From the Hinge Device to Curves of Linkages

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Abstract

We drew inspiration from a hinge device which is used to draw regular polygons, and then studied curves produced by points on linkages, and proved that all polynomial function curves can be produced by linkage curves. We put forward and solved two other problems, translating curves to linkages by construction, Taylor expansion and FFT, and translating linkages to curves by a magical physical method.

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1. The hinge device to draw regular polygons

There is a kind of hinge device described by one book [1]. It can draw regular n-gon $(n = 5,6,7,8,9,10)$, as in Figure 1, and meet the condition of AB = BC = CD = DE, quadrilateral ABFG is congruent to quadrilateral BCHK, where point D may move freely on line AG and E on BK. So it can guarantee ∠ABC = ∠BCD = ∠CDE when hinge changes shape.

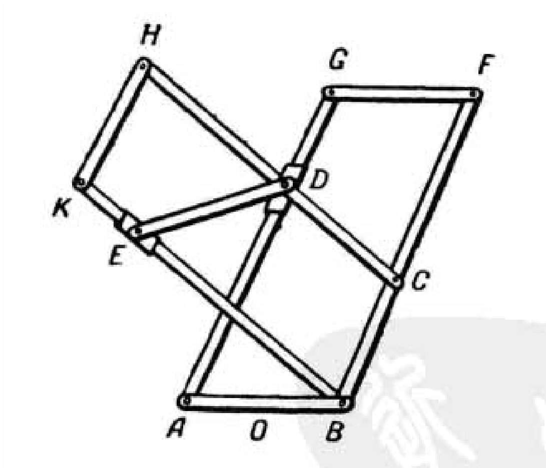

Figure 1

As Figure 2 shows, keeping line AB fixed on paper, sliding point D, E to special location, we can easily draw regular polygon from 5 to 10. The regular polygon's side length is equal to segment AB. When polygon is 9, $\angle Y_5AX = 60°$. When polygon is 10, $\angle Y_6AX = 36°$. We will not repeat to prove its correctness.

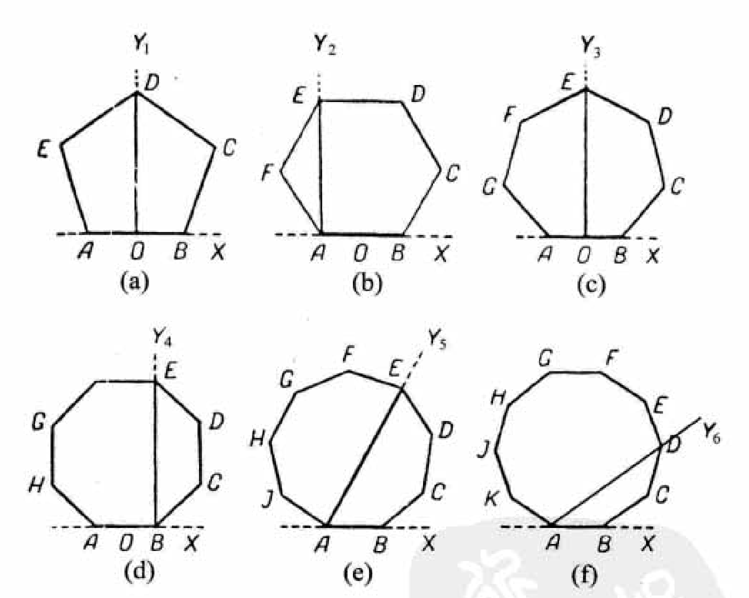

Figure 2

The writer only talked that the hinge device can draw regular polygons from 5 to 10. In fact it can draw any regular polygon except for mechanical problems. For example, when point A is coincided with point D, we can draw regular triangle. When point A is coincided with point E, we can draw square. If line BF can be extended unlimitedly, we can draw any regular polygon. It is obvious that is too correct to repeat.

2. Introduce to the problem of curves of linkages

Seeing the device above, I sighed for its fun and practicability. At the same time, I thought that the vertexes of linkage may be as useful as its sides. So I brought forward some questions.

