyhidi and Xiuxiong Chen for many helpful discussions during the course of this work. I am also indebted to Kefeng Liu and Xiaonan Ma for providing me with the version of Theorem 7 used here and kindly agreeing to write the necessary additional arguments in an appendix to this article. I would also like to thank Julien Keller, Yanir Rubinstein, and Richard Thomas for conversations about Bergman asymptotics and balanced embeddings.

2. When the balancing flows converge

In this section we will prove Theorem 4. We begin by describing balancing flow in terms of Kähler potentials. Given an embedding $\iota\colon X \to \mathbb{CP}^N$ and a Hermitian matrix $A \in i\mathfrak{u}(N+1)$, we view A as a vector field ξ_A on $\mathbb{CP}^N$ and consequently as an infinitesimal perturbation of ι. The corresponding infinitesimal change in $\iota^*\omega_{\mathrm{FS}}$ is given by the potential $\mathrm{tr}(A\mu)$ restricted to X via ι. (It suffices to prove this for $\mathbb{CP}^N$ itself, where it follows from the fact that $-i\mu\colon \mathbb{CP}^N \to \mathfrak{u}(N+1)$ is a moment map for the $\mathrm{U}(N+1)$-action.) Accordingly, the potential corresponding to balancing flow is the *balancing potential* $\beta = -\,\mathrm{tr}(\bar{\mu}_0\mu)$.

To obtain the correct asymptotics when considering embeddings via higher and higher powers L^k it is necessary to rescale the balancing flow to be generated by $-2\pi k^2\bar{\mu}_0$. Since the restriction of the Fubini–Study metric is also rescaled to remain in the fixed class $c_1(L)$, the corresponding balancing potential is $\beta_k = -2\pi k\,\mathrm{tr}(\bar{\mu}_0\mu)$.

Theorem 4 will follow directly from the next result.

Theorem 9. *Let $h_k \in \mathcal{B}_k$ be a sequence of Bergman metrics whose rescaled curvatures $\omega_k \in c_1(L)$ converge in C^∞ to a metric ω. Then the* *balancing potentials converge in C^∞ to the potential generating Calabi flow at ω:*

$$\beta_k(\omega_k) \to S(\omega) - \bar{S}.$$

(Here $\bar{S}$ is the mean value of the scalar curavture S.)

Proof. Let s_α be an orthonormal basis of $H^0(L^k)$ with respect to the L^2-inner-product determined by h_k and ω_k. Then the balancing potential is

$$\beta_k(\omega_k)(p) = 2\pi k \int_X \sum \left(\frac{\delta_{\alpha\beta}}{N_k+1} - \frac{(s_\alpha, s_\beta)(q)}{\rho_k(\omega_k)(q)} \right) \frac{(s_\beta, s_\alpha)(p)}{\rho_k(\omega_k)(p)} \frac{\omega_{\mathrm{FS}}^n(q)}{n!}.$$

Here $N_k + 1 = \dim H^0(L^k)$ which, by Riemann–Roch and Chern–Weil, is a polynomial with leading terms

$$N_k + 1 = V\left(k^n + \frac{\bar{S}}{2\pi} k^{n-1} + \cdots\right),$$

where $V = c_1(L)^n$.

Since ω_k converges in C^∞, we can apply the uniform asymptotic expansion of the Bergman functions $\rho_k(\omega_k)$ (Theorem 6) to conclude that $\frac{1}{k}\omega_{\mathrm{FS}} = \omega + O(k^2)$ and, moreover, that

$$\rho_k(\omega_k) = k^n + \frac{S(\omega_k)}{2\pi} k^{n-1} + O(k^{n-2}).$$

From here on in the proof, we write $\rho_k = \rho_k(\omega_k)$; similarly we write the operators appearing in Liu and Ma's theorem as $Q_k = Q_k(\omega_k)$.

Using the uniform expansion of the Bergman function, we have

$$\begin{aligned}
\beta_k(\omega_k) &= \frac{2\pi V k^{n+1}}{N_k+1} - \frac{2\pi k^{n+1}}{\rho_k(p)} \int_X K_k(p,q) \left(\frac{k^n}{\rho_k(q)} + O(k^{-2}) \right) \frac{\omega^n(q)}{n!} \\
&= \left(2\pi k - \bar{S} + O(k^{-1})\right) \\
&\qquad - \left(2\pi k - S(\omega_k) + O(k^{-1})\right) Q_k(1 + O(k^{-1})) \\
&= \left(S(\omega_k) - \bar{S}\right)\left(1 + O(k^{-1}) + Q_k(O(k^{-1}))\right)
\end{aligned}$$

since, by definition, $Q_k(1) = \rho_k/k^n = 1 + O(k^{-1})$. We will prove this converges to $S - \bar{S}$ by showing that the error $Q_k(O(k^{-1}))$ converges to zero in C^∞.

To control the term $Q_k(O(k^{-1}))$ we have to be a little careful, since the convergence $Q_k(f) \to f$ is not uniform. This can be seen for the heat kernel $\exp(-\Delta/k)$ itself by considering eigenfunctions of the Laplacian with higher and higher eigenvalues. Instead, for $f_k = O(k^{-1})$ note that by the $r = 0$ case of Theorem 7,

$$\begin{aligned}
\|Q_k(f_k))\|_{L^2} &\leq \frac{C}{k}\|f_k\|_{L^2} + \left\|\exp\left(-\frac{\Delta}{4\pi k}\right) f_k\right\|_{L^2} \\
&\leq \left(\frac{C}{k} + 1\right)\|f_k\|
\end{aligned}$$

where the last line follows because the heat kernel reduces L^2-norm. (Here we use the Laplacian and norms defined by the limiting metric ω.) So, indeed, $Q_k(f_k) \to 0$ in L^2. (Note we have used the uniformity in Theorem 7 to ensure that the constant C appearing here from the estimate for the asymptotics of the operators $Q_k(\omega_k)$ can be taken independent of k, since $\omega_k \to \omega$ in C^∞.)

For convergence in L^2_{2r}, write

$$f_k = k^{-1}F_1 + \cdots k^{-r+1}F_{r-1} + \hat{f}_k$$

where the F_j are functions independent of k and $\hat{f}_k = O(k^{-r})$. Now Theorem 8 guarantees that $k^{-j}Q_k(F_j) \to 0$ in C^∞, while Theorem 7 gives

$$\begin{aligned}
\left\| \Delta^r \left(Q_k(\hat{f}_k) \right) \right\|_{L^2} &\leq Ck^{r-1}\|\hat{f}_k\|_{L^2} + \left\| \exp\left(-\frac{\Delta}{4\pi k} \right) \left(\Delta^r \hat{f}_k \right) \right\|_{L^2} \\
&\leq Ck^{r-1}\|\hat{f}_k\|_{L^2} + \|\Delta^r \hat{f}_k\|_{L^2} \\
&\leq \left(Ck^{r-1} + 1 \right) \left\| \hat{f}_k \right\|_{L^2_{2r}}.
\end{aligned}$$

Since $\hat{f}_k = O(k^{-r})$, we see that $Q_k(\hat{f}_k) \to 0$ in L^2_{2r} and, hence, that $Q_k(f_k) \to 0$ in C^∞.

Recall that

$$\beta_k(\omega_k) = \left(S(\omega_k) - \bar{S} \right) \left(1 + O(k^{-1}) + Q_k(O(k^{-1})) \right).$$

Now combine the fact that $Q_k(O(k^{-1})) \to 0$ in C^∞ with the fact that $S(\omega_k) \to S(\omega)$ in C^∞ to complete the proof.

q.e.d.

We now give the proof of Theorem 4. Recall that $\omega \in c_1(L)$ is a given Kähler form, ω_k is the sequence of projective metrics in Tian's Theorem 1 and $\omega_k(t)$ is the balancing flow starting at ω_k. We assume that for each value of $t \in [0, T]$, $\omega_k(t)$ converges in C^∞ to a metric which we denote $\omega(t)$ and that moreover the convergence is C^1 in t. So, if we write $\partial\omega/\partial t = i\bar{\partial}\partial f(t)$ for a function $f(t)$ with $\omega(t)$-mean-value zero, then $\beta_k(\omega_k(t)) \to f(t)$. Now applying Theorem 9 to $\omega_k(t)$, it follows that $f(t) = S(\omega(t)) - \bar{S}$ and so $\omega(t)$ is a solution to Calabi flow on $[0, T]$ starting at ω.

3. Using Calabi flow to approximate balancing flow

We now move on to the proof of Theorem 5, a task which will take up the remainder of the article.

3.1. First-order approximation. We first explain how Calabi flow can be used to approximate the balancing flow metrics $\omega_k(t)$ to $O(k^{-1})$. Recall that h is a Hermitian metric in L with curvature $2\pi i\omega$. Since we are assuming Calabi flow exists, we have a path $h(t) = e^{\phi(t)}h$ of Hermitian metrics with curvatures $\omega(t)$ such that

$$\dot{\phi} = S(\omega(t)) - \bar{S}.$$

Let $\hat{h}_k(t) \in \mathcal{B}_k$ denote the k^{th} Bergman point of $h(t)$. In other words, take a basis s_α of $H^0(L^k)$ which is orthonormal with respect to the L^2-inner-product determined by $h(t)$ and $\omega(t)$; this defines a projective metric $\hat{h}_k(t)$ in L^k which is either characterised as the pull-back to L^k of the Fubini–Study metric from $\mathcal{O}(1) \to \mathbb{CP}^{N_k}$ via the embedding determined by s_α or, equivalently, as the unique metric for which $\sum |s_\alpha|^2 = 1$. Meanwhile, we denote by $h_k(t) \in \mathcal{B}_k$ the balancing flow starting at $\hat{h}_k(0)$. We will estimate the distance in $\mathcal{B}_k$ between $h_k(t)$ and $\hat{h}_k(t)$ using the symmetric Riemannian metric. It turns out to be convenient to rescale this by a power of k. Denote by d_k the distance function arising from using the metric given by the rescaled Killing form $k^{-(n+2)} \operatorname{tr} A^2$. (It can been shown that these norms converge in a certain sense to the L^2-norm on potentials, so this is a natural rescaling to consider.)

Our goal in this subsection is to prove the following result.

