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CONSTRAINED WILLMORE TORI IN THE 4–SPHERE

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Abstract

We prove that a constrained Willmore immersion of a 2–torus into the conformal 4–sphere S^4 is of "finite type", that is, has a spectral curve of finite genus, or of "holomorphic type" which means that it is super conformal or Euclidean minimal with planar ends in $\mathbb{R}^4 \cong S^4\backslash\{\infty\}$ for some point $\infty \in S^4$ at infinity. This implies that all constrained Willmore tori in S^4 can be constructed rather explicitly by methods of complex algebraic geometry. The proof uses quaternionic holomorphic geometry in combination with integrable systems methods similar to those of Hitchin's approach [**19**] to the study of harmonic tori in S^3.

1. Introduction

A conformal immersion of a Riemann surface is called a *constrained Willmore surface* if it is a critical point of the Willmore functional $\mathcal{W} = \int_M |\mathring{\mathrm{I\!I}}|^2 dA$ (with $\mathring{\mathrm{I\!I}}$ denoting the trace free second fundamental form) under compactly supported infinitesimal conformal variations, see [**23**, **27**, **8**, **5**]. The notion of constrained Willmore surfaces generalizes that of Willmore surfaces which are the critical points of $\mathcal{W}$ under all compactly supported variations. Because both the functional and the constraint of the above variational problem are conformally invariant, the property of being constrained Willmore depends only on the conformal class of the metric on the target space. This suggests an investigation within a Möbius geometric framework like the quaternionic projective model of the conformal 4–sphere used throughout the paper.

The space form geometries of dimension 3 and 4 occur in our setting as subgeometries of 4–dimensional Möbius geometry and provide several classes of examples of constrained Willmore surfaces, including constant mean curvature (CMC) surfaces in 3–dimensional space forms and minimal surfaces in 4–dimensional space forms. See [**5**] for an introduction to constrained Willmore surfaces including a derivation of the Euler–Lagrange equation for compact constrained Willmore surfaces.

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A prototype for our main theorem is the following result on harmonic maps, the solutions to another variational problem on Riemann surfaces.

Prototype Result. *Let $f\colon T^2 \to S^2$ be a harmonic map from a 2–torus T^2 to the 2–sphere S^2 with its standard metric. Then either*

- $\deg(f) = 0$ *and f is of "finite type", i.e., has a spectral curve of finite genus, or*
- $\deg(f) \neq 0$ *and f is conformal.*

That $\deg(f) \neq 0$ implies "holomorphic type" follows from a more general result by Eells and Wood [**11**]. That $\deg(f) = 0$ implies "finite type" has been proven by Pinkall and Sterling [**24**] and Hitchin [**19**]. In contrast to the conformal case when f itself is (anti–)holomorphic, parametrizing a harmonic map f of finite type involves holomorphic functions on a higher dimensional torus, the Jacobian of the *spectral curve*, an auxiliary compact Riemann surface attached to f.

The main theorem of the paper shows that the same dichotomy of "finite type" versus "holomorphic type" can be observed in case of constrained Willmore tori $f\colon T^2 \to S^4$ in the conformal 4–sphere S^4.

Main Theorem. *Let $f\colon T^2 \to S^4$ be a constrained Willmore immersion that is not Euclidean minimal with planar ends in $\mathbb{R}^4 \cong S^4\backslash\{\infty\}$ for some point at infinity $\infty \in S^4$. Then either*

- $\deg(\perp_f) = 0$ *and f is of "finite type", i.e., has a spectral curve of finite genus, or*
- $\deg(\perp_f) \neq 0$ *and f is super conformal,*

where $\deg(\perp_f)$ *is the degree of the normal bundle $\perp_f$ of f seen as a complex line bundle.*

Euclidean minimal tori with planar ends play a special role here since they can have both topologically trivial and non–trivial normal bundle. Constrained Willmore tori in the conformal 3–sphere S^3 occur in our setting as the special case of constrained Willmore immersions into S^4 that take values in a totally umbilic 3–sphere and, in particular, have trivial normal bundle.

The main theorem implies that every constrained Willmore torus $f\colon T^2 \to S^4$ can be parametrized quite explicitly by methods of complex algebraic geometry. If f is of "holomorphic type", that is, if f is super conformal or Euclidean minimal with planar ends, then f or its differential is given in terms of meromorphic functions on the torus itself: a super conformal torus is the twistor projection $\mathbb{CP}^3 \to \mathbb{HP}^1$ of an elliptic curve in $\mathbb{CP}^3$ and for a Euclidean minimal tori with planar ends there is a point $\infty \in S^4$ such that the differential of $f\colon T^2\backslash\{p_1,...,p_n\} \to \mathbb{R}^4 = S^4\backslash\{\infty\}$ is the real part of a meromorphic 1–form with 2^{nd}–order poles and no residues at the ends p_1,...,p_n. It should be noted that both super conformal and Euclidean minimal tori with planar ends are Willmore and, by the quaternionic Plücker formula [**13**], have Willmore energy $\mathcal{W} = 4\pi n$ for some integer $n \geq 2$.

If f is of "finite type", the algebraic geometry needed to parametrize the immersion is more involved: the immersion is then not given by holomorphic data on the torus itself, but can be interpreted as a periodic orbit of an algebraically completely integrable system whose phase space contains as an energy level the (generalized) Jacobian of a Riemann surface of finite genus, the spectral curve. This makes available the methods of "algebraic geometric" or "finite gap" integration from integrable systems theory and implies the existence of explicit parametrizations in terms of theta functions as in the special case of CMC tori in space forms [**1**]. The algebraic geometric reconstruction of general conformally immersed tori with finite spectral genus from their spectral data will be addressed in a forthcoming paper.

Our main theorem generalizes the following previous results:

- CMC tori in 3–dimensional space forms are finite type (Pinkall, Sterling 1989, [**24**]).
- Constrained Willmore in S^3 are of finite type (Schmidt 2002, [**25**]).
- Willmore tori in S^4 with topologically non–trivial normal bundle are super conformal or Euclidean minimal with planar ends (Leschke, Pedit, Pinkall 2003, [**20**]).

The fact that CMC tori are of finite type is closely related to the above prototype result on harmonic tori in S^2, because CMC surfaces are characterized by the harmonicity of their Gauss map $N\colon T^2 \to S^2$. In the early nineties this prototype result had been generalized to harmonic maps from T^2 into various other symmetric target spaces [**19**, **12**, **6**] which led to the conjecture that Willmore tori as well should be of finite type, because they are characterized by the harmonicity of their conformal Gauss map or mean curvature sphere congruence. This conjecture remained open for more than a decade until Martin Schmidt, on the last of over 200 pages of [**25**], gave a proof that constrained Willmore tori in S^3 are of finite type.

We investigate constrained Willmore tori by integrable systems methods similar to those in Hitchin's study [**19**] of harmonic tori in S^3. These provide a uniform, geometric approach to proving and generalizing the previous results mentioned above. The proof roughly consists of the following steps:

- Reformulation of the Euler–Lagrange equation describing constrained Willmore surface as a zero–curvature equation with spectral parameter. This zero–curvature formulation arises in the form of an associated family ∇^μ of flat connections on a trivial complex rank 4 bundle which depends on a spectral parameter $\mu \in \mathbb{C}_*$.
- Investigation of the holonomy representations $H^\mu\colon \Gamma \to \mathrm{SL}_4(\mathbb{C})$ that arise for the associated family ∇^μ of flat connections of constrained Willmore tori.
- Proof of the existence of a polynomial Killing field in case the holonomy H^μ is non–trivial. This implies that a constrained Willmore torus with non–trivial holonomy representation has a spectral curve of finite genus and hence is of "finite type".
- Proof that a constrained Willmore torus f is of "holomorphic type" if the family of holonomy representations H^μ of ∇^μ is trivial.

In order to make the strategy of [**19**] work for constrained Willmore tori in S^4 we apply quaternionic holomorphic geometry [**13**], in particular the geometric approach [**3**] to the spectral curve based on the Darboux transformation for conformal immersions into S^4. The main application of quaternionic methods is in the investigation of which holonomy representations $H^\mu\colon \Gamma \to \mathrm{SL}_4(\mathbb{C})$ are possible for the associated family ∇^μ of constrained Willmore tori. Understanding the possible holonomies is one of the major difficulties in adapting Hitchin's method to the study of constrained Willmore tori. The reason is that, compared to the $\mathrm{SL}_2(\mathbb{C})$–holonomies arising in the study of harmonic tori in S^3, in case of holomorphic families of $\mathrm{SL}_4(\mathbb{C})$–representations one has to cope with a variety of degenerate cases of collapsing eigenvalues.

This difficulty can be handled by applying two analytic results of quaternionic holomorphic geometry: the quaternionic Plücker formula [**13**] and the 1–dimensionality [**4**] of the spaces of holomorphic sections with monodromy corresponding to generic points of the spectral curve of a conformally immersed torus $f\colon T^2 \to S^4$ with topologically trivial normal bundle. The spectral curve as an invariant of conformally immersed tori was first introduced, for immersions into 3–space, by Taimanov [**26**] and Grinevich, Schmidt [**14**]. It is defined as the Riemann surface normalizing the Floquet–multipliers of a periodic differential operator attached to the immersion f. Geometrically this Riemann surface can be interpreted [**3**] as a space parameterizing generic Darboux transforms of f. In the following, the spectral curve will be referred to as the *multiplier spectral curve* Σ_{mult} of f in order to distinguish it from another Riemann surface that arises in our investigation of constrained Willmore tori.

In Section 2 of the paper we review the quaternionic projective approach to conformal surface theory in S^4 and introduce the associated family ∇^μ of flat connections of constrained Willmore immersions into

$S^4 = \mathbb{HP}^1$. This holomorphic family ∇^μ of flat connections allows to study spectral curves of constrained Willmore tori by investigating a holomorphic family of ordinary differential operators instead of the holomorphic family of elliptic partial differential operators needed to define the spectral curve of a general conformal immersion $f\colon T^2 \to S^4$ with trivial normal bundle.

In Section 3 we determine the types of holonomy representations H^μ that are possible for the associated family ∇^μ of constrained Willmore immersions $f\colon T^2 \to S^4$. It turns out that there are two essentially different cases: either all holonomies H^μ, $\mu \in \mathbb{C}_*$ have 1 as an eigenvalue of multiplicity 4 or, for generic $\mu \in \mathbb{C}_*$, the holonomy H^μ has non–trivial, simple eigenvalues. The latter occurs only if the normal bundle is trivial and allows to build a Riemann surface parametrizing the non–trivial eigenlines of the holonomy. In the following we call this Riemann surface the *holonomy spectral curve* Σ_{hol} of f.

In Section 4 we investigate the asymptotics of parallel sections for $\mu \to 0$ and ∞. This will be essential for proving the main theorem of the paper. The asymptotics shows that the holonomy spectral curve Σ_{hol}, whenever defined, essentially coincides with the multiplier spectral curve Σ_{mult}. In particular, Darboux transforms corresponding to points of the spectral curve are again constrained Willmore and Willmore if f itself is Willmore.

In Section 5, the main theorem is proven by separately discussing all possible cases of holonomy representations that occur for constrained Willmore tori. For constrained Willmore tori with non–trivial holonomy we prove the existence of a polynomial Killing field which implies that Σ_{hol} and hence Σ_{mult} can be compactified by adding points at infinity. The proof of the theorem is completed by showing that constrained Willmore tori with trivial holonomy are either super conformal or Euclidean minimal with planar ends.

In Section 6 we discuss a special class of constrained Willmore tori which is related to harmonic maps into S^2 and for which the holonomies of the constrained Willmore associated family reduce to $\mathrm{SL}(2,\mathbb{C})$–representations. This class includes CMC tori in $\mathbb{R}^3$ and S^3, Hamiltonian stationary Lagrangian tori in $\mathbb{C}^2 \cong \mathbb{H}$, and Lagrangian tori with conformal Maslov form in $\mathbb{C}^2 \cong \mathbb{H}$. In case the harmonic map $N\colon T^2 \to S^2$ related to such constrained Willmore torus $f\colon T^2 \to S^4$ is non–conformal, the above prototype result implies that the map N admits a spectral curve of finite genus. We show that this harmonic map spectral curve of N coincides with the spectral curve of the constrained Willmore immersion f.