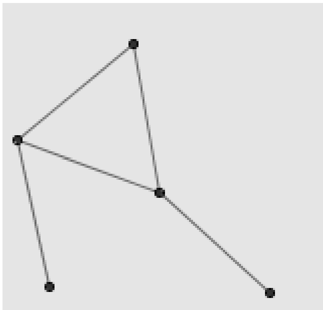

Figure 3[2]

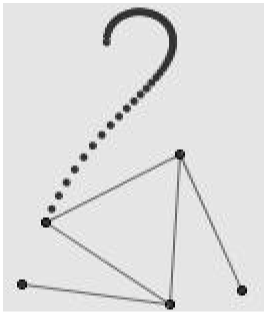

Figure 4

Fixing two points on the bottom on Figure 3, we can draw track of top point in movement. The result is as figure 4. (You can also open Figure 3 in other software to see its track.)

At first we shall define the linkage system. [3]It is one kind of device that is composed of at least one rod and two points. In order to facilitate composition, we consider that the points only exist at the both ends of the rod. We will mention it later. We abstract the linkage system to a graph G = (V, E). It is obviously a connected graph. For set of points V, $p: V \to R^2$. For a graph G, separate V to two disjoint nonempty sets of points (V_s, V_m), of which points in V_s are fixed and points in V_m are movable. For every elements in set of edges e = (u, v), let l_e = dis (u, v), and $\mathrm{L} = \{\mathrm{l_e} \mid \mathrm{e} \in \mathrm{E}\}$ which we call set of length. We call a graph G = (V, E) whose Vs, Vm and L define a linkage system.

In a linkage system, the track formed by movement caused by the point in V_m is called a linkage track. In order to facilitate discussing, we only consider the curve of one point. (Or we just need to consider curves of each point separately) The track $f: V \to R^2$ must meet the following conditions:

$$\forall u \in V_s,\ f(u) = p(u)$$
$$\forall e = (u, v) \in E,\ dis(f(u), f(v)) = l_e$$

Question one: what kind of track can be drawn by a linkage? How to draw the path by a linkage?
Question two: if we can get a class of curves by linkages, how can we get it?
Question three: if we have a linkage device, how can we get the linkage track?

Before solving the questions above, we had something to add. Just now, we prescribed that every points only exist at the both ends of the rod for convenience. Actually, it's a quite common situation where we have to break the rule. In order to discuss this situation, we take advantage of the stability of triangle to meet the requirement that the point only exist on the end of the rod. [3]

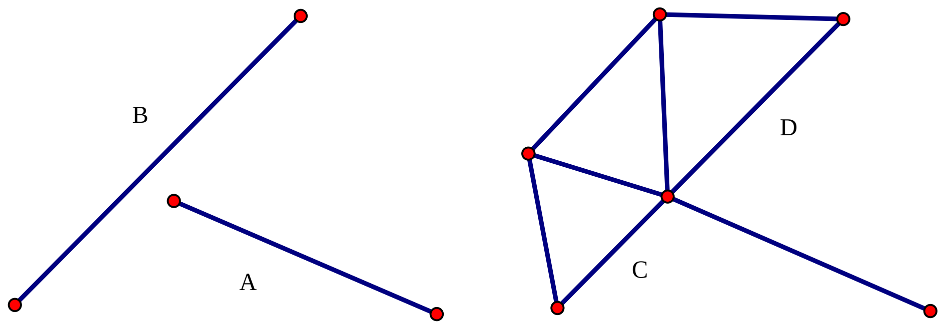

Figure 5

It has the same effect when we linked rod A to rod B, we broke rod B into rod C and rod D

3. Peaucellier Linkage

In the process of studying these issues, we found that there is a track which is very important, so we take it out, described it in a separated section, it is - the straight line.

We made a lot of linkage devices. However, after a long time, we could not make a device which can draw a straight line. Thus, we find Peaucellier Linkage. [5]