Proposition 10. *There is a constant C such that for all $t \in [0, T]$,*

$$d_k\left(h_k(t), \hat{h}_k(t)\right) \leq \frac{C}{k}.$$

Proof. We begin by considering the tangent vector to $\hat{h}_k(t)$. In general, given a smooth path $h(t) = e^{\phi(t)}h_0$ of positively curved Hermitian metrics, the infinitesimal change in L^2-inner-product on $H^0(L^k)$ corresponds to the Hermitian matrix $U = (U_{\alpha\beta})$ where

$$U_{\alpha\beta} = \int_X (s_\alpha, s_\beta)\left(k\dot{\phi} + \Delta\dot{\phi}\right)\frac{\omega^n}{n!}. \tag{2}$$

Here s_α is an $L^2(h^k(t), \omega(t))$-orthonormal basis of $H^0(L^k)$ and all relevant quantities are computed with respect to $h(t)$ and $\omega(t)$. The term $k\dot{\phi}$ here corresponds to the infinitesimal change to the fibrewise metric, while the term $\Delta\dot{\phi}$ corresponds to the infinitesimal change in volume form.

In our case, this gives that the tangent $U_k(t)$ to $\hat{h}_k(t)$ is the Hermitian matrix

$$U_k = \int_X (s_\alpha, s_\beta)\left(k(S - \bar{S}) + \Delta S\right)\frac{\omega^n}{n!},$$

where S is the scalar curvature of $\omega(t)$.

Meanwhile, the tangent to balancing flow through the same point $\hat{h}_k(t)$ is the Hermitian matrix

$$V_k = 2\pi k^2 \int_X \left(\frac{\delta_{\alpha\beta}}{N_k + 1} - \frac{(s_\alpha, s_\beta)}{\rho_k}\right)\frac{\omega_{\text{FS}}^n}{n!},$$

where ρ_k is the Bergman function for $h(t)$.

Using the asymptotic expansion of ρ_k and the fact that $\omega(t) = \frac{1}{k}\omega_{\mathrm{FS}} + O(k^{-2})$, there is an asymptotic expansion of V_k:

$$V_k = \int_X (s_\alpha, s_\beta)(k(S - \bar{S}) + O(1))\frac{\omega^n}{n!}. \tag{3}$$

So

$$U_k - V_k = \int_X (s_\alpha, s_\beta)\, O(1)\frac{\omega^n}{n!}.$$

It follows that we can write the norm of $U_k - V_k$ in terms of the operators $Q_k = Q_k(\omega(t))$ appearing in Liu and Ma's Theorem 7:

$$\frac{\mathrm{tr}\,(U_k - V_k)^2}{k^{n+2}} = \int_{X\times X} K_k(p,q)G_k(p)G_k(q) = \langle G_k, Q_k(G_k)\rangle_{L^2}$$

where $G_k = O(k^{-1})$.

Now, denoting the L^2-norm by $\|\cdot\|$,

$$\begin{aligned}
\langle G_k, Q_k(G_k)\rangle_{L^2} &\leq \|G_k\|\|Q_k(G_k)\|,\\
&\leq \|G_k\|\left(\frac{C}{k}\|G_k\| + \|\exp(-\Delta/(4\pi k))G_k\|\right),\\
&\leq \|G_k\|^2\left(\frac{C}{k} + 1\right),\\
&= O(k^{-2}).
\end{aligned}$$

(The penultimate inequality uses the fact that the heat kernel reduces the L^2-norm.) So $U_k - V_k$ is $O(k^{-1})$ in the rescaled symmetric Riemannian metric on $\mathcal{B}_k$. Moreover, the bound is uniform in t because of the uniformity in the asymptotic behavior of ρ_k and Q_k. This amounts to the infinitesimal version of the result we are aiming for.

In order to prove the actual result, let

$$f_k(t) = d_k\left(h_k(t), \hat{h}_k(t)\right)$$

denote the distance we are trying to control. Let $\tilde{h}_k(t)$ denote the balancing flow which at time $t = t_0$ passes through the point $\hat{h}(t_0)$. Our bound on $U_k - V_k$ says that $\hat{h}_k$ and $\tilde{h}_k$ are tangent to $O(k^{-1})$ at $t = t_0$. Now balancing flow is the downward gradient flow of a geodesically convex function, and hence is distance decreasing (this follows from the general moment-map description alluded to in §1.2; it was discovered prior to the moment-map interpretation by Paul [**24**] and Zhang [**38**]). So $\tilde{h}_k$ and h_k get closer and closer together. It follows that there is a constant C such that for all k, at $t = t_0$,

$$\frac{\mathrm{d}f_k}{\mathrm{d}t} \leq \frac{C}{k}.$$

However, t_0 was arbitrary, hence f_k has sub-linear growth on $[0,T]$ and, moreover, $f_k(0)=0$ so $f_k(t)\leq CT/k$ for all t. q.e.d.

3.2. Higher-order approximations. Unfortunately, the fact that the images of Calabi flow in $\mathcal{B}_k$ approximate balancing flow to $O(k^{-1})$ with respect to d_k is not sufficient for us to show that the balancing flows converge to Calabi flow. This is similar to the problem encountered by Donaldson, and we resolve it by the parabolic analogue of the trick appearing in §4.1 of [**12**]. Namely, we perturb the Calabi flow $h(t)=e^{\phi(t)}h$ by a polynomial in k^{-1} and consider instead a sequence of flows indexed by k. Let

$$\psi(k;t)=\phi(t)+\sum_{j=1}^{m}k^{-j}\eta_j(t)$$

where $\eta_j(t)$ are some judiciously chosen time-dependent potentials. Denote by $h(k;t)=e^{\psi(k;t)}h$ the corresponding sequence of paths of Hermitian metrics. Their curvatures give the perturbation

$$\omega(k;t)=\omega(t)+i\bar{\partial}\partial\sum_{j=1}^{m}k^{-j}\eta_j(t)$$

of Calabi flow on the level of Kähler forms. Note that for any given choice of η_j, $\omega(k;t)$ is positive for large enough k.

Let $h'_k(t)\in\mathcal{B}_k$ denote the k^{th} Bergman point of $h(k;t)$; i.e., given a basis s_α of $H^0(L^k)$ which is orthonormal with respect to the L^2-inner-product determined by $h(k;t)$ and $\omega(k;t)$, $h'_k(t)$ is pull-back of the Fubini–Study metric in $\mathcal{O}(1)\to\mathbb{CP}^{N_k}$ or, equivalently, the unique Hermitian metric in L^k such that $\sum|s_\alpha|^2=1$.

Our goal in this subsection is to prove:

Theorem 11. *For any m, there exist functions $\eta_1,\dots,\eta_m$ and a constant C such that the perturbed Calabi flow $h(k;t)=e^{\psi(k;t)}h$ with*

$$\psi(k;t)=\phi(t)+\sum_{j=1}^{m}k^{-j}\eta_j(t)$$

satisfies for all $t\in[0,T]$ and all k,

$$d_k\left(h_k(t),h'_k(t)\right)\leq\frac{C}{k^{m+1}}.$$

Proof. The proof is by induction with, Proposition 10 providing the case $m=0$. For clarity, we explain first the case $m=1$ in detail before moving to the general inductive step. So, let $\psi(k;t)=\phi(t)+k^{-1}\eta(t)$ for some η which we will now explain how to find.

Let $A_k(t)$ denote the Hermitian matrix corresponding to the tangent to the path $h'_k(t)$. From equation (2), we have that

$$A_k = \int_X (s_\alpha, s_\beta) \left(k(S(\omega(t)) - \bar{S}) + \dot{\eta} + \Delta S + O(k^{-1}) \right) \frac{\omega(k;t)^n}{n!}.$$

Here $S(\omega(t)) - \bar{S}$ is the tangent of the *unperturbed* Calabi flow $h(t)$ and the Lapalcian is that of the *unperturbed* metric $\omega(t)$; the $O(k^{-1})$ terms involve $k^{-1}\Delta\dot{\eta}$ and also the fact that in the full formula, the Laplacian of $\omega(k;t)$ should appear, but this agrees with Δ to $O(k^{-1})$.

Next, we compute the asymptotics of the Hermitian matrix $B_k(t)$ which is tangent to balancing flow through the point $h'_k(t)$. Let L_t denote the linearization of the scalar curvature map associated to $\omega(t)$; so $S(\omega(k;t)) = S(\omega(t)) + k^{-1}L_t(\eta(t)) + O(k^{-2})$. B_k is given, as in equation (3), by the following expression, where we have explicitly notated the $O(1)$ term by F:

$$\begin{aligned} B_k &= \int_X (s_\alpha, s_\beta)\Big(k\left(S(\omega(k;t) - \bar{S})\right) + F + O(k^{-1})\Big)\frac{\omega(k;t)^n}{n!} \\ &= \int_X (s_\alpha, s_\beta)\Big(k\left(S(\omega(t) - \bar{S})\right) - L_t(\eta(t)) + F + O(k^{-1})\Big)\frac{\omega(k;t)^n}{n!}. \end{aligned}$$

Here we use the uniformity in Theorem 6 along with the fact that $\omega(k;t) \to \omega(t)$ in C^∞ when expanding $\rho_k(\omega(k;t))$. It follows that

$$A_k - B_k = \int_X (s_\alpha, s_\beta)\Big[\dot{\eta} + L_t(\eta) - F + \Delta S + O(k^{-1})\Big]\frac{\omega(k;t)^n}{n!}.$$

Now we chose η to solve the Cauchy problem for the inhomogeneous, non-autonomous, linear, parabolic evolution equation:

$$\dot{\eta} + L_t(\eta) = F - \Delta S \tag{4}$$

for $t \in [0, T]$, with initial condition $\eta(0) = 0$. It is standard that equation (4) has a solution provided the spectra of the operators L_t are bounded below. The lower bound on the spectra ensures that for each t, $-L_t$ generates an analytic strongly continuous semi-group and from here the existence of a solution to equation (4) follows from semi-group theory. See, for example, the texts [**1**] or [**25**]. To verify that each of the operators L_t have only finitely many negative eigenvalues, we use the fact that

$$L_t(\eta) = \mathcal{D}^*\mathcal{D}(\eta) - (\nabla\eta, \nabla S)$$

which appears, for example, in [**10**]. For our purposes, all that matters in this expression is that the first term $\mathcal{D}^*\mathcal{D}$ is non-negative, elliptic, and of higher order than the second term involving gradients. This means we can connect L_t by a path of elliptic operators $L_t(s)$ to the non-negative operator $\mathcal{D}^*\mathcal{D}$:

$$L_t(s)(\eta) = \mathcal{D}^*\mathcal{D}(\eta) - s(\nabla\eta, \nabla S).$$

As s runs from 0 to 1, it is standard that only finitely many eigenvalues of $L_t(s)$ can become negative, proving that the spectrum of $L_t = L_t(1)$ is bounded below.

