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PROOF OF THE LABASTIDA-MARIÑO-OOGURI-VAFA CONJECTURE

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Abstract

Based on large N Chern-Simons/topological string duality, in a series of papers [**36**, **21**, **19**], Labastida, Mariño, Ooguri, and Vafa conjectured certain remarkable new algebraic structure of link invariants and the existence of infinite series of new integer invariants. In this paper, we provide a proof of this conjecture. Moreover, we also show these new integer invariants vanish at large genera.

1. Introduction

1.1. Overview. For decades, we have witnessed the great development of string theory and its powerful impact on the development of mathematics. There have been a lot of marvelous results revealed by string theory, which deeply relate different aspects of mathematics. All these mysterious relations are connected by a core idea in string theory called "duality." It was found that string theory on Calabi-Yau manifolds provided new insight in geometry of these spaces. The existence of a topological sector of string theory leads to a simplified model in string theory, the topological string theory.

A major problem in topological string theory is how to compute Gromov-Witten invariants. There are two major methods widely used: mirror symmetry in physics and localization in mathematics. Both methods are effective when genus is low but have trouble in dealing with higher genera due to the rapidly growing complexity during computation. However, when the target manifold is Calabi-Yau threefold, large N Chern-Simons/topological string duality opens a new gate to a complete solution of computing Gromov-Witten invariants at all genera.

The study of large N Chern-Simons/topological string duality was originated in physics by an idea that gauge theory should have a string theory explanation. In 1992, Witten [**46**] related topological string theory of T^*M of a three-dimensional manifold M to Chern-Simons

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gauge theory on M. In 1998, Gopakumar and Vafa [**11**] conjectured that, at large N, the open topological A-model of N D-branes on T^*S^3 is dual to closed topological string theory on resolved conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$. Later, Ooguri and Vafa [**36**] presented a picture of how to describe Chern-Simons invariants of a knot by open topological string theory on resolved conifold paired with Lagrangian associated with the knot.

Though large N Chern-Simons/topological string duality still remains open, there has been a lot of progress in this direction demonstrating the power of this idea. Even for the simplest knot, the unknot, the Mariño-Vafa formula [**32**, **25**] gives a beautiful closed formula for Hodge integrals up to three Chern classes of Hodge bundle. Furthermore, using topological vertex theory [**1**, **26**, **27**], one is able to compute Gromov-Witten invariants of any toric Calabi-Yau threefold by reducing the computation to a gluing algorithm of topological vertex. This thus leads to a closed formula of topological string partition function, a generating function of Gromov-Witten invariants, in all genera for any toric Calabi-Yau threefolds.

On the other hand, after Jones's famous work on polynomial knot invariants, there had been a series of polynomial invariants discovered (for example, [**15**, **9**]), the generalization of which was provided by quantum group theory [**42**] in mathematics and by the Chern-Simons path integral with the gauge group $SU(N)$ [**45**] in physics.

Based on the large N Chern-Simons/topological string duality, Ooguri and Vafa [**36**] reformulated knot invariants in terms of new integral invariants capturing the spectrum of M2 branes ending on M5 branes embedded in the resolved conifold. Later, Labastida, Mariño, and Vafa [**21**, **19**] refined the analysis of [**36**] and conjectured the precise integrality structure for open Gromov-Witten invariants. This conjecture predicts a remarkable new algebraic structure for the generating series of general link invariants and the integrality of the infinite family of new topological invariants. In string theory, this is a striking example that two important physical theories, topological string theory and Chern-Simons theory, exactly agree up to all orders. In mathematics this conjecture has interesting applications in understanding the basic structure of link invariants and three manifold invariants, as well as the integrality structure of open Gromov-Witten invariants. Recently, X.S. Lin and H. Zheng [**29**] verified the LMOV conjecture in several lower degree cases for some torus links.

In this paper, we give a complete proof of the Labastida-Mariño-Ooguri-Vafa conjecture for any link (we will briefly call it the LMOV conjecture). First, let us describe the conjecture and the main ideas of the proof. The details can be found in Sections 5 and 6.

1.2. Labastida-Mariño-Ooguri-Vafa conjecture. Let $\mathcal{L}$ be a link with L components and $\mathcal{P}$ be the set of all partitions. The Chern-Simons partition function of $\mathcal{L}$ is given by

$$Z_{\mathrm{CS}}(\mathcal{L};\, q,t) = \sum_{\vec{A}\in\mathcal{P}^L} W_{\vec{A}}(\mathcal{L};\, q,t) \prod_{\alpha=1}^{L} s_{A^\alpha}(x^\alpha) \tag{1.1}$$

for any arbitrarily chosen sequence of variables

$$x^\alpha = (x_1^\alpha, x_2^\alpha, \dots,)\,.$$

In (1.1), $W_{\vec{A}}(\mathcal{L})$ is the quantum group invariants of $\mathcal{L}$ labeled by a sequence of partitions $\vec{A} = (A^1, \dots, A^L) \in \mathcal{P}^L$ which correspond to the irreducible representations of quantized universal enveloping algebra $U_q(\mathfrak{sl}(N,\mathbb{C}))$, and $s_{A^\alpha}(x^\alpha)$ is the Schur function.

Free energy is defined to be

$$F = \log Z_{\mathrm{CS}}\,.$$

Using the plethystic exponential, one can obtain

$$F = \sum_{d=1}^{\infty}\sum_{\vec{A}\neq 0} \frac{1}{d} f_{\vec{A}}(q^d, t^d) \prod_{\alpha=1}^{L} s_{A^\alpha}\big((x^\alpha)^d\big)\,, \tag{1.2}$$

where

$$(x^\alpha)^d = \big((x_1^\alpha)^d, (x_2^\alpha)^d, \dots\big)\,.$$

Based on the duality between Chern-Simons gauge theory and topological string theory, Labastida, Mariño, Ooguri, and Vafa conjectured that $f_{\vec{A}}$ have the following highly nontrivial structures.

For any A, $B \in \mathcal{P}$, define the following function:

$$M_{AB}(q) = \sum_{\mu} \frac{\chi_A(C_\mu)\chi_B(C_\mu)}{\mathfrak{z}_\mu} \prod_{j=1}^{\ell(\mu)} (q^{-\mu_j/2} - q^{\mu_j/2})\,. \tag{1.3}$$

Conjecture (LMOV). For any $\vec{A} \in \mathcal{P}^L$,

(i). There exists $P_{\vec{B}}(q,t)$ for $\forall \vec{B} \in \mathcal{P}^L$, such that:

$$f_{\vec{A}}(q,t) = \sum_{|B^\alpha|=|A^\alpha|} P_{\vec{B}}(q,t) \prod_{\alpha=1}^{L} M_{A^\alpha B^\alpha}(q). \tag{1.4}$$

Furthermore, $P_{\vec{B}}(q,t)$ has the following expansion:

$$P_{\vec{B}}(q,t) = \sum_{g=0}^{\infty}\sum_{Q\in\mathbb{Z}/2} N_{\vec{B};\,g,Q}(q^{-1/2} - q^{1/2})^{2g-2} t^Q\,. \tag{1.5}$$

(ii). $N_{\vec{B};\,g,Q}$ are integers.

For the meaning of notations, the definition of quantum group invariants of links, and more details, please refer to Section 3.

This conjecture contains two parts:

- The existence of the special algebraic structure (1.5).
- The integrality of the new invariants $N_{\vec{B};\,g,Q}$.

If one looks at the right-hand side of (1.5), one will find it very interesting that the pole of $f_{\vec{A}}$ in $(q^{-1/2} - q^{1/2})$ is actually at most of order 1 for any link and any labeling partitions. However, by the calculation of quantum group invariants of links, the pole order of $f_{\vec{A}}$ might be going to ∞ when the degrees of labeling partitions go higher and higher. This miracle cancellation implies a very special algebraic structure of quantum group invariants of links and thus the Chern-Simons partition function. In Section 3.4.2, we include an example in the simplest setting showing that this cancellation has shed new light on the quantum group invariants of links.

1.3. Main ideas of the proof. In our proof of the LMOV conjecture, there are three new techniques.

- When dealing with the existence of the conjectured algebraic structure, one will encounter the problem of how to control the pole order of $(q^{-1/2} - q^{1/2})$. We consider the *framed partition function* $Z(\mathcal{L};\, q, t, \tau)$ of Chern-Simons invariants of links which satisfies the following cut-and-join equation:

$$\frac{\partial Z(\mathcal{L};\, q,t,\tau)}{\partial \tau} = \frac{u}{2}\sum_{\alpha=1}^{L}\sum_{i,j\geq 1}\left(ijp^{\alpha}_{i+j}\frac{\partial^2}{\partial p^{\alpha}_i \partial p^{\alpha}_j} + (i+j)p^{\alpha}_i p^{\alpha}_j \frac{\partial}{\partial p^{\alpha}_{i+j}}\right) Z(\mathcal{L};\, q,t,\tau)\,.$$

  Here $p^{\alpha}_n = p_n(x^{\alpha})$ are regarded as independent variables.

  However, a deeper understanding of this conjecture relies on the following log cut-and-join equation:

$$\frac{\partial F(\mathcal{L};\, q,t,\tau)}{\partial \tau} = \frac{u}{2}\sum_{\alpha=1}^{L}\sum_{i,j\geq 1}\left(ijp^{\alpha}_{i+j}\frac{\partial^2 F}{\partial p^{\alpha}_i \partial p^{\alpha}_j} + (i+j)p^{\alpha}_i p^{\alpha}_j \frac{\partial F}{\partial p^{\alpha}_{i+j}} + ijp^{\alpha}_{i+j}\frac{\partial F}{\partial p^{\alpha}_i}\frac{\partial F}{\partial p^{\alpha}_j}\right).$$

  This observation is based on the duality of Chern-Simons theory and open Gromov-Witten theory. The log cut-and-join equation is a non-linear ODE system and the non-linear part reflects the essential recursion structure of Chern-Simons partition function. The miracle cancellation of lower-order terms of $q^{-1/2} - q^{1/2}$ occurring in free energy can be indicated in the formulation of generating series of open Gromov-Witten invariants on the geometric side.

A powerful tool to control the pole order of $(q^{-1/2} - q^{1/2})$ through the log cut-and-join equation is developed in this paper as what we called *cut-and-join analysis*. An important feature of the cut-and-join equation shows that the differential equation at partition (d) can only have terms obtained from joining (please refer to section 5 for detailed description) while at (1^d) non-linear terms vanish and there are no joining terms. This special feature combined with the degree analysis will squeeze out the desired degree of $(q^{-1/2} - q^{1/2})$.

- We found a rational function ring which characterizes the algebraic structure of the Chern-Simons partition function and hence the (open) topological string partition function by duality.

  A similar ring characterizing closed topological string partition function first appears in the second author's work on the Gopakumar-Vafa conjecture [**37**, **38**]. The original observation in the closed case comes from the structure of the R-matrix in quantum group theory and the gluing algorithm in the topological vertex theory.

  However, the integrality in the case of open topological string theory is more subtle than the integrality of Gopakumar-Vafa invariants in the closed case. This is due to the fact that the reformulation of Gromov-Witten invariants as Gopakumar-Vafa invariants in the closed case is weighted by the power of curve classes, while in the open case the generating function is weighted by the labeling irreducible representation of $U_q(\mathfrak{sl}(N, \mathbb{C}))$. This subtlety had already been explained in [**21**].

  To overcome this subtlety, one observation is that quantum group invariants look Schur-function-like. This had already been demonstrated in the topological vertex theory (also see [**30**]). We refine the ring in the closed case and get the new ring $\mathfrak{R}(y;\, q, t)$ (cf. Section 6.1). Correspondingly, we consider a new generating series of $N_{\vec{B};\, g,Q}$, $T_{\vec{d}}$, as defined in (6.1).
- To prove $T_{\vec{d}} \in \mathfrak{R}(y;\, q, t)$, we combine with the multi-cover contribution and p-adic argument therein. Once we can prove that $T_{\vec{d}}$ lies in $\mathfrak{R}$, due to the pole structure of the ring $\mathfrak{R}$, the vanishing of $N_{\vec{B};\, g,Q}$ has to occur at large genera, we actually proved

$$\sum_{g=0}^{\infty} \sum_{Q \in \mathbb{Z}/2} N_{\vec{B};\, g,Q} (q^{-1/2} - q^{1/2})^{2g} t^Q \in \mathbb{Z}[(q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2, t^{\pm \frac{1}{2}}]\,.$$

The paper is organized as follows. In Section 2, we introduce some basic notation about partition and generalize this concept to simplify our calculation in the following sections. Quantum group invariants of links and main results are introduced in Section 3. In Section 4, we review some knowledge of Hecke algebra used in this paper. In Sections 5 and 6, we give the proof of Theorems 1 and 2 which answer the LMOV conjecture. In the last section, we discuss some problems related to LMOV conjecture for our future research.

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Before he passed away, Professor Xiao-Song Lin had been very interested in the LMOV conjecture and had been working on it. We would like to dedicate this paper to his memory.

2. Preliminary

2.1. Partition and symmetric function. A *partition* λ is a finite sequence of positive integers $(\lambda_1, \lambda_2, \cdots)$ such that

$$\lambda_1 \geq \lambda_2 \geq \cdots .$$

The total number of parts in λ is called the length of λ and denoted by $\ell(\lambda)$. We use $m_i(\lambda)$ to denote the number of times that i occurs in λ. The degree of λ is defined to be

$$|\lambda| = \sum_i \lambda_i .$$

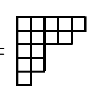
If $|\lambda| = d$, we say λ is a partition of d. We also use notation $\lambda \vdash d$. The automorphism group of λ, $\operatorname{Aut}\lambda$, contains all the permutations that permute parts of λ while still keeping it as a partition. Obviously, the order of $\operatorname{Aut}\lambda$ is given by

$$|\operatorname{Aut}\lambda| = \prod_i m_i(\lambda)! .$$

There is another way to rewrite a partition λ in the following format:

$$(1^{m_1(\lambda)} 2^{m_2(\lambda)} \cdots) .$$

A traditional way to visualize a partition is to identify a partition as a *Young diagram*. The Young diagram of λ is a two-dimensional graph with λ_j boxes on the j-th row, $j = 1, 2, \ldots, \ell(\lambda)$. All the boxes are put to fit the left-top corner of a rectangle. For example:

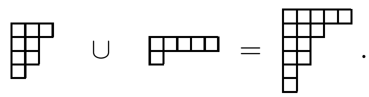
$$(5, 4, 2, 2, 1) = (12^2 45) =$$  .

For a given partition λ, denote by λ^t the conjugate partition of λ. The Young diagram of λ^t is transpose to the Young diagram of λ: the number of boxes on jth column of λ^t equals to the number of boxes on jth row of λ, where $1 \leq j \leq \ell(\lambda)$.

By convention, we regard a Young diagram with no box as the partition of 0 and use notation (0). Denote by $\mathcal{P}$ the set of all partitions. We can define an operation "$\cup$" on $\mathcal{P}$. Given two partitions λ and μ, $\lambda \cup \mu$ is the partition formed by putting all the parts of λ and μ together to form a new partition. For example,

$$(12^2 3) \cup (15) = (1^2 2^2 35).$$

Using a Young diagram, it looks like



The following number associated with a partition λ is used throughout this paper:

$$\mathfrak{z}_\lambda = \prod_j j^{m_j(\lambda)} m_j(\lambda)!\,, \qquad \kappa_\lambda = \sum_j \lambda_j(\lambda_j - 2j + 1)\,.$$

It's easy to see that

$$\kappa_\lambda = -\kappa_{\lambda^t}. \tag{2.1}$$

A power symmetric function of a sequence of variables $x = (x_1, x_2, \ldots)$ is defined by

$$p_n(x) = \sum_i x_i^n.$$

For a partition λ,

$$p_\lambda(x) = \prod_{j=1}^{\ell(\lambda)} p_{\lambda_j}(x).$$

It is well-known that every irreducible representation of symmetric group can be labeled by a partition. Let χ_λ be the character of the irreducible representation corresponding to λ. Each conjugate class of the symmetric group can also be represented by a partition μ such that the permutation in the conjugate class has cycles of length $\mu_1, \ldots, \mu_{\ell(\mu)}$. Schur function s_λ is determined by

$$s_\lambda(x) = \sum_{|\mu|=|\lambda|} \frac{\chi_\lambda(C_\mu)}{\mathfrak{z}_\mu} p_\mu(x) \tag{2.2}$$

where C_μ is the conjugate class of the symmetric group corresponding to partition μ.

2.2. Partitionable set and infinite series. The concept of partition can be generalized to the following *partitionable set*.

Definition 2.1. A countable set $(S, +)$ is called a partitionable set if

1). S is totally ordered;
2). S is an Abelian semi-group with summation "+"

3). the minimum element $\mathbf{0}$ in S is the zero-element of the semi-group, i.e., for any $a \in S$,

$$\mathbf{0} + a = a = a + \mathbf{0}.$$

For simplicity, we may briefly write S instead of $(S, +)$.

Example 2.2. The following sets are examples of partitionable set:

(1). The set of all nonnegative integers $\mathbb{Z}_{\geq 0}$;

(2). The set of all partitions $\mathcal{P}$. The order of $\mathcal{P}$ can be defined as follows: $\forall \lambda$, $\mu \in \mathcal{P}$, $\lambda \geq \mu$ iff $|\lambda| > |\mu|$, or $|\lambda| = |\mu|$ and there exists a j such that $\lambda_i = \mu_i$ for $i \leq j-1$ and $\lambda_j > \mu_j$. The summation is taken to be "$\cup$" and the zero-element is (0).