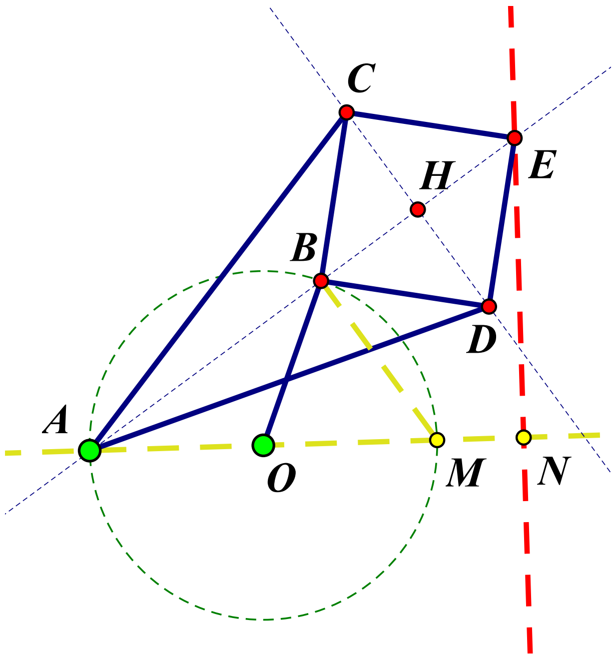

Figure 6

In the figure above, the solid lines represent linkage, the green points represent fixed points, the red points represent movable points, the red dashed line is formed in the process of moving the track by the point E. (After all that said, no state) And AC = AD = a，BC = CE = BD = DE = b，line OB is for any length， OA = OB. The following is the proof of its correctness.

Proof:

Draw line CH⊥BE cross at point H.

In this connecting rod system, point A, B, E, are on the same line obviously, BH = HE.

AB * AE = (AH – BH) * (AH + HE) = $AH^2 – BH^2$

= $(AC^2 – CH^2) – (BC^2 – CH^2) = a^2 – b^2$

Namely, AB * AE are fixed value.

Connect AO and extend it, cross ⊙O at point M.

Find a point N on AO to the left of A, to make AB * AE = AM * AN， connect EN.

∴ △AMB∽△AEN (SAS)。

∴∠ANE = ∠ABM = Rt∠

∴E is always on the line which passes N and is perpendicular to AO.

∴The track of moving E is a straight line.

Q. E. D.

4. Build the edifice of linkage construction

To solve the problems raised in the previous section, we thought: it is also a track, can we solve the problem of the linkage track by using the construction method which was used to draw picture by ruler and compasses? [4] When studying the construction used by ruler and compasses, we established the analytical principle that can be plotted by some types of basic structure that add, subtract or multiply two segments take inverse or square root of a segment. Then in the linkage construction, can we do it in the same way?

4.1 Coordinate axis

To be able to study linkage construction on the plane of rectangular coordinate system, we first establish a coordinate system with a linkage. At this point, there the previously mentioned Peaucellier Linkage is used.

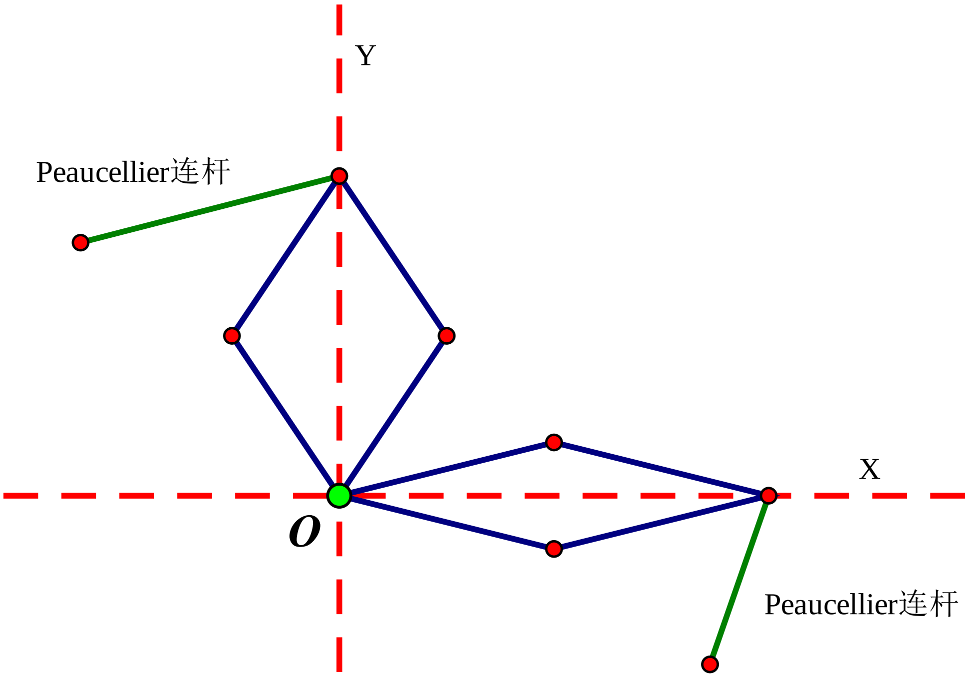

Figure 7

Green rods stand for Peaucellier Linkage, omitting the other six rods.