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2. Constrained Willmore Tori in S^4 and Their Associated Family

A characteristic property of constrained Willmore surfaces in $S^4 = \mathbb{HP}^1$ is the existence of an associated family of flat connections depending on a spectral parameter. This associated family of constrained Willmore surfaces is an essential ingredient in the proof of the main theorem. It is a direct generalization of the associated family [**13**, **20**] of Willmore surface in S^4. We show that parallel section of the associated family ∇^μ of flat connections of a constrained Willmore immersion f give rise to Darboux transforms of f that are again constrained Willmore.

2.1. Möbius geometry of surfaces in the 4–sphere. Throughout the paper we model 4–dimensional Möbius geometry as the geometry of the quaternionic projective line $\mathbb{HP}^1$, see [**7**] for a detailed introduction to the quaternionic approach to surface theory. In particular, we identify maps $f\colon M \to S^4$ from a Riemann surface M into the conformal 4–sphere with line subbundles $L \subset V$ of a trivial quaternionic rank 2 vector bundle V over M equipped with a trivial connection ∇. A map f is a *conformal immersion* if and only if its derivative $\delta = \pi\nabla_{|L} \in \Omega^1 \operatorname{Hom}(L, V/L)$, where $\pi\colon V \to V/L$ denotes the canonical projection, is nowhere vanishing and admits $J \in \Gamma(\operatorname{End}(L))$ and $\tilde{J} \in \Gamma(\operatorname{End}(V/L))$ with $J^2 = -\operatorname{Id}$ and $\tilde{J}^2 = -\operatorname{Id}$ such that

$$*\delta = \delta J = \tilde{J}\delta \tag{2.1}$$

with $*$ denoting the complex structure of T^*M, see Section 4.2 of [**7**] for details.

A fundamental object of surfaces theory in the conformal 4–sphere S^4 is the *mean curvature sphere congruence* (or conformal Gauss map) of a conformal immersion f. It is the unique congruence S of oriented 2–spheres in S^4 that pointwise touches f with the right orientation such that the mean curvature of each sphere $S(p)$, with respect to any compatible space form geometry, coincides with the mean curvature of the immersion f at the point $f(p)$ of contact.

In the quaternionic language, an oriented 2–sphere congruence is represented by a complex structure on V, that is, a section $S \in \Gamma(\operatorname{End}(V))$ satisfying $S^2 = -\operatorname{Id}$, with the 2–sphere at a point $p \in M$ corresponding to the eigenlines of S_p. Such a complex structure S on V gives rise to a decomposition $\nabla = \partial + \bar{\partial} + A + Q$ of the trivial connection ∇, where ∂ and $\bar{\partial}$ are S–complex linear holomorphic and anti–holomorphic structures and

$$A = \tfrac{1}{4}(S\nabla S + *\nabla S) \qquad \text{and} \qquad Q = \tfrac{1}{4}(S\nabla S - *\nabla S).$$

The so called *Hopf fields* A and Q of S are tensor fields $A \in \Gamma(K \operatorname{End}_-(V))$ and $Q \in \Gamma(\bar{K} \operatorname{End}_-(V))$, where $\operatorname{End}_-(V)$ denotes the bundle of endomorphisms of V that anti–commute with S and where we use the convention that a 1–form ω taking values in a quaternionic vector bundle (or its endomorphism bundle) equipped with a complex structure S is called of type K or $\bar{K}$ if $*\omega = S\omega$ or $*\omega = -S\omega$.

The mean curvature sphere congruence is characterized as the unique section $S \in \Gamma(\mathrm{End}(V))$ with $S^2 = -\,\mathrm{Id}$ that satisfies

$$SL = L, \qquad *\delta = S\delta = \delta S, \qquad \text{and} \qquad Q_{|L} = 0, \tag{2.2}$$

see Section 5.2 of [**7**]. The first two conditions express that, for every $p \in M$, the sphere S_p touches the immersion f at $f(p)$ with the right orientation. This is equivalent to the property that S induces the complex structures J and $\tilde{J}$ from (2.1) on the bundles L and V/L. The third condition singles out the mean curvature sphere congruence among all congruences of touching spheres. Given the first two conditions, the third one is equivalent to $\mathrm{im}(A) \subset L$.

The Hopf fields A and Q of the mean curvature sphere congruence S measure the local "defect" of the 2–sphere congruence S from being constant, that is, the defect of the immersion from being totally umbilic. A conformal immersion f is totally umbilic if both A and Q vanish identically. In case only one of the Hopf fields vanishes identically the immersion is called *super conformal* and is the twistor projection of a holomorphic curve in $\mathbb{CP}^3$, see Chapter 8 of [**7**].

The Möbius invariant quantity measuring the global "defect" of S from being constant is the Willmore functional $\mathcal{W}$ which, for a conformal immersion f of a compact surface M, can be expressed in terms of the Hopf fields by the formula

$$\mathcal{W} = 2\int_M < A \wedge *A > -2\pi \deg(\perp_f) = 2\int_M < Q \wedge *Q > +2\pi \deg(\perp_f), \tag{2.3}$$

where $<>$ denotes 1/4 of the real trace and $\deg(\perp_f)$ is the degree of the normal bundle of the immersion f.

2.2. Euler–Lagrange equation of constrained Willmore surfaces. The following proposition shows how the *Euler–Lagrange equation* describing compact constrained Willmore surfaces can be expressed in terms of the Hopf fields A and Q of the mean curvature sphere congruence S.

Proposition 2.1. *A conformal immersion $f\colon M \to S^4$ of a compact Riemann surface M is constrained Willmore if and only if there exists a 1–form $\eta \in \Omega^1(\mathcal{R})$ such that*

$$d^{\nabla}(2{*}A + \eta) = 0, \tag{2.4}$$

where $\mathcal{R} = \{B \in \mathrm{End}(V) \mid \mathrm{im}(B) \subset L \subset \ker(B)\}$.

A proof of the Euler–Lagrange equation for constrained Willmore immersions of compact surfaces can be found in [**5**]. The form η in (2.4) is the Lagrange multiplier of the underlying constrained variational problem. The vanishing of η corresponds to the case of Willmore surfaces which are characterized by $d^\nabla{*}A = 0$, see Chapter 6 of [**7**].

Equation (2.4) is equivalent to

$$d^\nabla(2{*}Q + \eta) = 0, \tag{2.5}$$

because $d^\nabla{*}Q = d^\nabla{*}A$. Every 1–form $\eta \in \Omega^1(\mathcal{R})$ with (2.4) satisfies $\eta \in \Gamma(K\mathcal{R}_+)$, i.e.,

$${*}\eta = S\eta = \eta S. \tag{2.6}$$

In fact, equation (2.4) implies $\delta \wedge (2{*}A + \eta) = 0$ and hence ${*}\eta = S\eta$, because ${*}A = SA$. Similarly, equation (2.5) implies ${*}\eta = \eta S$. Using $\nabla = \hat\nabla + A + Q$, where $\hat\nabla = \partial + \bar\partial$ is the S–commuting part of ∇, we obtain the decomposition

$$\underbrace{d^{\hat\nabla}\eta}_{+} + \underbrace{2Sd^{\hat\nabla}A + A \wedge \eta + \eta \wedge Q}_{-} = d^\nabla(2{*}A + \eta) = 0 \tag{2.7}$$

of (2.4) into S–commuting and anti–commuting parts, as usual denoted by $\pm$. This implies $d^{\hat\nabla}\eta = 0$ which is equivalent to $\eta\delta \in \Gamma(K^2 \operatorname{End}_+(L))$ $= \Gamma(K^2)$ being a holomorphic quadratic differential. In particular, if η does not vanish identically it vanishes at isolated points only. Moreover, by (2.7) and the analogous decomposed version of (2.5), if $\eta \not\equiv 0$ and one of the Hopf fields A and Q vanishes on some open set U, then both A and Q have to vanish on U, that is, on U the immersion is totally umbilic.

2.3. Uniqueness and non–uniqueness of the Lagrange multiplier η. For discussing the uniqueness of the Lagrange multiplier η in the Euler–Lagrange equation (2.4) of constrained Willmore surfaces we need the following quaternionic characterization of isothermic surfaces, see e.g. [**18**, **2**]: a conformal immersion $f\colon M \to S^4$ is *isothermic* if there is a non–trivial 1–form $\omega \in \Omega^1(\mathcal{R})$ with $d^\nabla\omega = 0$, where as above $\mathcal{R} = \{B \in \operatorname{End}(V) \mid \operatorname{im}(B) \subset L \subset \ker(B)\}$ as in Section 2.2. As above one can prove that every closed 1–form $\omega \in \Omega^1(\mathcal{R})$ satisfies $\omega \in \Gamma(K\mathcal{R}_+)$, that is,

$${*}\omega = S\omega = \omega S, \tag{2.8}$$

and that the quadratic differential $\omega\delta \in \Gamma(K^2 \operatorname{End}_+(L)) = \Gamma(K^2)$ is holomorphic. With this definition of isothermic surfaces, the following lemma is evident.

Lemma 2.2. *The Lagrange–multiplier η occurring in the Euler–Lagrange equation* (2.4) *of constrained Willmore surfaces is either unique or the surface is isothermic. In the latter case, the form η is unique up to adding a closed form $\omega \in \Omega^1(\mathcal{R})$.*

For isothermic surfaces that are not totally umbilic, the space of closed forms in $\Omega^1(\mathcal{R})$ is real 1–dimensional, see e.g. [**2**]. Examples of constrained Willmore surfaces for which the Lagrange–multiplier η is not unique are CMC surfaces in 3–dimensional space forms, cf. Sections 6.6 and 6.7. In fact, CMC tori with respect to 3–dimensional space form subgeometries are the only possible examples of constrained Willmore tori in the conformal 3–sphere with non–unique Lagrange multiplier η, cf. [**8**].

Examples of constrained Willmore tori whose Lagrange multiplier η is unique (namely $\eta \equiv 0$) are super conformal tori, see the discussion following equation (2.7). Examples of constrained Willmore tori in the 3–sphere which are non–isothermic (and hence not CMC with respect to any space form subgeometry) can be obtained from Pinkall's Hopf torus construction [**22**]: a Hopf torus, the preimage of a closed curve in S^2 under the Hopf fibration $S^3 \to S^2$, is never isothermic unless it is the Clifford torus. It is Willmore if the curve in S^2 is elastic [**22**] and it is constrained Willmore if the underlying curve is generalized elastic [**5**].

2.4. Associated family ∇^μ of flat connections of constrained Willmore surfaces in the 4–sphere. For a constrained Willmore immersion $f\colon M \to S^4$ with Lagrange multiplier $\eta \in \Omega^1(\mathcal{R})$ we denote by $A_\circ$ and $Q_\circ$ the 1–forms defined by $2{*}A_\circ = 2{*}A + \eta$ and $2{*}Q_\circ = 2{*}Q + \eta$. Like the Hopf fields A and Q they satisfy

$$\nabla S = 2{*}Q_\circ - 2{*}A_\circ,$$
$$\operatorname{im}(A_\circ) \subset L \text{ and } L \subset \ker(Q_\circ),$$
$${*}A_\circ = SA_\circ \text{ and } {*}\,Q_\circ = Q_\circ S.$$

However, in contrast to the Hopf fields, the forms $A_\circ$ and $Q_\circ$ do not anti–commute with S if η does not vanish identically.

The fundamental tool in our study of constrained Willmore tori is the *associated family*

$$\nabla^\mu = \nabla + (\mu - 1)\frac{1 - iS}{2}A_\circ + (\mu^{-1} - 1)\frac{1 + iS}{2}A_\circ, \tag{2.9}$$

of complex connections on the complex rank 4 bundle (V, i) which rationally depends on the *spectral parameter* $\mu \in \mathbb{C}_*$, where (V, i) denotes V seen as a complex vector bundle by restricting the scalar field to $\mathbb{C} = \operatorname{Span}_{\mathbb{R}}\{1, i\}$. In the following we call a 1–form ω to be of type $(1, 0)$ or $(0, 1)$ if ${*}\omega = \omega i$ or ${*}\omega = -\omega i$. Setting $A_\circ^{(1,0)} = \frac{1-iS}{2}A_\circ$ and $A_\circ^{(0,1)} = \frac{1+iS}{2}A_\circ$, formula (2.9) simplifies to

$$\nabla^\mu = \nabla + (\mu - 1)A_\circ^{(1,0)} + (\mu^{-1} - 1)A_\circ^{(0,1)}.$$

Lemma 2.3. *The connection ∇^μ is flat for every spectral parameter $\mu \in \mathbb{C}_*$.*

Proof. The curvature of ∇^μ is

$$\begin{aligned} R^{\nabla^\mu} &= (\mu-1)d^\nabla A_\circ^{(1,0)} + (\mu^{-1}-1)d^\nabla A_\circ^{(0,1)} \\ &\quad +(\mu-1)(\mu^{-1}-1)\left(A_\circ^{(1,0)}\wedge A_\circ^{(0,1)} + A_\circ^{(0,1)}\wedge A_\circ^{(1,0)}\right). \end{aligned}$$

From $d^\nabla * A_\circ = 0$ we obtain $d^\nabla A_\circ^{(1,0)} = d^\nabla A_\circ^{(0,1)} = \frac{1}{2}d^\nabla A_\circ = A_\circ \wedge A_\circ$ so that $R^{\nabla^\mu}=0$, because $(\mu-1)(\mu^{-1}-1)(A_\circ^{(1,0)}\wedge A_\circ^{(0,1)} + A_\circ^{(0,1)}\wedge A_\circ^{(1,0)}) = (2-\mu-\mu^{-1})(A_\circ\wedge A_\circ)$. q.e.d.