(3). $\mathcal{P}^n$. The order of $\mathcal{P}^n$ is defined similarly as (2): $\forall \vec{A}, \vec{B} \in \mathcal{P}^n$, $\vec{A} \geq \vec{B}$ iff $\sum_{i=1}^n |A^i| > \sum_{i=1}^n |B^i|$, or $\sum_{i=1}^n |A^i| = \sum_{i=1}^n |B^i|$ and there is a j such that $A^i = B^i$ for $i \leq j-1$ and $A^j > B^j$. Define

$$\vec{A} \cup \vec{B} = (A^1 \cup B^1, \ldots, A^n \cup B^n).$$

$((0), (0), \ldots, (0))$ is the zero-element. It's easy to check that $\mathcal{P}^n$ is a partitionable set.

Let S be a partitionable set. One can define partition with respect to S in the similar manner as that of $\mathbb{Z}_{\geq 0}$: a finite sequence of non-increasing non-minimum elements in S. We will call it an S-partition, $(\mathbf{0})$ the zero S-partition. Denote by $\mathcal{P}(S)$ the set of all S-partitions.

For an S-partition Λ, we can define the automorphism group of Λ in a similar way as that in the definition of traditional partition. Given $\beta \in S$, denote by $m_\beta(\Lambda)$ the number of times that β occurs in the parts of Λ; we then have

$$\operatorname{Aut}\Lambda = \prod_{\beta \in S} m_\beta(\Lambda)!\,.$$

Introduce the following quantities associated with Λ:

$$\mathfrak{u}_\Lambda = \frac{\ell(\Lambda)!}{|\operatorname{Aut}\Lambda|}\,, \qquad \Theta_\Lambda = \frac{(-1)^{\ell(\Lambda)-1}}{\ell(\Lambda)}\mathfrak{u}_\Lambda\,. \tag{2.3}$$

The following Lemma is quite handy when handling the expansion of generating functions.

Lemma 2.3. *Let S be a partitionable set. If $f(t) = \sum_{n \geq 0} a_n t^n$, then*

$$f\Big(\sum_{\beta \neq \mathbf{0},\, \beta \in S} A_\beta\, p_\beta(x)\Big) = \sum_{\Lambda \in \mathcal{P}(S)} a_{\ell(\Lambda)} A_\Lambda\, p_\Lambda(x)\, \mathfrak{u}_\Lambda\,,$$

where

$$p_\Lambda = \prod_{j=1}^{\ell(\Lambda)} p_{\Lambda_j}\,, \qquad A_\Lambda = \prod_{j=1}^{\ell(\Lambda)} A_{\Lambda_j}\,.$$

Proof. Note that

$$\Big(\sum_{\beta \in S,\, \beta \neq \mathbf{0}} \eta_\beta \Big)^n = \sum_{\Lambda \in \mathcal{P}(S),\, \ell(\Lambda)=n} \eta_\Lambda \mathfrak{u}_\Lambda \,.$$

Direct calculation completes the proof. q.e.d.

3. Labastida-Mariño-Ooguri-Vafa conjecture

3.1. Quantum trace. Let $\mathfrak{g}$ be a finite-dimensional complex semi-simple Lie algebra of rank N with Cartan matrix (C_{ij}). *Quantized universal enveloping algebra* $U_q(\mathfrak{g})$ is generated by $\{H_i, X_{+i}, X_{-i}\}$ together with the following defining relations:

$$[H_i, H_j] = 0\,, \quad [H_i, X_{\pm j}] = \pm C_{ij} X_{\pm j}\,, \quad [X_{+i}, X_{-j}] = \delta_{ij} \frac{q_i^{-H_i/2} - q_i^{H_i/2}}{q_i^{-1/2} - q_i^{1/2}}\,,$$

$$\sum_{k=0}^{1-C_{ij}} (-1)^k \left\{ \begin{array}{c} 1 - C_{ij} \\ k \end{array} \right\}_{q_i} X_{\pm i}^{1-C_{ij}-k} X_{\pm j} X_{\pm i}^k = 0\,, \quad \text{for all } i \neq j\,,$$

where

$$\{k\}_q = \frac{q^{-\frac{k}{2}} - q^{\frac{k}{2}}}{q^{-\frac{1}{2}} - q^{\frac{1}{2}}}\,, \qquad \{k\}_q! = \prod_{i=1}^{k} \{i\}_q\,,$$

and

$$\left\{ \begin{array}{c} a \\ b \end{array} \right\}_q = \frac{\{a\}_q \cdot \{a-1\}_q \cdots \{a-b+1\}_q}{\{b\}_q!}\,.$$

The ribbon category structure associated with $U_q(\mathfrak{g})$ is given by the following datum:

1) For any given two $U_q(\mathfrak{g})$-modules V and W, there is an isomorphism

$$\check{\mathcal{R}}_{V,W} : V \otimes W \to W \otimes V$$

satisfying

$$\check{\mathcal{R}}_{U\otimes V,W} = (\check{\mathcal{R}}_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes \check{\mathcal{R}}_{V,W}),$$
$$\check{\mathcal{R}}_{U,V\otimes W} = (\mathrm{id}_V \otimes \check{\mathcal{R}}_{U,W})(\check{\mathcal{R}}_{U,V} \otimes \mathrm{id}_W)$$

for $U_q(\mathfrak{g})$-modules U, V, W.

Given $f \in \mathrm{Hom}_{U_q(\mathfrak{g})}(U, \widetilde{U})$, $g \in \mathrm{Hom}_{U_q(\mathfrak{g})}(V, \widetilde{V})$, one has the following naturality condition:

$$(g \otimes f) \circ \check{\mathcal{R}}_{U,V} = \check{\mathcal{R}}_{\widetilde{U},\widetilde{V}} \circ (f \otimes g)\,.$$

2) There exists an element $K_{2\rho} \in U_q(\mathfrak{g})$, called the enhancement of $\check{\mathcal{R}}$, such that

$$K_{2\rho}(v \otimes w) = K_{2\rho}(v) \otimes K_{2\rho}(w)$$

for any $v \in V$, $w \in W$.

3) For any $U_q(\mathfrak{g})$-module V, the ribbon structure $\theta_V : V \to V$ associated to V satisfies

$$\theta_V^{\pm 1} = \mathrm{tr}_V \, \check{\mathcal{R}}_{V,V}^{\pm 1}.$$

The ribbon structure also satisfies the naturality condition

$$x \cdot \theta_V = \theta_{\widetilde{V}} \cdot x.$$

for any $x \in \mathrm{Hom}_{U_q(\mathfrak{g})}(V, \widetilde{V})$.

Definition 3.1. Given $z = \sum_i x_i \otimes y_i \in \mathrm{End}_{U_q(\mathfrak{g})}(U \otimes V)$, the quantum trace of z is defined by

$$\mathrm{tr}_V(z) = \sum_i \mathrm{tr}(y_i K_{2\rho}) x_i \in \mathrm{End}_{U_q(\mathfrak{g})}(U).$$

3.2. Quantum group invariants of links. Quantum group invariants of links can be defined over any complex simple Lie algebra $\mathfrak{g}$. However, in this paper, we only consider the quantum group invariants of links defined over $\mathfrak{sl}(N, \mathbb{C})$ (in the following context, we will briefly write $\mathfrak{sl}_N$) due to the current consideration for large N Chern-Simons/topological string duality.

Roughly speaking, a link is several disconnected S^1 embedded in S^3. A theorem of J. Alexander asserts that any oriented link is isotopic to the closure of some braid. A braid group $\mathcal{B}_n$ is defined by generators $\sigma_1, \cdots, \sigma_{n-1}$ and the defining relation

$$\begin{cases} \sigma_i\sigma_j = \sigma_j\sigma_i \,, & \text{if } |i-j| \geq 2\,; \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \,, & \text{if } |i-j| = 1\,. \end{cases}$$

Let $\mathcal{L}$ be a link with L components $\mathcal{K}_\alpha$, $\alpha = 1, \ldots, L$, represented by the closure of an element of braid group $\mathcal{B}_m$. We associate each $\mathcal{K}_\alpha$ an irreducible representation R_α of quantized universal enveloping algebra $U_q(\mathfrak{sl}_N)$, labeled by its highest weight Λ_α. Denote the corresponding module by V_{Λ_α}. The jth strand in the braid will be associated with the irreducible module $V_j = V_{\Lambda_\alpha}$, if this strand belongs to the component $\mathcal{K}_\alpha$. The braiding is defined through the following *universal R-matrix* of $U_q(\mathfrak{sl}_N)$:

$$\mathcal{R} = q^{\frac{1}{2}\sum_{i,j} C_{ij}^{-1} H_i \otimes H_j} \prod_{\text{positive root } \beta} \exp_q[(1-q^{-1}) E_\beta \otimes F_\beta]\,.$$

Here (C_{ij}) is the Cartan matrix and

$$\exp_q(x) = \sum_{k=0}^{\infty} q^{\frac{1}{4}k(k+1)} \frac{x^k}{\{k\}_q!}\,.$$

Define *braiding* by $\check{\mathcal{R}} = P_{12}\mathcal{R}$, where $P_{12}(v \otimes w) = w \otimes v$.

Now for a given link $\mathcal{L}$ of L components, one chooses a closed braid representative in braid group $\mathcal{B}_m$ whose closure is $\mathcal{L}$. In the case of no confusion, we also use $\mathcal{L}$ to denote the chosen braid representative in $\mathcal{B}_m$. We will associate each crossing by the braiding defined above. Let U, V be two $U_q(\mathfrak{sl}_N)$-modules labeling two outgoing strands of the crossing; the braiding $\check{R}_{U,V}$ (resp. $\check{R}^{-1}_{V,U}$) is assigned as in Figure 1.

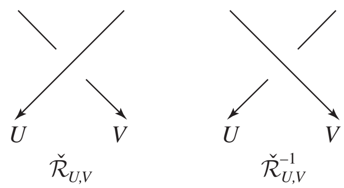
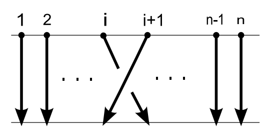


Figure 1. Assign crossing by $\check{\mathcal{R}}$.

The above assignment will give a representation of $\mathcal{B}_m$ on $U_q(\mathfrak{g})$-module $V_1 \otimes \cdots \otimes V_m$. Namely, for any generator, $\sigma_i \in \mathcal{B}_m$ (in the case of σ_i^{-1}, use $\check{\mathcal{R}}^{-1}_{V_i,V_{i+1}}$ instead),



define

$$h(\sigma_i) = \mathrm{id}_{V_1} \otimes \cdots \otimes \check{\mathcal{R}}_{V_{i+1},V_i} \otimes \cdots \otimes id_{V_m} .$$

Therefore, any link $\mathcal{L}$ will provide an isomorphism

$$h(\mathcal{L}) \in \mathrm{End}_{U_q(\mathfrak{sl}_N)}(V_1 \otimes \cdots \otimes V_m) .$$

For example, the link $\mathcal{L}$ in Figure 2 gives the following homomorphism:

$$h(\mathcal{L}) = (\check{\mathcal{R}}_{V,U} \otimes \mathrm{id}_U)(\mathrm{id}_V \otimes \check{\mathcal{R}}^{-1}_{U,U})(\check{\mathcal{R}}_{U,V} \otimes \mathrm{id}_U) .$$

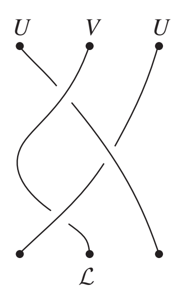


Figure 2. A braid representative for Hopf link

Let $K_{2\rho}$ be the enhancement of $\check{\mathcal{R}}$ in the sense of [**40**], where ρ is the half-sum of all positive roots of $\mathfrak{sl}_N$. The irreducible representation R_α is labeled by the corresponding partition A^α.

Definition 3.2. Given L labeling partitions $A^1, \ldots, A^L$, the quantum group invariant of $\mathcal{L}$ is defined as follows:

$$W_{(A^1,\ldots,A^L)}(\mathcal{L}) = q^{d(\mathcal{L})} \operatorname{tr}_{V_1\otimes\cdots\otimes V_m}(h(\mathcal{L}))\,,$$

where

$$d(\mathcal{L}) = -\frac{1}{2}\sum_{\alpha=1}^{L}\omega(\mathcal{K}_\alpha)(\Lambda_\alpha, \Lambda_\alpha + 2\rho) + \frac{1}{N}\sum_{\alpha<\beta}\operatorname{lk}(\mathcal{K}_\alpha, \mathcal{K}_\beta)|A^\alpha|\cdot|A^\beta|$$

and $\operatorname{lk}(\mathcal{K}_\alpha, \mathcal{K}_\beta)$ is the linking number of components $\mathcal{K}_\alpha$ and $\mathcal{K}_\beta$. A substitution of $t = q^N$ is used to give a two-variable framing independent link invariant.

Remark 3.3. In the above formula of $d(\mathcal{L})$, the second term on the right-hand side is meant to cancel unimportant terms involved with $q^{1/N}$ in the definition.

It will be helpful to extend the definition to allow some labeling partition to be the empty partition (0). In this case, the corresponding invariants will be regarded as the quantum group invariants of the link obtained by simply removing the components labeled by (0).

A direct computation shows that after removing the terms of $q^{\frac{1}{N}}$, $q^{d(\mathcal{L})}$ can be simplified as

$$q^{\sum_{\alpha=1}^{L}\kappa_{A^\alpha}w(\mathcal{K}_\alpha)/2}\cdot t^{\sum_{\alpha=1}^{L}|A^\alpha|w(\mathcal{K}_\alpha)/2}\,. \tag{3.1}$$

Example 3.4. The following examples are some special cases of quantum group invariants of links.

(1). If the link involved in the definition is the unknot $\bigcirc$,

$$W_A(\bigcirc; q, t) = \operatorname{tr}_{V_A}(\operatorname{id}_{V_A})$$

is equal to the quantum dimension of V_A, which will be denoted by $\dim_q V_A$.

(2). When all the components of $\mathcal{L}$ are associated with the fundamental representation, i.e., the labeling partition is the unique partition of 1, the quantum group invariant of $\mathcal{L}$ is related to the HOMFLY polynomial of the link, $P_{\mathcal{L}}(q,t)$, in the following way:

$$W_{(\square,\cdots,\square)}(\mathcal{L}; q, t) = t^{\operatorname{lk}(\mathcal{L})}\left(\frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}}}{q^{-\frac{1}{2}} - q^{\frac{1}{2}}}\right)P_{\mathcal{L}}(q,t)\,.$$

(3). If $\mathcal{L}$ is a disjoint union of L knots, i.e.,

$$\mathcal{L} = \mathcal{K}_1 \otimes \mathcal{K}_2 \otimes \cdots \otimes \mathcal{K}_L\,,$$

the quantum group invariants of $\mathcal{L}$ are simply the multiplication of quantum group invariants of $\mathcal{K}_\alpha$,

$$W_{(A^1,\ldots,A^L)}(\mathcal{L};\, q,t) = \prod_{\alpha=1}^{L} W_{A^\alpha}(\mathcal{K}_\alpha;\, q,t)\,.$$

3.3. Chern-Simons partition function. For a given link $\mathcal{L}$ of L components, we will fix the following notation in this paper. Given $\lambda \in \mathcal{P}$, $\vec{A} = (A^1, A^2, \ldots, A^L)$, $\vec{\mu} = (\mu^1, \mu^2, \ldots, \mu^L) \in \mathcal{P}^L$. Let $x = (x^1, \ldots, x^L)$ where x^α is a series of variables

$$x^\alpha = (x_1^\alpha, x_2^\alpha, \cdots)\,.$$

The following notation will be used throughout the paper:

$$[n]_q = q^{-\frac{n}{2}} - q^{\frac{n}{2}}\,, \qquad [\lambda]_q = \prod_{j=1}^{\ell(\lambda)} [\lambda_j]_q\,, \qquad \mathfrak{z}_{\vec{\mu}} = \prod_{\alpha=1}^{L} \mathfrak{z}_{\mu^\alpha}\,,$$

$$|\vec{A}| = (|A^1|, \ldots, |A^L|)\,, \qquad \| \vec{A} \| = \sum_{\alpha=1}^{L} |A^\alpha|\,, \qquad \ell(\vec{\mu}) = \sum_{\alpha=1}^{L} \ell(\mu^\alpha)\,,$$

$$\vec{A}^t = \big((A^1)^t, \ldots, (A^L)^t\big)\,, \quad \chi_{\vec{A}}(\vec{\mu}) = \prod_{\alpha=1}^{L} \chi_{A^\alpha}(C_{\mu^\alpha})\,, \quad s_{\vec{A}}(x) = \prod_{\alpha=1}^{L} s_{A^\alpha}(x^\alpha)\,.$$

Denote

$$1^{\vec{\mu}} = (1^{|\mu^1|}, \cdots, 1^{|\mu^L|})\,. \tag{3.2}$$

The *Chern-Simons partition function* can be defined to be the following generating function of quantum group invariants of $\mathcal{L}$:

$$Z_{\mathrm{CS}}(\mathcal{L}) = 1 + \sum_{\vec{A} \neq 0} W_{\vec{A}}(\mathcal{L};\, q,t) s_{\vec{A}}(x).$$

Define *free energy* as

$$F = \log Z = \sum_{\vec{\mu} \neq 0} F_{\vec{\mu}} p_{\vec{\mu}}(x)\,. \tag{3.3}$$

Similarly,

$$p_{\vec{\mu}}(x) = \prod_{\alpha=1}^{L} p_{\mu^\alpha}(x^\alpha)\,.$$

We rewrite the Chern-Simons partition function as

$$Z_{\mathrm{CS}}(\mathcal{L}) = 1 + \sum_{\vec{\mu} \neq 0} Z_{\vec{\mu}} p_{\vec{\mu}}(x)$$

where

$$Z_{\vec{\mu}} = \sum_{\vec{A}} \frac{\chi_{\vec{A}}(\vec{\mu})}{\mathfrak{z}_{\vec{\mu}}} W_{\vec{A}}. \tag{3.4}$$

By Lemma 2.3, we have

$$F_{\vec{\mu}} = \sum_{\Lambda \in \mathcal{P}(\mathcal{P}^L),\, |\Lambda| = \vec{\mu} \in \mathcal{P}^L} \Theta_\Lambda Z_\Lambda \,. \tag{3.5}$$

3.4. Main results.

3.4.1. Two theorems that answer the LMOV conjecture. Let $P_{\vec{B}}(q,t)$ be the function defined by (1.4), which can be determined by the following formula:

$$P_{\vec{B}}(q,t) = \sum_{|\vec{A}|=|\vec{B}|} f_{\vec{A}}(q,t) \prod_{\alpha=1}^{L} \sum_{\mu} \frac{\chi_{A^\alpha}(C_\mu)\chi_{B^\alpha}(C_\mu)}{\mathfrak{z}_\mu} \prod_{j=1}^{\ell(\mu^\alpha)} \frac{1}{q^{-\mu_j/2} - q^{\mu_j/2}} \,.$$

Theorem 1. *There exist topological invariants $N_{\vec{B};\,g,Q} \in \mathbb{Q}$ such that expansion* (1.5) *holds.*

Theorem 2. *Given any $\vec{B} \in \mathcal{P}^L$, the generating function of $N_{\vec{B};\,g,Q}$, $P_{\vec{B}}(q,t)$, satisfies*

$$(q^{-1/2} - q^{1/2})^2 P_{\vec{B}}(q,t) \in \mathbb{Z}\big[\big(q^{-1/2} - q^{1/2}\big)^2, t^{\pm 1/2}\big] \,.$$

It is clear that Theorems 1 and 2 answered the LMOV conjecture. Moreover, Theorem 2 implies, for fixed $\vec{B}$, $N_{\vec{B};\,g,Q}$ vanishes at large genera.

