A Proof on the Non-differentiability of Weierstrass Function in An Uncountable Dense Set

Boyu Chen

Adviser Zixiang Zhou

Shanghai Shibei High School

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Abstract

The Weierstrass function is the first function in history which is continuous everywhere, but differentiable nowhere. In this paper, we prove that the Weierstrass function with a condition weaker than Weierstrass's original one cannot be differentiable in an uncountable dense set in the framework of mathematical analysis.

Preface

Weierstrass function $f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$ is one of the most famous "pathological function" in mathematical history. Weierstrass proved that if $0 < a < 1$, b is an odd integer and $ab > 1 + \frac{3\pi}{2}$, then $f(x)$ is non-differentiable everywhere for $x \in \mathbb{R}$ [1]. This broke people's original guess that the continuous functions are always nearly differentiable, and makes the gradual development of real analysis from mathematics analysis.

In 1916, G.H.Hardy proved in the framework of real analysis that Weierstrass function $f(x)$ is non-differentiable everywhere for $x \in \mathbb{R}$ when $0 < a < 1$ and $ab \geqslant 1$[3]. But from the fundamental mathematics analysis, the best conclusion as far I know is that $f(x)$ is non-differentiable everywhere for $x \in \mathbb{R}$ when $0 < a < 1$, b is an odd integer and $ab > 1 + \frac{(1-a)\pi}{2}$[2].

In this paper, we will use a different method to prove that the function $f(x)$ is non-differentiable at least in an uncountable dense set when $0 < a < 1$, b is an integer greater than 5 and $ab > 1$. (The condition is a bit weaker than Weierstrass's original one).

The main theorem is as follow.

Theorem. Assume that the real number a and the natural number b satisfy the condition that $0 < a < 1, b \geqslant 6$ and $ab > 1$, then the function $f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$ is non-differentiable in an uncountable dense set.

Proof

First of all, let us prove three lemmas.

Lemma 1. Suppose $\{a_n\}$, $\{b_n\}$ satisfy $a_n > b_n > x'$, $\frac{b_n - x'}{a_n - b_n} = k$ is a constant for any positive integer n, and $a_n \to x'$, $b_n \to x'$. Moreover, suppose $\frac{\mathrm{d}f}{\mathrm{d}x}(x') = A$ exists. Then

$$\lim_{n\to\infty} \frac{f(a_n) - f(b_n)}{a_n - b_n} = A\ . \tag{1}$$

Proof.

$$\lim_{x\to x'} \frac{f(x) - f(x')}{x - x'} = \frac{\mathrm{d}f}{\mathrm{d}x}(x') = A. \tag{2}$$

Let $x = a_n$ and $x = b_n$ respectively, we have

$$\lim_{n\to\infty} \frac{f(a_n) - f(x')}{a_n - x'} = A, \quad \lim_{n\to\infty} \frac{f(b_n) - f(x')}{b_n - x'} = A, \tag{3}$$

since $a_n \to x'$, $b_n \to x'$. From the definition of limit, for any positive number ε, there must be an N such that for any $n > N$,

$$A - \varepsilon < \frac{f(a_n) - f(x')}{a_n - x'} < A + \varepsilon, \quad A - \varepsilon < \frac{f(b_n) - f(x')}{b_n - x'} < A + \varepsilon. \tag{4}$$

Hence

$$\begin{aligned}\frac{f(a_n) - f(b_n)}{a_n - b_n} &= \frac{\dfrac{f(a_n) - f(x')}{a_n - x'}(a_n - x') - \dfrac{f(b_n) - f(x')}{b_n - x'}(b_n - x')}{(a_n - x') - (b_n - x')} \\ &< \frac{(A+\varepsilon)(a_n - x') - (A-\varepsilon)(b_n - x')}{(a_n - x') - (b_n - x')} \\ &= A + \varepsilon\frac{(a_n - x') + (b_n - x')}{(a_n - x') - (b_n - x')} = A + \varepsilon + 2k\varepsilon. \end{aligned} \tag{5}$$

Similarly, $\dfrac{f(a_n) - f(b_n)}{a_n - b_n} > A - \varepsilon - 2k\varepsilon$. From the definition of limit,

$$\lim_{n\to\infty} \frac{f(a_n) - f(b_n)}{a_n - b_n} = A, \tag{6}$$

Lemma 1 is proved.