This associated family ∇^μ of flat connections has the symmetry

$$\nabla^{1/\bar\mu} = j^{-1}\nabla^\mu j \qquad \text{for all } \mu\in\mathbb{C}_* \tag{2.10}$$

with j denoting the complex anti–linear endomorphism of (V,i) given by right–multiplication with the quaternion j. For every $\mu\in S^1\subset\mathbb{C}_*$, the connection ∇^μ is therefore quaternionic.

Another holomorphic family of flat complex connections is given by

$$\tilde\nabla^\mu = \nabla + (\mu-1)Q_\circ^{(1,0)} + (\mu^{-1}-1)Q_\circ^{(0,1)}$$

with $Q_\circ^{(1,0)} = Q_\circ\frac{1-iS}{2}$ and $Q_\circ^{(0,1)} = Q_\circ\frac{1+iS}{2}$. Both families are in fact gauge equivalent:

$$\nabla^\mu = \big((\mu+1)-i(\mu-1)S\big)\circ\tilde\nabla^\mu\circ\big((\mu+1)-i(\mu-1)S\big)^{-1} \tag{2.11}$$

for all $\mu\in\mathbb{C}_*$.

As a consequence we obtain that in case of super conformal Willmore surfaces the connection ∇^μ is trivial for every $\mu\in\mathbb{C}_*$, because either $A\equiv 0$ or $Q\equiv 0$.

The family $\tilde\nabla^\mu$ of connections arises naturally by dualizing the associated family

$$(\nabla^\perp)^\mu = \nabla + (\mu-1)(A_\circ^\perp)^{(1,0)} + (\mu^{-1}-1)(A_\circ^\perp)^{(0,1)} \tag{2.12}$$

of the dual constrained Willmore surface $L^\perp\subset V^*$ seen as connection on the complex bundle $(V^*,-i)$: the immersion $f^\perp$ has the mean curvature sphere $S^\perp = S^*$ with Hopf fields $A^\perp = -Q^*$ and $Q^\perp = -A^*$. The form $2{*}A_\circ^\perp := 2{*}A^\perp + \eta^\perp$ with $\eta^\perp := -\eta^*$ is closed which shows that $f^\perp$ is again constrained Willmore. The complex bundle $(V^*,-i)$ carries the usual complex structure of the complex dual $(V,i)^*$ to (V,i) when applying the canonical identification between quaternionic and complex dual space, that is, the identification between $\alpha\in V^*$ and its complex part $\alpha_\mathbb{C}\in(V,i)^*$. The sign in $(V^*,-i)$ reflects the fact that V^* is made into a quaternionic right vector bundle by $\alpha\lambda := \bar\lambda\alpha$ for $\alpha\in V^*$ and $\lambda\in\mathbb{H}$.

2.5. Darboux transforms. The following lemma is essential for the next sections as it provides a link between the associated family ∇^μ of flat connections and quaternionic holomorphic geometry. Recall that a conformal immersion $f\colon M \to S^4 \cong \mathbb{HP}^1$ induces a unique quaternionic holomorphic structure [**13**] on the bundle V/L with the property that all parallel sections of the trivial connection on the bundle V project to holomorphic sections: the complex structure on V/L is $\tilde{J}$ from (2.1) while the holomorphic structure $D\colon \Gamma(V/L) \to \Gamma(\bar{K}V/L)$ is defined by $D\pi = (\pi\nabla)''$, where $\pi\colon V \to V/L$ is the canonical projection and $()''$ denotes the $\bar{K}$–part with respect to $\tilde{J}$, see [**3**] for details. The projection

$$\psi \in \Gamma(V) \quad \mapsto \quad \tilde{\psi} := \pi\psi \in \Gamma(V/L) \tag{2.13}$$

induces a 1–1–correspondence between sections of V with $\nabla\psi \in \Omega^1(L)$ and holomorphic sections of V/L; the section ψ with $\nabla\psi \in \Omega^1(L)$ and $\tilde{\psi} := \pi\psi$ is called the *prolongation* of the holomorphic section $\tilde{\psi}$. Existence and uniqueness of the prolongation ψ for a given holomorphic section $\tilde{\psi}$ immediately follows from the fact that $\delta = \pi\nabla_{|L}$ is nowhere vanishing. Flatness of ∇ implies that $\nabla\psi \in \Gamma(KL)$ for every $\psi \in \Gamma(V)$ with $\nabla\psi \in \Omega^1(L)$.

Lemma 2.4. *Let $f\colon M \to S^4$ be a constrained Willmore immersion and denote by ∇^μ its associated family of flat connections. Then every (local) ∇^μ–parallel section of V is the prolongation of a (local) holomorphic section of V/L.*

Proof. The lemma is an immediate consequence of the fact that $A_\circ$ takes values in L, because every ∇^μ–parallel section ψ of V satisfies

$$\nabla\psi = (1-\mu)A_\circ^{(1,0)}\psi + (1-\mu^{-1})A_\circ^{(0,1)}\psi \in \Omega^1(L). \qquad \text{q.e.d.}$$

A map $f^\sharp\colon M \to S^4$ is called a *Darboux transform* [**3**] of a conformal immersion $f\colon M \to S^4 \cong \mathbb{HP}^1$ if the corresponding line subbundle $L^\sharp \subset V$ is locally of the form $L^\sharp = \psi\mathbb{H}$ for ψ the prolongation of a nowhere vanishing holomorphic section of V/L. In case f is constrained Willmore we call $f^\sharp$ a ∇^μ*–Darboux transform* if there is $\mu \in \mathbb{C}_*$ such that the corresponding bundle is locally of the form $L^\sharp = \psi\mathbb{H}$ for ψ a ∇^μ–parallel section whose projection to V/L is nowhere vanishing.

Theorem 2.5. *Let $f\colon M \to S^4$ be a constrained Willmore immersion of a Riemann surface M. Every ∇^μ–Darboux transform $f^\sharp\colon M \to S^4$ of f is again constrained Willmore when restricted to the open subset of M over which it is immersed. In case f is Willmore and $f^\sharp$ is a ∇^μ–Darboux transform (for ∇^μ taken with $\eta \equiv 0$), then $f^\sharp$ is again Willmore where immersed.*

Proof. The proof uses notation and several results from [**2**]. If $f^\sharp$ is a Darboux transform of f, the corresponding line bundles satisfy

$V = L \oplus L^\sharp$ and locally $L^\sharp$ admits a nowhere vanishing section ψ with $\nabla\psi \in \Omega^1(L)$. In case ψ is ∇^μ–parallel for $\mu \in \mathbb{C}\backslash\{0,1\}$, by definition of ∇^μ we have

$$\nabla\psi\, b + *\nabla\psi = (2 * A_\circ)\psi = (2 * A + \eta)\psi \tag{2.14}$$

with $b \in \mathbb{C}$ defined by

$$b = \frac{2i}{1-\mu} - i = \frac{-2i}{1-\mu^{-1}} + i. \tag{2.15}$$

The endomorphisms $B,\ C \in \Gamma(\mathrm{End}(L^\sharp))$ in equation (74) of [**2**] then satisfy $(B + C)\psi = \psi b$ and $B + C$ is parallel, because b is constant. This shows that equation (75) of [**2**] is satisfied for $D := C$ so that $f^\sharp$ is again constrained Willmore. In particular, if f is Willmore and $\eta \equiv 0$, then $C \equiv 0$ and hence $D \equiv 0$ which proves that $f^\sharp$ is also Willmore.

q.e.d.

2.6. Global Darboux transforms of conformal tori in the 4–sphere. In case the underlying surface M is a torus $T^2 = \mathbb{C}/\Gamma$, global Darboux transforms of f are obtained from prolongations of holomorphic sections with *monodromy* of V/L: these are sections $\psi \in \Gamma(\tilde{V})$ of the pullback $\tilde{V}$ of V to the universal covering $\mathbb{C}$ of T^2 which transform by

$$\gamma^*\psi = \psi h_\gamma, \qquad \gamma \in \Gamma \tag{2.16}$$

for some *multiplier* $h \in \mathrm{Hom}(\Gamma, \mathbb{H}^*)$ and have derivative $\nabla\psi$ with values in the pullback $\tilde{L}$ of L. Multiplying ψ by a quaternionic constant $\lambda \in \mathbb{H}_*$ yields the same Darboux transform while the multiplier h gets conjugated $\lambda^{-1}h\lambda$. Because the group Γ of deck transformations of the torus is abelian it is therefore sufficient to consider prolongations of holomorphic section with complex multiplier $h \in \mathrm{Hom}(\Gamma, \mathbb{C}_*)$.

For a constrained Willmore torus $f\colon T^2 \to S^4 \cong \mathbb{HP}^1$, one can obtain global ∇^μ–Darboux transforms from holomorphic sections with monodromy of V/L whose prolongation $\psi \in \Gamma(\tilde{V})$ is ∇^μ–parallel for some $\mu \in \mathbb{C}_*$. Every ∇^μ–parallel section $\psi \in \Gamma(\tilde{V})$ satisfying (2.16) for some $h \in \mathrm{Hom}(\Gamma, \mathbb{C}_*)$ gives rise to a ∇^μ–parallel (complex) line subbundle of V and vice versa. Such line subbundles correspond to 1–dimensional invariant subspaces of the holonomy representation, i.e., to simultaneous eigenlines of $H^\mu_p(\gamma)$ for all $\gamma \in \Gamma$. These will be studied in the following section.

3. Holonomy of Constrained Willmore Tori

The characterization of constrained Willmore tori in S^4 in terms of the associated family ∇^μ of flat connections puts us in a situation that is quite familiar in integrable systems theory. What one usually does when encountering a family of flat connections over the torus is to investigate its holonomy representations and, in particular, the eigenlines of the holonomy. In general, investigating the holonomies of a family of flat connections on a complex rank 4 bundle over the torus $T^2 = \mathbb{C}/\Gamma$ is more involved than in case of rank 2 bundles (like for harmonic tori in S^2 or S^3, see [**19**] or Section 6.4 below) because one has to deal with various possible configurations of collapsing eigenvalues. In the present section we show that for the associated family ∇^μ of constrained Willmore tori only few of these configurations do actually occur.

3.1. Main result of the section. Before stating the main result of the section we collect the relevant properties of the holonomy representation for a family ∇^μ of flat connections on a surface M:

- Because all ∇^μ are flat, for a fixed $p \in M$ and fixed $\mu \in \mathbb{C}_*$, the holonomy $H^\mu_p(\gamma) \in \mathrm{GL}_\mathbb{C}(V_p)$ depends only on the homotopy class of closed curves based at the point p. The holonomy is thus a representation $\gamma \in \Gamma \mapsto H^\mu_p(\gamma) \in \mathrm{GL}_\mathbb{C}(V_p)$ of the group Γ of deck transformations.
- For fixed $p \in M$ and $\gamma \in \Gamma$, the holonomies $H^\mu_p(\gamma)$ depend holomorphically on the spectral parameter $\mu \in \mathbb{C}_*$.
- The holonomies for different points on the torus are conjugated; the eigenvalues of $H^\mu_p(\gamma)$ are therefore independent of $p \in M$, only the eigenlines change when changing $p \in M$.