The method in this paper may apply to the general complex simple Lie algebra $\mathfrak{g}$ instead of only considering $\mathfrak{sl}(N,\mathbb{C})$. We will put this in our future research. If this is the case, it might require a more generalized duality picture in physics which will definitely be very interesting to consider and reveal much deeper relations between Chern-Simons gauge theory and geometry of moduli space. The extension to some other gauge group has already been done in [**41**], where non-orientable Riemann surfaces are involved. For the gauge group $SO(N)$ or $Sp(N)$, a more complete picture appeared in the recent work of [**6**, **7**], and therein a BPS structure of the colored Kauffman polynomials was also presented. We would like to see that the techniques developed in our paper extend to these cases.

The existence of (1.5) has its deep root in the duality between large N Chern-Simons/topological string duality. As already mentioned in the introduction, by the definition of quantum group invariants, $P_{\vec{B}}(q,t)$ might have very high order of pole at $q = 1$, especially when the degree of $\vec{B}$ goes higher and higher. However, the LMOV conjecture claims that the pole at $q = 1$ is at most of order 2 for any $\vec{B} \in \mathcal{P}^L$. Any term that has power of $q^{-1/2} - q^{1/2}$ lower than -2 will be canceled! Without the motivation of Chern-Simons/topological string duality, this mysterious cancelation is hardly able to be realized from knot-theory point of view.

3.4.2. An application to knot theory. We now discuss applications to knot theory, following [**21**, **19**].

Consider associating the fundamental representation to each component of the given link $\mathcal{L}$. As discussed above, the quantum group invariant of $\mathcal{L}$ will reduce to the classical HOMFLY polynomial $P_{\mathcal{L}}(q,t)$ of $\mathcal{L}$ except for a universal factor. HOMFLY has the following expansion:

$$P_{\mathcal{L}}(q,t) = \sum_{g\geq 0} p_{2g+1-L}(t)(q^{-\frac{1}{2}} - q^{\frac{1}{2}})^{2g+1-L}\,. \tag{3.6}$$

The lowest power of $q^{-1/2} - q^{1/2}$ is $1-L$, which was proved in [**28**] (or one may directly derive it from Lemma 5.1). After a simple algebra calculation, one will have

$$F_{(\square,\dots,\square)} = \left(\frac{t^{-1/2} - t^{1/2}}{q^{-1/2} - q^{1/2}}\right) \sum_{g\geq 0} \widetilde{p}_{2g+1-L}(t)(q^{-\frac{1}{2}} - q^{\frac{1}{2}})^{2g+1-L}\,. \tag{3.7}$$

Lemma 5.3 states that

$$\widetilde{p}_{1-L}(t) = \widetilde{p}_{3-L}(t) = \cdots = \widetilde{p}_{L-3}(t) = 0\,,$$

which implies that the $p_k(t)$ are completely determined by the HOMFLY polynomial of its sub-links for $k = 1-L, 3-L, \dots, L-3$.

Now, we only look at $\widetilde{p}_{1-L}(t) = 0$. A direct comparison of the coefficients of $F = \log Z_{\mathrm{CS}}$ immediately leads to the following theorem proved by Lickorish and Millett [**28**].

Theorem 3.5 (Lickorish-Millett)**.** *Let $\mathcal{L}$ be a link with L components. Its HOMFLY polynomial*

$$P_{\mathcal{L}}(q,t) = \sum_{g\geq 0} p^{\mathcal{L}}_{2g+1-L}(t)\left(q^{-\frac{1}{2}} - q^{\frac{1}{2}}\right)^{2g+1-L}$$

satisfies

$$p^{\mathcal{L}}_{1-L}(t) = t^{-\mathrm{lk}}\left(t^{-\frac{1}{2}} - t^{\frac{1}{2}}\right)^{L-1} \prod_{\alpha=1}^{L} p_0^{\mathcal{K}_\alpha}(t)$$

where $p_0^{\mathcal{K}_\alpha}(t)$ is a HOMFLY polynomial of the αth component of the link $\mathcal{L}$ with $q=1$.

In [**28**], Lickorish and Millett obtained the above theorem by skein analysis. Here as the consequence of the higher-order cancellation phenomenon, one sees how easily it can be achieved. Note that we only utilize the vanishing of $\widetilde{p}_{1-L}$. If one is ready to carry out the calculation of more vanishing terms, one can definitely get much more information about algebraic structure of HOMFLY polynomial. Similarly, a lot of the deep relation of quantum group invariants can be obtained by the cancellation of higher order poles.

3.4.3. Geometric interpretation of the new integer invariants. The following interpretation is taken from the physics literature and a string theory point of view [**21**, **36**].

Quantum group invariants of links can be expressed as the vacuum expectation value of Wilson loops which admit a large N expansion in physics. It can also be interpreted as a string theory expansion. This leads to a geometric description of the new integer invariants $N_{\vec{B};\,g,Q}$ in terms of open Gromov-Witten invariants (also see [**19**] for more details).

The geometric picture of $f_{\vec{A}}$ is proposed in [**21**]. One can rewrite the free energy as

$$F = \sum_{\vec{\mu}} \sqrt{-1}^{\ell(\vec{\mu})} \sum_{g=0}^{\infty} \lambda^{2g-2+\ell(\vec{\mu})} F_{g,\vec{\mu}}(t) p_{\vec{\mu}} \,. \tag{3.8}$$

The quantities $F_{g,\vec{\mu}}(t)$ can be interpreted in terms of the Gromov-Witten invariants of Riemann surface with boundaries. It was conjectured in [**36**] that for every link $\mathcal{L}$ in S^3, one can canonically associate a lagrangian submanifold $\mathcal{C}_{\mathcal{L}}$ in the *resolved conifold*

$$\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1 \,.$$

The assignment implies that $b_1(\mathcal{C}_{\mathcal{L}}) = L$, the number of components of $\mathcal{L}$. Let γ_α, $\alpha = 1, \dots, L$, be one-cycles representing a basis for $H_1(\mathcal{C}_{\mathcal{L}}, \mathbb{Z})$. Denote by $\mathcal{M}_{g,h,Q}$ the moduli space of Riemann surfaces of genus g and h holes embedded in the resolved conifold. There are h_α holes ending on the non-trivial cycles γ_α for $\alpha = 1, \dots, L$. The product of symmetric groups

$$\Sigma_{h_1} \times \Sigma_{h_2} \times \cdots \times \Sigma_{h_L}$$

acts on the Riemann surfaces by exchanging the h_α holes that end on γ_α. The integer $N_{\vec{B};\,q,t}$ is then interpreted as

$$N_{\vec{B};\,q,t} = \chi(\mathbf{S}_{\vec{B}}(H^*(\mathcal{M}_{g,h,Q}))) \tag{3.9}$$

where $\mathbf{S}_{\vec{B}} = \mathbf{S}_{B^1} \otimes \cdots \otimes \mathbf{S}_{B^L}$, and $\mathbf{S}_{B^\alpha}$ is the Schur functor.

The recent progress in the mathematical definitions of open Gromov-Witten invariants [**18**, **23**, **24**] may be used to put the above definition on a rigorous setting.

4. Hecke algebra and cabling

4.1. Centralizer algebra and Hecke algebra representation. We review some facts about centralizer algebra and Hecke algebra representation and their relation to the representation of braid group.

Denote by V the fundamental representation of $U_q(\mathfrak{sl}_N)$. We will reserve V to denote the fundamental representation of $U_q(\mathfrak{sl}_N)$ from now on. Let

$$\{K_i^{\pm 1}, E_i, F_i : 1 \le i \le N-1\}$$

be the standard generators of the quantized universal enveloping algebra $U_q(\mathfrak{sl}_N)$. Under a suitable basis $\{X_1, \ldots, X_N\}$ of V, the fundamental representation is given by the matrices

$$\begin{aligned} E_i &\longmapsto E_{i,i+1} \\ F_i &\longmapsto E_{i+1,i} \\ K_i &\longmapsto q^{-1/2} E_{i,i} + q^{1/2} E_{i+1,i+1} + \sum_{i \neq j} E_{jj} \end{aligned}$$

where $E_{i,j}$ denotes the $N \times N$ matrix with 1 at the (i,j)-position and 0 elsewhere. Direct calculation shows

$$K_{2\rho}(X_i) = q^{-\frac{N+1-2i}{2}} X_i \tag{4.1}$$

and

$$q^{-\frac{1}{2N}} \check{\mathcal{R}}(X_i \otimes X_j) = \begin{cases} q^{-1/2} X_i \otimes X_j, & i = j\,, \\ X_j \otimes X_i, & i < j\,, \\ X_j \otimes X_i + (q^{-1/2} - q^{1/2}) X_i \otimes X_j, & i > j\,. \end{cases}$$

The *centralizer algebra* of $V^{\otimes n}$ is defined as the following:

$$\mathcal{C}_n = \mathrm{End}_{U_q(\mathfrak{sl}_N)}(V^{\otimes n}) = \left\{ x \in \mathrm{End}(V^{\otimes n}) : \, xy = yx, \forall y \in U_q(\mathfrak{sl}_N) \right\}\,.$$

Hecke algebra $\mathcal{H}_n(q)$ of type A_{n-1} is the complex algebra with $n-1$ generators $g_1, \ldots, g_{n-1}$, together with the following defining relations:

$$\begin{aligned} &g_i g_j = g_j g_i\,, && |i-j| \geq 2, \\ &g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}\,, && i = 1, 2, \ldots, n-2, \\ &(g_i - q^{-1/2})(g_i + q^{1/2}) = 0\,, && i = 1, 2, \ldots, n-1. \end{aligned}$$

Remark 4.1. Here we use $q^{-1/2}$ instead of q to adapt our notation to the definition of quantum group invariants of links. Note that when $q = 1$, the Hecke algebra $\mathcal{H}_n(q)$ is just the group algebra $\mathbb{C}\Sigma_n$ of symmetric group Σ_n. When N is large enough, $\mathcal{C}_n$ is isomorphic to the Hecke algebra $\mathcal{H}_n(q)$. $\mathcal{H}_n(q)$ has a canonical decomposition:

$$\mathcal{H}_n(q) = \bigoplus_{A \vdash n} \mathcal{H}_A(q) \tag{4.2}$$

with each $\mathcal{H}_A(q)$ being a matrix algebra.

A very important feature of the homomorphism

$$h : \mathbb{C}\mathcal{B}_n \longrightarrow \mathcal{C}_n$$

is that h factors through $\mathcal{H}_n(q)$ via

$$q^{-\frac{1}{2N}} \sigma_i \mapsto g_i \mapsto q^{-\frac{1}{2N}} h(\sigma_i)\,. \tag{4.3}$$

It is well-known that the irreducible modules S^λ (*Specht module*) of $\mathcal{H}_n(q)$ are in one-to-one correspondence to the partitions of n.

Any permutation π in symmetric group Σ_n can express as a product of transpositions

$$\pi = s_{i_1} s_{i_2} \cdots s_{i_l} \,.$$

If l is minimal, we say π has length $\ell(\pi) = l$ and

$$g_\pi = g_{i_1} g_{i_2} \cdots g_{i_l} \,.$$

It is not difficult to see that g_π is well-defined. All of such $\{g_\pi\}$ form a basis of $\mathcal{H}_n(q)$.

Minimal projection $\mathfrak{S}$ is an element in $\mathcal{C}_n$ such that $\mathfrak{S}V^{\otimes n}$ is some irreducible representation $\mathfrak{S}_\lambda$. The minimal projections of Hecke algebras are well studied (for example, [**14**]), which is a $\mathbb{Z}(q^{\pm 1})$-linear combination $\{q^{\frac{1}{2}} g_i\}$.

4.2. Quantum dimension.

4.2.1. Explicit formula. An explicit formula for the quantum dimension of any irreducible representation of $U_q(\mathfrak{sl}_N)$ can be computed via decomposing $V^{\otimes n}$ into permutation modules.

A *composition* of n is a sequence of non-negative integer

$$\mathfrak{b} = (\mathfrak{b}_1, \mathfrak{b}_2, \ldots)$$

such that

$$\sum_{i \geq 1} \mathfrak{b}_i = n \,.$$

We will write it as $\mathfrak{b} \vDash n$. The largest j such that $\mathfrak{b}_j \neq 0$ is called the *end* of $\mathfrak{b}$ and is denoted by $\ell(\mathfrak{b})$.

Let $\mathfrak{b}$ be a composition such that $\ell(\mathfrak{b}) \leq N$. Define $M^{\mathfrak{b}}$ to be the subspace of $V^{\otimes n}$ spanned by the vectors

$$X_{j_1} \otimes \cdots \otimes X_{j_n} \tag{4.4}$$

such that X_i occurs precisely $\mathfrak{b}_i$ times. It is clear that $M^{\mathfrak{b}}$ is an $\mathcal{H}_n(q)$-module and is called a *permutation module*. Moreover, by the explicit matrix formula of $\{E_i, F_i, K_i\}$ acting on V under the basis $\{X_i\}$, we have

$$M^{\mathfrak{b}} = \{X \in V^{\otimes n} : \, K_i(X) = q^{-\frac{\mathfrak{b}_i - \mathfrak{b}_{i+1}}{2}} X\} \,.$$

The following decomposition is very useful:

$$V^{\otimes n} = \bigoplus_{\mathfrak{b} \vDash n, \, \ell(\mathfrak{b}) \leq N} M^{\mathfrak{b}} \,.$$

Let A be a partition and V_A the irreducible representation labeled by A. The *Kostka number* $K_{A\mathfrak{b}}$ is defined to be the weight of V_A in $M^{\mathfrak{b}}$, i.e.,

$$K_{A\mathfrak{b}} = \dim(V_A \cap M^{\mathfrak{b}}) \,.$$

The Schur function has the following formulation through Kostka numbers:

$$s_A(x_1,\dots,x_N)=\sum_{\mathfrak{b}\vDash|A|,\,\ell(\mathfrak{b})\le N}K_{A\mathfrak{b}}\prod_{j=1}^N x_j^{\mathfrak{b}_j}\,.$$

By (4.1), $K_{2\rho}$ is acting on $M^{\mathfrak{b}}$ as a scalar $\prod_{j=1}^N q^{-\frac12(N+1-2j)\mathfrak{b}_j}$. Thus

$$\begin{aligned}\dim_q V_A&=\mathrm{tr}_{V_A}\,\mathrm{id}_{V_A}\\&=\sum_{\mathfrak{b}\vDash|A|}\dim(V_A\cap M^{\mathfrak{b}})\prod_{j=1}^N q^{-\frac12(N+1-2j)\mathfrak{b}_j}\\&=s_A\Big(q^{\frac{N-1}{2}},\dots,q^{-\frac{N-1}{2}}\Big).\end{aligned}\tag{4.5}$$

By (2.2),

$$\begin{aligned}\dim_q V_A&=\sum_{|\mu|=|A|}\frac{\chi_A(C_\mu)}{\mathfrak{z}_\mu}p_\mu\Big(q^{\frac{N-1}{2}},\dots,q^{-\frac{N-1}{2}}\Big)\\&=\sum_{|\mu|=|A|}\frac{\chi_A(C_\mu)}{\mathfrak{z}_\mu}\prod_{j=1}^{\ell(\mu)}\frac{t^{-\mu_j/2}-t^{\mu_j/2}}{q^{-\mu_j/2}-q^{\mu_j/2}}\,.\end{aligned}\tag{4.6}$$

Here in the last step, we use the substitution $t=q^N$.

4.2.2. An expansion of the Mariño-Vafa formula. Here we give a quick review of the Mariño-Vafa formula [**32**, **25**] for the convenience of Knot theorists. For details, please refer [**25**].