With this choice of η, tracing through the argument used in the proof of Proposition 10 we see that

$$\frac{\operatorname{tr}(A_k - B_k)^2}{k^{n+2}} = \int_{X\times X} K_k(p,q)G_k(p)G_k(q) = \langle G_k, Q_k(G_k)\rangle_{L^2}$$

where this time $G_k = O(k^{-2})$. As before, it follows from Liu and Ma's theorem that there is a constant C such that, for all $t \in [0,T]$,

$$\frac{\operatorname{tr}(A_k - B_k)^2}{k^{n+2}} \leq \frac{C}{k^4}.$$

Throughout, we have expanded $\rho_k(\omega(k;t))$ and $Q_k(\omega(k;t))$ using the uniformity in Theorems 6 and 7 along with the fact that $\omega(k;t) \to \omega(t)$ in C^∞, uniformly for $t \in [0,T]$. This is the infinitesimal version of the result with $m = 1$. As in the proof of Proposition 10, this implies that there is a constant C such that

$$d_k\left(h_k(t), h_k'(t)\right) \leq \frac{C}{k^2}$$

and so the result with $m = 1$ is true.

For general m, we work iteratively and assume we have selected η_j for $j = 1, \dots m-1$ solving a collection of linear parabolic evolution equations to be specified. Let

$$\psi(k;t) = \phi(t) + \sum_{j=1}^{m} k^{-j}\eta_j$$

where we will find η_m presently. Using equation (2), we have that the tangent to the path $h_k'(t)$ is

$$A_k = \int_X (s_\alpha, s_\beta)\left(k(S - \bar{S}) + \sum_{j=0}^{m-1} k^{-j}\dot{\eta}_{j+1} + \Delta' S + \sum_{j=1}^{m} k^{-j}\Delta'\dot{\eta}_j\right)\frac{\omega^n(k;t)}{n!}$$

where $S = S(\omega(t))$ and Δ' is the Laplacian of the metric $\omega(k;t)$.

The Laplacian depends analytically on the metric, meaning that we can write Δ' as a power series in k^{-1}:

$$\Delta' = \Delta_0 + k^{-1}\Delta_1 + \cdots$$

where Δ_0 is the Laplacian of the unperturbed metric $\omega(t)$ and Δ_r depends only on η_j for $j = 1, \dots r$. This means that the Laplacian terms in the integrand for A_k expands further as

$$\begin{aligned}\Delta' S &= \sum_{j=0}^{m-1} k^{-j}\Delta_j S + O(k^{-m}),\\ \sum_{j=1}^{m} k^{-j}\Delta'\dot\eta_j &= \sum_{j+r=1}^{m-1} k^{-j-r}\Delta_r\dot\eta_j + O(k^{-m}).\end{aligned}$$

Crucially, the choice of η_m only affects the $O(k^{-m})$ terms in these two expansions and no lower-order terms. So, up to $O(k^{-m-1})$, the only contribution of η_m to A_k is the term involving $k^{-m+1}\dot\eta_m$. Hence, we can write

$$A_k = \int_X (s_\alpha, s_\beta)\left(k(S-\bar S) + \sum_{j=0}^{m-1} k^{-j}M_j + k^{-m+1}\dot\eta_m + O(k^{-m})\right)\frac{\omega^n(k;t)}{n!}$$

where $S = S(\omega(t))$ and the M_j are determined by the η_j for $j < m$.

Meanwhile, as in equation (3),

$$B_k = \int_X (s_\alpha, s_\beta)\Big(k\left(S(\omega(k;t)) - \bar S\right) + \Phi_k\Big)\frac{\omega(k;t)^n}{n!}$$

where Φ_k is built out of the Bergman function $\rho_k(\omega(k;t))$ by a combination of ρ_k^{-1} and errors introduced by replacing ω_{FS} with $\omega(k;t)$. Theorem 6 says that ρ_k has an asymptotic expansion in which the coefficients are polynomials in the curvature of $\omega(k;t)$. Consequently, Φ_k has an asymptotic expansion, this time in increasing powers of k^{-1}, and again the coefficients are polynomials in the curvature of $\omega(k;t)$. It follows that the first contribution of η_m to Φ_k occurs at $O(k^{-m})$. In addition, scalar curvature depends analytically on the metric, so again we have that the only contribution of η_m to $S(\omega(k;t))$ occurs at $O(k^{-m})$ and here the contribution is precisely $k^{-m}L_t(\eta_m)$. So we can write B_k as

$$\int_X (s_\alpha, s_\beta)\left(k(S-\bar S) + \sum_{j=0}^{m-2} k^{-j}F_j + k^{-m+1}(F_m - L_t(\eta_m)) + O(k^{-m})\right)$$

where $S = S(\omega(t))$ and all F_j for $j < m$ are determined by η_j for $j < m$.

To complete the proof, we assume that we have chosen the $\eta_1, \dots, \eta_{m-1}$ so that the terms of $O(k^{-m+2})$ in $A_k - B_k$ cancel and, moreover, so that $\eta_j(0) = 0$. This amounts to solving a sequence of parabolic Cauchy problems of the form (4) in which the inhomogeneous term in the j^{th} equation involves the solutions to all previous equations. Assuming this is done, we are left with

$$A_k - B_k = \int_X (s_\alpha, s_\beta)\Big[k^{-m+1}(\dot\eta_m + L_t(\eta_m) + M_m - F_m) + O(k^{-m})\Big]\frac{\omega^n(k;t)}{n!}$$

for M_m and F_m depending only on η_j for $j < m$. Choosing η_m to solve the parabolic equation

$$\dot{\eta}_m + L(\eta_m) = F_m - M_m$$

with $\eta_m(0) = 0$ gives

$$A_k - B_k = \int_X (s_\alpha, s_\beta) O(k^{-m}) \frac{\omega(k;t)^n}{n!},$$

and from here the proof proceeds via Liu and Ma's theorem precisely as above. q.e.d.

4. Analytic estimates

The previous section produced, for a given integer m, a sequence of flows $\omega(k;t)$ for which $\omega(k;t) \to \omega(t)$ as $k \to \infty$ in C^∞ and such that the k^{th} Bergman point $h'_k(t)$ of $\omega(k;t)$ satisfies $d_k(h'_k(t), h_k(t)) = O(k^{-m-1})$. To complete the proof that $\omega_k(t) \to \omega(t)$ in C^∞, we use the fact that, in the regions of $\mathcal{B}_k$ of interest to us at least, $k^{(r/2)+1+n} d_k$ uniformly controls the C^r norm on the curvature tensors of Bergman metrics. It is precisely this power of k appearing in front of d_k which makes the higher-order approximations of Theorem 11 necessary.

The first step in controlling the C^r norm, carried out in this section, is to prove some analytitc estimates which reduce the problem to controlling the norm of the matrix $\bar{\mu}$. The main estimate we use is due to Donaldson [**12**]. We give here a brief description of the relevant part of §3.2 of [**12**]. In order to avoid worrying about powers of k at every step here, when proving the estimates we use for each k the *large* metrics in the class $kc_1(L)$. Then, at the end it is a simple matter to rescale to metrics in the fixed class and take care of the powers of k at a single stroke.

Fix a reference metric $\omega_0 \in c_1(L)$ and denote $\tilde{\omega}_0 = k\omega_0 \in kc_1(L)$. We say another metric $\tilde{\omega} \in kc_1(L)$ has *R-bounded geometry in C^r* if $\tilde{\omega} > R^{-1}\tilde{\omega}_0$ and

$$\|\tilde{\omega} - \tilde{\omega}_0\|_{C^r} < R$$

where the norm $\|\cdot\|_{C^r}$ is that determined by the metric $\tilde{\omega}_0$. Given a basis $\{s_\alpha\}$ for $H^0(L^k)$, we get an embedding $X \subset \mathbb{CP}^{N_k}$ and hence a metric $\tilde{\omega} = \omega_{\text{FS}}|_X$. Equivalently, $2\pi i\tilde{\omega}$ is the curvature of the unique metric on L^k for which $\sum |s_\gamma|^2 = 1$. We say that the basis $\{s_\alpha\}$, or the corresponding point in $\mathcal{B}_k$, has R-bounded geometry if the metric $\tilde{\omega}$ does.

Given a basis $\{s_\alpha\}$ and a Hermitian matrix $A = (A_{\alpha\beta})$, define

$$H_A = \sum A_{\alpha\beta}(s_\alpha, s_\beta)$$

where we have taken the inner-product here using the pull-back of the Fubini–Study metric for which $\sum |s_\gamma|^2 = 1$. Note that $H_A = \operatorname{tr}(A\mu)$

restricted to X and so is the potential giving the infinitesimal deformation corresponding to A of the restriction of the Fubini–Study metric to X. As a final piece of notation, we denote by $\|A\|_{\mathrm{op}}$ the maximum of the moduli of the eigenvalues of A and by $\|A\| = \sqrt{\operatorname{tr} A^2}$ the norm of A with respect to the Killing form. The first estimate we want is the following:

Proposition 12 (Donaldson [**12**]). *There is a constant C such that for all points of $\mathcal{B}_k$ with R-bounded geometry in C^r and any Hermitian matrix A,*

$$\|H_A\|_{C^r} \leq C\|\bar{\mu}\|_{\mathrm{op}}\|A\|,$$

where $\bar{\mu} = \int_X \mu\,\tilde{\omega}^n/n!$ is computed using the embedding corresponding to the point of $\mathcal{B}_k$ and the C^r-norm is taken with respect to the fixed reference metric $\tilde{\omega}_0$.

The key point is that C depends only on R and r, but not on k. This is proved more or less explicitly in the course of the proof of Lemma 24 of [**12**], even though the end result is not stated in quite the form we give here. Accordingly, we give only a sketch proof here, giving nearly word-for-word parts of the proof of lemma 24 of [**12**].

Sketch of proof of Proposition 12. First, we recall the following standard estimate. Let Z be a compact complex Hermitian manifold, $E \to X$ a Hermitian holomorphic vector bundle, and $P \subset Z$ a differentiable (real) submanifold. There is a constant C such that for any $\sigma \in H^0(E, Z)$,

$$\|\sigma\|_{C^r(P)} \leq C\|\sigma\|_{L^2(Z)}. \tag{5}$$

Moreover, provided that the data Z, P, E has bounded local geometry in C^r in a suitable sense, C can be taken to be independent of the particular manifolds and bundles involved.