In case the underlying surface is a torus $T^2 = \mathbb{C}/\Gamma$, the group Γ of Deck transformations is abelian and the holonomies $H^\mu_p(\gamma_1)$ and $H^\mu_p(\gamma_2)$ commute for all $\gamma_1, \gamma_2 \in \Gamma$. For fixed $p \in T^2$ and $\mu \in \mathbb{C}_*$, the eigenspaces of the holonomy $H^\mu_p(\gamma_1)$ for one $\gamma_1 \in \Gamma$ are thus invariant subspaces of all other holonomies $H^\mu_p(\gamma_2)$, $\gamma_2 \in \Gamma$. In particular, simple eigenspaces are eigenspaces of all holonomies. More generally, every eigenspace of $H^\mu_p(\gamma_1)$ for one $\gamma_1 \in \Gamma$ contains a simultaneous eigenline of $H^\mu_p(\gamma_2)$ for all $\gamma_2 \in \Gamma$. The restriction of the holonomy representation $H^\mu_p(\gamma)$ to such a simultaneous eigenline is a multiplier $h \in \mathrm{Hom}(\Gamma, \mathbb{C}_*)$ which is the monodromy of the ∇^μ–parallel sections $\psi \in \Gamma(\tilde V)$ whose value ψ_p at p is contained in the eigenline (cf. Section 2.6).

Proposition 3.1. *Let $f\colon T^2 \to S^4$ be a constrained Willmore torus. The holonomy representations of the associated family ∇^μ of f belong to one of the following three cases:*

I. *there is $\gamma \in \Gamma$ such that, away from isolated $\mu \in \mathbb{C}_*$, the holonomy $H^\mu_p(\gamma)$ has 4 distinct eigenvalues which are non–constant as functions of μ,*
II. *all holonomies $H^\mu_p(\gamma)$ have a 2–dimensional common eigenspace with eigenvalue 1 and there is $\gamma \in \Gamma$ such that, away from isolated $\mu \in \mathbb{C}_*$, the holonomy $H^\mu_p(\gamma)$ has 2 simple eigenvalues which are non–constant as functions of μ, or*

III. *all* $H_p^\mu(\gamma)$, $\gamma \in \Gamma$ *have* 1 *as an eigenvalue of multiplicity 4. More precisely, either*
   (a) all holonomies are trivial, i.e., $H_p^\mu(\gamma) = \mathrm{Id}$ *or*
   (b) generic holonomies are non–semisimple with two 2×2 *Jordan–blocks.*

If the immersion f *has topologically non–trivial normal bundle, it belongs to Case III.*

The proposition will be proven in Sections 3.3 and 3.5. The cases of immersions with trivial and non–trivial normal bundle are treated separately, because the situation in both cases is quite different and so is the analysis needed in the proof. In the quaternionic model, the normal bundle of an immersion is the bundle $\mathrm{Hom}_-(L, V/L)$, where "$-$" denotes the homomorphisms anti–commuting with J and $\tilde{J}$. The degree of the normal bundle for an immersion of a compact surface is

$$\deg(\perp_f) = \deg(\mathrm{Hom}_-(L, V/L)) = 2\deg(V/L) + \deg(K),$$

where the last equality holds because the differential δ of f is a nowhere vanishing section of $K\,\mathrm{Hom}_+(L, V/L)$. In particular, the normal bundle of an immersed torus is trivial if and only if the induced quaternionic holomorphic line bundle V/L has degree zero.

All cases described in Proposition 3.1 do actually occur:

- examples for Case I are Willmore tori with $\eta \equiv 0$ that are neither super conformal nor Euclidean minimal with planar ends for some point ∞ at infinity (see Corollary 5.2),
- examples for Case II are CMC tori in $\mathbb{R}^3$ (see Section 6),
- examples for Case IIIa are super conformal tori (see Section 2.4), and
- examples for Case IIIb are Euclidean minimal tori with planar ends for which the surfaces in the minimal surface associated family have translational periods (see [**20**]).

3.2. Non–trivial eigenvalues of H^μ. For fixed $\gamma \in \Gamma$, away from isolated spectral parameters $\mu \in \mathbb{C}_*$ the eigenvalues of the holonomy $H^\mu(\gamma)$ are locally given by holomorphic functions $\mu \mapsto \lambda(\mu)$: to see this note that

$$\{(\lambda, \mu) \in \mathbb{C}_* \times \mathbb{C}_* \mid f(\lambda, \mu) = 0\} \qquad \text{with} \qquad f(\lambda, \mu) = \det(\lambda - H^\mu(\gamma))$$

is a 1–dimensional analytic subset of $\mathbb{C}_* \times \mathbb{C}_*$ whose non–empty intersection with $\mathbb{C}_* \times \{\mu\}$ for $\mu \in \mathbb{C}_*$ consists of up to 4 points. Denote by X the Riemann surface normalizing this analytic set and by $\mu\colon X \to \mathbb{C}_*$ its projection to the μ–coordinate. The holomorphic function μ is then a branched covering whose number of sheets is between 1 and 4 and coincides with the generic number of different eigenvalues of $H^\mu(\lambda)$. Away from the branch points of μ, the different sheets of the analytic set are therefore locally graphs of holomorphic functions $\mu \mapsto \lambda(\mu)$ describing the eigenvalues of $H^\mu(\gamma)$.

If one of the local holomorphic functions $\mu \mapsto \lambda(\mu)$ that describe the eigenvalues of the holonomy $H^\mu(\gamma)$ for some $\gamma \in \Gamma$ is constant, that is, if there is $\lambda \in \mathbb{C}$ that is eigenvalue of $H^\mu(\gamma)$ for all μ in an open subset of $\mathbb{C}_*$, then λ is eigenvalue for all $\mu \in \mathbb{C}_*$ and the following lemma implies that $\lambda \equiv 1$.

Lemma 3.2. *If $\lambda \in \mathbb{C}_*$ is an eigenvalue of $H^\mu(\gamma)$ for all μ, then $\lambda = 1$. The multiplicity of $\lambda = 1$ as simultaneous eigenvalue of $H^\mu(\gamma)$ for all $\mu \in \mathbb{C}_*$ is even.*

Proof. The first statement follows from the fact that $H^\mu(\gamma) = \mathrm{Id}$ for $\mu = 1$. The second statement is a consequence of the quaternionic symmetry (2.10) of ∇^μ for $\mu \in S^1$. It implies that, for $\mu \in S^1$, the multiplicity of 1 as an eigenvalue of $H^\mu_p(\gamma)$ is even. The same holds for all $\mu \in \mathbb{C}_*$, because the minimal kernel dimension of the holomorphic family $\mu \mapsto H^\mu_p(\gamma) - \mathrm{Id}$ of endomorphisms is generic and attained away from isolated points, see Proposition 3.3 below. q.e.d.

We denote by *non–trivial eigenvalues* the eigenvalues that are not equal to 1 and therefore locally given by non–constant functions $\mu \mapsto \lambda(\mu)$. In the proof of Proposition 3.1 we will see that non–trivial eigenvalues are generically simple, i.e., have algebraic multiplicity 1. The corresponding eigenlines are in the following called *non–trivial eigenlines.*

In the investigations of the present paper we will frequently apply this proposition a proof of which can be found in [**4**] (see Proposition 3.1 there):

Proposition 3.3. *For a family of Fredholm operators that holomorphically depends on a parameter in a connected complex manifold X, the minimal kernel dimension is generic and attained away from an analytic subset $Y \subset X$. In case X is 1–dimensional, Y is a set of isolated points and the holomorphic vector bundle defined by the kernels over $X \backslash Y$ holomorphically extends through the isolated points Y with higher dimensional kernel. If the index of the operators is zero, the set of x for which they are non–invertible is locally given as the vanishing locus of one holomorphic function.*

3.3. Proof of Proposition 3.1 in the non–trivial normal bundle case. The proof in case of non–trivial normal bundle is analogous to that of Lemma 3.1 in [**20**]. We show that 1 is an eigenvalue of multiplicity 4 for all holonomies of ∇^μ. Because the degree of the quaternionic holomorphic line bundle $V^*/L^\perp \cong L^{-1}$ induced by the dual constrained Willmore torus $f^\perp$ is $\deg(V^*/L^\perp) = -\deg(V/L)$ and the holonomy representation of $(\nabla^\perp)^\mu$, by (2.11) and (2.12), is equivalent to the dual representation of the holonomy of ∇^μ, we assume without loss of generality (if necessary by passing to the dual surface $f^\perp$) that the degree of the normal bundle and therefore of V/L is negative.

If the multiplicity of 1 as an eigenvalue was not 4 for all holonomies, by Lemma 3.2, there had to be a non–constant holomorphic map $h\colon U \to \mathrm{Hom}(\Gamma, \mathbb{C}_*)$ defined on an open subset $U \subset \mathbb{C}_*$ such that, for every $\mu \in U$, there is a non–trivial ∇^μ–parallel section $\psi^\mu \in \Gamma(\tilde{V})$ with $H^\mu(\gamma)\psi^\mu = \psi^\mu h^\mu_\gamma$ for all $\gamma \in \Gamma$. Projecting the ψ^μ to V/L yields a family of holomorphic sections with monodromy all of whose multipliers are different and which are therefore linearly independent. But the existence of such an infinite dimensional space of holomorphic sections with monodromy of V/L contradicts the quaternionic Plücker formula with monodromy according to which

$$\mathcal{W}(V/L) \geq -n \deg(V/L)$$

in case there exists an n–dimensional linear system with monodromy, see Appendix of [**3**].

The quaternionic symmetry (2.10) of ∇^μ for $\mu \in S^1$ implies that in the non–semisimple case all Jordan blocks are 2×2 (because they are 2×2 for $\mu \in S^1$ so that the holomorphic family $H^\mu(\gamma) - \mathrm{Id}$ of endomorphisms squares to zero for all $\mu \in S^1$ and hence everywhere).

q.e.d.

3.4. The multiplier spectral curve. The proof of Proposition 3.1 in the trivial normal bundle case requires ideas from quaternionic holomorphic geometry related to the spectral curve of a conformally immersed torus $f\colon T^2 \to S^4$ with trivial normal bundle. The spectral curve of f, in the following also called the *multiplier spectral curve* Σ_{mult}, is the Riemann surface normalizing its *spectrum*, the complex analytic set that consists of all complex multipliers

$$h \in \mathrm{Hom}(\Gamma, \mathbb{C}_*) \cong \mathbb{C}_* \times \mathbb{C}_*$$

for which there exists a non–trivial holomorphic section with monodromy h of the quaternionic holomorphic line bundle V/L, cf. [**3**, **4**]. The idea of defining a spectral curve for conformal immersions is due to Taimanov [**26**], Grinevich, and Schmidt [**14**] who give a slightly different (but equivalent, cf. [**2**]) definition of the spectral curve for immersions $f\colon T^2 \to \mathbb{R}^3$ which is based on the Euclidean concept of Weierstrass representation.

In order to justify the definition of Σ_{mult} one has to verify that the possible multipliers form a 1–dimensional complex analytic set. In [**4**] this is proven by asymptotic analysis of a holomorphic family of elliptic operators. In addition it is shown that Σ_{mult} has one or two ends (depending on whether its genus is infinite or finite) and one or two connected components each of which contains an end. Moreover, the minimal vanishing order of the functions describing the spectrum is one at generic points which implies that, away from isolated points $\sigma \in \Sigma_{mult}$, the space of holomorphic sections with monodromy h^σ of V/L is complex 1–dimensional.

Because the kernels of a holomorphic 1–parameter family of elliptic operators form a holomorphic vector bundle which holomorphically extends through the isolated points with higher dimensional kernel, see Proposition 3.3, we obtain [**4**] a unique line subbundle $\mathcal{L}$ of the trivial bundle $\Sigma_{mult} \times \Gamma(\widetilde{V/L})$ equipped with the C^∞–topology each fiber $\mathcal{L}_\sigma$ of which is contained in (and generically coincides with) the space of holomorphic sections with monodromy h^σ of V/L. This defines [**3**] a map

$$F\colon T^2 \times \Sigma_{mult} \to \mathbb{CP}^3, \qquad (p,\sigma) \mapsto \psi^\sigma(p)\mathbb{C},$$

where ψ^σ denotes the prolongation of a non–trivial element of $\mathcal{L}_\sigma$. For a fixed point $p \in T^2$ on the torus, the map $\sigma \in \Sigma_{mult} \to F(p,\sigma)$ is holomorphic. For fixed $\sigma \in \Sigma_{mult}$ in the spectral curve, the twistor projection of $p \mapsto F(p,\sigma)$ to $\mathbb{HP}^1$ is a *singular Darboux transform* of f, that is, a Darboux transform defined away from the finitely many points p at which ψ^σ is contained in L.