Let $\overline{M}_{g,n}$ denote the Deligne-Mumford moduli stack of stable curves of genus g with n marked points. Let $\pi:\overline{M}_{g,n+1}\to\overline{M}_{g,n}$ be the universal curve, and let ω_π be the relative dualizing sheaf. The Hodge bundle $\mathbb{E}=\pi_*\omega_\pi$ is a rank g vector bundle over $\overline{M}_{g,n}$. Let $s_i:\overline{M}_{g,n}\to\overline{M}_{g,n+1}$ denote the section of π which corresponds to the ith marked point, and let $\mathbb{L}_i=s_i^*\omega_\pi$. A Hodge integral is an integral of the form

$$\int_{\overline{M}_{g,n}}\psi_1^{j_1}\cdots\psi_n^{j_n}\lambda_1^{k_1}\cdots\lambda_g^{k_g}$$

where $\psi_i=c_1(\mathbb{L}_i)$ is the first Chern class of $\mathbb{L}_i$ and $\lambda_j=c_j(\mathbb{E})$ is the jth Chern class of the Hodge bundle. Let

$$\Lambda_g^\vee(u)=u^g-\lambda_1u^{g-1}+\cdots+(-1)^g\lambda_g$$

be the Chern polynomial of $\mathbb{E}^\vee$, the dual of the Hodge bundle.

Define

$$\mathcal{C}_{g,\mu}(\tau) = -\frac{\sqrt{-1}^{\ell(\mu)}}{|\operatorname{Aut}(\mu)|}[\tau(\tau+1)]^{\ell(\mu)-1}\prod_{i=1}^{\ell(\mu)}\frac{\prod_{a=1}^{\mu_i-1}(\mu_i\tau+a)}{(\mu_i-1)!} \cdot \int_{\overline{M}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(-\tau-1)\Lambda_g^\vee(\tau)}{\prod_{i=1}^{\ell(\mu)}(1-\mu_i\psi_i)}.$$

Note that

$$\mathcal{C}_{0,\mu}(\tau) = -\frac{\sqrt{-1}^{\ell(\mu)}}{|\operatorname{Aut}(\mu)|}[\tau(\tau+1)]^{\ell(\mu)-1}\prod_{i=1}^{\ell(\mu)}\frac{\prod_{a=1}^{\mu_i-1}(\mu_i\tau+a)}{(\mu_i-1)!} \cdot \int_{\overline{M}_{0,l(\mu)}} \frac{1}{\prod_{i=1}^{\ell(\mu)}(1-\mu_i\psi_i)}. \tag{4.7}$$

The coefficient of the leading term in τ is

$$-\frac{\sqrt{-1}^{\ell(\mu)}}{|\operatorname{Aut}\mu|}\prod_j \frac{\mu_j^{\mu_j}}{\mu_j!}\cdot|\mu|^{\ell(\mu)-3}. \tag{4.8}$$

The Mariño-Vafa formula gives the following identity:

$$\sum_\mu p_\mu(x)\sum_{g\geq 0} u^{2g-2+\ell(\mu)}\mathcal{C}_{g,\mu}(\tau) = \log\Big(\sum_A s_A(q^\rho)q^{\frac{1}{2}\kappa_A\tau}s_A(x)\Big), \tag{4.9}$$

where $q = e^{\sqrt{-1}u}$, $q^\rho = (q^{-1/2}, q^{-3/2}, \cdots, q^{-n+1/2}, \cdots)$.

Let

$$\log\Big(\sum_A s_A(q^\rho)s_A(x)q^{\frac{\kappa_A\tau}{2}}\Big) = \sum_\mu G_\mu p_\mu(x).$$

Then

$$\begin{aligned}
G_\mu =& \sum_{|\Lambda|=\mu}\Theta_\Lambda\prod_{\alpha=1}^{\ell(\Lambda)}\sum_{A^\alpha}\frac{\chi_{A^\alpha}(C_{\Lambda^\alpha})}{\mathfrak{z}_{\Lambda^\alpha}}s_{A^\alpha}(q^\rho)q^{\frac{\kappa_{A^\alpha}\tau}{2}}\\
=&G_\mu(0) + p_{1^\mu}(q^\rho)\sum_{p\geq 1}\frac{(\frac{u\tau}{2})^p}{p!}\sum_{|\Lambda|=\mu}\Theta_\Lambda\prod_{\alpha=1}^{\ell(\Lambda)}\sum_{\sum_\alpha p_\alpha=p}\sum_{A^\alpha}\frac{\chi_{A^\alpha}(C_{\Lambda^\alpha})}{\mathfrak{z}_{\Lambda^\alpha}}\frac{\chi_{A^\alpha}(C_{1^{\Lambda^\alpha}})}{\mathfrak{z}_{1^{\Lambda^\alpha}}}\kappa_{A^\alpha}^{p_\alpha}\\
&+\sum_{p\geq 1}\frac{(\frac{u\tau}{2})^p}{p!}\sum_{|\Lambda|=\mu}\Theta_\Lambda\sum_{\Omega\neq(1^\Lambda)}\prod_{\alpha=1}^{\ell(\Lambda)}\sum_{\sum_\alpha p_\alpha=p}\sum_{A^\alpha}\frac{\chi_{A^\alpha}(C_{\Lambda^\alpha})}{\mathfrak{z}_{\Lambda^\alpha}}\frac{\chi_{A^\alpha}(C_{\Omega^\alpha})}{\mathfrak{z}_{\Omega^\alpha}}p_{\Omega^\alpha}(q^\rho)\kappa_{A^\alpha}^{p_\alpha}.
\end{aligned} \tag{4.10}$$

The second summand of the above formula gives the non-vanishing leading term in τ which is equal to (4.8). Therefore, we have

$$\sum_{|\Lambda|=\mu}\Theta_\Lambda\prod_{\alpha=1}^{\ell(\Lambda)}\sum_{\sum_\alpha p_\alpha=p}\sum_{A^\alpha}\frac{\chi_{A^\alpha}(C_{\Lambda^\alpha})}{\mathfrak{z}_{\Lambda^\alpha}}\frac{\chi_{A^\alpha}(C_{1^{\Lambda^\alpha}})}{\mathfrak{z}_{1^{\Lambda^\alpha}}}\kappa_{A^\alpha}^{p_\alpha}\neq 0 \tag{4.11}$$

for $\forall p \geq |\mu| + \ell(\mu) - 2$.

4.3. Cabling Technique. Given irreducible representations $V_{A^1}, \dots, V_{A^L}$ to each component of link $\mathcal{L}$, let $|A^\alpha| = d_\alpha$, $\vec{d} = (d_1, \dots, d_L)$. The cabling braid of $\mathcal{L}$, $\mathcal{L}_{\vec{d}}$, is obtained by substituting d_α parallel strands for each strand of $\mathcal{K}_\alpha$, $\alpha = 1, \dots, L$.

Using cabling of the $\mathcal{L}$ gives a way to take trace in the vector space of tensor product of fundamental representation. To get the original trace, one has to take certain projection, which is the following lemma from [**29**].

Lemma 4.2 ([**29**], Lemma 3.3)**.** *Let $V_i = \mathfrak{S}_i V^{\otimes d_i}$ for some minimal projection $\mathfrak{S}_i$ and $\mathfrak{S}_i = \mathfrak{S}_j$ if the ith and jth strands belong to the same knot. Then*

$$\mathrm{tr}_{V_1 \otimes \cdots V_m}(h(\mathcal{L})) = \mathrm{tr}_{V^{\otimes n}}\left(h(\mathcal{L}_{(d_1,\dots,d_L)}) \cdot \mathfrak{S}_1 \otimes \cdots \otimes \mathfrak{S}_n\right).$$

where m is the number of strands belonging to $\mathcal{L}$, $n = \sum_{\alpha=1}^L d_\alpha r_\alpha$, and r_α is the number of strands belong to $\mathcal{K}_\alpha$, the αth component of $\mathcal{L}$.

5. Pole structure

5.1. Pole structure of quantum group invariants. By an observation from the action of $\check{\mathcal{R}}$ on $V \otimes V$, we define

$$\widetilde{X}_{(i_1,\dots,i_n)} = q^{\frac{\#\{(j,k)|j<k, i_j>i_k\}}{2}} X_{i_1} \otimes \cdots \otimes X_{i_n}.$$

$\{\widetilde{X}_{(i_1,\dots,i_n)}\}$ form a basis of $V^{\otimes n}$. By (4.3), $\forall g_j \in \mathcal{H}_n(q)$, we have

(5.1)

$$q^{\frac{1}{2}} g_j \widetilde{X}_{(\dots,i_j,i_{j+1},\dots)} = \begin{cases} \widetilde{X}_{(\dots,i_{j+1},i_j,\dots)}, & i_j \leq i_{j+1}, \\ q\widetilde{X}_{(\dots,i_{j+1},i_j,\dots)} + (1-q)\widetilde{X}_{(\dots,i_j,i_{j+1},\dots)}, & i_j \geq i_{j+1}. \end{cases}$$

Lemma 5.1. *Let $\bigcirc$ be the unknot. Given any $\vec{A} = (A^1, \dots, A^L) \in \mathcal{P}^L$,*

$$\lim_{q \to 1} \frac{W_{\vec{A}}(\mathcal{L}; q, t)}{W_{\vec{A}}(\bigcirc^{\otimes L}; q, t)} = \prod_{\alpha=1}^{L} \xi_{\mathcal{K}_\alpha}(t)^{d_\alpha}, \tag{5.2}$$

where $|A^\alpha| = d_\alpha$, $\mathcal{K}_\alpha$ is the αth component of $\mathcal{L}$, and $\xi_{\mathcal{K}_\alpha}(t)$, $\alpha = 1, \dots, L$, are independent of $\vec{A}$.

Proof. Choose $\beta \in \mathcal{B}_m$ such that $\mathcal{L}$ is the closure of β, the total number of crossings of $\mathcal{L}_{\vec{d}}$ is even, and the last L strands belongs to distinct L components of $\mathcal{L}$. Let r_α be the number of the strands which belong to $\mathcal{K}_\alpha$. $n = \sum_\alpha d_\alpha r_\alpha$ is equal to the number of components in the cabling link $\mathcal{L}_{\vec{d}}$.

Let

$$\mathcal{Y} = \mathrm{tr}_{V^{\otimes \sum_{\alpha=1}^L d_\alpha(r_\alpha - 1)}}\left(h(\mathcal{L}_{\vec{d}})\right). \tag{5.3}$$

$\mathcal{Y}$ is both a central element of $\mathrm{End}_{U_q(\mathfrak{sl}_N)}\left(V^{\otimes(d_1+\cdots+d_L)}\right)$ and a $\mathbb{Z}[q^{\pm 1}, t^{\pm\frac{1}{2}}]$-matrix under the basis $\{\widetilde{X}_{(i_1,\ldots,i_n)}\}$.

On the other hand,

$$\mathfrak{S}_{\vec{A}} = \mathfrak{S}_{A^1} \otimes \cdots \otimes \mathfrak{S}_{A^L}$$

is a $\mathbb{Z}[t^{\pm\frac{1}{2}}]\big(q^{\pm 1}\big)$-matrix under the basis $\{\widetilde{X}_{(i_1,\ldots,i_n)}\}$. By Schur's lemma, we have

$$\mathfrak{S}_{\vec{A}} \circ \mathcal{Y} = \epsilon \cdot p_{\vec{A}},$$

where $\epsilon \in \mathbb{Z}[t^{\pm\frac{1}{2}}](q^{\pm 1})$ is an eigenvalue of $\mathcal{Y}$. Since $\mathbb{Z}[q^{\pm 1}, t^{\pm\frac{1}{2}}]$ is a UFD for transcendental q, ϵ must stay in $\mathbb{Z}[q^{\pm 1}, t^{\pm\frac{1}{2}}]$. By Lemma 4.2,

$$\begin{aligned}\mathrm{tr}_{V^{\otimes n}}\left(\mathfrak{S}_{A^1}^{\otimes r_1} \otimes \cdots \otimes \mathfrak{S}_{A^L}^{\otimes r_L} \circ h(\mathcal{L}_{\vec{d}})\right) &= \mathrm{tr}_{V^{\otimes(d_1+\cdots d_L)}}(\mathfrak{S}_{\vec{A}} \circ \mathcal{Y}) \\ &= \epsilon \cdot \mathrm{tr}_{V^{\otimes(d_1+\cdots d_L)}}(\mathfrak{S}_{\vec{A}}) \\ &= \epsilon \cdot W_{\vec{A}}(\bigcirc^{\otimes L}) \,. \end{aligned} \tag{5.4}$$

The definition of quantum group invariants of $\mathcal{L}$ gives

$$W_{\vec{A}}\big(\mathcal{L};\, q, t\big) = q^{\sum_\alpha \kappa_{A^\alpha} w(\mathcal{K}_\alpha)/2} t^{\sum_\alpha d_\alpha w(\mathcal{K}_\alpha)/2} \cdot \epsilon \cdot W_{\vec{A}}(\bigcirc^{\otimes L};\, q, t). \tag{5.5}$$

When $q \to 1$, $\mathcal{L}_{\vec{d}}$ reduces to an element in symmetric group Σ_n of $\| \vec{d} \|$ cycles. Moreover, when $q \to 1$, the calculation is actually taken in individual knot component while the linking of different components have no effect. By Example 3.4(1) and 3.4(3), we have

$$W_{\vec{A}}(\bigcirc^{\otimes L};\, q, t) = \prod_{\alpha=1}^{L} W_{A^\alpha}(\bigcirc;\, q, t)\,.$$

This implies

$$\lim_{q\to 1} \frac{W_{\vec{A}}(\mathcal{L};\, q, t)}{W_{\vec{A}}(\bigcirc^{\otimes L};\, q, t)} = \prod_{\alpha=1}^{L} \lim_{q\to 1} \frac{W_{A^\alpha}(\mathcal{K}_\alpha;\, q, t)}{W_{A^\alpha}(\bigcirc;\, q, t)}\,. \tag{5.6}$$

Let us consider the case when $\mathcal{K}$ is a knot. A is the partition of d associated with $\mathcal{K}$, and $\mathcal{K}_d$ is the cabling of $\mathcal{K}$. Each component of $\mathcal{K}_d$ is a copy of $\mathcal{K}$. $q \to 1$, $\mathcal{K}_d$ reduces to an element in Σ_{dr}. To calculate the $\mathcal{Y}$, it is then equivalent to discussing d disjoint unions of $\mathcal{K}$. Say $\mathcal{K}$ has r strands. Consider

$$\mathcal{Y}_0 = \mathrm{tr}_{V^{\otimes(r-1)}}\, h(\mathcal{K})\,.$$

The eigenvalue of $\mathcal{Y}_0$ is then $\frac{P_{\mathcal{K}}(1,t)}{t^{w(\mathcal{K})}}$, where $P_{\mathcal{K}}(q,t)$ is the HOMFLY polynomial for $\mathcal{K}$. Denote $\xi_{\mathcal{K}}(t) = P_{\mathcal{K}}(1,t)$, we have

$$\lim_{q\to 1} \frac{W_A(\mathcal{K};\, q, t)}{W_A(\bigcirc;\, q, t)} = \xi_{\mathcal{K}}(t)^{|A|}\,. \tag{5.7}$$

Combined with (5.6), the proof is completed. q.e.d.

5.2. Symmetry of quantum group invariants. Define

$$\phi_{\vec{\mu}}(q) = \prod_{\alpha=1}^{L} \prod_{j=1}^{\ell(\mu^\alpha)} (q^{-\mu_j^\alpha/2} - q^{\mu_j^\alpha/2}) .$$

Comparing (3.3) and (1.2), we have

$$\begin{aligned} F_{\vec{\mu}} &= \sum_{d|\vec{\mu}} \frac{1}{d} \sum_{\vec{A}} \frac{\chi_{\vec{A}}(\vec{\mu}/d)}{\mathfrak{z}_{\vec{\mu}/d}} f_{\vec{A}}\left(q^d, t^d\right) \\ &= \sum_{d|\vec{\mu}} \frac{1}{d\cdot \mathfrak{z}_{\vec{\mu}/d}} \sum_{\vec{A}} \chi_{\vec{A}}(\vec{\mu}/d) \sum_{\vec{B}} P_{\vec{B}}(q^d, t^d) \prod_{\alpha=1}^{L} M_{A^\alpha B^\alpha}(q^d) \\ &= \sum_{d|\vec{\mu}} \frac{1}{d} \cdot \frac{\phi_{\vec{\mu}/d}(q^d)}{\mathfrak{z}_{\vec{\mu}/d}} \sum_{\vec{B}} \chi_{\vec{B}}(\vec{\mu}/d) P_{\vec{B}}(q^d, t^d) \\ &= \phi_{\vec{\mu}}(q) \sum_{d|\vec{\mu}} \frac{1}{d\cdot \mathfrak{z}_{\vec{\mu}/d}} \sum_{\vec{B}} \chi_{\vec{B}}(\vec{\mu}/d) P_{\vec{B}}(q^d, t^d) , \end{aligned}$$

and

$$\frac{F_{\vec{\mu}}}{\phi_{\vec{\mu}}(q)} = \sum_{d|\vec{\mu}} \frac{1}{d\cdot \mathfrak{z}_{\vec{\mu}/d}} \sum_{\vec{B}} \chi_{\vec{B}}(\vec{\mu}/d) P_{\vec{B}}(q^d, t^d) . \tag{5.8}$$

Apply the Möbius inversion formula,

$$P_{\vec{B}}(q,t) = \sum_{\vec{\mu}} \frac{\chi_{\vec{B}}(\vec{\mu})}{\phi_{\vec{\mu}}(q)} \sum_{d|\vec{\mu}} \frac{\mu(d)}{d} F_{\vec{\mu}/d}(q^d, t^d) \tag{5.9}$$

where $\mu(d)$ is the Möbius function defined as follows:

$$\mu(d) = \begin{cases} (-1)^r, & \text{if } d \text{ is a product of } r \text{ distinct prime numbers;} \\ 0, & \text{otherwise.} \end{cases}$$

To prove the existence of formula (1.5), we need to prove:

- Symmetry of $P_{\vec{B}}(q,t) = P_{\vec{B}}(q^{-1},t)$.
- The lowest degree $\left(q^{-1/2} - q^{1/2}\right)$ in $P_{\vec{B}}$ is no less than -2.