Now expand any real number x' in base b, that is

$$x' = \alpha_0 + \frac{\alpha_1}{b} + \frac{\alpha_2}{b^2} + \frac{\alpha_3}{b^3} + \cdots + \frac{\alpha_n}{b^n} + \cdots, \tag{7}$$

where $\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_n, \cdots$ are all integers such that $0 \leqslant \alpha_i \leqslant b-1$ for $i \geqslant 1$ and α_i is not always equal to $b-1$ when i is large enough. Similarly, we can also expand $\dfrac{1}{3}$ in base b, which is

$$\frac{1}{3} = \frac{\beta_1}{b} + \frac{\beta_2}{b^2} + \frac{\beta_3}{b^3} + \cdots + \frac{\beta_n}{b^n} + \cdots \tag{8}$$

with $0 \leqslant \beta_i \leqslant b-1$ for $i \geqslant 1$. Since $b \geqslant 6$, we always have $\beta_1 \geqslant 2$ and $\beta_1 \leqslant \dfrac{b}{3}$. We call a real number x' a decidable number if there are infinitely many

$$\alpha_{n_1}, \alpha_{n_2}, \alpha_{n_3}, \cdots, \alpha_{n_i}, \cdots \tag{9}$$

in $\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_n, \cdots$ satisfying $\alpha_{n_i} < \beta_1 - 2$ or $\alpha_{n_i} > b - \beta_1 - 3$.

Lemma 2. For any decidable number x', there are infinitely many positive integers k such that

$$\left|\cos\left(b^{k-1}\pi x' + \frac{2\pi}{b}\right)\right| > 0.5 \,. \tag{10}$$

Proof. We only need to prove that for any decidable number x', there are infinitely many k such that

$$b^{k-1}x' \in \big(m - \frac{1}{3} - \frac{2}{b}, m + \frac{1}{3} - \frac{2}{b}\big), \tag{11}$$

where m is an integer decided by k. Using (8), we have

$$m + \frac{1}{3} - \frac{2}{b} = m + \frac{\beta_1 - 2}{b} + \frac{\beta_2}{b^2} + \frac{\beta_3}{b^3} + \cdots + \frac{\beta_n}{b^n} + \cdots, \tag{12}$$

and

$$m - \frac{1}{3} - \frac{2}{b} = m - 1 + \frac{b - \beta_1 - 3}{b} + \frac{b - 1 - \beta_2}{b^2} + \frac{b - 1 - \beta_3}{b^3} + \cdots + \frac{b - 1 - \beta_n}{b^n} + \cdots. \tag{13}$$

Since x' is a decidable number, we can find infinitely many numbers

$$\alpha_{n_1}, \alpha_{n_2}, \alpha_{n_3}, \cdots, \alpha_{n_i}, \cdots \tag{14}$$

which satisfy $\alpha_{n_i} < \beta_1 - 2$ or $\alpha_{n_i} > b - \beta_1 - 3$. Taking $k = n_i$, we have

$$b^{k-1}x' = (b^{n_i-1}\alpha_0 + b^{n_i-2}\alpha_1 + b^{n_i-3}\alpha_2 + \cdots + \alpha_{n_i-1}) + \big(\frac{\alpha_{n_i}}{b} + \frac{\alpha_{n_i+1}}{b^2} + \cdots\big). \tag{15}$$

If $\alpha_{n_i} < \beta_1 - 2$, then let $m = b^{n_i-1}\alpha_0 + b^{n_i-2}\alpha_1 + b^{n_i-3}\alpha_2 + \cdots + \alpha_{n_i-1}$ and we have

$$m - \frac{1}{3} - \frac{2}{b} < m \leqslant b^{k-1}x' < m + \frac{\beta_1 - 2}{b} \leqslant m + \frac{1}{3} - \frac{2}{b}. \tag{16}$$

Otherwise $\alpha_{n_i} > b - \beta_1 - 3$, then let $m - 1 = b^{n_i-1}\alpha_0 + b^{n_i-2}\alpha_1 + b^{n_i-3}\alpha_2 + \cdots + \alpha_{n_i-1}$ and we have

$$m - \frac{1}{3} - \frac{2}{b} < b^{k-1}x' < m \leqslant m + \frac{1}{3} - \frac{2}{b}. \tag{17}$$

Lemma2 is true because there are infinitely many such n_i.

Lemma 3. For a given integer $b \geq 6$, the set of its decidable numbers is dense in $\mathbb{R}$ and has the same cardinality as $\mathbb{R}$.