The set of possible multipliers is invariant under complex conjugation, because multiplying a holomorphic section with monodromy $h \in \mathrm{Hom}(\Gamma, \mathbb{C}_*)$ by the quaternion j yields a holomorphic section with monodromy $\bar{h}$. By lifting the map $h \mapsto \bar{h}$ to the normalization Σ_{mult} we obtain an anti–holomorphic involution $\rho\colon \Sigma_{mult} \to \Sigma_{mult}$ which has no fixed points because

$$F(p, \rho(\sigma)) = F(p,\sigma)j.$$

Let φ^σ be a nowhere vanishing local holomorphic section of the bundle $\mathcal{L} \to \Sigma_{mult}$, i.e., φ^σ is a family of holomorphic sections in $\Gamma(\widetilde{V/L})$ which holomorphically depends on σ and satisfies

$$\gamma^*\varphi^\sigma = \varphi^\sigma h^\sigma_\gamma$$

for all $\gamma \in \Gamma$. Taking the derivative $\frac{\partial}{\partial x}$ with respect to an arbitrary chart x of Σ_{mult} yields

$$\gamma^* \tfrac{\partial \varphi^\sigma}{\partial x} = \tfrac{\partial \varphi^\sigma}{\partial x} h^\sigma_\gamma + \varphi^\sigma \tfrac{\partial h^\sigma_\gamma}{\partial x}$$

so that φ^σ and $\frac{\partial \varphi^\sigma}{\partial x}$ span a 2–dimensional linear system with non–semisimple monodromy of V/L (if at a point of the spectral curve $\frac{\partial h^\sigma_\gamma}{\partial x}$ vanishes for all $\gamma \in \Gamma$ one has to take higher derivatives). The following lemma shows that this linear system is generically the unique 2–dimensional linear system with monodromy of V/L for which h^σ is the only eigenvalue of the monodromy (like φ^σ generically spans the space of holomorphic sections with monodromy h^σ of V/L).

Lemma 3.4. *For a generic point $\sigma \in \Sigma_{mult}$ in the multiplier spectral curve of an immersed torus $f\colon T^2 \to S^4$ with trivial normal bundle, there is a unique 2–dimensional linear systems with monodromy of V/L for which h^σ is the only eigenvalue of the monodromy.*

Proof. Firstly, we prove that every connected component of Σ_{mult} contains a point σ_0 that admits a unique 2–dimensional linear system with monodromy whose only eigenvalue is h^{σ_0}. Secondly, using Proposition 3.3 we deduce that generic points $\sigma \in \Sigma_{mult}$ admit a unique 2–dimensional linear system with monodromy whose only eigenvalue is h^σ.

Step 1: Let $\sigma_0 \in \Sigma_{mult}$ be a point corresponding to a regular point of the spectrum, the analytic set of possible multipliers of holomorphic sections with monodromy. The spectrum is then locally given as the vanishing locus of a holomorphic function that vanishes to first order at h^{σ_0}. Because the minimal vanishing order of holomorphic functions describing the spectrum is greater or equal to the dimension of the space of holomorphic sections with monodromy, the space of holomorphic sections with monodromy h^{σ_0} is 1–dimensional. We can assume that σ_0 is chosen such that non–trivial holomorphic sections with monodromy h^{σ_0} have no zeros, because every component of Σ_{mult} by Lemma 4.9 of [**4**] contains such a regular point. Denote by ∇ the quaternionic connection of V/L rendering this space of holomorphic sections parallel. Then d^∇ makes KV/L into a quaternionic holomorphic line bundle of degree 0. Because ∇ is flat, it maps holomorphic sections with monodromy of V/L to holomorphic sections with monodromy of KV/L so that the spectrum of V/L is included in the spectrum of KV/L. This shows that both spectra coincide, because the spectrum of a quaternionic holomorphic line bundle of degree 0 over a torus is a 1–dimensional analytic set that is either irreducible or has two irreducible components interchanged under complex conjugation, see [**4**]. In particular, there is a local holomorphic function describing the spectrum of KV/L that vanishes to first order at h^{σ_0} so that the space of holomorphic sections with monodromy h^{σ_0} of KV/L is also 1–dimensional. This implies the uniqueness of the 2–dimensional linear system with monodromy of V/L for which the only eigenvalue of the monodromy is h^{σ_0}: let φ_1, φ_2 be holomorphic sections of $\widetilde{V/L}$ and $t \in \mathrm{Hom}(\Gamma, \mathbb{C})$ such that

$$\gamma^*\varphi_1 = \varphi_1 h^{\sigma_0}_\gamma \qquad \text{and} \qquad \gamma^*\varphi_2 = \varphi_2 h^{\sigma_0}_\gamma + \varphi_1 t_\gamma h^{\sigma_0}_\gamma$$

for all $\gamma \in \Gamma$. Then φ_1 is unique up to scaling, because it is a holomorphic section with monodromy h^{σ_0}, and φ_2 is unique up to scaling and adding a multiple of φ_1, because $\nabla\varphi_2$ is a holomorphic section with monodromy h^{σ_0} of KV/L.

Step 2: Analogous to Section 2.3 of [**4**], the 2–dimensional linear systems with non–semisimple monodromy of V/L correspond to

2–dimensional spaces of solutions to

$$(*) \qquad D_\omega \varphi_1 = 0 \qquad \text{and} \qquad D_\omega \varphi_2 + (\varphi_1 \eta)'' = 0,$$

where ω, $\eta \in \mathrm{Hom}(\Gamma, \mathbb{C}) \cong \mathrm{Harm}(\mathbb{C}/\Gamma, \mathbb{C})$, $\eta \neq 0$, and $D_\omega \varphi_1 = D\varphi_1 + (\varphi_1 \omega)''$: given a solution φ_1, $\varphi_2 \in \Gamma(V/L)$ to $(*)$ with $\varphi_1 \neq 0$, $\varphi_2 \neq 0$, and $\eta \neq 0$, then

$$(\tilde{\varphi}_1, \tilde{\varphi}_2) = (\varphi_1, \varphi_2) e^{\int \omega} \begin{pmatrix} 1 & \int \eta \\ 0 & 1 \end{pmatrix}$$

span a 2–dimensional linear system with monodromy of V/L for which h with $h_\gamma = e^{\int_\gamma \omega}$ is the only eigenvalue of the monodromy. Clearly, non–trivial solutions to $(*)$ can only exist if h belongs to the spectrum. Denote by $\tilde{\Sigma}$ the "logarithmic spectral curve", the normalization of the space of ω for which h belongs to the spectrum. We consider now the holomorphic family

$$D_{\omega,\eta} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} D_\omega \varphi_1 \\ D_\omega \varphi_2 + (\varphi_1 \eta)'' \end{pmatrix}$$

of elliptic operators parametrized over $\tilde{\Sigma} \times (\mathrm{Harm}(\mathbb{C}/\Gamma, \mathbb{C}) \backslash \{0\}) \cong \tilde{\Sigma} \times (\mathbb{C}^2 \backslash \{0\})$. The fact that for every h in the spectrum there exists at least one 2–dimensional linear system with non–semisimple monodromy and eigenvalue h implies that for every $\omega \in \tilde{\Sigma}$ there exists at least a line worth of $\eta \in (\mathrm{Harm}(\mathbb{C}/\Gamma, \mathbb{C}) \backslash \{0\})$ for which $D_{\omega,\eta}$ has a 2–dimensional kernel. On the other hand, we have proven above the existence of a multiplier h that admits a unique 2–dimensional linear system with monodromy, i.e., there exists $\omega \in \tilde{\Sigma}$ admitting a unique (up to scale) η for which $\ker(D_{\omega,\eta})$ is 2–dimensional. Proposition 3.3 now implies that the set of ω, η for which $D_{\omega,\eta}$ has a 2–dimensional kernel is a non–empty 2–dimensional analytic set in $\tilde{\Sigma} \times (\mathbb{C}^2 \backslash \{0\})$ which projects to a 1–dimensional analytic subset of $\tilde{\Sigma} \times \mathbb{CP}^1$. To complete the proof we have to show that, apart from components of the form $\{\omega\} \times \mathbb{CP}^1$, the normalization of this 1–dimensional analytic set is a graph over $\tilde{\Sigma}$. Assume this was not the case. Then, the normalization has one component X that is neither of the form $\{\omega\} \times \mathbb{CP}^1$ nor a part of the graph over $\tilde{\Sigma}$ that corresponds to the "generic" 2–dimensional linear system whose monodromy has only one eigenvalue (as described before the statement of the lemma).

The projection to $\tilde{\Sigma}$ would map this additional component X onto a connected component of $\tilde{\Sigma}$ (for every Riemann surface Y, the image of a connected component of the normalization of a 1–dimensional analytic set in $Y \times \mathbb{CP}^1$ under the projection to Y is either a point or a connected component of Y, because if the projection is not constant, by compactness of $\mathbb{CP}^1$ it is a finitely sheeted branched covering). But this is impossible, because every connected component of $\tilde{\Sigma}$ contains a regular point at which the corresponding holomorphic sections with monodromy of V/L are nowhere vanishing so that, as seen in Step 1, there is a unique 2–dimensional linear system with monodromy belonging to the respective eigenvalue. q.e.d.

3.5. Proof of Proposition 3.1 in the trivial normal bundle case. We fix $\gamma \in \Gamma$ for which $H^\mu(\gamma)$ generically has the maximal number of different eigenvalues so that the branched covering $\mu\colon X \to \mathbb{C}_*$ has the maximal number of sheets, where X as in Section 3.2 denotes the Riemann surface normalizing the 1–dimensional analytic subset of $\mathbb{C}_* \times \mathbb{C}_*$ given by $f(\lambda,\mu) = 0$ with $f(\lambda,\mu) = \det(\lambda - H^\mu(\gamma))$. Recall that the number of sheets is 4 in case all eigenvalues of $H^\mu(\gamma)$ are distinct and 1, 2, or 3 if the discriminant of the characteristic polynomial of $H^\mu_p(\tilde\gamma)$ vanishes identically for all $\tilde\gamma \in \Gamma$.

In case the number of sheets is 4 we are in Case I of the above list: away from isolated parameters μ the holonomy $H^\mu_p(\gamma)$ has then 4 different eigenvalues which, by Lemma 3.2, are non–constant as functions of μ.

If the number of sheets is 1 we are in Case III of the above list: for every $\mu \in \mathbb{C}_*$ the only eigenvalue of the $\mathrm{SL}(4,\mathbb{C})$–holonomy is then a fourth root of unity and hence equal to 1, because $\nabla^{\mu=1}$ is trivial. As in the non–trivial normal bundle case, the statement about the non–semisimple holonomy is an immediate consequence of the quaternionic symmetry (2.10) of ∇^μ for $\mu \in S^1$.

If the number of sheets is 3 we are in Case II of the above list: away from a discrete set of points, the dimension of the generalized eigenspaces $\ker((\lambda - H^\mu(\gamma))^2)$ is constant on connected components of Σ, cf. Proposition 3.3. The Riemann surface Σ is thus the disconnected sum of one sheet that corresponds to a double eigenvalue of $H^\mu(\gamma)$ and a hyper–elliptic surface that parametrizes its simple eigenvalues. The quaternionic symmetry (2.10) implies that for generic $\mu \in S^1$ the holonomy $H^\mu(\gamma)$ has 2 simple eigenvalues which are complex conjugate and one real eigenvalue of geometric multiplicity 2. The corresponding eigenspaces are invariant under all holonomies $H^\mu(\tilde\gamma)$, $\tilde\gamma \in \Gamma$. In particular, because there is no holonomy with 4 different eigenvalues, for all $\tilde\gamma \in \Gamma$ the restriction of $H^\mu(\tilde\gamma)$ to the 2–dimensional eigenspace of $H^\mu(\gamma)$ is a multiple of identity. As explained in Section 3.4, there is only a discrete set of complex multipliers $h \in \mathrm{Hom}(\Gamma,\mathbb{C}_*)$ for which the space of holomorphic sections with monodromy h of V/L has dimension greater or equal 2. Because ∇^μ–parallel sections that correspond to simultaneous eigenlines of the holonomy project to holomorphic sections with monodromy of V/L, see Lemma 2.4, we obtain that the double eigenvalues of $H^\mu(\tilde\gamma)$, $\tilde\gamma \in \Gamma$ are locally constant as function of μ and hence, by Lemma 3.2, equal to 1. The same lemma shows that the simple eigenvalues are non–constant as functions of μ so that we are in Case II of the list.

To complete the proof of the proposition it remains to show that the branched covering $\mu\colon \Sigma \to \mathbb{C}_*$ cannot be 2–sheeted. It is impossible that the Riemann surface Σ is the disconnected sum of two sheets that correspond to a simple and a triple eigenvalue, respectively, because this would contradict the quaternionic symmetry (2.10) for $\mu \in S^1$. Thus, if Σ is a 2–sheeted branched covering, by Proposition 3.3 the generalized eigenspaces of the holonomies $H^\mu(\gamma)$ define a rank 2 bundle over Σ.