Combining (5.8), (3.5), and (3.4), it's not difficult to find that the first property on the symmetry of $P_{\vec{B}}$ follows from the following lemma.

Lemma 5.2. $W_{\vec{A}^t}(q,t) = (-1)^{\|\vec{A}\|} W_{\vec{A}}(q^{-1},t)$.

Proof. The following irreducible decomposition of $U_q(\mathfrak{sl}_N)$ modules is well-known:

$$V^n = \bigoplus_{B\vdash n} d_B V_B ,$$

where $d_B = \chi_B(C_{(1^n)})$.

Let $d_{\vec{A}} = \prod_{\alpha=1}^{L} d_{A^\alpha}$. Combined with Lemma 4.2 and the eigenvalue of $\mathcal{Y}$ in Lemma 5.1 and (3.1), we have

$$W_{(\square,\ldots,\square)}(L_{\vec{d}}) = \sum_{|\vec{A}|=\vec{d}} W_{\vec{A}}(\mathcal{L}) d_{\vec{A}}$$

$$= \sum_{|\vec{A}|=\vec{d}} d_{\vec{A}}\, q^{\sum_\alpha \kappa_{A^\alpha} w(\mathcal{K}_\alpha)/2} \cdot t^{\sum_\alpha |A^\alpha| w(\mathcal{K}_\alpha)/2} \epsilon_{\vec{A}} \cdot \prod_{\alpha=1}^{L} \dim_q V_{A^\alpha}\,, \tag{5.10}$$

Where $\epsilon_{\vec{A}}$ is the eigenvalue of $\mathcal{Y}$ on $\bigotimes_{\alpha=1}^{L} V_{A^\alpha}$ as defined in the proof of Lemma 5.1. Here if we change A^α to $(A^\alpha)^t$, we have $\kappa_{(A^\alpha)^t} = -\kappa_{A^\alpha}$, which is equivalent to keeping A^α while changing q to q^{-1}.

Note that $W_{(\square,\ldots,\square)}(L_{\vec{d}})$ is essentially a HOMFLY polynomial of $\mathcal{L}_{\vec{d}}$ by Example 3.4. From the expansion of HOMFLY polynomial (3.6) and Example 3.4 (2), we have

$$W_{(\square,\ldots,\square)}(L_{\vec{d}};\, q^{-1}, t) = (-1)^{\sum_\alpha |A_\alpha|} W_{(\square,\ldots,\square)}(L_{\vec{d}};\, q, t)\,. \tag{5.11}$$

However, one can generalize the definition of quantum group invariants of links in the following way. Note that in the definition of quantum group invariants, the enhancement of $\check{\mathcal{R}}$, $K_{2\rho}$, acts on X_i (see (4.1)) as a scalar $q^{-\frac{1}{2}(N+1-2i)}$. We can actually generalize this scalar to any z_i^α where α corresponds the strands belonging to the αth component (cf. [**42**]). It's not difficult to see that (5.10) still holds. The quantum dimension of V_{A^α} thus becomes $s_{A^\alpha}(z_1^\alpha, \ldots, z_N^\alpha)$ obtained in the same way as (4.5).

We rewrite the above generalized version of quantum group invariants of links as $W_{\vec{A}}(\mathcal{L};\, q, t; z)$, where $z = \{z^\alpha\}$. (5.11) becomes

$$W_{(\square,\ldots,\square)}(L_{\vec{d}};\, q^{-1}, t; -z) = (-1)^{\sum_\alpha |A_\alpha|} W_{(\square,\ldots,\square)}(L_{\vec{d}};\, q, t; z), \tag{5.12}$$

Now, combining (5.12), (5.10), and (5.11), we obtain

$$\sum_{\vec{A}^t} d_{\vec{A}^t}\, q^{-\sum \kappa_{(A^\alpha)^t} w(\mathcal{K}_\alpha)/2} \cdot t^{\sum |(A^\alpha)^t| w(\mathcal{K}_\alpha)/2} \epsilon_{\vec{A}^t}(q^{-1};\, -z) \prod_{\alpha=1}^{L} s_{(A^\alpha)^t}(-z^\alpha)$$
$$= (-1)^{\sum_\alpha |A_\alpha|} \cdot \sum_{\vec{A}} d_{\vec{A}}\, q^{\sum \kappa_{A^\alpha} w(\mathcal{K}_\alpha)/2} \cdot t^{\sum |A^\alpha| w(\mathcal{K}_\alpha)/2} \epsilon_{\vec{A}}(q;\, z) \prod_{\alpha=1}^{L} s_{A^\alpha}(z^\alpha). \tag{5.13}$$

Note the following facts:

$$s_{A^t}(-z) = (-1)^{\ell(A)} s_A(z)\,, \tag{5.14}$$

$$d_{\vec{A}^t} = d_{\vec{A}}\,., \tag{5.15}$$

where the second formula follows from

$$\chi_{A^t}(C_\mu) = (-1)^{|\mu|-\ell(\mu)} \chi_A(C_\mu)\,. \tag{5.16}$$

Apply (5.14) and (5.15) to (5.13). Let $z_i^\alpha = q^{-\frac{1}{2}(N+1-2i)}$; then using $-z$ instead of z is equivalent to substituting q by q^{-1} while keeping t in the definition of quantum group invariants of links. This can be seen by comparing

$$p_\mu(z_i^\alpha = q^{-\frac{N-2i+1}{2}}) = \prod_{j=1}^{\ell(\mu)} \frac{t^{-\mu_j/2} - t^{\mu_j/2}}{q^{-\mu_j/2} - q^{\mu_j/2}}$$

with

$$p_\mu(-z^\alpha) = (-1)^{\ell(\mu)} p_\mu(z^\alpha)\,.$$

Therefore, we have

$$\epsilon_{\vec{A}^t}(q^{-1}, t) = \epsilon_{\vec{A}}(q, t)\,. \tag{5.17}$$

By the formula of quantum dimension, it is easy to verify that

$$W_{\vec{A}^t}(\bigcirc^L;\, q, t) = (-1)^{\|\vec{A}\|} W_{\vec{A}}(\bigcirc^L;\, q^{-1}, t), \tag{5.18}$$

Combining (5.4), (5.17), and (5.18), the proof of the Lemma is then completed. q.e.d.

By Lemma 5.2, we have the following expansion about $P_{\vec{B}}$:

$$P_{\vec{B}}(q,t) = \sum_{g\geq 0} \sum_{Q\in\mathbb{Z}/2} N_{\vec{B},g,Q} (q^{-1/2} - q^{1/2})^{2g-2N_0} t^Q$$

for some N_0. We will show $N_0 \leq 1$.

Let $q = e^u$. The pole order of $(q^{-1/2} - q^{1/2})$ in $P_{\vec{B}}$ is the same as the pole order of u.

Let $f(u)$ be a Laurent series in u. Denote $\deg_u f$ to be the lowest degree of u in the expansion of u in f.

Combined with (5.8), $N_0 \leq 1$ follows from the following lemma.

Lemma 5.3. $\deg_u F_{\vec{\mu}} \geq \ell(\vec{\mu}) - 2$.

Lemma 5.3 can be proved through the following cut-and-join analysis.

5.3. Cut-and-join analysis.

5.3.1. Cut-and-join operators. Let $\vec{\tau} = (\tau_1, \cdots, \tau_L)$, and substitute

$$W_{\vec{A}}(\mathcal{L};\, q, t; \vec{\tau}) = W_{\vec{A}}(\mathcal{L};\, q, t) \cdot q^{\sum_{\alpha=1}^L \kappa_{A^\alpha}\tau_\alpha/2}$$

in the Chern-Simons partition function; then we have the framed partition function

$$Z(\mathcal{L};\, q, t, \vec{\tau}) = 1 + \sum_{\vec{A}\neq 0} W_{\vec{A}}(\mathcal{L};\, q, t, \vec{\tau}) \cdot s_{\vec{A}}(x)\,.$$

Similarly, we have the framed free energy

$$F(\mathcal{L};\, q, t, \vec{\tau}) = \log Z(\mathcal{L};\, q, t, \vec{\tau})\,.$$

We also defined the framed version of $Z_{\vec{\mu}}$ and $F_{\vec{\mu}}$ as follows:

$$Z(\mathcal{L};\, q,t,\vec{\tau}) = 1 + \sum_{\vec{\mu}\neq 0} Z_{\vec{\mu}}(q,t,\vec{\tau}) \cdot p_{\vec{\mu}}(x)\,,$$

$$F(\mathcal{L};\, q,t,\vec{\tau}) = \sum_{\vec{\mu}\neq 0} F_{\vec{\mu}}(q,t,\vec{\tau}) p_{\vec{\mu}}(x)\,.$$

One important fact of these framing series is that they satisfy the cut-and-join equation which will give a good control of $F_{\vec{\mu}}$.

Define exponential cut-and-join operator $\mathfrak{E}$,

$$\mathfrak{E} = \sum_{i,j\geq 1}\left(ijp_{i+j}\frac{\partial^2}{\partial p_i \partial p_j} + (i+j)p_i p_j \frac{\partial}{\partial p_{i+j}}\right), \tag{5.19}$$

and log cut-and-join operator $\mathfrak{L}$,

$$\mathfrak{L}F = \sum_{i,j\geq 1}\left(ijp_{i+j}\frac{\partial^2 F}{\partial p_i \partial p_j} + (i+j)p_i p_j \frac{\partial F}{\partial p_{i+j}} + ijp_{i+j}\frac{\partial F}{\partial p_i}\frac{\partial F}{\partial p_j}\right). \tag{5.20}$$

Here $\{p_i\}$ are regarded as independent variables. Schur function $s_A(x)$ is then a function of $\{p_i\}$.

Schur function s_A is an eigenfunction of exponential cut-and-join with eigenvalue κ_A [**8**, **12**, **49**]. Therefore, $Z(\mathcal{L};\, q,t,\vec{\tau})$ satisfies the exponential cut-and-join equation

$$\frac{\partial Z(\mathcal{L};\, q,t,\vec{\tau})}{\partial \tau_\alpha} = \frac{u}{2}\mathfrak{E}_\alpha Z(\mathcal{L};\, q,t,\vec{\tau})\,, \tag{5.21}$$

or equivalently, we also have the log cut-and-join equation

$$\frac{\partial F(\mathcal{L};\, q,t,\vec{\tau})}{\partial \tau_\alpha} = \frac{u}{2}\mathfrak{L}_\alpha F(\mathcal{L};\, q,t,\vec{\tau})\,. \tag{5.22}$$

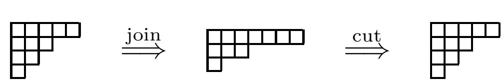
In the above notation, $\mathfrak{E}_\alpha$ and $\mathfrak{L}_\alpha$ correspond to variables $\{p_i\}$ which take value of $\{p_i(x^\alpha)\}$.

(5.22) specializes to $\vec{\mu}$ will be of the following form:

$$\frac{\partial F_{\vec{\mu}}}{\partial \tau_\alpha} = \frac{u}{2}\left(\sum_{|\vec{\nu}|=|\vec{\mu}|,\, \ell(\vec{\nu})=\ell(\vec{\mu})\pm 1} \alpha_{\vec{\mu}\vec{\nu}} F_{\vec{\nu}} + \text{nonlinear terms}\right), \tag{5.23}$$

where $\alpha_{\vec{\mu}\vec{\nu}}$ is some constant and $\vec{\nu}$ is obtained by cutting or jointing of $\vec{\mu}$.

Recall that given two partitions A and B, we say A is a cutting of B if one cuts a row of the Young diagram of B into two rows and reforms it into a new Young diagram which happens to be the Young diagram of A, and we also say B is a joining of A. For example, $(7,3,1)$ is a joining of $(5,3,2,1)$ where we join 5 and 2 to get 7 boxes. Using a Young diagram, it looks like



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In (5.23), cutting and joining happens only for the αth partition.

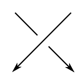
5.3.2. Degree of u. By (3.5), it is easy to see through induction that for two links $\mathcal{L}_1$ and $\mathcal{L}_2$,

$$F_{(\square,\cdots,\square)}(\mathcal{L}_1 \otimes \mathcal{L}_2) = 0\,. \tag{5.24}$$

For simplicity of writing, we denote this by

$$F_{(\square,\cdots,\square)}(\mathcal{L}) = F^\circ(\mathcal{L}).$$

Note that when we put the labeling irreducible representation by the fundamental ones, quantum group invariants of links reduce to HOMFLY polynomials except for a universal factor. Therefore, if we apply skein relation, the following version of skein relation can be obtained. Let positive crossing



used later *appear between two different components*, $\mathcal{K}_1$ and $\mathcal{K}_2$, of the link. Denote $lk = lk(\mathcal{K}_1, \mathcal{K}_2)$; then

$$F^\circ\left(\diagdown\!\!\!\!\!\diagup\right) - F^\circ\left(\diagup\!\!\!\!\!\diagdown\right) = t^{-lk+\frac{1}{2}}(q^{-\frac{1}{2}} - q^{\frac{1}{2}})F^\circ\left()\;(\right) \tag{5.25}$$

We want to claim that

$$\deg_u F^\circ(\mathcal{L}) \geq L - 2\,, \tag{5.26}$$

where L is the number of components of $\mathcal{L}$.

First, for a knot $\mathcal{K}$, a simple computation shows that

$$\deg_u F_\square(\mathcal{K}) = -1\,.$$

Using induction, we may assume that claim (5.26) holds for $L \leq k$. When $L = k+1$, by (5.25),

$$\deg_u\left(F^\circ\left(\diagdown\!\!\!\!\!\diagup\right) - F^\circ\left(\diagup\!\!\!\!\!\diagdown\right)\right) = 1 + \deg_u\left(F^\circ\left()\;(\right)\right) \geq k-1\,. \tag{5.27}$$

However, if

$$\deg_u\left(F^\circ\left(\diagdown\!\!\!\!\!\diagup\right)\right) < k-1\,,$$

this will imply that in the procedure of unlinking $\mathcal{L}$, the lowest-degree term of u in F° are always the same. However, this unlinking will lead to F° equal to 0 due to (5.24), which is a contradiction! This implies that if the number of components of $\mathcal{L}_1$ is greater than the number of components of $\mathcal{L}_2$, we always have

$$\deg_u F^\circ(\mathcal{L}_1) > \deg_u F^\circ(\mathcal{L}_2).$$

Therefore, we proved the claim (5.26).

5.3.3. Induction procedure of cut-and-join analysis. Let $\mathfrak{S}_A$ be the minimal projection from $\mathcal{H}_n \to \mathcal{H}_A$, and

$$\mathfrak{P}_\mu = \sum_A \chi_A(C_\mu)\mathfrak{S}_A\,.$$

Since the Hecke algebra H_n and the group algebra of symmetric group $\mathbb{C}[S_n]$ have the same branching rules, by taking $q \to 1$ and noting that $\mathbb{C}[S_n] \cong \mathrm{End}_{\mathfrak{sl}_N}(V^{\otimes n})$, one decomposes the tensor products $\mathfrak{P}_{(\mu_1)} \otimes \cdots \otimes \mathfrak{P}_{(\mu_{\ell(\mu)})}$ and $\mathfrak{P}_\mu$ into minimal idempotents, which have the same multiplicity $\chi_A(C_\mu)$ on $\mathfrak{S}_A$. This is a direct corollary of taking trace on a generic element with diagonal entries $x_1,\ldots,x_N$ in the Cartan subgroup and comparing symmetric power functions $p_\mu(x_1,\cdots,x_N)$ and $\prod_{i=1}^{\ell(\mu)} p_{\mu_i}(x_1,\cdots,x_N)$.

Given partitions $\mu^i = (\mu^i_1,\ldots,\mu^i_{\ell_i})$, $i = 1,\ldots,L$. We define

$$\hat{Z}_{\vec{\mu}} = Z_{\vec{\mu}} \cdot \mathfrak{z}_{\vec{\mu}}, \qquad \hat{F}_{\vec{\mu}} = F_{\vec{\mu}} \cdot \mathfrak{z}_{\vec{\mu}}\,.$$

Let $\mathcal{L}$ be the closure of a braid β with writhe number 0. Cable the ith component of β, β_i, by substituting $\ell(\mu^i)$ parallel strands for each strand of β_i. The $\ell(\mu^i)$ parallel components are colored by partitions $(\mu^i_1),\ldots,(\mu^i_{\ell(\mu^i)})$. We take the closure of this new braid and obtain a new link, denoted by $\mathcal{L}^{(c)}_{\vec{\mu}}$. We will call it the $\vec{\mu}$-colored cabling of $\mathcal{L}$.