Using Peaucellier Linkage, it is easy to make two points of two diamonds, respectively, move in the X axis and Y axis. Thus, we have constructed the linkage plane rectangular coordinate system.

In addition, in order to transfer to the length of Y axis to X axis, the diamond of Y axis will be the O-centered clockwise 90°, as shown in the following linkage system, which OA = OC = OD = OF = BA = BC = ED = EF = ℓ, AD = CF = $\sqrt{2}$ ℓ. In this way, we successfully moved segment OB on X axis to the segment OE on Y axis.

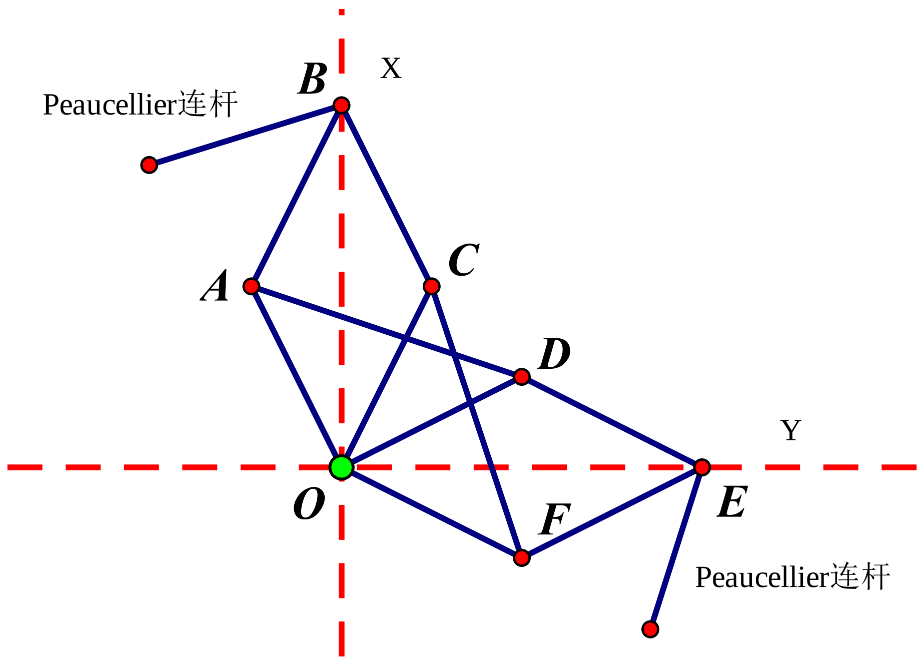

Figure 8

Green rods stand for Peaucellier Linkage, omitting the other six rods.

4.2 Calculator

With the coordinate axes, we are now to construct a variety of calculators to calculate the points on the X axis. (We will then find that we only need to know how to calculate the points on X axis) In the calculator, we will make extensive use of parallelograms, because they are suitable for transferring the length as well as adding vectors.

4. 2. 1 Addition calculator

a) Plus constant c to coordinates of a point

As Figure, in order to get y by adding a constant c to x, we fixed two points A and B, then we can obtain the equation y = x + c by constructing a parallelogram. The correctness is obvious, and we can not do this by only one parallelogram.