Proof. The set of decidable numbers belongs to $\mathbb{R}$. Let $C = \{\alpha \,|\, 0 \leqslant \alpha < \beta_1 - 2 \text{ or } b - \beta_1 - 3 < \alpha \leqslant b - 1\}$. Let c be the number of elements in C. Let φ be a one to one correspondence between C and $\{0, 1, 2, \cdots, c-1\}$which maps $b-1$ to $c-1$. Let S be the set of all x' whose $\alpha_1, \alpha_2, \cdots$ satisfy that either $\alpha_k < \beta_1 - 2$ or $\alpha_k > b - \beta_1 - 3$ for any $k \geqslant 1$. Obviously any element of S is a decidable number. For $x' \in S$, let

$$\psi(x') = \alpha_0 + \frac{\varphi(\alpha_1)}{c} + \frac{\varphi(\alpha_2)}{c^2} + \frac{\varphi(\alpha_3)}{c^3} + \cdots + \frac{\varphi(\alpha_n)}{c^n} + \cdots. \tag{18}$$

It is a one to one correspondence between S and $\mathbb{R}$. Hence S and $\mathbb{R}$ have the same cardinality.

Meanwhile, for any real number $x = \sum_{j=0}^{\infty} \frac{a_j}{b^j}$ and any positive number ε, let n be large enough such that $b^{-n} < \varepsilon$. Assume the integer c satisfies $c < \beta_1 - 2$ or $c > b - \beta_1 - 3$, then

$$x' = \sum_{j=0}^{n} \frac{a_j}{b^j} + \sum_{j=n+1}^{\infty} \frac{c}{b^j} \tag{19}$$

is a decidable number, and

$$|x' - x| \leqslant \sum_{j=n+1}^{\infty} \frac{|a_j - c|}{b^j} \leqslant \sum_{j=n+1}^{\infty} \frac{b-1}{b^j} \leqslant \varepsilon. \tag{20}$$

So the set of decidable number is dense in $\mathbb{R}$. Lemma 3 is proved.

Proof of the theorem. Let x' be a decidable number. Assume that $f(x)$ had a finite derivative at x', then

$$\lim_{x \to x'} \frac{f(x) - f(x')}{x - x'} = A. \tag{21}$$

Let $a_k = x' + \frac{2}{b^k}$. Obviously $a_k \to x'$, so $\lim_{n\to\infty} \frac{f(a_n) - f(x')}{a_n - x'} = A$. On the other hand,

$$\begin{aligned}
\frac{f(a_k) - f(x')}{a_k - x'} &= \frac{b^k}{2} \sum_{n=0}^{\infty} a^n \Big[\cos\Big(b^n \pi \big(x' + \frac{2}{b^k}\big)\Big) - \cos(b^n \pi x') \Big] \\
&= \frac{b^k}{2} \sum_{n=0}^{k-1} a^n \Big[\cos\Big(b^n \pi \big(x' + \frac{2}{b^k}\big)\Big) - \cos(b^n \pi x') \Big] \\
&+ \frac{b^k}{2} \sum_{n=k}^{\infty} a^n \Big[\cos\Big(b^n \pi \big(x' + \frac{2}{b^k}\big)\Big) - \cos(b^n \pi x') \Big].
\end{aligned} \tag{22}$$

Since $\cos\Big(b^n \pi \big(x' + \frac{2}{b^k}\big)\Big) - \cos(b^n \pi x') = 0$ when $n \geqslant k$, we have

$$\begin{aligned}
\frac{f(a_k) - f(x')}{a_k - x'} &= \frac{b^k}{2} \sum_{n=0}^{k-1} a^n \Big[\cos\Big(b^n \pi \big(x' + \frac{2}{b^k}\big)\Big) - \cos(b^n \pi x') \Big] \\
&= -\sum_{n=0}^{k-1} a^n b^k \sin(b^n \pi x' + b^{n-k}\pi) \sin(b^{n-k}\pi).
\end{aligned} \tag{23}$$

Hence

$$\lim_{k\to\infty} \sum_{n=0}^{k-1} a^n b^k \sin(b^n \pi x' + b^{n-k}\pi) \sin(b^{n-k}\pi) = -A. \tag{24}$$

Similarly, by setting $a_k = x' + \frac{4}{b^k}, b_k = x' + \frac{2}{b^k}$, Lemma 1 leads to

$$\lim_{k\to\infty} \sum_{n=0}^{k-1} a^n b^k \sin(b^n \pi x' + 3b^{n-k}\pi) \sin(b^{n-k}\pi) = -A. \tag{25}$$