We apply this to the manifold $Z = X \times \overline{X}$ where $\overline{X}$ is X with the opposite complex structure. The Hermitian metric on L^k induces a connection in L^k; one component of the connection recovers the holomorphic structure on $L^k \to X$, while the other component makes $\overline{L}^k \to \overline{X}$ into a holomorphic line bundle. Let $E \to Z$ be the tensor product of the pull-back of L^k from the first factor and $(\overline{L}^k)^*$ from the second. Given a Bergman metric in $\mathcal{B}_k$, we take the obvious induced Kähler metric on Z and Hermitian metric in E. Let P denote the diagonal in Z. We will use the estimate (5) in this situation along with the fact that the constant can be taken independently of the Bergman metric used, provided it has R-bounded geometry in C^r.

A holomorphic section s of $L^k \to X$ defines a holomorphic section $\tilde{s}$ of $(\overline{L}^k)^* \to \overline{X}$ via the bundle isomorphism given by the fibre metric.

Thus for any Hermitian matrix A, we get a holomorphic section

$$\sigma_A = \sum A_{\alpha\beta}\, s_\alpha \otimes \tilde{s}_\beta$$

of E over Z. We have

$$\|\sigma_A\|^2_{L^2(Z)} = \sum A_{\alpha\beta}\overline{A}_{\alpha'\beta'}\langle s_\alpha, s'_\alpha\rangle\langle s_\beta, s'_\beta\rangle$$

(where $\langle\cdot,\cdot\rangle$ denotes the L^2-inner-product). In matrix notation this reads

$$\|\sigma_A\|^2_{L^2(Z)} = \mathrm{tr}\,(A\bar{\mu}A^*\bar{\mu}^*)\,.$$

There is a standard inequality that for Hermitian matrices G, F,

$$\mathrm{tr}(FGFG) \leq \|F\|^2\|G\|^2_{\mathrm{op}}, \tag{6}$$

which here gives

$$\|\sigma_A\|_{L^2(Z)} \leq \|\bar{\mu}\|_{\mathrm{op}}\|A\|.$$

Now, over P, the metric on L^k defines a C^∞ trivialization of E and the function $H_A = \sum A_{\alpha\beta}(s_\alpha, s_\beta)$ is just the restriction of σ_A to the diagonal in this trivialization. Hence, by the inequality (5), we have

$$\|H_A\|_{C^r(X)} \leq C\|\sigma_A\|_{L^2(Z)} \leq C\|\bar{\mu}\|_{\mathrm{op}}\|A\|.$$

q.e.d.

We can rephrase this result by saying that under certain conditions, the Riemannian distance on $\mathcal{B}_k$ controls the C^{r-2}-norm on Kähler forms. To make this precise, let $\tilde{\omega}(s)$ for $s \in [0,1]$ denote a path of Kähler forms in $\mathcal{B}_k$. We denote by L the length of the path, measured using *large* symmetric Riemannian metric on $\mathcal{B}_k$, i.e., the metric corresponding to the Killing form $\mathrm{tr}\,A^2$.

Lemma 13. *If all the metrics $\tilde{\omega}(s)$ for $s \in [0,1]$ have R-bounded geometry in C^r and also satisfy $\|\bar{\mu}\|_{\mathrm{op}} < K$, then*

$$\|\tilde{\omega}(0) - \tilde{\omega}(1)\|_{C^{r-2}} < CKL,$$

where the C^{r-2} norm is taken with respect to the reference metric $\tilde{\omega}_0$.

Proof. Let $A(s)$ denote the Hermitian matrix which is tangent to the given path. We have that

$$\left\|\frac{\partial\tilde{\omega}}{\partial t}\right\|_{C^{r-2}} = \left\|i\partial\bar{\partial}H_{A(s)}\right\|_{C^{r-2}} \leq CK\|A\|.$$

The result now follows by integrating along the path. q.e.d.

Of course, to apply this lemma we need to find regions in $\mathcal{B}_k$ which consist of R-bounded metrics and also for which $\|\bar{\mu}\|_{\mathrm{op}}$ is uniformly controlled. In this direction, we prove the following simple lemma. We denote by $d = k^{n+2}d_k$ the unscaled symmetric Riemannian metric on $\mathcal{B}_k$.

Lemma 14. *Let $\tilde{\omega}_k \in \mathcal{B}_k$ be a sequence of metrics with $R/2$-bounded geometry in C^{r+2} and such that $\|\bar{\mu}(\tilde{\omega}_k)\|_{\mathrm{op}}$ is uniformly bounded. Then there is a constant C such that if $\tilde{\omega} \in \mathcal{B}_k$ has $d(\tilde{\omega}_k, \tilde{\omega}) < C$, then $\tilde{\omega}$ has R-bounded geometry in C^r.*

Proof. There is a Hermitian matrix B such that $\tilde{\omega} = e^B \cdot \tilde{\omega}_k$; note that $d(\tilde{\omega}_k, \tilde{\omega}) = \|B\|$. Let s_α be a basis for $H^0(L^k)$ defining the embedding corresponding to $\tilde{\omega}_k$ and chosen, moreover, so that $B = \mathrm{diag}(\lambda_\alpha)$ is diagonal in this basis. (This can be done thanks to $\mathrm{U}(N+1)$-invariance.) Then $\tilde{\omega} = \tilde{\omega}_k + i\bar{\partial}\partial v$ where

$$e^v = \sum e^{2\lambda_\alpha} |s_\alpha|^2.$$

Because $\tilde{\omega}_k$ has $R/2$-bounded geometry in C^{r+2} and $\|\bar{\mu}(\tilde{\omega}_k)\|_{\mathrm{op}}$ uniformly bounded, Proposition 12 implies that there is a constant c such that for any α,

$$\left\| |s_\alpha|^2 \right\|_{C^{r+2}} < c.$$

(Apply the result to the matrix A with a single 1 as entry α on the diagonal and zeros elsewhere.) It follows that

$$\|e^v\|_{C^{r+2}} \leq c e^{2 \max |\lambda_\alpha|} \leq c e^{2\|B\|}.$$

So a bound on $\|B\|$ gives uniform control of $\|\tilde{\omega} - \tilde{\omega}_k\|_{C^r}$. Hence, there is a C such that when $\|B\| = d(\tilde{\omega}, \tilde{\omega}_k) < C$, $\tilde{\omega}$ has R-bounded geometry in C^r. q.e.d.

5. Projective estimates

Our final task is to control $\|\bar{\mu}\|_{\mathrm{op}}$. For the first lemma in this direction, we consider the situation from Tian's Theorem 1; so h is a Hermitian metric in L with positive curvature defining a Kähler form ω and $h_k \in \mathcal{B}_k$ is the sequence of Bergman metrics in L corresponding to an $L^2(h^k, \omega)$-orthonormal basis of $H^0(L^k)$.

Lemma 15. *$\|\bar{\mu}(h_k) - 1_k\|_{\mathrm{op}} \to 0$, where $1_k \in i\mathfrak{u}(N_k + 1)$, is the identity matrix. Moreover, this convergence is uniform in ω in the sense that there is an integer s such that if ω runs over a set of metrics bounded in C^s and for which ω is bounded below, then the convergence is uniform.*

Proof. We borrow another trick we learned from [**12**]. Let s_α be a basis for $H^0(L^k)$ determining ω_k. Given a continuous function $F\colon X \to \mathbb{R}$, set A_F to be the Hermitian matrix with entries

$$(A_F)_{\alpha\beta} = \int_X (s_\alpha, s_\beta) F \, \frac{\omega^n}{n!}.$$

Then $\|A_F\|_{\mathrm{op}} \leq \|F\|_{C^0}$. This is because the map $A_F\colon H^0(L^k) \to H^0(L^k)$ factors through the space V of all L^2-integrable sections as $A_F = \pi \circ M_F \circ j$ where $j\colon H^0(L^K) \to V$ is the inclusion, M_F is multiplication by F, and $\pi\colon V \to H^0(L^k)$ is orthogonal projection in V.

We are interested in the matrix

$$\begin{aligned}\bar{\mu}_{\alpha\beta} &= \int_X \frac{(s_\alpha, s_\beta)}{\rho_k}\frac{\omega_{\mathrm{FS}}^n}{n!}\\ &= \int_X (s_\alpha, s_\beta)(1+O(k^{-1}))\frac{\omega^n}{n!}\end{aligned}$$

where we have used the asymptotic expansion of ρ_k and $\omega = \frac{1}{k}\omega_{\mathrm{FS}} + O(k^{-2})$. So $\bar{\mu} - 1_k$ is the matrix associated to a function $F_k\colon X \to \mathbb{C}$ with $\|F_k\|_{C^0} = O(k^{-1})$. Hence, $\|\bar{\mu} - 1_k\|_{\mathrm{op}} = O(k^{-1})$. The convergence is uniform in ω because the asymptotic expansion of ρ_k is. q.e.d.

Remark 16. As with the asymptotics of ρ_k and Q_k, the fact the convergence of Lemma 15 is uniform allows us to pass from a single metric ω to a sequence $\omega(k)$ which converges in C^∞. If we denote by $\omega_k(k) \in \mathcal{B}_k$ the k^{th} standard projective approximation to $\omega(k)$, then it follows from the uniformity that $\|\bar{\mu}(\omega_k(k)) - 1_k\|_{\mathrm{op}} \to 0$

The remainder of this section is devoted to controlling $\|\bar{\mu}\|_{\mathrm{op}}$ in terms of the Riemannian distance in the Bergman space. Our arguments will apply simultaneously to all $\mathcal{B}_k$ without k playing a role. Accordingly, until the end of the section we drop the reference to k and work on the Bergman space $\mathcal{B} \cong \mathrm{GL}(N+1)/\,\mathrm{U}(N+1)$ associated to a given subvariety $X \subset \mathbb{CP}^N$.

Given a point $b \in \mathcal{B}$ and tangent vector $A \in T_b\mathcal{B} \cong i\mathfrak{u}(N+1)$, we differentiate $\bar{\mu}\colon \mathcal{B} \to i\mathfrak{u}(N+1)$ at b to obtain $\mathrm{d}\bar{\mu}(A) \in i\mathfrak{u}(N+1)$. The first fact we need—which appears, for example, in [**26**]—is the relationship between $\mathrm{d}\bar{\mu}(A)$ and the extrinsic geometry of the embedding $X \subset \mathbb{CP}^N$ corresponding to $b \in \mathcal{B}$. Let ξ_A denote the vector field on $\mathbb{CP}^N$ corresponding to A. Let ξ_A^{TX} denote the component of $\xi_A|_X$ which is tangent to X and $\xi_A^\perp$ the component which is perpendicular. Finally let $(\cdot,\cdot)$ denote the Fubini–Study inner product on tangent vectors.