In case all holonomies $H^\mu(\tilde\gamma)$, $\tilde\gamma \in \tilde\Gamma$ are diagonalizable, every vector in this rank 2 bundle is an eigenvector of $H^\mu(\tilde\gamma)$ for all $\tilde\gamma \in \tilde\Gamma$, because otherwise there would be $\tilde\gamma \in \tilde\Gamma$ for which $H^\mu(\tilde\gamma)$ has four distinct eigenvalues so that we were in Case I. Hence, every fiber of this rank 2 bundle gives rise to a 2–dimensional space of holomorphic section with monodromy of V/L. But this is impossible, because the eigenvalues of $H^\mu(\gamma)$ are non–constant as function of μ while higher dimensional spaces of holomorphic section with monodromy h of V/L can only exists for isolated $h \in \mathrm{Hom}(\Gamma, \mathbb{C}_*)$, see Section 3.4.

We can therefore chose $\gamma \in \Gamma$ for which $H^\mu(\gamma)$ generically has two double eigenvalues with geometric multiplicity 1 and algebraic multiplicity 2. All holonomies $H^\mu(\tilde\gamma)$, $\tilde\gamma \in \Gamma$ then leave the rank 2 bundle defined by the generalized eigenspaces of $H^\mu(\gamma)$ invariant and all their restrictions to this rank 2 bundle have a double eigenvalue. In other words, taking projections to V/L of the ∇^μ–parallel sections which correspond to the rank 2 bundle, a generic point $\sigma \in \Sigma$ gives rise to a 2–dimensional linear system with monodromy of V/L for which h^σ is the only eigenvalue.

By Lemma 3.4, for generic σ such linear system is unique so that if ψ^μ denotes a holomorphic family of ∇^μ–parallel sections with monodromy of V, the sections $\frac{\partial \psi^\mu}{\partial \mu}$ are also ∇^μ–parallel (because generically the projections to V/L of ψ^μ and $\frac{\partial \psi^\mu}{\partial \mu}$ span the unique 2–dimensional linear system with monodromy belonging to the corresponding multiplier, see the discussion before the statement of Lemma 3.4). Taking the derivative of $\nabla^\mu \psi^\mu = 0$ with respect to μ then yields $A_\circ^{(1,0)}\psi^\mu - \mu^{-2} A_\circ^{(0,1)}\psi^\mu = 0$ so that $A_\circ \psi^\mu = 0$. Hence, all ψ^μ are constant sections of V and all holonomies are trivial so that the number of sheets of $\mu\colon \Sigma \to \mathbb{C}_*$ is 1. This completes the proof, because it shows that the number of sheets of $\mu\colon \Sigma \to \mathbb{C}_*$ can never be 2. q.e.d.

3.6. The holonomy spectral curve. We show that a constrained Willmore torus that belongs to Case I or II of Proposition 3.1 gives rise to a Riemann surface parametrizing the non–trivial eigenlines of the holonomies $H_p^\mu(\gamma)$, $\gamma \in \Gamma$.

Lemma 3.5. *Let $A(\mu)$ be a family of complex $n \times n$–matrices depending holomorphically on a parameter $\mu \in U$ in a connected open set $U \subset \mathbb{C}$. If the eigenvalues of $A(\mu)$ that are non–constant as functions of μ are generically simple, there exists a unique (up to isomorphism) Riemann surface Σ with holomorphic maps $\mu\colon \Sigma \to U$ and $\mathcal{E}\colon \Sigma \to \mathbb{CP}^{n-1}$ such that μ is a branched covering and*

a) *for every $\sigma \in \Sigma$, the line $\mathcal{E}(\sigma)$ is an eigenline of $A(\mu(\sigma))$ and*
b) *for generic $\mu_1 \in U$, the $\mathcal{E}(\sigma_0),\dots,\mathcal{E}(\sigma_{\tilde n})$ with $\{\sigma_1, \dots, \sigma_{\tilde n}\} = \mu^{-1}\{\mu_1\}$ are mutually distinct and coincide with the eigenlines of $A(\mu_1)$ that belong to non–constant eigenvalues.*

In particular, the map $\mu\colon \Sigma \to U$ is a branched covering whose number of sheets equals the number of eigenlines that belong to non–constant eigenvalues.

Proof. Denote by $g(\lambda, \mu) = \det(\lambda - A(\mu))$ the characteristic polynomial of $A(\mu)$. After dividing by linear factors $(\lambda - \lambda_j)$ belonging to eigenvalues that are constant in μ we obtain a function $\tilde g(\lambda, \mu)$ whose discriminant, as a polynomial in λ, does not vanish identically. We define Σ as the Riemann surface normalizing the 1–dimensional analytic set

$$\{(\lambda, \mu) \in \mathbb{C} \times U \mid \tilde g(\lambda, \mu) = 0\}.$$

The projection $\mu\colon \Sigma \to U$ is then a branched covering and Σ is the unique Riemann surface equipped with holomorphic functions μ and λ such that, for generic $\mu_1 \in U$, the non–constant eigenvalues of $A(\mu_1)$ are given by $\lambda(\sigma_j)$, $j = 1, \dots, \tilde n$ with $\{\sigma_1, \dots, \sigma_{\tilde n}\} = \mu^{-1}\{\mu_1\}$.

The family of matrices $\lambda(\sigma)\,\mathrm{Id} - A(\mu(\sigma))$ depends holomorphically on $\sigma \in \Sigma$ and generically, away from a set of isolated points, has a 1–dimensional kernel which is an eigenline of $A(\mu(\sigma))$. Because Σ is complex 1–dimensional, the line bundle $\mathcal{E}(\sigma) = \ker(\lambda(\sigma) - A(\mu(\sigma)))$ defined over generic points extends holomorphically through the isolated points with a higher dimensional kernel, cf. Proposition 3.3.

By construction, Σ with μ and $\mathcal{E}$ satisfies a) and b) in the statement of the lemma. The uniqueness of Σ with μ and $\mathcal{E}$ follows from the above uniqueness property of Σ with μ and λ, because the eigenline map $\mathcal{E}$ of $A(\mu)$ allows to recover the holomorphic function λ describing the eigenvalues. q.e.d.

For constrained Willmore tori belonging to Case I or II of Proposition 3.1, the preceding lemma allows to define a Riemann surface, in the following called the *holonomy spectral curve* Σ_{hol}, that parametrizes the non–trivial eigenlines of the holonomies $H^\mu(\gamma)$: we define Σ_{hol} as the 2 or 4–sheeted branched covering of $\mathbb{C}_*$ obtained from Lemma 3.5 applied to $A(\mu) = H^\mu_p(\gamma)$ with fixed $p \in T^2$ and $\gamma \in \Gamma$ for which $H^\mu_p(\gamma)$ has the maximal number of non–trivial eigenvalues. Because the holonomies for different $\gamma \in \Gamma$ commute, the uniqueness part of Lemma 3.5 shows that

Σ_{hol} is independent of the choice of γ. Moreover, the Riemann surface Σ_{hol} does not depend on the choice of p, because changing the point p on the torus amounts to conjugate the holonomy (but the Riemann surface Σ in the proof of Lemma 3.5 is defined purely in terms of eigenvalues of $A(\mu)$). What does depend on the point $p \in T^2$ of the torus is the eigenline curve $\mathcal{E}_p \colon \Sigma_{hol} \to \mathbb{CP}^3$.

For every point $\sigma \in \Sigma_{hol}$, the line $\mathcal{E}_p(\sigma)$ is invariant under the holonomy representation $\gamma \in \Gamma \mapsto H_p^{\mu(\sigma)}(\gamma)$. This defines a holomorphic map $h \colon \Sigma_{hol} \to \mathrm{Hom}(\Gamma, \mathbb{C}_*)$ whose image is contained in the set of multipliers of holomorphic sections with monodromy of V/L. This map lifts to a holomorphic map $\iota \colon \Sigma_{hol} \to \Sigma_{mult}$ which turns out to be injective and almost surjective, see Theorem 4.5. By definition, the eigenline curve $\mathcal{E}$ is related to the map F defined in Section 3.4 by $\mathcal{E}_p(\sigma) = F(\iota(\sigma), p)$ for all $\sigma \in \Sigma_{hol}$. The map F can therefore be seen as a generalization of the holonomy eigenline curve $\mathcal{E}$ of the constrained Willmore associated family ∇^μ to arbitrary conformal immersions $f \colon T^2 \to S^4$ with trivial normal bundle.

The map $\iota \colon \Sigma_{hol} \to \Sigma_{mult}$ interchanges the fixed point free, anti–holomorphic involutions on Σ_{hol} and Σ_{mult}: under ι, the involution ρ on Σ_{hol} induced by the symmetry (2.10) via

$$\mathcal{E}_{\rho(\sigma)} = \mathcal{E}_\sigma j$$

corresponds to the involution ρ on Σ_{mult}. This fixed points free involution ρ on Σ_{hol} covers the involution $\mu \mapsto 1/\bar{\mu}$ of the μ–plane.

4. The Asymptotics of ∇^μ–parallel Sections

The proof of the main theorem in Section 5 requires some control over the asymptotic behavior of ∇^μ–parallel sections for $\mu \to 0$ or ∞. This is provided by Proposition 4.1 of the present section. As an immediate application of Proposition 4.1 we show that the holonomy spectral curve essentially coincides with the multiplier spectral curve in case they are both defined, that is, for constrained Willmore tori belonging to Cases I and II of Proposition 3.1.

4.1. Main result of the section. Because of the symmetry (2.10) it is sufficient to understand the asymptotic behavior of parallel sections for $\mu \to \infty$. We approach this problem by investigating the sections $\psi \in \Gamma(\tilde{V})$ of the pullback $\tilde{V}$ of V to the universal covering $\mathbb{C}$ of $T^2 = \mathbb{C}/\Gamma$ that satisfy

$$\nabla\psi \in \Omega^1(\tilde{L}) \qquad \text{and} \qquad (A_\circ\psi)^{(1,0)} = 0, \tag{4.1}$$

where $\tilde{L}$ denotes the pullback of L to the universal covering. Equation (4.1) is an asymptotic version of $\nabla^\mu\psi^\mu = 0$ for $\mu \to \infty$; examples of solutions to (4.1) are the prolongations of holomorphic sections that are suitable limits of ∇^μ–parallel sections for $\mu \to \infty$.

The following proposition summarizes Lemmas 4.3 and 4.4 below.

Proposition 4.1. *Let $f\colon T^2 \to S^4$ be a constrained Willmore torus. If f is neither super conformal nor Euclidean minimal with planar ends and $\eta \equiv 0$, then the complex dimension of the space of solutions to* (4.1) *is at most two.*

For the rest of the section we assume that f is not super conformal and, in particular, $A_\circ$ does not vanish identically. Because $A_\circ$ takes values in L, its rank is then at most one. Away from its isolated zeros the rank of $A_\circ$ is one and the kernel bundle $\check{L} = \ker(A_\circ)$ smoothly extends through the isolated zeroes to a line bundle over T^2. To see this note that, because $\delta\eta$ is a holomorphic quadratic differential, either $\eta \not\equiv 0$ so that η and hence $A_\circ$ have no zeroes at all, or $\eta \equiv 0$ so that $A_\circ = A$ itself is holomorphic, see Proposition 22 of [**7**].

Denote by U the open set of points where $A_{|L}$ does not vanish or, equivalently, where $V = L \oplus \check{L}$. The set U is non–empty: otherwise, by Lemma 22 of [**7**], the Hopf field A had to vanish identically which is impossible because then $\eta \equiv 0$ (see the discussion following (2.7)) so that $A_\circ \equiv 0$. If $\check{L} = \ker(A_\circ)$ is constant the subset U is dense, because it is the complement of the set of points where the immersion f goes through $\check{L}$.