Let $|\mu^\alpha| = d_\alpha$, $\alpha = 1,\ldots,L$. By (3.4) and Lemma 4.2,

$$\begin{aligned}
\hat{Z}_{\vec{\mu}}(\mathcal{L}) &= \sum_{\vec{A}} \chi_{\vec{A}}(C_{\vec{\mu}}) \mathrm{Tr}\Big(h(\mathcal{L}_{(d_1,\ldots,d_L)}) \cdot \mathfrak{S}_{A^1} \otimes \cdots \otimes \mathfrak{S}_{A^L}\Big) \\
&= \mathrm{Tr}\Big(h(\mathcal{L}_{(d_1,\ldots,d_L)}) \cdot \mathfrak{P}_{\mu^1} \otimes \cdots \otimes \mathfrak{P}_{\mu^L}\Big) \\
&= \mathrm{Tr}\Big(h(\mathcal{L}_{(d_1,\ldots,d_L)}) \cdot \otimes_{\alpha=1}^L \otimes_{i=1}^{\ell(\mu^\alpha)} \mathfrak{P}_{\mu^\alpha_i}\Big) \\
&= \hat{Z}_{\big((\mu^1_1),\cdots,(\mu^1_{\ell(\mu^1)}),\cdots,(\mu^L_{\ell(\mu^L)})\big)}(\mathcal{L}^{(c)}_{\vec{\mu}})\,.
\end{aligned}$$

Combine (3.5), nad we have the following *colored cabling formulas*:

$$\hat{Z}_{(\mu^1,\cdots,\mu^L)}(\mathcal{L}) = \hat{Z}_{\big((\mu^1_1),\cdots,(\mu^1_{\ell_1}),\cdots,(\mu^L_1),\cdots,(\mu^L_{\ell_L})\big)}(\mathcal{L}^{(c)}_{\vec{\mu}})\,, \tag{5.28}$$

$$\hat{F}_{(\mu^1,\cdots,\mu^L)}(\mathcal{L}) = \hat{F}_{\big((\mu^1_1),\cdots,(\mu^1_{\ell_1}),\cdots,(\mu^L_1),\cdots,(\mu^L_{\ell_L})\big)}(\mathcal{L}^{(c)}_{\vec{\mu}})\,. \tag{5.29}$$

Let $\vec{\tau} = (\tau_1, 0, \cdots, 0)$. Similar to (4.10), we give the following formula:

$$\begin{aligned}
F_{\vec{\mu}} &= \sum_{|\Lambda|=\vec{\mu}} \Theta_\Lambda \prod_{\alpha=1}^{\ell(\Lambda)} \sum_{\vec{A^\alpha}} \frac{\chi_{\vec{A^\alpha}}(\vec{\Lambda^\alpha})}{\mathfrak{z}_{\vec{\Lambda^\alpha}}} W_{\vec{A^\alpha}}(q,t) q^{\frac{\kappa_{A^\alpha_1}\tau_1}{2}} \\
&= \sum_{p\geq 0} \frac{(\tau_1 u/2)^p}{p!} \sum_{|\Lambda|=\vec{\mu}} \Theta_\Lambda \prod_{\alpha=1}^{\ell(\Lambda)} \sum_{\sum_\alpha p_\alpha = p} \sum_{\vec{A^\alpha}} \frac{\chi_{\vec{A^\alpha}}(\vec{\Lambda^\alpha})}{\mathfrak{z}_{\vec{\Lambda^\alpha}}} W_{\vec{A^\alpha}}(q,t) k^{p_\alpha}_{A^\alpha_1}\,.
\end{aligned}$$

By lemma 5.1, $W_{\vec{A}}(\mathcal{L})$ and $W_{\vec{A}}(\bigcirc^{\otimes L})$ have the same lowest degree in u. The coefficient of $u^{\ell(\vec{\mu})-2}$ in the expansion of $F_{\vec{\mu}}$ contains at least a non-zero term with certain positive power in τ_1 by the non-vanishing result of (4.11).

We have proved that $\deg_u F_{(\square,\ldots,\square)} \geq L-2$. Assume that if $|\mu^\alpha| \leq d$, $\deg_u F_{\vec{\mu}} \geq \ell(\vec{\mu}) - 2$. By the colored cabling formula (5.29), the general case can be reduced to that all the colors associated with the knot components are of type (n) for some n. Without loss of generality, let $\big((n_1), \ldots, (n_L)\big)$ such that $n_1 = d+1$ and $n_i \leq d$ for $i \geq 2$. If we can prove

$$\deg_u F_{\big((n_1),\ldots,(n_L)\big)} \geq L-2\,,$$

the proof is completed by induction. Consider the cut-and-join equation for τ_1:

$$\begin{aligned}&\frac{\partial F_{\big((d,1),(n_2),\cdots,(n_L)\big)}}{\partial \tau_1}\\ &= \frac{u}{2}\Big((d+1)F_{\big((d+1),(n_2)\cdots,(n_L)\big)} + \sum_{i+j=d, i\geq j\geq 1} \beta_{i,j} F_{\big((i,j,1),(n_2),\ldots,(n_L)\big)}\Big) + *\end{aligned}$$

where $\beta_{i,j}$ are some constants and $*$ represents some non-linear terms in the cut-and-join equation. The crucial observation of these non-linear terms are that their degrees in u are equal to $L-1$. By assumption, $\deg_u F_{\big((i,j,1),(n_2),\ldots,(n_L)\big)} \geq L$. Comparing the degree in u on both sides of the equation, we have

$$\deg_u F_{\big((d+1),(n_2),\ldots,(n_L)\big)} \geq (L+1) - 2 - 1 = L-2\,,$$

which completes the induction.

Define

$$\widetilde{F}_{\vec{\mu}} = \frac{F_{\vec{\mu}}}{\phi_{\vec{\mu}}(q)}, \qquad \widetilde{Z}_{\vec{\mu}} = \frac{Z_{\vec{\mu}}}{\phi_{\vec{\mu}}(q)}\,.$$

Lemma 5.3 directly implies the following:

Corollary 5.4. *$\widetilde{F}$ are of the following form:*

$$\widetilde{F}_{\vec{\mu}}(q,t) = \sum_{\textit{finitely many } n_\alpha} \frac{a_\alpha(t)}{[n_\alpha]^2} + \textit{ polynomial.}$$

Remark 5.5. Combining the above Corollary, (3.7), and (3.5), we have

$$\widetilde{F}_{(\square,\ldots,\square)}(q,t) = \frac{a(t)}{[1]^2} + \text{ polynomial.} \tag{5.30}$$

5.4. Framing and pole structures. Let $\delta_n = \sigma_1 \cdots \sigma_{n-1}$. $\{\delta_n; n \in \mathbb{Z}\}$ generate the skein of the annulus [**43**] (for a systematic discussion of δ_n, one may refer to [**2**]). $h(\delta_n^n)$ is an central element in $\mathcal{H}_n(q)$ satisfying (refer to [**29**, **3**] for the proof)

$$h(\delta_n^n) \cdot \mathfrak{S}_A = q^{\frac{1}{2}\kappa_A} \mathfrak{S}_A \,.$$

Let π_A be the central idempotent labeled by partition A according to the decomposition (4.2). The eigenvalues of $\delta_n \cdot \pi_A$ are either 0 or $q^{\frac{1}{2n}\kappa_A}$ multiplied by an nth root of unity. Let ζ_A denote the character of the irreducible representation of $\mathcal{H}_n(q)$. Then

$$\zeta_A(h(\delta_n)) = a_A \cdot q^{\frac{1}{2n}\kappa_A}$$

for some rational number a_A. Taking $q \to 1$, $h(\delta_n)$ degenerates to a permutation of partition type (n), which implies $a_A = \chi_A(C_{(n)})$. Note that in this case, only those Young diagrams that have the shape of a hook will contribute non-zero a_A.

Let $\vec{d} = \big((d_1), \ldots, (d_L)\big)$. Because of (5.28) and (5.29), we can simply deal with all the color of one row without loss of generality. Take framing $\tau_\alpha = -n_\alpha + \frac{1}{d_\alpha}$ and choose a braid group representative of $\mathcal{L}$ such that the writhe number of $\mathcal{L}_\alpha$ is n_α. Denoting by $\vec{\tau} = (\tau_1, \ldots, \tau_L)$,

$$\begin{aligned}
\hat{Z}_{\vec{d}}\Big(\mathcal{L}; q, t; \vec{\tau}\Big) &= \sum_{\vec{A}} \chi_{\vec{A}}(C_{\vec{d}}) W_{\vec{A}}(\mathcal{L}; q, t) q^{\frac{1}{2}\sum_{\alpha=1}^L \kappa_{A^\alpha}\left(-n_\alpha + \frac{1}{d_\alpha}\right)} \\
&= t^{\frac{1}{2}\sum_\alpha d_\alpha n_\alpha} \mathrm{Tr}\Big(h(\mathcal{L}_{\vec{d}}) \sum_{\vec{A}} \chi_{\vec{A}}(C_{\vec{d}}) q^{\frac{1}{2}\sum_\alpha \kappa_{A^\alpha}\frac{1}{d_\alpha}} \otimes_{\alpha=1}^L \mathfrak{S}_{A^\alpha}\Big) \\
&= t^{\frac{1}{2}\sum_\alpha d_\alpha n_\alpha} \mathrm{Tr}\Big(h(\mathcal{L}_{\vec{d}}) \sum_{\vec{A}} \chi_{\vec{A}}(C_{\vec{d}}) h(\delta_{d_1} \otimes \cdots \otimes \delta_{d_L}) \otimes_{\alpha=1}^L \mathfrak{S}_{A^\alpha}\Big) \\
&= t^{\frac{1}{2}\sum_\alpha d_\alpha n_\alpha} \mathrm{Tr}\Big(h(\mathcal{L}_{\vec{d}}) \cdot h(\otimes_{\alpha=1}^L \delta_{d_\alpha}) \cdot \mathfrak{P}_{(1)}^{(d_1)} \otimes \cdots \otimes \mathfrak{P}_{(1)}^{(d_L)}\Big) \,.
\end{aligned}$$

Here $\mathfrak{P}_{(1)}^{(d_\alpha)}$ means that in the projection we use $q^{d_\alpha}, t^{d_\alpha}$ instead of using q, t. If we denote

$$\mathcal{L} * Q_{\vec{d}} = h(\mathcal{L}_{\vec{d}} \cdot \delta_{\vec{d}}) \cdot \mathfrak{P}_{(1)}^{(d_1)} \otimes \cdots \otimes \mathfrak{P}_{(1)}^{(d_L)} \,,$$

we have

$$\hat{Z}_{\vec{d}}\Big(\mathcal{L}; q, t; \frac{1}{\vec{d}}\Big) = t^{\frac{1}{2}\sum_{\alpha=1}^L (d_\alpha - 1)} \mathfrak{H}(\mathcal{L} * Q_{\vec{d}}) \,. \tag{5.31}$$

Here $\mathfrak{H}$ is the HOMFLY polynomial which is normalized as

$$\mathfrak{H}(\text{unknot}) = \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \,.$$

With the above normalization, for any given link $\mathcal{L}$, we have

$$[1]^L \cdot \mathfrak{H}(\mathcal{L}) \in \mathbb{Q}\big[[1]^2, t^{\pm\frac{1}{2}}\big] \,.$$

Substituting q by q^{d_α} in the corresponding component, it leads to the following:

$$\prod_{\alpha=1}^{L}[d_\alpha]\cdot \hat{Z}_{\vec{d}}(\mathcal{L};q,t;\vec{\tau})\in\mathbb{Q}\big[[1]^2,t^{\pm\frac{1}{2}}\big]\,. \tag{5.32}$$

On the other hand, given any frame $\vec{\omega}=(\omega_1,\dots,\omega_L)$,

$$\begin{aligned}\hat{Z}_{\vec{\mu}}(\mathcal{L};\,q,t;\vec{\omega}) &= \sum_{\vec{A}}\chi_{\vec{A}}(C_{\vec{\mu}})W_{\vec{A}}(\mathcal{L};\,q,t;\vec{\omega})\\ &= \sum_{\vec{A}}\chi_{\vec{A}}(C_{\vec{\mu}})\sum_{\vec{\nu}}\frac{\chi_{\vec{A}}(C_{\vec{\nu}})}{\mathfrak{z}_{\vec{\nu}}}\hat{Z}_{\vec{\nu}}(\mathcal{L};\,q,t)q^{\frac{1}{2}\sum_\alpha\kappa_{A^\alpha}\omega_\alpha}\,.\end{aligned}$$

Exchanging the order of summation, we have the following *convolution formula*:

$$\hat{Z}_{\vec{\mu}}(\mathcal{L};\,q,t;\vec{\omega})=\sum_{\vec{\nu}}\frac{\hat{Z}_{\vec{\nu}}(\mathcal{L};\,q,t)}{\mathfrak{z}_{\vec{\nu}}}\sum_{\vec{A}}\chi_{\vec{A}}(C_{\vec{\mu}})\chi_{\vec{A}}(C_{\vec{\nu}})q^{\frac{1}{2}\sum_\alpha\kappa_{A^\alpha}\omega_\alpha}\,. \tag{5.33}$$

Return to (5.32). This property holds for arbitrary choice of n_α, $\alpha=1,\dots,L$. By the convolution formula, the coefficients of possible other poles vanish for arbitrary integer n_α. q^{n_α} is involved through certain polynomial relations since κ_{A^α} are even numbers by direct computation and summations on the r.h.s of the above convolution formula w.r.t $\vec{\nu}$ and $\vec{A}$ are finite. This implies the coefficients for those possible poles other than that produced by $\frac{1}{\prod_{\alpha=1}^{L}[n_\alpha]}$ are simply zero. Therefore, (5.32) holds for any frame.

Consider $\hat{Z}_{\vec{d}}(\mathcal{L})$, where each component of $\mathcal{L}$ is labeled by a young diagram of one row. As discussed above, this assumption does not lose any generality.

Let $\mathcal{K}$ be a component of $\mathcal{L}$ labeled by the color (c). If we multiply $\widetilde{Z}_{\vec{d}}(\mathcal{L})$ by $[c]^2$, we call it normalizing $\widetilde{Z}_{\vec{d}}(\mathcal{L})$ w.r.t $\mathcal{K}$, denoted by $\widetilde{Z}^{(\mathcal{K})}_{\vec{d}}(\mathcal{L})$; if $\mathcal{K}$ is not a component of $\mathcal{L}$, let $\widetilde{Z}^{(\mathcal{K})}_{\vec{d}}(\mathcal{L})=\widetilde{Z}_{\vec{d}}(\mathcal{L})$.

Similarly, if $\mathcal{K}$ is a component of $\mathcal{L}$ colored by (c), we call $[c]^2\widetilde{F}_{\vec{d}}(\mathcal{L})$ normalizing $\widetilde{F}_{\vec{d}}(\mathcal{L})$ w.r.t $\mathcal{K}$ and denote it by $\widetilde{F}^{(\mathcal{K})}_{\vec{d}}(\mathcal{L})$; if $\mathcal{K}$ is not a component of $\mathcal{L}$, define $\widetilde{F}^{(\mathcal{K})}_{\vec{d}}(\mathcal{L})=\widetilde{F}_{\vec{d}}(\mathcal{L})$. Combining with (5.30), if $\mathcal{K}$ is a component of $\mathcal{L}$, one obtains the following fact:

$$\widetilde{F}^{(\mathcal{K})}_{\square,\dots,\square}(q,t)\in\mathbb{Q}\big[[1]^2,t^{\pm\frac{1}{2}}\big]\,. \tag{5.34}$$

Comparing the definition of above normalized invariants w.r.t $\mathcal{K}$ to (3.5), the computation can be passed to normalized invariants w.r.t $\mathcal{K}$:

$$\widetilde{F}^{(\mathcal{K})}_{\vec{d}}(\mathcal{L}) = \sum_{|\Lambda|=\vec{d}} \Theta_\Lambda \prod_{j=1}^{\ell(\Lambda)} \widetilde{Z}^{(\mathcal{K})}_{\Lambda_j}(\mathcal{L})\,. \tag{5.35}$$

Again, choose a braid representative with the writhe number of the ith component equal to n_i and choose framing $\vec{\tau} = (-n_1+\frac{1}{d_1}, \ldots, -n_L+\frac{1}{d_L})$. Combining (5.31), if $\mathcal{K}$ is a component of $\mathcal{L}$, we have

$$\widetilde{F}^{(\mathcal{K})}_{\vec{d}}(\mathcal{L}; q, t; \vec{\tau}) = \widetilde{F}^{(\mathcal{K})}_{\square,\ldots,\square}(\mathcal{L} * Q_{\vec{d}})\,, \tag{5.36}$$

which is a polynomial of $[1]^2$ and $t^{\pm\frac{1}{2}}$ since $\forall k$, $[k]^2$ is a polynomial of $[1]^2$. As discussed above, arbitrary choice of n_i leads to the conclusion that $\widetilde{F}^{(\mathcal{K})}_{\vec{d}}(\mathcal{L}; q, t)$ is polynomial of $[1]^2$ and $t^{\pm\frac{1}{2}}$ for the canonical framing.