Lemma 17. *For any pair of Hermitian matrices $A, B \in i\mathfrak{u}(N+1)$,*

$$\mathrm{tr}(B\,\mathrm{d}\bar{\mu}(A)) = \int_X (\xi_A^\perp, \xi_B^\perp)\,\frac{\omega_{\mathrm{FS}}^n}{n!}.$$

Proof.

$$\begin{aligned}\mathrm{tr}(B\,\mathrm{d}\bar{\mu}(A)) &= \int_X \mathrm{tr}(B\,\mathrm{d}\mu(A))\,\frac{\omega_{\mathrm{FS}}^n}{n!} + \int_X \mathrm{tr}(B\mu)\frac{L_{\xi_A}(\omega_{\mathrm{FS}}^n)}{n!}\\ &= \int_X \Big((\xi_A, \xi_B) - H_B\Delta H_A\Big)\frac{\omega_{\mathrm{FS}}^n}{n!}\\ &= \int_X \Big((\xi_A, \xi_B) - (\xi_A^{TX}, \xi_B^{TX})\Big)\frac{\omega_{\mathrm{FS}}^n}{n!}.\end{aligned}$$

Here the various equalities all follow from the fact that $-i\mu$ is a moment map for the $\mathrm{U}(N+1)$-action on $\mathbb{CP}^N$; we have

$$\begin{aligned}\mathrm{tr}(B\,\mathrm{d}\mu(A)) &= \omega_{\mathrm{FS}}(J\xi_A, \xi_B)\\ &= (\xi_A, \xi_B);\\ L_{\xi_A}\omega_{\mathrm{FS}} &= 2i\bar\partial\partial\,(\mathrm{tr}\,A\mu)\\ &= 2i\bar\partial\partial H_A;\\ (L_{\xi_A}\omega_{\mathrm{FS}}^n)\,|_X &= -\Delta H_A\,(\omega_{\mathrm{FS}}^n|_X)\,.\end{aligned}$$

q.e.d.

We now continue with a series of identities and estimates in projective geometry which provide the pieces needed to control $\|\bar\mu\|_{\mathrm{op}}$.

Lemma 18. *Let $A, B \in i\mathfrak{u}(N+1)$ be Hermitian matrices. At every point of $\mathbb{CP}^N$,*

$$H_A H_B + (\xi_A, \xi_B) = \mathrm{tr}(AB\mu).$$

Proof. By $\mathrm{U}(N+1)$ equivariance, it suffices to consider the point $p = [1\colon 0\colon \cdots : 0]$. Let $(x_1, \ldots, x_N) \mapsto [1\colon x_1\colon \cdots : x_N]$ be unitary coordinates. At p, $\mu(p)$ has a single non-zero entry which is a one in the top left corner. Hence, $H_A(p) = A_{00}$, $H_B(p) = B_{00}$ and

$$\mathrm{tr}(AB\mu) = A_{00}B_{00} + A_{01}B_{10} + \cdots + A_{0N}B_{N0}.$$

Meanwhile, at p, the coordinate vectors ∂_i are orthonormal, while $\xi_A = A_{01}\partial_1 + \cdots + A_{0N}\partial_N$ and similarly for ξ_B. Putting the pieces together and using $B^* = B$ gives the result. q.e.d.

Remark 19. As an aside, it is interesting to compare this result with the analytic estimate in Proposition 12. It follows from Lemma 18 that for any A, at every point of X,

$$\begin{aligned}H_A^2 + |\nabla H_A|^2 &= H_A^2 + |\xi_A^{TX}|^2\\ &\leq H_A^2 + |\xi_A|^2\\ &= \mathrm{tr}(A^2\mu)\\ &\leq \|A\|^2.\end{aligned}$$

So the C^1 case of Proposition 12, $\|H_A\|_{C^1} \leq \|A\|$, comes "for free" from projective geometry with no need to use analysis (and with no need to involve $\|\bar\mu\|_{\mathrm{op}}$ in the bound).

Lemma 20. *For any Hermitian matrices $A, B \in i\mathfrak{u}(N+1)$,*

$$\mathrm{tr}(B\,\mathrm{d}\bar\mu(A)) + \langle H_A, H_B\rangle_{L^2_1(X)} = \mathrm{tr}(AB\bar\mu).$$

Proof. From Lemma 18 we have, at every point of X,

$$H_A H_B + (\xi_A^{TX}, \xi_B^{TX}) + (\xi_A^\perp, \xi_B^\perp) = \mathrm{tr}(AB\mu).$$

Now use the identity $\xi_A^{TX} = \nabla H_A$, integrate over X, and apply Lemma 17. q.e.d.

Remark 21. This identity fits into Donaldson's "double quotient" picture described in §2.1 of [**12**]; we explain this here, although we make no direct use of this observation later. Donaldson considers the infinite-dimensional space $\mathcal{X} = \Gamma(L^k)^{N_k+1}$ (where Γ denotes *smooth* sections). Given a Hermitian metric h in L with positive curvature $2\pi i\omega$, $\mathcal{X}$ is formally a Kähler manifold, where the Riemannian metric is given by the $L^2(h^k, \omega)$-inner-product on sections of L^k and the complex structure is given by multiplication of sections by i. Two groups act on $\mathcal{X}$, one finite-dimensional the other infinite-dimensional. The finite dimensional group is $\mathrm{GL}(N_k + 1)$ which acts on $\mathcal{X}$ by Kähler isometries, mixing the $\Gamma(L^k)$ factors in the obvious way. The infinite-dimensional group is the group $\mathcal{G}$ of Hermitian bundle maps $L \to L$ which preserve the Chern connection of h; this acts preserving the L^2-inner-product on $\Gamma(L^k)$ and hence by it acts by Kähler isometries on $\mathcal{X}$. While the complexification of $\mathcal{G}$ doesn't exist, one can still make sense of the complex "orbits" in $\mathcal{X}$.

Assume that (h, ω) come from a projective embedding defined via a basis $\underline{s} = (s_0, \dots s_{N_k}) \in \mathcal{X}$. Given a Hermitian matrix $A \in i\mathfrak{u}(N_k + 1)$, we get an infinitesimal change in the basis $\underline{s}$, i.e., a tangent vector $V_A \in T_{\underline{s}}\mathcal{X}$. Let $P \subset T_{\underline{s}}\mathcal{X}$ denote the tangent space to the complex "orbit" through $\underline{s}$. We can decompose V_A into two components, the part V_A' which is in P and the part V_A'' which is orthogonal. Doing likewise for a second Hermitian matrix B, we have the obvious identity

$$(V_A, V_B) = (V_A', V_B') + (V_A'', V_B'').$$

Proposition 19 in [**12**] gives V_A' explicitly in terms of H_A and using this one can write out the terms in this identity, giving

$$\begin{aligned}
(V_A, V_B) &= \mathrm{tr}(AB\bar{\mu}), \\
(V_A', V_B') &= \langle H_A, H_B \rangle_{L_1^2(X)}, \\
(V_A'', V_B'') &= \mathrm{tr}(B\, \mathrm{d}\bar{\mu}(A)).
\end{aligned}$$

So Lemma 20 amounts to the orthogonal decomposition $T_{\underline{s}}\mathcal{X} = P \oplus P^\perp$. Meanwhile, the equality $(V_A'', V_B'') = \mathrm{tr}(B\, \mathrm{d}\bar{\mu}(A))$ is a consequence of the fact that balancing flow is the downward gradient flow of the Kempf–Ness function associated to the finite-dimensional moment-map problem that remains after taking the symplectic reduction by the action of $\mathcal{G}$.

Lemma 22. *For any Hermitian matrix $A \in i\mathfrak{u}(N + 1)$,*

$$\|H_A\|^2_{L_1^2} \leq \|A\|^2 \|\bar{\mu}\|_{\mathrm{op}}.$$

Proof. It follows from Lemma 20 that

$$\|H_A\|^2_{L_1^2} = \mathrm{tr}(A^2\bar{\mu}) - \mathrm{tr}(A\, \mathrm{d}\bar{\mu}(A)).$$

From Lemma 17,

$$\operatorname{tr}(A\, \mathrm{d}\bar{\mu}(A)) = \int_X |\xi_A^\perp|^2 \frac{\omega_{\mathrm{FS}}^n}{n!} > 0.$$

Hence,

$$\|H_A\|^2_{L^2_1} \leq \operatorname{tr}(A^2 \bar{\mu}) \leq \|A\|^2 \|\bar{\mu}\|_{\mathrm{op}},$$

where the second inequality follows from inequality (6). q.e.d.

Lemma 23. *For any Hermitian matrix $A \in i\mathfrak{u}(N+1)$,*

$$\|\mathrm{d}\bar{\mu}(A)\| \leq 2\|A\|\|\bar{\mu}\|_{\mathrm{op}}.$$

Proof. From Lemma 20 with $B = \mathrm{d}\bar{\mu}(A)$,

$$\|\mathrm{d}\bar{\mu}(A)\|^2 = \operatorname{tr}\left(\mathrm{d}\bar{\mu}(A)^2\right) = \operatorname{tr}(A\, \mathrm{d}\bar{\mu}(A)\bar{\mu}) - \langle H_A, H_{\mathrm{d}\bar{\mu}(A)}\rangle_{L^2_1}.$$

Now apply Cauchy–Schwarz, Lemma 22, and inequality (6) to deduce

$$\|\mathrm{d}\bar{\mu}(A)\|^2 \leq 2\|A\|\|\mathrm{d}\bar{\mu}(A)\|\|\bar{\mu}\|_{\mathrm{op}}.$$

q.e.d.

Finally, we are in a position to control $\|\bar{\mu}\|_{\mathrm{op}}$ in terms of Riemannian distance on $\mathcal{B}$.

Proposition 24. *Let $b_0, b_1 \in \mathcal{B}$ and let $d(b_0, b_1)$ denote the Riemannian distance between b_0 and b_1. Then*

$$\|\bar{\mu}(b_1)\|_{\mathrm{op}} \leq e^{2d(b_0,b_1)} \|\bar{\mu}(b_0)\|_{\mathrm{op}}.$$

Proof. Let A generate the geodesic e^{tA} in $\mathcal{B}$ joining b_0 and b_1 so that $\|A\| = d(b_0, b_1)$. As we run along the geodesic from b_0 to b_1, the rate of change of $\|\bar{\mu}\|_{\mathrm{op}}$ is at most

$$\|\mathrm{d}\bar{\mu}(A)\|_{\mathrm{op}} \leq \|\mathrm{d}\bar{\mu}(A)\| \leq 2\|A\|\|\bar{\mu}\|_{\mathrm{op}},$$

by Lemma 23, and so the growth is sub-exponential. q.e.d.