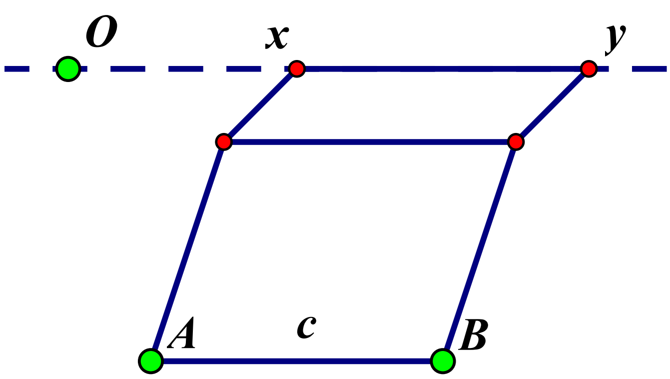

Figure 9

b) Add up the coordinates of two points

Using parallelogram law, we can not only add up two points on X-axis but also add up any two vectors, using the constructed parallelogram as described above. As figure, it is obvious that $\overrightarrow{Oz} = \overrightarrow{Ox} + \overrightarrow{Oy}$.

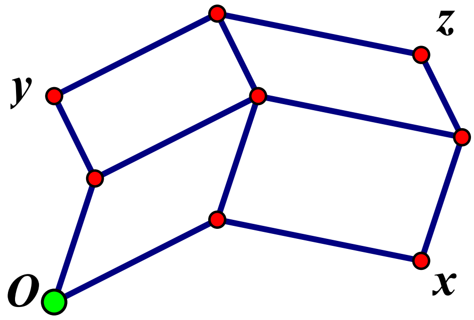

Figure 10

4. 2. 2 Reciprocal calculator

To facilitate the introduction to multiplication calculator, we presented reciprocal calculator first. In the chapter of Peaucellier Linkage, we have proved y * x = $a^2 - b^2$ in Figure 11, so $y = \frac{a^2 - b^2}{x}$. As long as we take appropriate values of a and b so as to make $a^2 - b^2 = 1$, we can get y which is the inverse of x.

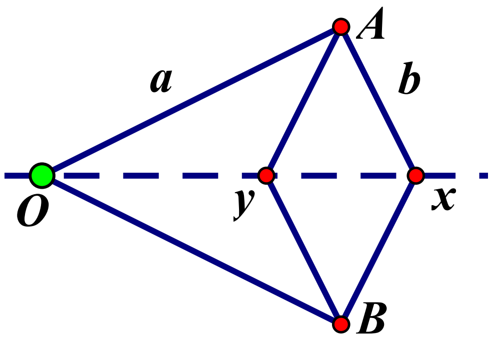

Figure 11

4. 2. 3 Multiplication calculator

a) Multiply a constant c to coordinates of a point

Let OA = c * OB, and construct a parallelogram ABxC. It was clear that △ Obx ∽ △ OAy. Therefore y = x * c.

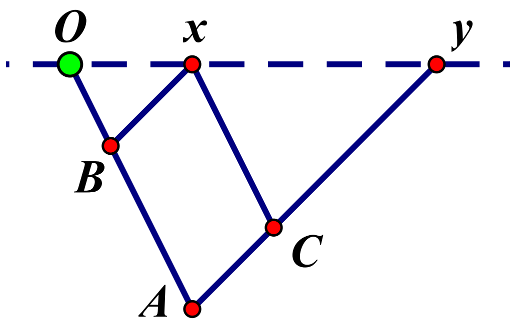

Figure 12

b) Multiply two coordinates

Multiplying two coordinates is not an easy thing, we had thought for a long time but there is no direct way. But we find another way which brought us to

$$x * y = \frac{(x+y)^2 - (x-y)^2}{4}$$

So, as long as we have a square calculator, we can get x * y. However, the square calculator racks our brains. Once again, an inspiration led us to

$$\frac{1}{a-1} - \frac{1}{a+1} = \frac{2}{a^2-1}$$

Finally we got

$$x * y = \frac{1}{\frac{2}{x+y-1} - \frac{2}{x+y+1}} - \frac{1}{\frac{2}{x-y-1} - \frac{2}{x-y+1}}$$

Fantastic! The problem of multiplying two coordinates has been transformed into a reciprocal problem at last! And what make us happy is the problem of reciprocal is not so difficult and can be directly solved. Using the reciprocal calculator described previously, the problem was also solved.

5. Solve the Question One

Question one: what kind of track can be drawn by a linkage? How to draw the path by a linkage?

According to the previous section, we could make addition, multiplication, reciprocal operations to the points on X axis with linkage. Then, what does it means to the curves of linkages? Operations above enable us to construct a point $T(f(x,y),0)$. Then we construct a point S which always lies on (x, y). Finally we fixed point T at origin points, the track of moving S is just the curve $f(x,y)=0$. This idea is amazing but obviously correct, so we don't prove it here.