Therefore,

$$\begin{aligned}
& \lim_{k\to\infty} \sum_{n=0}^{k-1} a^n b^k \cos(b^n \pi x' + 2b^{n-k}\pi) \sin^2(b^{n-k}\pi) \\
= \; & \frac{1}{2} \lim_{k\to\infty} \sum_{n=0}^{k-1} a^n b^k \sin(b^n \pi x' + 3b^{n-k}\pi) \sin(b^{n-k}\pi) \\
- \; & \frac{1}{2} \lim_{k\to\infty} \sum_{n=0}^{k-1} a^n b^k \sin(b^n \pi x' + b^{n-k}\pi) \sin(b^{n-k}\pi) \\
= \; & \frac{1}{2}(-A + A) = 0.
\end{aligned} \tag{26}$$

For each decidable number x', Lemma 2 implies that there are either infinitely many k such that $\cos\left(b^{k-1}\pi x' + \frac{2\pi}{b}\right) > 0.5$, or infinitely many k such that $\cos\left(b^{k-1}\pi x' + \frac{2\pi}{b}\right) < -0.5$.

Suppose there are infinitely many k such that $\cos\left(b^{k-1}\pi x' + \frac{2\pi}{b}\right) > 0.5$, then for any $k \in \mathbb{N}^+$,

$$\begin{aligned}
& \sum_{n=0}^{k-1} a^n b^k \cos(b^n \pi x' + 2b^{n-k}\pi) \sin^2(b^{n-k}\pi) \\
\geqslant \; & a^{k-1} b^k \cos\left(b^{k-1}\pi x' + \frac{2\pi}{b}\right) \sin^2\left(\frac{\pi}{b}\right) - \sum_{n=0}^{k-2} a^n b^k \sin^2(b^{n-k}\pi) \\
> \; & a^{k-1} b^k \cos\left(b^{k-1}\pi x' + \frac{2\pi}{b}\right) \sin^2\left(\frac{\pi}{b}\right) - \sum_{n=0}^{k-2} a^n b^{2n-k} \pi^2 \\
= \; & b(ab)^{k-1} \sin^2\left(\frac{\pi}{b}\right) \cos\left(b^{k-1}\pi x' + \frac{2\pi}{b}\right) - \frac{[(ab^2)^{k-1} - 1]\pi^2}{(ab^2 - 1)b^k} \\
> \; & (ab)^{k-1} \left[b \sin^2\left(\frac{\pi}{b}\right) \cos\left(b^{k-1}\pi x' + \frac{2\pi}{b}\right) - \frac{\pi^2}{b(ab^2 - 1)} \right].
\end{aligned} \tag{27}$$

Noticing that $\left(\frac{\pi}{b \sin\left(\frac{\pi}{b}\right)}\right)^2 < \left(\frac{\pi}{2}\right)^2 < \frac{b-1}{2} < \frac{ab^2 - 1}{2}$, we have

$$\frac{1}{2} b \sin^2\left(\frac{\pi}{b}\right) > \frac{\pi^2}{b(ab^2 - 1)}. \tag{28}$$

Hence there are infinitely many k such that

$$b\sin^2\left(\frac{\pi}{b}\right)\cos\left(b^{k-1}\pi x' + \frac{2\pi}{b}\right) - \frac{\pi^2}{b(ab^2-1)} > \lambda, \tag{29}$$

where $\lambda = \frac{1}{2}b\sin^2\left(\frac{\pi}{b}\right) - \frac{\pi^2}{b(ab^2-1)} > 0$ is a constant. However, $\lambda(ab)^{k-1}$ diverges as $k \to \infty$, which contradicts (26).

Similarly, if there are infinitely many k such that $\cos\left(b^{k-1}\pi x' + \frac{2\pi}{b}\right) < -0.5$, then for each $k \in \mathbb{N}^+$,

$$\begin{aligned} & \sum_{n=0}^{k-1} a^n b^k \cos(b^n \pi x' + 2b^{n-k}\pi)\sin^2(b^{n-k}\pi) \\ < \quad & (ab)^{k-1}\left[b\sin^2\left(\frac{\pi}{b}\right)\cos\left(b^{k-1}\pi x' + \frac{2\pi}{b}\right) + \frac{\pi^2}{b(ab^2-1)}\right]. \end{aligned} \tag{30}$$

Similar to (32), there are infinitely many k such that

$$b\sin^2\left(\frac{\pi}{b}\right)\cos\left(b^{k-1}\pi x' + \frac{2\pi}{b}\right) + \frac{\pi^2}{b(ab^2-1)} < -\lambda. \tag{31}$$

This also contradicts (26) since $-\lambda(ab)^{k-1}$ diverges as $k \to \infty$.

Therefore, the function $f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$ cannot be differentiable at any decidable number. Lemma 3 implies that $f(x)$ is non-differentiable in an uncountable dense set.

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