6. Completing the proof of Theorem 5

Now, finally, all the pieces are in place to prove our main result. We begin by recalling our notation. Let h be a Hermitian metric in L with positive curvature $2\pi i\omega$; denote by $\omega(t)$ the Calabi flow starting at ω. Let ι_k be the embedding of X defined by a basis of $H^0(L^k)$ which is orthonormal with respect to the L^2 inner product defined by $h(0)$ and $\omega(0)$. Let $\omega_k = \frac{1}{k}\iota_k^*\omega_{\mathrm{FS}}$ denote the standard sequence of projective approximations to $\omega(0)$. Let $\iota_k(t)$ solve the balancing flow (1) and let $\omega_k(t) = \frac{1}{k}\iota_k(t)^*\omega_{\mathrm{FS}}$. We must prove first that $\omega_k(t) \to \omega(t)$ in C^∞.

Recall that in Theorem 11, for any given integer m we constructed a sequence of flows $\omega(k;t)$ which satisfies

1) $\omega(k;t) \to \omega(t)$ in C^∞ as $k \to \infty$;

2) $d_k(h_k(t), h'_k(t)) \leq Ck^{-m-1}$ where $h_k(t)$ is the balancing flow, $h'_k(t)$ denotes the k^{th} standard projective approximation to $\omega(k;t)$, and d_k is the scaled Riemannian distance function on $\mathcal{B}_k$ corresponding to the Killing form $k^{-n-2}\operatorname{tr} A^2$ on Hermitian matrices.

Let $\omega'_k(t) \in c_1(L)$ denote the (rescaled) Kähler form corresponding to $h'_k(t)$. It follows from point 1 above and the uniformity of the asymptotic expansion of ρ_k that $\omega'_k(t) \to \omega(t)$ in C^∞. We will show $\omega_k(t) \to \omega(t)$ by proving that

$$\|\omega_k(t) - \omega'_k(t)\|_{C^r(\omega(t))} \to 0$$

as $k \to \infty$. To do this we will control the C^r norm on metrics by the distance d_k in $\mathcal{B}_k$ along the geodesics joining $h'_k(t)$ to $h_k(t)$. We can do this by Lemma 13, provided all the points have R-bounded geometry in C^{r+2} and all have $\|\bar{\mu}\|_{\text{op}}$ uniformly controlled as well.

We begin with the control of $\|\bar{\mu}\|_{\text{op}}$. By Lemma 15 and Remark 16, $\|\bar{\mu}\|_{\text{op}}$ is controlled for $h'_k(t)$ uniformly in k. Now we apply Proposition 24, for which we need $h_k(t)$ to be a uniformly bounded distance from $h'_k(t)$ in the *unscaled* distance $d = k^{n+2}d_k$. Provided we take $m \geq n+1$ this holds, giving that $\|\bar{\mu}\|_{\text{op}}$ is uniformly bounded along the geodesics joining $h'_k(t)$ to $h_k(t)$.

Next, we establish that the points of these geodesics have R-bounded geometry in C^{r+2}. We use $\omega(t)$ as our reference metric and apply Lemma 14 to the sequence $\omega'_k(t)$ (but with r replaced by $r+2$). We have already observed that the part of the hypothesis concerning $\|\bar{\mu}\|_{\text{op}}$ is satisfied and the metrics $\omega'_k(t)$ certainly have $R/2$-bounded geometry in C^{r+4}, since they converge in C^∞ to $\omega(t)$. Now, provided we take $m \geq n+2$, the *unscaled* distance $d(h'_k(t), h_k(t))$ tends to zero, and so, for sufficiently large k, all points on the geodesics joining $h'_k(t)$ and $h_k(t)$ have R-bounded geometry in C^{r+2}.

Finally, we can apply Corollary 13. This tells us that for the *unscaled* metrics there is some constant M such that

$$\|k\omega_k(t) - k\omega'_k(t)\|_{C^r(k\omega(t))} \leq Mk^{n+2}d_k(h_k(t), h'_k(t)) \leq MCk^{n+1-m}.$$

Rescaling this inequality, we see that

$$\|\omega_k(t) - \omega'_k(t)\|_{C^r(\omega(t))} \leq MCk^{(r/2)+1+n-m}.$$

So provided we take $m > \frac{r}{2} + 1 + n$, we obtain that $\omega_k(t)$ converges to $\omega(t)$ in C^r.

For $t \in [0,T]$, $\{\omega(t)\}$ is a compact set of metrics; from here it is easy to check that the convergence $\omega_k(t) \to \omega(t)$ is uniform in t. There are various places where uniformity must be checked. First, we have used $\omega(t)$ to define the C^r-norms, but the compactness ensures all these norms are uniformly equivalent. Second, we have applied asymptotic expansions for ρ_k and Q_k, but these are both uniform as t varies, by uniformity in the relevant Theorems 6, 7, and 8. We also must check that $\omega(k;t)$ converges uniformly to $\omega(t)$. This holds since, for given r, there are only finitely many perturbations η_j present. Finally, if we denote by $\dot{\phi}_k$ the potential for the t-derivative of $\omega_k(t)$, it follows from Theorem 9 that $\dot{\phi}_k(t) \to S(\omega(t)) - \bar{S}$ in C^∞. This is uniform for t in a compact interval, again by the uniformity of the asymptotics in Theorems 6, 7, and 8.

Appendix: Asymptotics of the operators Q_k

Written by Kefeng Liu and Xiaonan Ma. This note is a continuation of [**17**] providing a technical result needed in the preceding article of Fine. We refer to [**15**], [**17**], and Fine's paper for the context of the problem. We also refer the readers to the recent book [**20**] for more information on the Bergman kernel.

We begin by recalling the basic setting and notation in [**17**], which we will use freely throughout.

Let (X, ω, J) be a compact Kähler manifold with $\dim_{\mathbb{C}} X = n$, and let (L, h^L) be a holomorphic Hermitian line bundle on X. Let ∇^L be the holomorphic Hermitian connection on (L, h^L) with curvature R^L. We assume that

$$\frac{\sqrt{-1}}{2\pi} R^L = \omega. \tag{7}$$

Let $g^{TX}(\cdot,\cdot) := \omega(\cdot, J\cdot)$ be the Riemannian metric on TX induced by ω, J. Let dv_X be the Riemannian volume form of (TX, g^{TX}); then $dv_X = \omega^n/n!$. Let $d\nu$ be any volume form on X. Let η be the positive function on X defined by

$$dv_X = \eta \, d\nu. \tag{8}$$

The L^2–scalar product $\langle\ \ \rangle_\nu$ on $C^\infty(X, L^p)$, the space of smooth sections of L^p, is given by

$$\langle \sigma_1, \sigma_2 \rangle_\nu := \int_X \langle \sigma_1(x), \sigma_2(x) \rangle_{h^{L^p}} \, d\nu(x)\,. \tag{9}$$

Let $P_{\nu,p}(x, x')$ $(x, x' \in X)$ be the smooth kernel of the orthogonal projection from $(C^\infty(X, L^p), \langle\ \ \rangle_\nu)$ onto $H^0(X, L^p)$, the space of the holomorphic sections of L^p on X, with respect to $d\nu(x')$. Following [**15**, §4], set

(10)
$$K_p(x, x') := |P_{\nu,p}(x, x')|^2_{h_x^{L^p} \otimes h_{x'}^{L^{p*}}}, \quad R_p := (\dim H^0(X, L^p))/\mathrm{Vol}(X, \nu),$$

where $\mathrm{Vol}(X, \nu) := \int_X d\nu$. Set $\mathrm{Vol}(X, dv_X) := \int_X dv_X$.

Let Q_{K_p} be the integral operator associated to K_p which is defined for $f \in C^\infty(X)$,

$$Q_{K_p}(f)(x) := \frac{1}{R_p}\int_X K_p(x,y)f(y)d\nu(y). \tag{11}$$

Let Δ be the (positive) Laplace operator on (X, g^{TX}) acting on the functions on X. We denote by $|\ \ |_{L^2}$ the L^2-norm on the function on X with respect to dv_X.

The following result is an improvement of [**17**, Theorem 0.1], where it is proved for $q = 0$ and $q = 1$.

Theorem 25. *For and $q \in \mathbb{N}$, there exists a constant $C > 0$ such that for any $f \in C^\infty(X)$, $p \in \mathbb{N}$,*

$$\left|\left(\Big(\frac{\Delta}{p}\Big)^q Q_{K_p} - \frac{\mathrm{Vol}(X,\nu)}{\mathrm{Vol}(X,dv_X)}\Big(\frac{\Delta}{p}\Big)^q \eta \exp\Big(-\frac{\Delta}{4\pi p}\Big)\right) f\right|_{L^2} \leqslant \frac{C}{p}\,|f|_{L^2}\,. \tag{12}$$

Moreover, (12) *is uniform in that there is an integer s such that if all data $h^L, d\nu$ run over a set which is bounded in C^s-topology and that g^{TX}, dv_X are bounded from below, then the constant C is independent of $h^L, d\nu$.*

Proof. Let $E = \mathbb{C}$ be the trivial holomorphic line bundle on X. Let h^E the metric on E defined by $|\mathbf{e}|^2_{h^E} = 1$, where $\mathbf{e}$ is the canonical unity element of E. We identify canonically L^p to $L^p \otimes E$ by section $\mathbf{e}$.