Now a linkage can draw curves which only contain functions in need of operations above. Namely, any polynomial function $F(x,y)=\sum_{i=0}^{n}\sum_{j=0}^{n}c_{i,j}x^{i}y^{j}=0, c_{i,j}\in R, n\in N$ can be drawn with a linkage. And we can conclude with happiness that all curves which can be drawn with ruler and compasses can be drawn with linkage, and that some curves which cannot be drawn with ruler and compasses can be drawn with linkage, too.

Another funny thing is that Kempe thought you can even use a linkage to sign your name! [6] Because many curves can be fitted into polynomial functions. However, even if your name is simple, you have to use a lot of robs to make the linkage.

However, it is a pity that we haven't found necessary and sufficient condition of drawing functions with linkage. Hopefully we will solve the problem in later edition!

6. Solve the Question Two

Question two: if we can get a class of curves by linkages, how can we get it?

6. 1 Use FFT [7]

At first, I must declare that it is just an approximation algorithm, briefly introduced as follow.

ff (t) is a periodic complex function whose Fourier expansion of complex index form is:

$$ff(t) = x(t) + iy(t) = \sum_{n=-\infty}^{+\infty} F_n e^{in\omega_0 t}$$

We define F_n as Fourier coefficient which is a complex vector. Get answers from:

$$F_n = \frac{1}{T}\int_0^T ff(t)e^{-in\omega_0 t}$$

Discrete Fourier Transform (DFT): discretize *ff (t)* in one period.

Fast Fourier Transform (FFT): a fast algorithm for DFT.

$r_p(t)$ represents any 3-dimension function, and its projection on X axis and YOZ plane is $r_{px}(t)$ and $r_{pyz}(t)$. Separately make 1-dimension and 2-dimension FFT to the actual curve of $r_p(t)$ to get harmonic components as follow:

$$r_{px}(t) = \sum_{n=-\infty}^{+\infty} D_n e^{k\zeta_n} e^{kn\omega t}$$

$$r_{pyz}(t) = \sum_{n=-\infty}^{+\infty} D_n^{'} e^{k\zeta_n^{'}} e^{kn\omega t}$$

D_n and D_n' represent amplitudes of the n-th item. According to geometric relations:

$$\begin{cases} R_n = \sqrt{D_n^2 + D_n^{'2}}, n \in N^* \\ \phi_n = \arccos(\frac{D_n}{R_n}\cos(\zeta_n + n\omega t)), n \in N^* \\ \varphi_n = \arctan(\frac{R_{nx}}{R_{ny}}), n \in N \\ R_{-n} = \sqrt{D_{-n}^2 + D_{-n}^{'2} - 2D_{-n}^2 D_{-n}^{'2}\left|\sin(\zeta_{-n} - n\omega t)\right|\sin(\zeta_n^{'} - \zeta_{-n}^{'} + 2n\omega t)}, n \in N^* \\ \phi_{-n} = \arccos(\frac{D_{-n}}{R_{-n}}\cos(\zeta_{-n} - n\omega t)), n \in N^* \\ \varphi_{-n} = \arctan(\frac{R_{-nx}}{R_{-ny}}), n \in N^* \end{cases}$$

ϕ is the angle of *R* and X axis, and ψ is the angle of *R* and Y axis.

Geometric meaning of 1-st Fourier series: A linear function related to given angle of rotation.

Geometric meaning of 2-nd Fourier series: synthesis of a series of frequency components of circle motion with different periods.

Using complex notation, 2-dimension curves of linkages is $F(\omega) = x(\omega) + iy(\omega)$. Let the device rotate at an even speed, $F(t) = x(t) + iy(t)$.

Since most curves of linkages are closed, $F(T+t) = F(t)$.