Let ψ be a solution to (4.1) defined on U. Using $d^\nabla A_\circ = 2A_\circ \wedge A_\circ$, differentiation of $*A_\circ\psi = -A_\circ\psi i$ yields

$$- * A_\circ \wedge \nabla\psi = -2A_\circ \wedge A_\circ\psi i + A_\circ \wedge \nabla\psi i.$$

Because $\nabla\psi$ takes values in L and $(A_\circ)_{|L} = A_{|L}$ this implies

$$A_\circ \wedge (-S\nabla\psi + \nabla\psi i - 2A_\circ\psi i) = 0.$$

On the open set U where $A_{|L}$ has no zeros, every form $\alpha \in \Gamma(KL)$ with $A_\circ \wedge \alpha = 0$ has to vanish identically. Since $\nabla\psi \in \Gamma(KL)$, on U every solution ψ to (4.1) satisfies

$$(\nabla\psi)^{(0,1)} = A_\circ\psi. \tag{4.2}$$

4.2. The case that $A_\circ \not\equiv 0$ and $\check{L} = \ker(A_\circ)$ is non–constant. If $\check{L} = \ker(A_\circ)$ is non–constant, the corresponding map into the 4–sphere is called a 2–step *Bäcklund transformation* of L (see e.g. [**2**] for a detailed discussion of Bäcklund transformations).

Lemma 4.2. *Let $f\colon M \to S^4$ be a constrained Willmore immersion with $A_\circ \not\equiv 0$ for which $\check{L} = \ker(A_\circ)$ is non–constant. The corresponding map $\check{f}$ into S^4 is then conformal and, on the open set U where $V = L \oplus \check{L}$, it is constrained Willmore and admits a 1–form $\check{\eta}$ with $\operatorname{im}(\check{\eta}) \subset \check{L} \subset \ker(\check{\eta})$ such that the form $2{*}\check{Q}_\circ = 2{*}\check{Q} + \check{\eta}$ is closed and satisfies $2{*}\check{Q}_\circ = 2{*}A_\circ$.*

The last formula shows $L = \operatorname{im}(\check{Q}_\circ)$ which, in the language of [**2**], implies that L is a 2–step backward Bäcklund transformation of $\check{L}$.

Proof. For every $\varphi \in \Gamma(\check{L})$ we have $*A_\circ\varphi = 0$ and therefore

$$0 = d^\nabla(*A_\circ\varphi) = \underbrace{d^\nabla(*A_\circ)}_{=0}\varphi - *A_\circ \wedge \nabla\varphi.$$

Hence $\check{f}$ is a (possibly branched) conformal immersion, because $A_\circ \wedge \check{\delta} = 0$ with $\check{\delta} = \pi_{V/\check{L}}\nabla_{|\check{L}}$ denoting the derivative of $\check{f}$. With respect to the splitting $L \oplus \check{L}$ on U, the connection ∇ and the mean curvature sphere S of L can be written as

$$\nabla = \begin{pmatrix} \nabla^L & \check{\delta} \\ \delta & \check{\nabla} \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} J & B \\ 0 & \tilde{J} \end{pmatrix}, \tag{4.3}$$

where J and $\tilde{J}$ are complex structures on L and $\check{L}$ with $*\delta = \delta J = \tilde{J}\delta$ and where $B \in \Gamma(\operatorname{Hom}(\tilde{L}, L))$ with $JB + B\tilde{J} = 0$. The derivative of S is

$$\nabla S = \begin{pmatrix} \nabla^L J - B\delta & \nabla B + \check{\delta}\tilde{J} - J\check{\delta} \\ 0 & \check{\nabla}\tilde{J} + \delta B \end{pmatrix}.$$

The mean curvature sphere condition $Q_{|L} = 0$ now becomes that $\nabla^L J - B\delta$ is left K (and right $\bar{K}$) with respect to J. Moreover, because $(A_\circ)_{|\check{L}} = 0$ and $(Q_\circ)_{|L} = 0$, the identity $\nabla S = 2{*}Q_\circ - 2{*}A_\circ$ implies

$$2{*}A_\circ = \begin{pmatrix} -\nabla^L J + B\delta & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad 2{*}Q_\circ = \begin{pmatrix} 0 & \nabla B + \check{\delta}\tilde{J} - J\check{\delta} \\ 0 & \check{\nabla}\tilde{J} + \delta B \end{pmatrix}. \tag{4.4}$$

From $A_\circ \wedge \check{\delta} = 0$ we obtain $*\check{\delta} = -J\check{\delta}$ (because $\nabla^L J - B\delta$ is right–$\bar{K}$) and the mean curvature sphere of $\check{L}$ is

$$\check{S} = \begin{pmatrix} -J & 0 \\ \check{B} & -\check{J} \end{pmatrix}, \tag{4.5}$$

where $\check{J}$ is the complex structure on $\check{L}$ with $*\check{\delta} = -\check{\delta}\check{J}$ and where $\check{B} \in \Gamma(\operatorname{Hom}(L, \check{L}))$ with $\check{J}\check{B} + \check{B}J = 0$. Now

$$\nabla\check{S} = \begin{pmatrix} -\nabla^L J + \check{\delta}\check{B} & 0 \\ \nabla\check{B} + \check{J}\delta - \delta J & -\check{\nabla}\check{J} - \check{B}\check{\delta} \end{pmatrix}$$

and the condition $\operatorname{im}(\check{A}) \subset \check{L}$ that $\check{S}$ is the mean curvature sphere of $\check{L}$ becomes that $-\nabla^L J + \check{\delta}\check{B}$ is left K (and right $\bar{K}$) with respect to J.

The Hopf field $\check{Q}$ of $\check{L}$ is given by

$$2{*}\check{Q} = \frac{1}{2}(\nabla\check{S} + \check{S}{*}\nabla\check{S}) = \begin{pmatrix} -\nabla^L J + \check{\delta}\check{B} & 0 \\ * & 0 \end{pmatrix}. \tag{4.6}$$

The condition that S is mean curvature sphere of L is equivalent to $(\nabla^L J)'' = B\delta$ with $()''$ denoting the $\bar{K}$–part with respect to J. Similarly, that $\check{S}$ is mean curvature sphere of $\check{L}$ is equivalent to $(\nabla^L J)'' = \check{\delta}\check{B}$. This implies

$$B\delta = \check{\delta}\check{B}. \tag{4.7}$$

By (4.4), (4.6) and (4.7), the 1–form $\check{\eta} := 2{*}A_\circ - 2{*}\check{Q}$ satisfies $\operatorname{im}(\check{\eta}) \subset \check{L} \subset \ker(\check{\eta})$ and $2{*}\check{Q}_\circ = 2{*}\check{Q} + \check{\eta}$ is closed because

$$2{*}\check{Q}_\circ = 2{*}A_\circ. \tag{4.8}$$

This shows that, on U, $\check{L}$ is constrained Willmore. q.e.d.

Lemma 4.3. *Let $f\colon T^2 \to S^4$ be a constrained Willmore immersion with $A \not\equiv 0$, $Q \not\equiv 0$ for which $\check{L} = \ker(A_\circ)$ is non–constant. Then, the space of solutions to* (4.1) *defined on the universal covering of T^2 is at most (complex) 2–dimensional. In case f is Willmore with $\eta \equiv 0$ but neither super conformal nor Euclidean minimal with planar ends, the space of solutions to* (4.1) *is 0–dimensional if $AQ \equiv 0$ and 1–dimensional if $AQ \not\equiv 0$.*

Examples of Willmore immersions with $AQ \equiv 0$ are Willmore surfaces contained in a totally umbilic 3–sphere S^3, minimal surfaces in the 4–sphere equipped with its standard metric, or minimal surfaces in 4–dimensional hyperbolic space, see Chapter 10 of [**7**]. In case of Willmore surfaces with $AQ \not\equiv 0$, the unique solution to (4.1) corresponds to a 4–step Willmore Bäcklund transformation.

Proof. We first prove that, on the open set U where $L \oplus \check{L}$, a local solution ψ to (4.1) satisfies $\check{S}\psi = -\psi i$ where $\check{S}$ is the mean curvature sphere of $\check{L}$, see (4.5). Writing $\psi = (\psi_1, \psi_2)$ with respect to $L \oplus \check{L}$, the equation ${*}A_\circ\psi = -A_\circ\psi i$ implies $J\psi_1 = \psi_1 i$. By (4.5) we thus have $\check{S}\psi + \psi i \in \Gamma(\check{L})$. Using (4.8) we obtain

$$\begin{aligned}\nabla(\check{S}\psi + \psi i) &= (2{*}\check{Q}_\circ - 2{*}\check{A}_\circ)\psi + \check{S}\nabla\psi + \nabla\psi i \\ &= (2{*}A_\circ - 2{*}\check{A}_\circ)\psi + \check{S}\nabla\psi + \nabla\psi i = -2{*}\check{A}_\circ\psi + \check{B}\nabla\psi \in \Omega^1(\check{L}),\end{aligned} \tag{4.9}$$

where the last identity holds because, by (4.2), we have $2 * A_\circ\psi - {*}\nabla\psi + \nabla\psi i = 0$. Because $\check{L}$ is non–constant, this implies $\check{S}\psi + \psi i = 0$ or, equivalently, $\check{S}\psi = -\psi i$.

If f is Willmore with $\eta \equiv 0$, equation (4.4) implies $B = 0$ because S anti–commutes with $A = A_\circ$ and therefore, by (4.7), $\check{B} = 0$. Plugging this and $\check{S}\psi + \psi i = 0$ into (4.9) yields $\check{A}\psi = 0$, that is, ψ is a section of the complex line bundle $\{v \in \ker(\check{A}) \mid \check{S}v = -vi\}$. The space of solutions to (4.1) is thus at most 1–dimensional, because for any two solutions ψ, φ there is a non–empty open set $U' \subset U$ and a complex function g on U' such that $\psi = \varphi g$. Hence, $\nabla\psi = \nabla\varphi g + \varphi dg$ and, taking the projection π to V/L, we have $(\pi\varphi)dg = 0$ which shows that g is constant (because the holomorphic section $\pi\varphi$ of V/L vanishes at isolated points only) and ψ and φ are linearly dependent on T^2 because they are linearly dependent on the open subset $U' \subset U$.

If $\eta \equiv 0$ and $AQ \equiv 0$, then every section ψ solving (4.1) has to vanish identically: because $Q \not\equiv 0$, the 2–step (forward) Bäcklund transformations $\check{L} = \ker(A)$ is at the same time a 2–step (backward) Bäcklund transformations, that is, $\check{L} = \hat{L} = \operatorname{im}(Q)$. Hence $\ker(\check{A}) = L$ by Theorem 8 of [**7**]. Because f is immersed, a section $\psi \in \Gamma(L)$ with $\nabla\psi \in \Omega^1(L)$ has to vanish identically. This proves the statement about the Willmore case with $\eta \equiv 0$.

The rest of the proof deals with the case that $\eta \not\equiv 0$. Assume that, on U, there are two (complex) linearly independent solutions ψ and φ to (4.1). Then, on an open and dense subset $U' \subset U$ both section are pointwise linearly independent: if there were not, there had to be an open set $\tilde{U}$ and a complex function g on $\tilde{U}$ such that $\psi = \varphi g$. But this is impossible, because, by the same argument as above, g then had to be constant and prolongations of holomorphic sections that are linearly dependent on an open set are linearly dependent everywhere. We now prove that on the set U there cannot be more solutions to (4.1) than the complex 2–dimensional space spanned by ψ and φ. Assume $\tilde{\psi}$ was another solution. Then, on the open set U' where ψ, φ pointwise span the subbundle $\{v \in V \mid \check{S}v = -vi\}$, there would be complex valued function g_1, g_2 such that $\tilde{\psi} = \psi g_1 + \varphi g_2$. The functions g_1 and g_2 are holomorphic, because, by (4.2), all solutions to (4.1) are holomorphic with respect to the complex holomorphic structure $(\nabla - A_\circ)^{(0,1)}$. Taking the projection of $\nabla\tilde{\psi} = \nabla\psi g_1 + \nabla\varphi g_2 + \psi dg_1 + \varphi dg_2$ to V/L shows that if one of the functions g_1 and g_2 is constant, the other has to be constant as well. To prove the claim we have to show that g_1 and g_2 are both constant. Assume that this was not the case, i.e., that both functions are non–constant. The projection of $\nabla\tilde{\psi}$ to V/L then yields $(\pi\psi) = (\pi\varphi)h$ with h the meromorphic function defined by $h dg_1 = -dg_2$. This would force h to be constant and ψ and φ to be linearly dependent, because the quotient of two holomorphic sections of the quaternionic holomorphic line bundle V/L with non–trivial Hopf field $Q \not\equiv 0$ has to be constant if it is complex holomorphic (recall that, by the discussion following (2.7), the Hopf field Q cannot vanish on any open subset of U because $\eta \not\equiv 0$ and A is nowhere vanishing on U). Hence, both g_1 and g_2 have to be constant and the space of local solutions to (4.1) defined on the open set U' is 2–dimensional, because on U' every solution $\tilde{\psi}$ is linearly dependent to ψ and φ.