Let h^E_ω be the metric on E defined by $|\mathbf{e}|^2_{h^E_\omega} = \eta^{-1}$. Let $\langle\ \ \rangle_\omega$ be the Hermitian product on $C^\infty(X, L^p \otimes E) = C^\infty(X, L^p)$ induced by h^L, h^E_ω, dv_X as in (9). If $P_{\omega,p}(x,x')$, $(x, x' \in X)$ denotes the smooth kernel of the orthogonal projection $P_{\omega,p}$ from $(C^\infty(X, L^p \otimes E), \langle\cdot,\cdot\rangle_\omega)$ onto $H^0(X, L^p \otimes E) = H^0(X, L^p)$ with respect to $dv_X(x)$. By [**17**, (11)], we have

$$P_{\nu,p}(x,x') = \eta(x')\, P_{\omega,p}(x,x'). \tag{13}$$

For $f \in C^\infty(X)$, set

$$\begin{aligned} K_{\omega,p}(x,x') &= |P_{\omega,p}(x,x')|^2_{(h^{L^p}\otimes h^E_\omega)_x \otimes (h^{L^{p*}}\otimes h^{E*}_\omega)_{x'}},\\ (K_{\omega,p}f)(x) &= \int_X K_{\omega,p}(x,y)f(y)dv_X(y). \end{aligned} \tag{14}$$

Then by [**17**, (15)], we have

$$Q_{K_p}(f)(x) = \frac{1}{R_p}\int_X K_{\omega,p}(x,y)\eta(x)f(y)dv_X(y). \tag{15}$$

Now we use the normal coordinate as in [**17**]. Then under our identification, $P_{\omega,p}(Z,Z')$ is a function on $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| \leqslant \varepsilon$; we denote it by $P_{\omega,p,x_0}(Z,Z')$ with complex values.

Note that $|P_{\omega,p,x_0}(Z,Z')|^2 = P_{\omega,p,x_0}(Z,Z')\overline{P_{\omega,p,x_0}(Z,Z')}$; thus from [**17**, (19),(20)], (14), there exist $J'_r(Z,Z')$ polynomials in Z,Z' such that

$$\begin{aligned}(16)\quad &\Big|\frac{1}{p^q}\Delta_Z^q\Big(\frac{1}{p^{2n}}K_{\omega,p,x_0}(Z,Z') \\ &\qquad -\Big(1+\sum_{r=2}^{k}\frac{1}{p^{r/2}}J'_r(\sqrt{p}Z,\sqrt{p}Z')\Big)e^{-\pi p|Z-Z'|^2}\Big)\Big|_{C^0(X)} \\ &\leqslant Cp^{-(k+1)/2}(1+|\sqrt{p}Z|+|\sqrt{p}Z'|)^N\exp(-C_0\sqrt{p}|Z-Z'|)+O(p^{-\infty}).\end{aligned}$$

(There is a misprint in [**17**, (25)], we need to move a factor $\frac{1}{p^{2n}}$ into the parenthesis; thus $\frac{1}{p^{2n+1}}\Delta_Z\Big(K_{\omega,p,x_0}\cdots$ therein should be read as $\frac{1}{p}\Delta_Z\Big(\frac{1}{p^{2n}}K_{\omega,p,x_0}\cdots)$

For a function $f\in C^\infty(X)$, we denote it as $f_{x_0}(Z)$ a family (with parameter x_0) of function of Z in the normal coordinate near x_0.

Observe that in the normal coordinate, we denote by $g_{ij}(Z)=\langle e_i,e_j\rangle_Z$ with $e_i=\frac{\partial}{\partial Z_i}$, and let $(g^{ij}(Z))$ be the inverse of the matrix $(g_{ij}(Z))$. If Γ_{ij}^l is the connection form of ∇^{TX} with respect to the basis $\{e_i\}$, then we have $(\nabla^{TX}_{e_i}e_j)(Z)=\Gamma^l_{ij}(Z)e_l$. Set $\Delta_0=-\sum_{j=1}^{2n}\frac{\partial^2}{\partial Z_j^2}$; then

$$(17)\qquad \Delta_Z=\Delta_0-\sum_{i,j=1}^{2n}\Big((g^{ij}(Z)-1)\tfrac{\partial^2}{\partial Z_i\partial Z_j}-g^{ij}(Z)\Gamma^k_{ij}(Z)\tfrac{\partial}{\partial Z_k}\Big).$$

As $g^{ij}(Z)=1+O(|Z|^2),\Gamma^k_{ij}(Z)=O(|Z|)$ (cf. [**20**, (1.2.19), (4.1.102)]), by recurrence, we know

(18)

$$\Delta_Z^q e^{-\pi p|Z-Z'|^2}=\Delta_0^q e^{-\pi p|Z-Z'|^2}+\sum_{i=0}^{2}h_i(Z,\sqrt{p},\sqrt{p}Z,\sqrt{p}Z')e^{-\pi p|Z-Z'|^2}.$$

Here $h_i(Z,a,x,y)$ are polynomials on a,x,y, and the degree on a is $\leqslant 2q-2+i$; moreover, the coefficients of $h_i(Z,a,x,y)$ as a function on Z is C^∞ and $O(|Z|^i)$. Thus,

(19)

$$p^{-q}\Delta_Z^q e^{-\pi p|Z-Z'|^2}=p^{-q}(\Delta_0^q e^{-\pi p|Z-Z'|^2})|_{Z=0}+p^{-q}h(\sqrt{p},\sqrt{p}Z')e^{-\pi p|Z'|^2},$$

and $h(a,Z')$ is a polynomial on a and Z', and its degree on a is $\leqslant 2q-2$.

From (16), (19), and [**17**, (27)],

(20)
$$\left|p^{-n-q}\Delta^q K_{\omega,p}f - p^{n-q}\int_{|Z'|\leqslant\varepsilon}(\Delta_0^q e^{-\pi p|Z-Z'|^2})|_{Z=0}f_{x_0}(Z')dv_X(Z')\right|_{L^2} \leqslant \frac{C}{p}\,|f|_{L^2}\,.$$

Let $e^{-u\Delta}(x,x')$ be the smooth kernel of the heat operator $e^{-u\Delta}$ with respect to $dv_X(x')$. By [**17**, (35)], there exist $\phi_{i,x_0}(Z')$ such that uniformly for $x_0\in X$, $Z'\in T_{x_0}X, |Z'|\leqslant\varepsilon$, we have the following asymptotic expansion when $u\to 0$:

(21)
$$\left|\frac{\partial^l}{\partial u^l}\Big(e^{-u\Delta}(0,Z') - (4\pi u)^{-n}\Big(1+\sum_{i=1}^k u^i\phi_{i,x_0}(Z')\Big)e^{-\frac{1}{4u}|Z'|^2}\Big)\right|_{C^0(X)} = O(u^{k-n-l+1}).$$

Observe that

$$\begin{aligned}\Delta^q\exp\Big(-\frac{\Delta}{4\pi p}\Big) &= (-1)^q(\tfrac{\partial}{\partial u}e^{-u\Delta})\Big|_{u=\frac{1}{4\pi p}},\\ p^n(\Delta_0^q e^{-\pi p|Z-Z'|^2})|_{Z=0} &= \Big(\Delta_0^q\exp\Big(-\frac{\Delta_0}{4\pi p}\Big)\Big)(0,Z')\\ &= (-1)^q(\tfrac{\partial^q}{\partial u^q}e^{-u\Delta_0})\Big|_{u=\frac{1}{4\pi p}}(0,Z'). \qquad (22)\end{aligned}$$

By (20), (21), (22), and [**17**, (27)], we have

$$\left|p^{-q}\Big(p^{-n}\Delta^q K_{\omega,p} - \Delta^q\exp\Big(-\frac{\Delta}{4\pi p}\Big)\Big)f\right|_{L^2} \leqslant \frac{C}{p}\,|f|_{L^2}\,. \qquad (23)$$

Thus, we have proved (12) when $\eta=1$.

If $\eta\neq 1$, set

$$\begin{aligned}K_{\eta,\omega,p,q}(x,y) &= \langle d\eta(x), d_x\Delta_x^{q-1}K_{\omega,p}(x,y)\rangle_{g^{T^*X}},\\ (K_{\eta,\omega,p,q}f)(x) &= \int_X K_{\eta,\omega,p,q}(x,y)f(y)dv_X(y).\end{aligned} \qquad (24)$$

Then from [**17**, (19),(20),(27)], (18), and (24), we get

(25)
$$\Big|p^{-n-q}K_{\eta,\omega,p,q}f - p^{n-q}\int_{|Z'|\leqslant\varepsilon}\sum_{i=1}^{2n}(\tfrac{\partial}{\partial Z_i}\eta)(x_0,0)(\tfrac{\partial}{\partial Z_i}\Delta_0^{q-1}e^{-\pi p|Z-Z'|^2})|_{Z=0}f_{x_0}(Z')dv_X(Z')\Big|_{L^2} \leqslant \frac{C}{p}\,|f|_{L^2}\,,$$

where C is independent on p.

By [**17**, (33)], we get the analogue of [**17**, (36)]

$$\begin{aligned}(26)\quad &\Big|\frac{\partial^l}{\partial u^l}\Big[\langle d\eta(x_0), d_{x_0}e^{-u\Delta}\rangle_{g^{T^*X}}(0,Z')\\ &\quad -(4\pi u)^{-n}\sum_{i=1}^{2n}(\frac{\partial}{\partial Z_i}\eta)(x_0,0)\frac{Z'_i}{2u}\Big(1+\sum_{i=1}^{k}u^i\phi_{i,x_0}(Z')\Big)\Big)e^{-\frac{1}{4u}|Z'|^2}\\ &-(4\pi u)^{-n}\sum_{i=1}^{k}u^i\langle d\eta(x_0),(d_{x_0}\Phi_i)(0,Z')\rangle e^{-\frac{1}{4u}|Z'|^2}\Big]\Big|_{C^0(X)}=O(u^{k-n-l+\frac{1}{2}}).\end{aligned}$$

From (25), (26), and [**17**, (27)],

$$(27)\qquad \Big|p^{-q}\Big(p^{-n}K_{\eta,\omega,p,q}-\langle d\eta, d\Delta^{q-1}\exp(-\frac{\Delta}{4\pi p})\rangle\Big)f\Big|_{L^2}\leqslant\frac{C}{p}\,|f|_{L^2}\,.$$

Finally,

$$\begin{aligned}(28)\quad (\Delta^q(\eta K_{\omega,p}))(x,y)&=\eta(x)\Delta_x^qK_{\omega,p}(x,y)\\ &\quad -2\langle d\eta(x), d_x\Delta_x^{q-1}K_{\omega,p}(x,y)\rangle_{g^{T^*X}}+\widetilde{K}_{\omega,p},\end{aligned}$$

where $\widetilde{K}_{\omega,p}$ has $\leqslant 2q-2$ derivative on $K_{\omega,p}(x,y)$; thus

$$(29)\qquad \Big|\widetilde{K}_{\omega,p}f\Big|_{L^2}\leqslant C\left(|K_{\omega,p}f|_{L^2}+\left|\Delta_x^{q-1}K_{\omega,p}f\right|_{L^2}\right).$$

Note also $R_p=\frac{\mathrm{Vol}(X,dv_X)}{\mathrm{Vol}(X,\nu)}p^n+O(p^{n-1})$. From (15), (23), and (27)–(29), we get (12).