To summarize, we thus proved the following proposition:

Proposition 5.6. *With notations as above, we have*

$$\prod_{\alpha=1}^{L} [d_\alpha] \cdot \hat{Z}_{\vec{d}}(\mathcal{L}; q, t) \in \mathbb{Q}[[1]^2, t^{\pm\frac{1}{2}}]; \tag{5.37}$$

$$[d_\alpha]^2 \widetilde{F}_{\vec{d}}(\mathcal{L}; q, t) \in \mathbb{Q}\big[[1]^2, t^{\pm\frac{1}{2}}\big], \forall \alpha\,. \tag{5.38}$$

For a given $\vec{d}$, denote by $D_{\vec{d}}$ the $\gcd\{d_1, \ldots, d_L\}$, and choose a braid group element representative for $\mathcal{L}$ such that its αth component has writhe number n_α. For any $k|D_{\vec{d}}$, we choose $\vec{\tau} = (-n_1+\frac{1}{k}, \ldots, -n_L+\frac{1}{k})$.

$$\begin{aligned}
\hat{Z}_{\vec{d}}(\mathcal{L}; q, t; \vec{\tau}) &= \sum_{\vec{A}} \chi_{\vec{A}}(C_{\vec{d}}) W_{\vec{A}}(\mathcal{L}; q, t) q^{\frac{1}{2}\sum_\alpha \kappa_{A^\alpha}(-n_\alpha+\frac{1}{k})} \\
&= t^{\frac{1}{2}\sum_\alpha d_\alpha n_\alpha} \mathrm{Tr}\Big(h(\mathcal{L}_{\vec{d}}) \sum_{\vec{A}} \chi_{\vec{A}}(C_{\vec{d}}) q^{\frac{1}{2}\sum_\alpha \kappa_{A^\alpha}\frac{1}{k}} \otimes_{\alpha=1}^{L} \mathfrak{S}_{A^\alpha}\Big) \\
&= t^{\frac{1}{2}\sum_\alpha d_\alpha n_\alpha} \mathrm{Tr}\Big(h\Big(\mathcal{L}_{\vec{d}} \cdot \otimes_{\alpha=1}^{L} \delta_{d_\alpha}^{\frac{d_\alpha}{k}}\Big) \cdot \otimes_{\alpha=1}^{L} \mathfrak{P}^{(k)}_{(d_\alpha/k)}\Big) \\
&= t^{\frac{1}{2}\sum_\alpha d_\alpha n_\alpha} \hat{Z}_{\vec{d}/k}\Big(h\Big(\mathcal{L}_{\vec{d}} \cdot \otimes_{\alpha=1}^{L} \delta_{d_\alpha}^{\frac{d_\alpha}{k}}\Big); q^k, t^k\Big)\,.
\end{aligned} \tag{5.39}$$

This implies that $\hat{Z}_{\vec{d}}$ is a rational function of $q^{\pm\frac{1}{2}k}$ and $t^{\pm\frac{1}{2}k}$ for any $k|D_{\vec{d}}$, and hence a rational function of $q^{\pm\frac{1}{2}D_{\vec{d}}}$ and $t^{\pm\frac{1}{2}D_{\vec{d}}}$. Proposition 5.6 implies that the principal part of $\widetilde{F}_{\vec{d}}$ is a possible summation of $\frac{a(t)}{[k]^2}$, where k divides each d_α, or equivalently, $k|D_{\vec{d}}$. Therefore, the principal part of $\widetilde{F}_{\vec{d}}$ must be of the form $\frac{a(t)}{[D_{\vec{d}}]^2}$. For any $k|D_{\vec{d}}$, we have the following structure:

$$\widetilde{F}_{\vec{d}}(\mathcal{L};q,t)=\frac{H_{\vec{d}/k}(\mathcal{L}_{\vec{d}}\cdot\otimes_{\alpha=1}^{L}\delta_{d_\alpha}^{\frac{d_\alpha}{k}};t^k)}{[D_{\vec{d}}]^2}+\text{polynomial}\,. \tag{5.40}$$

Note that, in the above formula, $H_{\vec{d}/k}(\mathcal{L}_{\vec{d}}\cdot\otimes_{\alpha=1}^{L}\delta_{d_\alpha}^{\frac{d_\alpha}{k}};t^k)$ is independent of the choice of $k|D_{\vec{d}}$ and can be expressed as $H_{\vec{d}/D_{\vec{d}}}(t^{D_{\vec{d}}})$. $H_{\vec{d}/D_{\vec{d}}}(t)$ is a function of t which only depends on $\vec{d}/D_{\vec{d}}$ and $\mathcal{L}$.

Once again, due to arbitrary choice of n_α, we know the above pole structure of $\widetilde{F}_{\vec{d}}$ holds for any frame.

The above discussion w.r.t the color $\vec{d}$ for any $\mathcal{L}$ is equivalent to the general coloring according to (5.28) and (5.29), which leads to the following proposition:

Proposition 5.7. *Notation is as above. Assume $\mathcal{L}$ is labeled by the color $\vec{\mu}=(\mu^1,\dots,\mu^L)$. Denote by $D_{\vec{\mu}}$ the greatest common divisor of $\{\mu^1_1,\dots,\mu^1_{\ell(\mu^1)},\dots,\mu^i_j,\dots,\mu^L_{\ell(\mu^L)}\}$. $\widetilde{F}_{\vec{\mu}}$ has the following structure:*

$$\widetilde{F}_{\vec{\mu}}(q,t)=\frac{H_{\vec{\mu}/D_{\vec{\mu}}}(t^{D_{\vec{\mu}}})}{[D_{\vec{\mu}}]^2}+f(q,t)\,,$$

where $f(q,t)\in\mathbb{Q}\big[[1]^2,t^{\pm\frac{1}{2}}\big]$.

Remark 5.8. In Proposition 5.7, it is very interesting to interpret in topological string side that $H_{\vec{\mu}/D_{\vec{\mu}}}(t)$ only depends on $\vec{\mu}/D_{\vec{\mu}}$ and $\mathcal{L}$. The principal term is generated due to summation of counting rational curves and independent choice of k in the labeling color $k\cdot\vec{\mu}/D_{\vec{\mu}}$. This phenomenon simply tells us that contributions of counting rational curves in the labeling color $k\cdot\vec{\mu}/D_{\vec{\mu}}$ are through multiple cover contributions of $\vec{\mu}/D_{\vec{\mu}}$.

6. Integrality

6.1. A ring characterizes the partition function. Let $\vec{d}=(d_1,\dots,d_L)$, $y=(y^1,\dots,y^L)$. Define

$$T_{\vec{d}}=\sum_{\vec{B}=\vec{d}}s_{\vec{B}}(y)P_{\vec{B}}(q,t)\,. \tag{6.1}$$

By the calculation in Appendix A, one will get

$$T_{\vec{d}}=q^{|\vec{d}|}\sum_{k|\vec{d}}\frac{\mu(k)}{k}\sum_{\mathfrak{A}\in\mathcal{P}(\mathcal{P}^n),\|\mathfrak{A}\|=\vec{d}/k}\Theta_{\mathfrak{A}}W_{\mathfrak{A}}(q^k,t^k)s_{\mathfrak{A}}(z^k). \tag{6.2}$$

However,

$$\begin{aligned} T_{\vec{d}} &= \sum_{|\vec{B}|=\vec{d}} s_{\vec{B}}(y) P_{\vec{B}}(q,t) \\ &= \sum_{g=0}^{\infty} \sum_{Q\in\mathbb{Z}/2} \bigg(\sum_{|\vec{B}|=\vec{d}} N_{\vec{B};\,g,Q} s_{\vec{B}}(y) \bigg) (q^{1/2}-q^{-1/2})^{2g-2} t^Q \,. \end{aligned}$$

Denote by $\Omega(y)$ the space of all integer coefficient symmetric functions in y. Since Schur functions form a basis of $\Omega(y)$ over $\mathbb{Z}$, $N_{\vec{B};\,g,Q} \in \mathbb{Z}$ will follow from

$$\sum_{|\vec{B}|=\vec{d}} N_{\vec{B};\,g,Q} s_{\vec{B}}(y) \in \Omega(y)\,.$$

Let $v = [1]_q^2$. It's easy to see that $[n]_q^2$ is a monic polynomial of v with integer coefficients. The following ring is very crucial in characterizing the algebraic structure of the Chern-Simons partition function, which will lead to the integrality of $N_{\vec{B};\,g,Q}$.

$$\mathfrak{R}(y;v,t) = \left\{ \frac{a(y;v,t)}{b(v)} \,:\, a(y;v,t) \in \Omega(y)[v,t^{\pm 1/2}], b(v) = \prod_{n_k} [n_k]_q^2 \in \mathbb{Z}[v] \right\}.$$

If we slightly relax the condition in the ring $\mathfrak{R}(y;\,v,t)$, we have the following ring which is convenient in the p-adic argument in the following subsection.

$$\mathcal{M}(y;\,q,t) = \left\{ \frac{f(y;\,q,t)}{b(v)} \,:\, f \in \Omega(y)[q^{\pm 1/2}, t^{\pm 1/2}], b(v) = \prod_{n_k} [n_k]_q^2 \in \mathbb{Z}[v] \right\}.$$

Given $\frac{f(y;\,q,t)}{b(v)} \in \mathcal{M}(y;\,q,t)$, if $f(y;\,q,t)$ is a primitive polynomial in terms of $q^{\pm 1/2}$, $t^{\pm 1/2}$ and Schur functions of y, we say $\frac{f(y;\,q,t)}{b(v)}$ is primitive.

6.2. Multi-cover contribution and p-adic argument.

Proposition 6.1. $T_{\vec{d}}(y;\,q,t) \in \mathfrak{R}(y;v,t)$.

Proof. Recall the definition of quantum group invariants of links. By the formula of universal R-matrix, it's easy to see that

$$W_{\vec{A}}(\mathcal{L}) \in \mathcal{M}(y;\,q,t)\,.$$

Since we have already proven the existence of the pole structure in the LMOV conjecture, Proposition 6.1 will be naturally satisfied if we can prove

$$T_{\vec{d}}(y;\,q,t) \in \mathcal{M}(y;\,q,t)\,. \tag{6.3}$$

Before diving into the proof of (6.3), let's do some preparation. For

$$\forall \mathfrak{A} = (\vec{A}_1, \vec{A}_2, \dots) \in \mathcal{P}(\mathcal{P}^L)\,,$$

define

$$\mathfrak{A}^{(d)} = \Big(\underbrace{\vec{A}_1, \dots, \vec{A}_1}_{d}, \underbrace{\vec{A}_2, \dots, \vec{A}_2}_{d}, \dots \Big) .$$

Lemma 6.2. *If $\Theta_{\mathfrak{A}} = \frac{c}{d}$, $d > 1$ and $\gcd(c, d) = 1$, then for any $r|d$, we can find $\mathfrak{B} \in \mathcal{P}(\mathcal{P}^L)$ such that $\mathfrak{A} = \mathfrak{B}^{(r)}$.*

Proof. Let $\ell = \ell(\mathfrak{A})$; then we have

$$\Theta_{\mathfrak{A}} = \frac{(-1)^{\ell(\mathfrak{A})-1}}{\ell(\mathfrak{A})} u_{\mathfrak{A}} = \frac{(-1)^{\ell(\mathfrak{A})-1}(\ell(\mathfrak{A})-1)!}{|\operatorname{Aut}\mathfrak{A}|} .$$

Let

$$\mathfrak{A} = \Big(\underbrace{\vec{A}_1, \dots, \vec{A}_1}_{m_1}, \dots, \underbrace{\vec{A}_n, \dots, \vec{A}_n}_{m_n} \Big) ,$$

so $\ell(\mathfrak{A}) = m_1 + \cdots + m_n$. Note that

$$u_{\mathfrak{A}} = \binom{\ell}{m_1,\, m_2, \cdots,\, m_n} .$$

Let $\eta = \gcd(m_1, m_2, \cdots, m_n)$. We have $\mathfrak{A} = \widetilde{\mathfrak{A}}^{(\eta)}$, where

$$\widetilde{\mathfrak{A}} = \Big(\underbrace{\vec{A}_1, \dots, \vec{A}_1}_{m_1/\eta}, \dots, \underbrace{\vec{A}_n, \dots, \vec{A}_n}_{m_n/\eta} \Big) .$$

By Corollary A.4 , $\frac{\ell}{\eta} | u_{\mathfrak{A}}$, and $\gcd(c, d) = 1$, one has $d|\eta$. We can take $\mathfrak{B} = \widetilde{\mathfrak{A}}^{(\frac{\eta}{r})}$. This completes the proof. q.e.d.

Remark 6.3. By the choice of η in the above proof, we know $|\vec{A}_\alpha|$ is divisible by η for any α.

By Lemma 6.2 and (6.2), we know $T_{\vec{d}}$ is of form $f(y;\, q, t)/k$, where $f \in \mathcal{M}(y;\, q, t)$, $k|\vec{d}$. We will show k is in fact 1.

Given $\frac{r}{s}\frac{f(y;\, q,t)}{b(v)}$ where

$$\frac{f(y;\, q,t)}{b(v)} \in \mathcal{M}(y;\, q,t)$$

is primitive, define

$$\operatorname{Ord}_p\left(\frac{r}{s}\frac{f(y;\, q,t)}{b(v)}\right) = \operatorname{Ord}_p\left(\frac{r}{s}\right) . \tag{6.4}$$

Lemma 6.4. *Given $\mathfrak{A} \in \mathcal{P}\left(\mathcal{P}^L\right)$, p any prime number, and $f_{\mathfrak{A}}(y;\, q,t) \in \mathcal{M}(y;\, q,t)$, we have*

$$\operatorname{Ord}_p\Big(\Theta_{\mathfrak{A}^{(p)}} f_{\mathfrak{A}^{(p)}}(y;\, q,t) - \frac{1}{p}\Theta_{\mathfrak{A}} f_{\mathfrak{A}}(y^p;\, q^p, t^p)\Big) \geq 0 .$$

Proof. Assume $\Theta_{\mathfrak{A}} = \frac{b}{p^r \cdot a}$, where $\gcd(p^r \cdot a, b) = 1$, $p \nmid a$. In (6.5), minus is taken except for one case: $p = 2$ and $r = 0$, in which the calculation is very simple and the same result holds. Therefore, we only show the general case in which the minus sign is taken. By Lemma 6.2, one can choose $\mathfrak{B} \in \mathcal{P}(\mathcal{P}^L)$ such that $\mathfrak{A} = \mathfrak{B}^{(p^r)}$. Note that $f_{\mathfrak{A}} = f_{\mathfrak{B}}^{p^r}$.

Let $s = \ell(\mathfrak{B})$ and

$$\mathfrak{B} = \Big(\underbrace{\vec{B}_1, \ldots, \vec{B}_1}_{s_1}, \ldots, \underbrace{\vec{B}_k, \ldots, \vec{B}_n}_{s_k} \Big).$$

Since $\ell(\mathfrak{A}^{(p)}) = p \cdot \ell(\mathfrak{A})$, by Theorem A.2,

$$\begin{aligned}
&\mathrm{Ord}_p \left(\Theta_{\mathfrak{A}^{(p)}} - \frac{1}{p} \Theta_{\mathfrak{A}} \right) \\
&= \mathrm{Ord}_p \frac{1}{p \cdot \ell(\mathfrak{A})} \left[\binom{p^{r+1} s}{p^{r+1} s_1, \cdots, p^{r+1} s_k} - \binom{p^r s}{p^r s_1, \cdots, p^r s_k} \right] && (6.5)\\
&\geq 2(r+1) - (1+r) \\
&> 0\,. && (6.6)
\end{aligned}$$

By Theorem A.6,

$$\begin{aligned}
&\mathrm{Ord}_p \left[\frac{1}{p} \Theta_{\mathfrak{A}} \Big(f_{\mathfrak{A}^{(p)}}(y;\, q, t) - f_{\mathfrak{A}}(y^p;\, q^p, t^p) \Big) \right] \\
&= \mathrm{Ord}_p \left[\frac{b}{p^{r+1} a} \Big(f_{\mathfrak{B}}(y;\, q, t)^{p^{r+1}} - f_{\mathfrak{B}}(y^p;\, q^p, t^p)^{p^r} \Big) \right] \\
&\geq 0\,.
\end{aligned}$$

Apply the above two inequalities to

$$\mathrm{Ord}_p \left(\Theta_{\mathfrak{A}^{(p)}} f_{\mathfrak{A}^{(p)}}(y;\, q, t) - \frac{1}{p} \Theta_{\mathfrak{A}} f_{\mathfrak{A}}(y^p;\, q^p, t^p) \right) = \mathrm{Ord}_p$$
$$\left[\left(\Theta_{\mathfrak{A}^{(p)}} - \frac{1}{p} \Theta_{\mathfrak{A}} \right) f_{\mathfrak{A}^{(p)}}(y;\, q, t) + \frac{1}{p} \Theta_{\mathfrak{A}} \Big(f_{\mathfrak{A}^{(p)}}(y;\, q, t) - f_{\mathfrak{A}}(y^p;\, q^p, t^p) \Big) \right].$$

The proof is completed. q.e.d.

Applying the above Lemma, we have

$$\mathrm{Ord}_p \left(\Theta_{\mathfrak{A}^{(p)}} W_{\mathfrak{A}^{(p)}}(q, t) s_{\mathfrak{A}^{(p)}}(z) - \frac{1}{p} \Theta_{\mathfrak{A}} W_{\mathfrak{A}}(q^p, t^p) s_{\mathfrak{A}}(z^p) \right) \geq 0\,. \tag{6.7}$$

Let

$$\Phi_{\vec{d}}(y;\, q, t) = \sum_{\mathfrak{A} \in \mathcal{P}(\mathcal{P}^n),\, \|\mathfrak{A}\| = \vec{d}} \Theta_{\mathfrak{A}} W_{\mathfrak{A}}(q, t) s_{\mathfrak{A}}(z)\,.$$

By Lemma 6.2, one has

$$\left\{ \mathfrak{B} : \| \mathfrak{B} \| = p\, \vec{d} \text{ and } \mathrm{Ord}_p(\Theta_{\mathfrak{B}}) < 0 \right\} = \left\{ \mathfrak{A}^{(p)} : \| \mathfrak{A} \| = \vec{d} \right\}.$$

Therefore, by (6.7),

$$\begin{aligned}
&\mathrm{Ord}_p\left(\Phi_{p\vec{d}}(y;\, q,t)-\frac{1}{p}\Phi_{\vec{d}}(y^p;q^p,t^p)\right)\\
&=\sum_{\|\mathfrak{A}\|=\vec{d}}\mathrm{Ord}_p\left(\Theta_{\mathfrak{A}^{(p)}}W_{\mathfrak{A}^{(p)}}(q,t)s_{\mathfrak{A}^{(p)}}(z)-\frac{1}{p}\Theta_{\mathfrak{A}}W_{\mathfrak{A}}(q^p,t^p)s_{\mathfrak{A}}(z^p)\right)\\
&\geq 0\,.
\end{aligned}$$

We have thus proven the following Lemma.