When $D_n = 0$ and $\zeta_n = 0$, we can get 2-dimension Fourier series expressions with equations above :

$$F(t) = \sum_{n=1}^{\infty} \begin{matrix} ((jD_0^{'} \cos\zeta_0^{'} + kD_0^{'} \sin\zeta_0^{'}) + j(D_n^{'} \cos(\zeta_n^{'} + n\omega t) + \\ D_{-n}^{'} \cos(\zeta_{-n}^{'} - n\omega t)) + k(D_n^{'} \sin(\zeta_n^{'} + n\omega t) + D_{-n}^{'} \sin(\zeta_{-n}^{'} - n\omega t)))) \end{matrix}$$

Arrange the expression and transform it into XOY coordinate system:

$$F(t) = \sum_{n=-\infty}^{+\infty} D_n^{'}(\cos(\zeta_n^{'} + n\omega t) + i\sin(\zeta_n^{'} + n\omega t)) = \sum_{n=-\infty}^{\infty} D_n^{'} e^{i\zeta_n^{'}} e^{in\omega t}$$

ω is the fundamental frequency of Fourier expansion formula, D_n' is the amplitude of the n-th harmonic component, ζ_n' is the initial phase of the n-th harmonic component.

Then I will explain these by an example.

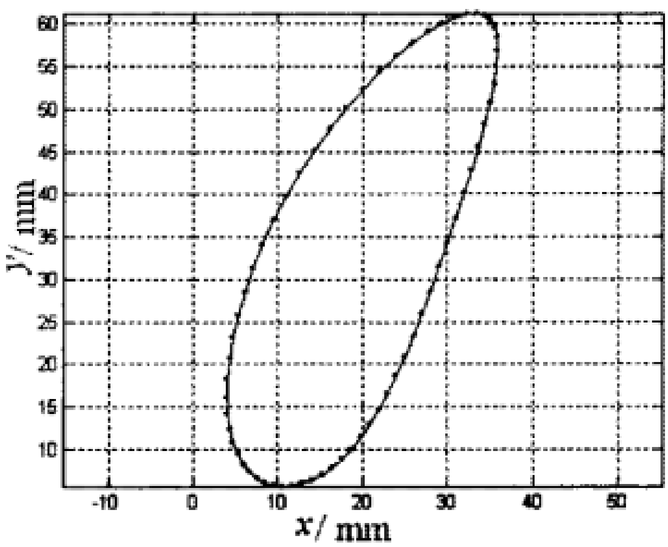

Figure 13

The figure above is produced by a linkage system, which contain 64 sampling points. Make 2-dimension FFT to these sampling points to get the following table:

n	+2	+1	0	-1	-2	-3
D'_n(mm)	1.44	20.22	37.36	10.00	1.40	0.26
ζ'_n(°)	-99.99	4.87	58.06	127.57	177.38	175.04

Then we can conveniently write out the approximate expression of the curve of the linkage as follow:

$$F(t) = D'_0\, e^{i\zeta'_0} + D'_1\, e^{i\zeta'_1} e^{i\omega t} + D'_{-1}\, e^{i\zeta'_{-1}} e^{-i\omega t} + D'_2\, e^{i\zeta'_2} e^{2i\omega t} + D'_{-2}\, e^{i\zeta'_{-2}} e^{-2i\omega t} + D'_{-3}\, e^{i\zeta'_{-3}} e^{-3i\omega t}$$

$$\dot{F}(t) = iD'_1\, \omega e^{i\zeta'_1} e^{i\omega t} - iD'_{-1}\, \omega e^{i\zeta'_{-1}} e^{-i\omega t} + 2iD'_2\, \omega e^{i\zeta'_2} e^{2i\omega t} - 2iD'_{-2}\, \omega e^{i\zeta'_{-2}} e^{-2i\omega t} - 3iD'_{-3}\, \omega e^{i\zeta'_{-3}} e^{-3i\omega t}$$

$$\ddot{F}(t) = -1D'_1\, \omega^2 e^{i\zeta'_1} e^{i\omega t} - D'_{-1}\, \omega^2 e^{i\zeta'_{-1}} e^{-i\omega t} - 4D'_2\, \omega^2 e^{i\zeta'_2} e^{2i\omega t} - 4D'_{-2}\, \omega^2 e^{i\zeta'_{-2}} e^{-2i\omega t} - 9D'_{-3}\, \omega^2 e^{i\zeta'_{-3}} e^{-3i\omega t}$$

The following table gives us the real value and the approximate value of the first 8 sampling points. We find this method most effective.