To get the last part of Theorem 25, as we noticed in [**9**, §4.5], the constants in [**17**, (19)] will be uniformly bounded under our condition, and thus we can take C in (12), (27), and (29) independent of h^L, $d\nu$. q.e.d.

We have also C^m estimates.

Theorem 26. *For $m\in\mathbb{N}$, there exists a constant $C>0$ such that for any $f\in C^\infty(X)$, $p\in\mathbb{N}$,*

$$(30)\qquad \Big|Q_{K_p}f-\frac{\mathrm{Vol}(X,\nu)}{\mathrm{Vol}(X,dv_X)}\eta f\Big|_{C^m(X)}\leqslant\frac{C}{p}\,|f|_{C^m(X)}\,.$$

Again the constant C here is uniform bounded in the sense after (12).

Proof. Now we replace (16) by the following equation, which is again from [**17**, (19)]:

$$\begin{aligned}(31)\quad &\Big|\frac{1}{p^{2n}}K_{\omega,p,x_0}(Z,Z')\\ &\qquad -\Big(1+\sum_{r=2}^{k}p^{-r/2}J'_r(\sqrt{p}Z,\sqrt{p}Z')\Big)e^{-\pi p|Z-Z'|^2}\Big)\Big|_{C^m(X)}\\ &\leqslant Cp^{-(k+1)/2}(1+|\sqrt{p}Z|+|\sqrt{p}Z'|)^N\exp(-C_0\sqrt{p}|Z-Z'|)+O(p^{-\infty}).\end{aligned}$$

Here $C^m(X)$ is the C^m norm for the parameter $x_0 \in X$. Thus,

$$\begin{aligned}(32)\quad &\big|p^{-n}K_{\omega,p}f\\ &-p^n\int_{|Z'|\leqslant\varepsilon}\Big(1+\sum_{r=2}^{k}p^{-r/2}J'_r(0,\sqrt{p}Z')\Big)e^{-\pi p|Z'|^2}f_{x_0}(Z')dv_X(Z')\Big|_{C^m(X)}\\ &\qquad\qquad\leqslant Cp^{-(k+1)/2}\,|f|_{C^m(X)}\,.\end{aligned}$$

But as in the proof of [**2**, theorem 2.29. (2)], we get

(33)
$$\begin{aligned}&\left|p^n\int_{|Z'|\leqslant\varepsilon}J'_r(0,\sqrt{p}Z')e^{-\pi p|Z'|^2}f_{x_0}(Z')dv_X(Z')\right|_{C^m(X)}\leqslant C\,|f|_{C^m(X)}\,,\\ &\left|p^n\int_{|Z'|\leqslant\varepsilon}e^{-\pi p|Z'|^2}f_{x_0}(Z')dv_X(Z')-f(x_0)\right|_{C^m(X)}\leqslant Cp^{-1}\,|f|_{C^m(X)}\,.\end{aligned}$$

From (32), (33), we get

$$(34)\qquad \big|p^{-n}K_{\omega,p}f-f\big|_{C^m(X)}\leqslant C\,|f|_{C^m(X)}\,.$$

Now by (15),

$$(35)\qquad (Q_{K_p}f)(x)=\frac{1}{R_p}\eta(x)(K_{\omega,p}f)(x).$$

From (34), (35), we get (30).

As the constant C in [**17**, (19)] is uniformly bounded under our condition, thus the constant C in (31) (and so (30)) is uniformly bounded. q.e.d.

References

[1] H. Amann, *Linear and quasilinear parabolic problems, Vol. I*, volume 89 of *Monographs in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1995. Abstract linear theory.

[2] N. Berline, E. Getzler & M. Vergne, *Heat kernels and Dirac operators*. Grundlehren Text Editions, Springer-Verlag, Berlin, 2004 corrected reprint of the 1992 original.

[3] J.-P. Bourguignon, P. Li & S.-T. Yau, *Upper bound for the first eigenvalue of algebraic submanifolds*, Comment. Math. Helv., **69**(2):199–207, 1994.

[4] E. Calabi, *Extremal Kähler metrics*, in *Seminar on Differential Geometry*, volume 102 of Ann. of Math. Stud., pages 259–290, Princeton Univ. Press, Princeton, N.J., 1982.

[5] E. Calabi & X.X. Chen, *The space of Kähler metrics, 2*, J. Differential Geom., **61**(2):173–193, 2002.

[6] D. Catlin, *The Bergman kernel and a theorem of Tian*, in *Analysis and geometry in several complex variables (Katata, 1997), Trends Math.*, pages 1–23, Birkhäuser Boston, Boston, MA, 1999.

[7] X.X. Chen & W.Y. He, *The Calabi flow on toric Fano surface*, arXiv:0807.3984v2.

[8] X.X. Chen & W.Y. He, *On the Calabi flow*, Amer. J. Math., **130**(2):539–570, 2008.

[9] X. Dai, K. Liu & X. Ma, *On the asymptotic expansion of Bergman kernel*, J. Differential Geom., **72**(1):1–41, 2006.

[10] S.K. Donaldson, *Remarks on gauge theory, complex geometry and* 4*-manifold topology*, in *Fields Medallists' lectures*, volume 5 of World Sci. Ser. 20th Century Math., pages 384–403, World Sci. Publ., River Edge, NJ, 1997.

[11] S.K. Donaldson, *Symmetric spaces, Kähler geometry and Hamiltonian dynamics*, in *Northern California Symplectic Geometry Seminar*, volume 196 of Amer. Math. Soc. Transl. Ser. 2, pages 13–33, Amer. Math. Soc., Providence, RI, 1999.

[12] S.K. Donaldson, *Scalar curvature and projective embeddings*, I. J. Differential Geom., **59**(3):479–522, 2001.

[13] S.K. Donaldson, *Scalar curvature and stability of toric varieties*, J. Differential Geom., **62**(2):289–349, 2002.

[14] S.K. Donaldson, *Scalar curvature and projective embeddings*, II. Q.J. Math., **56**(3):345–356, 2005.

[15] S.K. Donaldson, *Some numerical results in complex differential geometry*, Pure Appl. Math. Q., **5**(2):571–618, 2009.

[16] A. Fujiki, *Moduli space of polarized algebraic manifolds and Kähler metrics* [translation of Sûgaku **42** (1990), no. 3, 231–243; MR 1073369 (92b:32032)], Sugaku Expositions, **5**(2):173–191, 1992.

[17] K. Liu & X. Ma, *A remark on: "Some numerical results in complex differential geometry" by S.K. Donaldson*, Math. Res. Lett., **14**(2):165–171, 2007.

[18] Z. Lu, *On the lower order terms of the asymptotic expansion of Tian-Yau-Zelditch* Amer. J. Math., **122**(2):235–273, 2000.

[19] H. Luo, *Geometric criterion for Gieseker-Mumford stability of polarized manifolds*, J. Differential Geom., **49**(3):577–599, 1998.

[20] X. Ma & G. Marinescu, *Holomorphic Morse inequalities and Bergman kernels*, volume 254 of Progress in Mathematics, Birkhäuser Verlag, Basel, 2007.

[21] T. Mabuchi, *Some symplectic geometry on compact Kähler manifolds. I*, Osaka J. Math., **24**(2):227–252, 1987.

[22] T. Mabuchi, *Stability of extremal Kähler manifolds*, Osaka J. Math., **41**(3):563–582, 2004.

[23] T. Mabuchi, *An energy-theoretic approach to the Hitchin-Kobayashi correspondence for manifolds. I*, Invent. Math., **159**(2):225–243, 2005.

[24] S.T. Paul, *Geometric analysis of Chow Mumford stability*, Adv. Math., **182**(2):333–356, 2004.

[25] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, volume 44 of Applied Mathematical Sciences, Springer-Verlag, New York, 1983.

[26] D.H. Phong & J. Sturm, *Stability, energy functionals & Kähler-Einstein metrics*, Comm. Anal. Geom., **11**(3):565–597, 2003.

[27] D.H. Phong & J. Sturm, *Scalar curvature, moment maps & the Deligne pairing*, Amer. J. Math., **126**(3):693–712, 2004.

[28] W.-D. Ruan, *Canonical coordinates and Bergmann metrics*, Comm. Anal. Geom., **6**(3):589–631, 1998.

[29] S. Semmes, *Complex Monge-Ampère and symplectic manifolds*, Amer. J. Math., **114**(3):495–550, 1992.

[30] G. Tian, *On a set of polarized Kähler metrics on algebraic manifolds*, J. Differential Geom., **32**(1):99–130, 1990.

[31] G. Tian, *On Calabi's conjecture for complex surfaces with positive first Chern class*, Invent. Math., **101**(1):101–172, 1990.

[32] G. Tian, *Kähler-Einstein metrics with positive scalar curvature*, Invent. Math., **130**(1):1–37, 1997.

[33] V. Tosatti & B. Weinkove, *The Calabi flow with small initial energy*, Math. Res. Lett., **14**(6):1033–1039, 2007.

[34] X. Wang, *Moment map, Futaki invariant and stability of projective manifolds*, Comm. Anal. Geom., **12**(5):1009–1037, 2004.

[35] S.-T. Yau, *Nonlinear analysis in geometry*, volume 33 of *Monographies de L'Enseignement Mathématique [Monographs of L'Enseignement Mathématique]*, L'Enseignement Mathématique, Geneva, 1986. Série des Conférences de l'Union Mathématique Internationale [Lecture Series of the International Mathematics Union], 8.

[36] S.-T. Yau, *Open problems in geometry*, in *Differential geometry: partial differential equations on manifolds* (Los Angeles, CA, 1990), volume 54 of Proc. Sympos. Pure Math., pages 1–28, Amer. Math. Soc., Providence, RI, 1993.

[37] S. Zelditch, *Szego kernels and a theorem of Tian*, Internat. Math. Res. Notices, (6):317–331, 1998.

[38] S. Zhang, *Heights and reductions of semi-stable varieties*, Compositio Math., **104**(1):77–105, 1996.

Département de Mathématique
Université Libre de Bruxelles CP218
Boulevard du Triomphe
Bruxelles 1050, Belgique

E-mail address: joel.fine@ulb.ac.be

Center of Mathematical Science
Zhejiang University China and
Department of Mathematics UCLA
Los Angeles, CA 90095

E-mail address: liu@math.ucla.edu

Université Denis Diderot—Paris 7
UFR de Mathématiques
Case 7012, Site Chevaleret
75205 Paris Cedex 13
France
E-mail address: ma@math.jussieu.fr