Lemma 6.5. *For any prime number p and $\vec{d}$,*

$$\mathrm{Ord}_p\left(\Phi_{p\vec{d}}(y;\, q,t)-\frac{1}{p}\Phi_{\vec{d}}(y^p;q^p,t^p)\right)\geq 0\,.$$

For any $p|\vec{d}$, by (6.2),

$$\begin{aligned}
T_{\vec{d}}&=q^{|\vec{d}|}\sum_{k|\vec{d}}\frac{\mu(k)}{k}\sum_{\mathfrak{A}\in\mathcal{P}(\mathcal{P}^L),\,\|\mathfrak{A}\|=\vec{d}/k}\Theta_{\mathfrak{A}}W_{\mathfrak{A}}(q^k,t^k)s_{\mathfrak{A}}(z^k)\\
&=q^{|\vec{d}|}\Bigg(\sum_{k|\vec{d},\,p\nmid k}\frac{\mu(k)}{k}\Phi_{\vec{d}/k}(y^n;\,q^n,t^n)\\
&\quad+\sum_{k|\vec{d},\,p\nmid k}\frac{\mu(pk)}{pk}\Phi_{\vec{d}/(pk)}(y^{pk};\,q^{pk},t^{pk})\Bigg)\\
&=q^{|\vec{d}|}\sum_{k|\vec{d},\,p\nmid k}\frac{\mu(k)}{k}\left(\Phi_{\vec{d}/k}(y^k;\,q^k,t^k)-\frac{1}{p}\Phi_{\vec{d}/(pk)}(y^{pk};\,q^{pk},t^{pk})\right).
\end{aligned}$$

By Lemma 6.5,

$$\mathrm{Ord}_p\,T_{\vec{d}}\geq 0\,. \tag{6.8}$$

By the arbitrary choice of p, we prove that $T_{\vec{d}}\in\mathcal{M}(y;\,q,t)$, and hence

$$T_{\vec{d}}\in\mathfrak{R}(y;v,t)\,.$$

The proof of Proposition 6.1 is completed. q.e.d.

6.3. Integrality. By (5.9) and Proposition 5.7 (note that $\phi_{\vec{\mu}/d}(q^d, t^d) = \phi_{\vec{\mu}}(q,t)$),

$$\begin{aligned}
P_{\vec{B}}(q,t) &= \sum_{\vec{\mu}} \frac{\chi_{\vec{B}}(\vec{\mu})}{\phi_{\vec{\mu}}(q)} \sum_{d|\vec{\mu}} \frac{\pi(d)}{d} F_{\vec{\mu}/d}(q^d, t^d) \\
&= \sum_{\vec{\mu}} \chi_{\vec{B}}(\vec{\mu}) \sum_{d|\vec{\mu}} \frac{\pi(d)}{d} \widetilde{F}_{\vec{\mu}/d}(q^d, t^d) \\
&= \sum_{\vec{\mu}} \chi_{\vec{B}}(\vec{\mu}) \sum_{d|D_{\vec{\mu}}} \frac{\pi(d)}{d} \frac{H_{\vec{\mu}/D_{\vec{\mu}}}(t^{D_{\vec{\mu}}})}{[D_{\vec{\mu}}]^2} + \text{polynomial} \\
&= \sum_{\vec{\mu}} \chi_{\vec{B}}(\vec{\mu}) \delta_{1,D_{\vec{\mu}}} \frac{H_{\vec{\mu}/D_{\vec{\mu}}}(t^{D_{\vec{\mu}}})}{[D_{\vec{\mu}}]^2} + \text{polynomial}\,,
\end{aligned}$$

where $\delta_{1,n}$ equals 1 if $n = 1$ and 0 otherwise. It implies that $P_{\vec{B}}$ is a rational function which only has pole at $q = 1$. In the above computation, we used a fact of Möbius inversion,

$$\sum_{d|n} \frac{\pi(d)}{d} = \delta_{1,n}\,.$$

Therefore, for each $\vec{B}$,

$$\sum_{g=0}^{\infty} \sum_{Q \in \mathbb{Z}/2} N_{\vec{B};\,g,Q} (q^{-1/2} - q^{1/2})^{2g} t^Q \in \mathbb{Q}[(q^{-1/2} - q^{1/2})^2, t^{\pm\frac{1}{2}}]\,.$$

On the other hand, by Proposition 6.1, $T_{\vec{d}} \in \mathfrak{R}(y;\, v, t)$ and $\mathrm{Ord}_p\, T_{\vec{d}} \geq 0$ for any prime number p. We have

$$\sum_{|\vec{B}| = \vec{d}} N_{\vec{B};\,g,Q}\, s_{\vec{B}}(y) \in \Omega(y)\,.$$

This implies $N_{\vec{B};\,g,Q} \in \mathbb{Z}$.

Combining the above discussions, we have

$$\sum_{g=0}^{\infty} \sum_{Q \in \mathbb{Z}/2} N_{\vec{B};\,g,Q} (q^{-1/2} - q^{1/2})^{2g} t^Q \in \mathbb{Z}[(q^{-1/2} - q^{1/2})^2, t^{\pm 1/2}]\,.$$

The proof of Theorem 2 is completed.

7. Concluding remarks and future research

In this section, we briefly discuss some interesting problems related to string duality which may be approached through the techniques developed in this paper.

7.1. Duality from a mathematical point of view. Let $\mathbf{p} = (\mathbf{p}^1, \dots, \mathbf{p}^L)$, where $\mathbf{p}^\alpha = (p_1^\alpha, p_2^\alpha, \dots,)$. Define the generating series of open Gromov-Witten invariants

$$F_{g,\vec{\mu}}(t,\tau) = \sum_\beta K^\beta_{g,\vec{\mu}}(\tau) e^{\int_\beta \omega}$$

where ω is the Kähler class of the resolved conifold, τ is the framing parameter, and

$$t = e^{\int_{\mathbb{P}^1} \omega}, \quad \text{and} \quad e^{\int_\beta \omega} = t^Q .$$

Consider the following generating function:

$$F(\mathbf{p};\, u, t;\, \tau) = \sum_{g=0}^\infty \sum_{\vec{\mu}} u^{2g-2+\ell(\vec{\mu})} F_{g,\vec{\mu}}(t;\, \tau) \prod_{\alpha=1}^L p^\alpha_{\mu^\alpha} .$$

It satisfies the log cut-and-join equation

$$\frac{\partial F(\mathbf{p};\, u, t;\, \tau)}{\partial \tau} = \frac{u}{2} \sum_{\alpha=1}^L \mathfrak{L}_\alpha F(\mathbf{p};\, u, t;\, \tau) .$$

Therefore, duality between Chern-Simons theory and open Gromov-Witten theory reduces to verifying the uniqueness of the solution of the cut-and-join equation.

The cut-and-join equation for the Gromov-Witten side comes from the degeneracy and gluing procedure, while uniqueness of the cut-and-join system should in principle be obtained from the verification at some initial value. However, it seems very difficult to find a suitable initial value. For example, in the case of topological vertex theory, the cut-and-join system has singularities when the framing parameter takes value at 0, -1, ∞ while these points are the possible ones to evaluate at. One solution might be through studying the Riemann-Hilbert problem on controlling the monodromy at three singularity points. When the case goes beyond, the situation will be even more complicated. A universal method of handling uniqueness is required for the final proof of the Chern-Simons/topological string duality conjecture.

A new hope might be found in our development of cut-and-join analysis. In the log cut-and-join equation (5.22), the non-linear terms reveal the important recursion structure. For the uniqueness of the cut-and-join equation, it will appear as the vanishing of all non-linear terms. We will put this in our future research.

7.2. Other related problems. There are many other problems related to our work on the LMOV conjecture. We briefly list some problems that we are working on.

The volume conjecture was proposed by Kashaev in [**16**] and reformulated by [**33**]. It relates the volume of hyperbolic 3-manifolds to the limits of quantum invariants. This conjecture was later generalized to complex case [**34**] and to incomplete hyperbolic structures [**13**]. The study of this conjecture is still at a rather primitive stage [**35**, **17**, **48**, **10**].

The LMOV conjecture has shed new light on volume conjecture. The cut-and-join analysis we developed in this paper combined with rank-level duality in Chern-Simons theory seems to provide a new way to prove the existence of the limits of quantum invariants.

There are also other open problems related to the LMOV conjecture. For example, quantum group invariants satisfy skein relations which must have some implications on the topological string side, as mentioned in [**19**]. One could also rephrase a lot of unanswered questions in knot theory in terms of open Gromov-Witten theory. We hope that the relation between knot theory and open Gromov-Witten theory will be explored much more in detail in the future. This will definitely open many new avenues for future research.

Appendix A. Appendix

Here we carry out the calculation in Section 6.1.

$$\begin{aligned}
T_{\vec{d}} &= \sum_{|\vec{B}|=\vec{d}} s_{\vec{B}}(y) P_{\vec{B}}(q,t) \\
&= \sum_{|\vec{B}|=\vec{d}} \frac{s_{\vec{B}}(y)\chi_{\vec{B}}(\vec{\mu})}{\phi_{\vec{\mu}}(q)} \sum_{k|\vec{\mu}} \frac{\mu(k)}{k} F_{\vec{\mu}/k}\left(q^k, t^k\right) \\
&= \sum_{k|\vec{\mu},|\vec{\mu}|=\vec{d}} \frac{\mu(k)}{k} \frac{p_{\vec{\mu}}(y)}{\phi_{\vec{\mu}}(q)} \sum_{\Lambda\in\mathcal{P}(\mathcal{P}^n),\,|\Lambda|=\vec{\mu}/k} \Theta_\Lambda Z_\Lambda(q^k,t^k) \\
&= \sum_{k|\vec{\mu},|\vec{\mu}|=\vec{d}} \frac{\mu(k)}{k} \sum_{\|\Lambda\|=\vec{\mu}/k} \Theta_\Lambda \phi_\Lambda^{-1}(q^k) p_\Lambda(y^k) Z_\Lambda(q^k,t^k) \\
&= \sum_{k|\vec{\mu},|\vec{\mu}|=\vec{d}} \frac{\mu(k)}{k} \sum_{\|\Lambda\|=\vec{\mu}/k} \Theta_\Lambda \phi_\Lambda^{-1}(q^k) p_\Lambda(y^k) \prod_{\beta=1}^{\ell(\Lambda)} \sum_{\vec{\mathfrak{A}}_\beta} \frac{\chi_{\vec{\mathfrak{A}}_\beta}(\vec{\Lambda}_\beta)}{\mathfrak{z}_{\vec{\Lambda}_\beta}} W_{\vec{\mathfrak{A}}_\beta}(q^k) \\
&= \sum_{k|\vec{d}} \frac{\mu(k)}{k} \sum_{|\vec{\mathfrak{A}}_\beta|=|\vec{\Lambda}_\beta|=\vec{d}_\beta/k} \frac{(-1)^{\ell(\Lambda)-1}}{\ell(\Lambda)} \cdot \mathfrak{u}_\Lambda \\
&\qquad \times \prod_{\beta=1}^{\ell(\Lambda)} \left\{ \phi_{\vec{\Lambda}_\beta}^{-1}(q^k) p_{\vec{\Lambda}_\beta}(y^k) \sum_{\vec{\mathfrak{A}}_\beta} \frac{\chi_{\vec{\mathfrak{A}}_\beta}(\vec{\Lambda}_\beta)}{\mathfrak{z}_{\vec{\Lambda}_\beta}} W_{\vec{\mathfrak{A}}_\beta}(q^k,t^k) \right\} \\
&= \sum_{k|\vec{\mu},|\vec{\mu}|=\vec{d}} \frac{\mu(k)}{k} \sum_{\mathfrak{A}=(\vec{\mathfrak{A}}_1,\ldots),\,\|\mathfrak{A}\|=\vec{d}/k} \frac{(-1)^{\ell(\mathfrak{A})-1}}{\ell(\mathfrak{A})} \cdot \mathfrak{u}_{\mathfrak{A}} \\
&\qquad \times \prod_{\beta=1}^{\ell(\mathfrak{A})} \left\{ W_{\vec{\mathfrak{A}}_\beta}(q^k,t^k) \sum_{\vec{\Lambda}_\beta} \frac{\chi_{\vec{\mathfrak{A}}_\beta}(\vec{\Lambda}_\beta)}{\mathfrak{z}_{\vec{\Lambda}_\beta}} \phi_{\vec{\Lambda}_\beta}^{-1}(q^k) p_{\vec{\Lambda}_\beta}(y^k) \right\} \\
&= q^{|\vec{d}|} \sum_{k|\vec{d}} \frac{\mu(k)}{k} \sum_{\mathfrak{A}\in\mathcal{P}(\mathcal{P}^L),\,\|\mathfrak{A}\|=\vec{d}/k} \Theta_{\mathfrak{A}} W_{\mathfrak{A}}(q^k,t^k) s_{\mathfrak{A}}(z^k)
\end{aligned}$$

where

$$p_n(z) = p_n(y)\cdot p_n(x_i = q^{i-1})\,.$$

A.1.

Lemma A.1. *p is prime, $r \geq 1$. Then*

$$\binom{p^r a}{p^r b} - \binom{p^{r-1}a}{p^{r-1}b} \equiv 0 \mod (p^{2r}).$$

Proof. Consider the ratio

$$\frac{\binom{p^r a}{p^r b}}{\binom{p^{r-1}a}{p^{r-1}b}} = \frac{\prod_{k=1}^{p^r b} \frac{(a-b)p^r+k}{k}}{\prod_{k=1}^{p^{r-1}b} \frac{(a-b)p^{r-1}+k}{k}}$$

$$= \prod_{\substack{\gcd(k,p)=1,\\ 1\le k\le p^r b}} \frac{(a-b)p^r+k}{k}$$

$$\equiv 1 + p^r(a-b) \sum_{\substack{\gcd(k,p)=1,\\ 1\le k\le p^r b}} \frac{1}{k} \mod (p^{2r})\,.$$

Let

$$A_p(n) = \sum_{\substack{1\le k\le n,\\ \gcd(k,p)=1}} \frac{1}{k}\,.$$

Therefore, the proof of the lemma can be completed by showing

$$A_p(p^r b) = \frac{p^r c}{d}$$

for some c, d such that $\gcd(d,p) = 1$.

If $\gcd(k,p) = 1$, there exist α_k, β_k such that

$$\alpha_k k + \beta_k p^r = 1\,.$$

Let

$$B_p(n) = \sum_{1\le k\le n, \gcd(k,p)=1} k\,.$$

By the above formula,

$$\begin{aligned} A_p(p^r b) &\equiv b A_p(p^r) \mod (p^r) \\ &\equiv b B_p(p^r) \mod (p^r)\,, \end{aligned}$$

and

$$\begin{aligned} B_p(p^r) &= \sum_{k=1}^{p^r} k - p \sum_{k=1}^{p^{r-1}} k \\ &= \frac{p^r(p^r+1)}{2} - p\frac{p^{r-1}(p^{r-1}+1)}{2} \\ &= \frac{p^{2r-1}(p-1)}{2}\,. \end{aligned}$$

Here, we have $2r - 1 \ge r$ since $r \ge 1$. The proof is then completed. q.e.d.

The following theorem is a simple generalization of the above Lemma.

Theorem A.2. *$\sum_{i=1}^n a_i = a$, p is prime, $r \geq 1$; then*

$$\binom{p^r a}{p^r a_1, \cdots, p^r a_n} - \binom{p^r a}{p^{r-1} a_1, \cdots, p^{r-1} a_n} \equiv 0 \mod (p^{2r})$$

Proof. We have

$$\begin{aligned}
&\binom{p^r a}{p^r a_1, \cdots, p^r a_n} - \binom{p^r a}{p^{r-1} a_1, \cdots, p^{r-1} a_n} \\
&\qquad = \prod_{k=1}^n \binom{p^r(a - \sum_{i=1}^{k-1} a_i)}{p^r a_k} - \prod_{k=1}^n \binom{p^{r-1}(a - \sum_{i=1}^{k-1} a_i)}{p^{r-1} a_k} \\
&\qquad \equiv 0 \mod (p^{2r}) .
\end{aligned}$$

In the last step, we used the Lemma A.1. The proof is completed. q.e.d.

Lemma A.3. *We have*

$$\frac{a}{\gcd(a,b)} \Bigg| \binom{a}{b} .$$

Proof. Notice that

$$\binom{a}{b} = \frac{a}{b} \binom{a-1}{b-1},$$

i.e.,

$$\frac{b}{\gcd(a,b)} \binom{a}{b} = \frac{a}{\gcd(a,b)} \binom{a-1}{b-1} .$$

However,

$$\gcd\left(\frac{a}{\gcd(a,b)}, \frac{b}{\gcd(a,b)} \right) = 1 ,$$

so

$$\frac{a}{\gcd(a,b)} \Bigg| \binom{a}{b} .$$

q.e.d.

This direct leads to the following corollary.

Corollary A.4. *$a = a_1 + a_2 + \cdots + a_n$. Then*

$$\frac{a}{\gcd(a_1, a_2, \cdots, a_n)} \Bigg| \binom{a}{a_1,\ a_2, \cdots,\ a_n} .$$

Lemma A.5. *$a, r \in \mathbb{N}$, p is a prime number, then*

$$a^{p^r} - a^{p^{r-1}} \equiv 0 \mod (p^r) .$$

Proof. If a is p, since $p^{r-1} \geq r$, the claim is true. If $\gcd(a,p)=1$, by Fermat theorem, $a^{p-1} \equiv 1 \mod (p)$. We have

$$\begin{aligned} a^{p^r} - a^{p^{r-1}} &= a^{p^{r-1}}\Big((kp+1)^{p^{r-1}(p-1)} - 1\Big) \\ &= a^{p^{r-1}} \sum_{i=1}^{p^{r-1}} (kp)^i \binom{p^{r-1}}{i} \\ &\equiv 0 \mod (p^r)\,. \end{aligned}$$

Here in the last step, we used Lemma A.3. q.e.d.

A direct consequence of the above lemma is the following theorem.

Theorem A.6. *Given $f(y;\, q,t) \in \mathcal{M}(y;\, q,t)$, we have*

$$\mathrm{Ord}_p\Big(f(y;\, q,t)^{p^{r+1}} - f(y^p;\, q^p, t^p)^{p^r}\Big) \geq r+1\,.$$

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