(cm)

位置 / 坐标	1	2	3	4	5	6	7	8
x 原曲线	31.83	32.73	33.55	34.29	34.93	35.43	35.76	35.91
x 计算值 (n=3)	31.90	32.78	33.59	34.31	34.92	35.40	35.70	35.84
y 原曲线	39.36	42.84	45.64	48.32	50.82	53.12	55.18	56.96
y 计算值 (n=3)	40.01	42.86	45.63	48.27	50.75	53.04	55.09	56.89

6. 2 Use Taylor expansion

Just on the night before replying in the east area, we put up with another method. Although it is also an approximate method.

It is convinced that if a function is smooth enough, we can use Taylor expansion to get the information near a point. And its expansion is a polynomial function and a small quantity.

As we've talked before, we can make addition, subtraction, multiplication, reciprocal operation, so we can get the polynomial function easily. As for the small quantity, we ignore it, or rather, leave it to mechanical precision.

7. Solve the Question Three

Question three: if we have a linkage device, how can we get the linkage track?

“It’s a pleasure to cooperate with physicians.” My classmate provides me with a magic physical method.

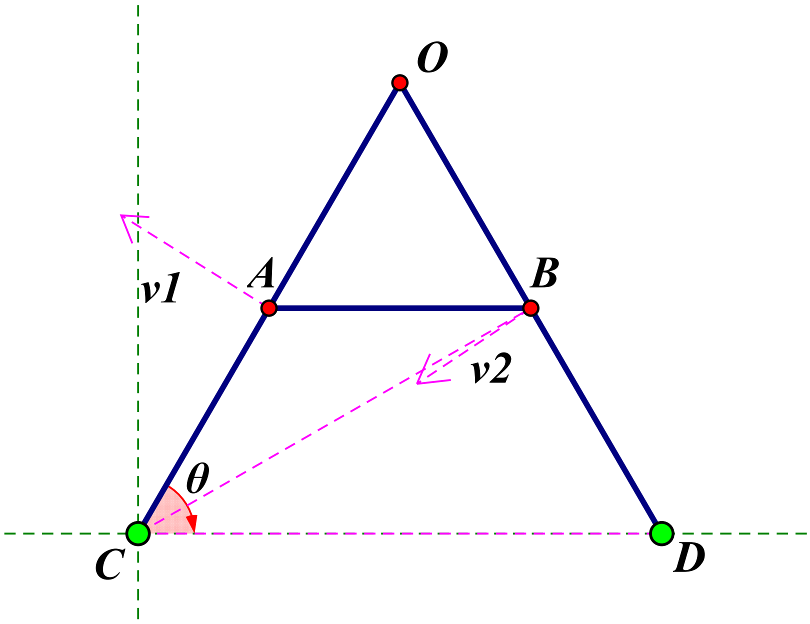

Figure 14

Figure 14 is the same as figure 2 and 3.

In figure 14, point C and D are fixed, AC = AB = BD = OA = OB = 1, CD = 2. To get the track of point O, we let $\theta = \angle ACD$. Since point A makes circle movement, its velocity v_1 must be perpendicular to rod AC. Also, point B's velocity is v2 and perpendicular to rod BD.

Out of physical knowledge, we know that for every point on AB, its projection of velocity to the rod AB is equal. So $v_1 \cos\theta_1 = v_2 \cos\theta_2$. Also, every point on the midperpendicalar of AB has the velocity of the average of A and B.

So, we can write the velocity (include is value and direction) of point O with only v_1 and θ. Then get answer by integration. In this way, we get its parametric equation as follow :

$$\begin{cases} X = \dfrac{(2+3\cos\theta+\sqrt{3}\sin\theta)\sqrt{5-4\cos\theta}-(2\sqrt{3}-\sqrt{3}\cos\theta-\sin\theta)\sqrt{4\cos\theta-1}}{4\sqrt{5-4\cos\theta}} \\ Y = \dfrac{(2\sqrt{3}-\sqrt{3}\cos\theta+3\sin\theta)\sqrt{5-4\cos\theta}+(2-\cos\theta+\sqrt{3}\sin\theta)\sqrt{4\cos\theta-1}}{4\sqrt{5-4\cos\theta}} \end{cases}$$

$$0 \le \theta \le \frac{\pi}{3}$$

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