Because prolongations of holomorphic sections are uniquely determined by their values on an open set this implies that the space of global solutions to (4.1) defined on the universal covering of T^2 is at most (complex) 2–dimensional. q.e.d.

4.3. The case that $A_\circ \not\equiv 0$ and $\check L = \ker(A_\circ)$ is constant. The following lemma is the analogue to Lemma 4.3 in the case that $\check L = \ker(A_\circ)$ is constant. It should be noted that every constrained Willmore torus in S^4 for which $\check L = \ker(A_\circ)$ is constant belongs to Case II or III of Proposition 3.1, because a constant section of $\ker(A_\circ)$ is ∇^μ–parallel for every $\mu \in \mathbb{C}_*$. A detailed discussion of constrained Willmore surfaces with constant $\check L = \ker(A_\circ)$ is given in Section 6.

Lemma 4.4. *For a constrained Willmore torus $f\colon T^2 \to S^4$ with the property that $\eta \not\equiv 0$ and $\check L = \ker(A_\circ)$ is constant, the space of solutions to* (4.1) *is (complex) 2–dimensional.*

The case that $\eta \equiv 0$ and $\check L = \ker(A_\circ)$ is constant which is excluded from the lemma corresponds to minimal surfaces with planar ends in the Euclidean space $\mathbb{R}^4 = S^4 \backslash \{\infty\}$ defined by $\infty = \check L = \ker(A)$, see Section 6.3.

Proof. As in the proof of Lemma 4.2, on the open set U where $V = L \oplus \check L$ the connection ∇ and the mean curvature sphere S of L take the form

$$(4.10) \qquad \nabla = \begin{pmatrix} \nabla^L & 0 \\ \delta & \check\nabla \end{pmatrix} \qquad \text{and} \qquad S = \begin{pmatrix} J & B \\ 0 & \tilde J \end{pmatrix},$$

where J and $\tilde J$ are complex structures on L and $\check L$ with $*\delta = \delta J = \tilde J \delta$ and where $B \in \Gamma(\mathrm{Hom}(\check L, L))$ with $JB + B\tilde J = 0$. The derivative of S is

$$(4.11) \qquad \nabla S = \begin{pmatrix} \nabla^L J - B\delta & \nabla B \\ 0 & \check\nabla \tilde J + \delta B \end{pmatrix}$$

and, by $\nabla S = 2{*}Q_\circ - 2{*}A_\circ$, the form $A_\circ$ is given by

$$(4.12) \qquad 2{*}A_\circ = \begin{pmatrix} -\nabla^L J + B\delta & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $\psi = (\psi_1, \psi_2)$ be a solution to (4.1) defined on U. Then $J\psi_1 = \psi_1 i$ and therefore $(\nabla^L J)\psi_1 = \nabla^L \psi_1 i - J\nabla^L \psi_1$. Hence, taking the $(0,1)$–part of $\nabla\psi = \nabla^L\psi_1$ yields

$$(\nabla\psi)^{(0,1)} = \tfrac{1}{2}(\nabla^L\psi_1 + J\nabla^L\psi_1 i) = \tfrac{1}{2}J(\nabla^L J)\psi_1.$$

On the other hand, on U, equation (4.1) implies (4.2) so that by (4.12)

$$(\nabla\psi)^{(0,1)} = \tfrac{1}{2}J(\nabla^L J)\psi_1 - \tfrac{1}{2}JB\delta\psi_1.$$

Together, the last two equations imply that solutions to (4.1) with $\psi_1 \not\equiv 0$ can only exist if $B \equiv 0$ which is impossible because, by (4.12), $B \equiv 0$ is equivalent to $A_\circ$ anti–commuting with S which again is equivalent to $\eta \equiv 0$. q.e.d.

4.4. Relation between holonomy and multiplier spectral curve. As an application of Proposition 4.1 we show now that the holonomy spectral curve Σ_{hol} essentially coincides with the multiplier spectral curve Σ_{mult} provided they are both defined. This is the case for constrained Willmore tori belonging to Case I or II of Proposition 3.1.

Theorem 4.5. *Let $f\colon T^2 \to S^4$ be a constrained Willmore torus for which both Σ_{hol} and Σ_{mult} are defined. Then the holomorphic map $\iota\colon \Sigma_{hol} \to \Sigma_{mult}$ is an injective immersion whose image is Σ_{mult} with finitely many points removed. The multipliers of the removed points or their conjugates belong to holomorphic sections whose prolongations solve* (4.1)*. In particular, all but finitely many points in Σ_{mult} give rise to (possibly singular) Darboux transforms which are again constrained Willmore where they are immersed. In case f is Willmore, all Darboux transforms belonging to points of Σ_{mult} are again Willmore.*

As suggested by the theorem, we will in the following not distinguish between Σ_{hol} and its image $\iota(\Sigma_{hol})$ under $\iota\colon \Sigma_{hol} \to \Sigma_{mult}$. Theorem 5.1 below shows that $\Sigma_{mult}\backslash\Sigma_{hol}$ consists of at most four points.

Corollary 4.6. *If $f\colon T^2 \to S^4$ is a Willmore torus with $\eta \equiv 0$ and $AQ \equiv 0$, then $\Sigma_{hol} = \Sigma_{mult}$ in case they are both defined.*

Proof. This follows from Lemma 4.3 according to which the space of solutions to (4.1) is 0–dimensional if $AQ \equiv 0$. q.e.d.

Proof of Theorem 4.5. If the map $\iota\colon \Sigma_{hol} \to \Sigma_{mult}$ is not injective, there is a non–empty open subset of Σ_{mult} all points of which have several preimages. In particular, there is $\sigma \in \Sigma_{mult}$ that has two preimages belonging to different parameters μ_1 and $\mu_2 \in \mathbb{C}_*$ and for which the space of holomorphic section with monodromy h^σ is 1–dimensional, see Section 3.4. The prolongation ψ of such a holomorphic section satisfies

$$\nabla\psi = (1-\mu_l)(A_\circ\psi)^{(1,0)} + (1-\mu_l^{-1})(A_\circ\psi)^{(0,1)}$$

for $l = 1$ and 2. Because $\mu_1 \neq \mu_2$ this implies $A_\circ\psi = 0$ which yields

$$0 = d^\nabla(*A_\circ\psi) = -*A_\circ \wedge \nabla\psi.$$

Because $\nabla\psi$ takes values in KL and $(A_\circ)_{|L} = A_{|L}$ is right $\bar{K}$ and does not vanish on the non–empty open set U where $V = L \oplus \check{L}$, this forces ψ to be constant on U and hence everywhere. But this contradicts the assumption that the multiplier h^σ is non–trivial so that the map ι is injective and in particular unbranched.

The image $\iota(\Sigma_{hol})$ is an open subset of Σ_{mult}. If Σ_{mult} is not connected it has two connected components which get interchanged by ρ and, because ι is compatible with the involutions ρ the image $\iota(\Sigma_{hol})$ intersects both components. In order to prove the lemma we show that the boundary of $\iota(\Sigma_{hol})$ in Σ_{mult} consist of points whose multipliers or their conjugates belong to global solutions to (4.1). Assume there is a sequence of points $\sigma_n \in \iota(\Sigma_{hol})$ converging to $\sigma_0 \in \Sigma_{mult}\backslash\iota(\Sigma_{hol})$. The corresponding sequence of parameters $\mu_n \in \mathbb{C}_*$ cannot be contained in a bounded subset of $\mathbb{C}_*$: otherwise we could assume by passing to a subsequence that μ_n converges to some $\mu_0 \in \mathbb{C}_*$. But then $\nabla^{\mu_n}\psi^{\sigma_n} = 0$ would imply $\nabla^{\mu_0}\psi^{\sigma_0} = 0$ which contradicts the assumption that σ_0 is not contained in $\iota(\Sigma_{hol})$; here ψ^σ denotes the prolongation of a local holomorphic section defined near σ_0 of the line bundle $\mathcal{L}$ from Section 3.4, that is, every ψ^σ is the prolongation of a holomorphic section $\pi\psi^\sigma$ with monodromy h^σ of V/L and $\sigma \mapsto \pi\psi^\sigma$ depends holomorphically on σ.

Hence μ_n is not contained in a bounded subset of $\mathbb{C}_*$ and, by passing to a subsequence and possibly applying the anti–holomorphic involutions ρ of Σ_{hol} and Σ_{mult}, we can assume that μ_n converges to ∞. Because

$$\nabla\psi^{\sigma_n} = (1-\mu_n)(A_\circ\psi^{\sigma_n})^{(1,0)} + (1-\mu_n^{-1})(A_\circ\psi^{\sigma_n})^{(0,1)}$$

for all n, while $\nabla\psi^{\sigma_n} \to \nabla\psi^{\sigma_0}$ and $A_\circ\psi^{\sigma_n} \to A_\circ\psi^{\sigma_0}$ for $n \to \infty$, we obtain

$$(A_\circ\psi^{\sigma_0})^{(1,0)} = 0.$$

This shows that for every point σ in the boundary of $\iota(\Sigma_{hol})$ the multiplier h^σ or $\bar{h}^\sigma = h^{\rho(\sigma)}$ admits a holomorphic section of V/L whose prolongation solves (4.1). By Proposition 4.1, there are at most two multipliers belonging to solutions to (4.1), because holomorphic sections of V/L with different monodromies are linearly independent over $\mathbb{C}$. The boundary of $\iota(\Sigma_{hol})$ thus consists of finitely many points and hence coincides with its complement $\Sigma_{mult}\backslash\iota(\Sigma_{hol})$.

By Theorem 2.5, the (possibly singular) Darboux transforms corresponding to points of Σ_{hol} are constrained Willmore or Willmore, respectively. In the Willmore case, by continuity this holds for all points of Σ_{mult}. q.e.d.

The fact that the spectral curve Σ_{mult} of a constrained Willmore torus belonging to Case I or II has finite genus can be immediately deduced from Theorem 4.5 (in Section 5 we more generally prove the existence of a polynomial Killing field): in the infinite genus case the spectral curve Σ_{mult} of a conformal torus in S^4 has only one end (cf. Section 3.4) which in particular is fixed under the anti–holomorphic involution ρ. But for a constrained Willmore torus of Case I or II, Theorem 4.5 implies that the ends of Σ_{mult} are also ends of Σ_{hol} and hence interchanged under ρ (because μ tends to 0 or ∞ at the ends of Σ_{hol} and ρ covers the involution $\mu \mapsto 1/\bar{\mu}$).

4.5. Constant Darboux transforms belonging to points of the spectral curve. For conformally immersed tori $f\colon T^2 \to S^4$ with trivial normal bundle, the normalization map $h\colon \Sigma_{mult} \to \mathrm{Hom}(\Gamma, \mathbb{C}_*)$ of the spectrum has a special multiple point at the trivial multiplier $h = 1 \in \mathrm{Hom}(\Gamma, \mathbb{C}_*)$. This singularity is characteristic for spectral curves of quaternionic holomorphic line bundles of degree 0 that are induced by immersed tori. Among the points desingularizing this singularity one is especially interested in the points corresponding to constant Darboux transforms of f. In the following we investigate the number of such points in case of constrained Willmore tori in S^4. For general conformal immersions $f\colon T^2 \to S^4$ very little is known about this number.

Lemma 4.7. *Let $f\colon T^2 \to S^4$ be a constrained Willmore immersion of a torus which belongs to Case I or II of Proposition* 3.1*.*

- *If $\check{L} = \ker(A_\circ)$ is non–constant, the only points in Σ_{mult} that correspond to constant Darboux transforms are the 2 or 4 points $\mu^{-1}(\{1\}) \subset \Sigma_{hol}$.*
- *If $\check{L} = \ker(A_\circ)$ is constant, then $\mu^{-1}(\{1\}) \subset \Sigma_{hol}$ consists of 2 points corresponding to constant Darboux transforms. Moreover, every constant Darboux transform that corresponds to a point in $\Sigma_{mult}\backslash\mu^{-1}(\{1\})$ is contained in $\check{L} = \ker(A_\circ)$.*

Proof. Because $\nabla^{\mu=1}$ is trivial, the points in the fiber $\mu^{-1}(\{1\}) \subset \Sigma_{hol}$ correspond to constant Darboux transforms. The set $\mu^{-1}(\{1\})$ is invariant under the fixed point free involution ρ and therefore consists of 2 or 4 points. For immersions whose holonomy representation belongs to Case II, for example if $\check{L} = \ker(A_\circ)$ is constant, Σ_{hol} is a 2–fold branched covering of $\mathbb{C}_*$ and $\mu^{-1}(\{1\})$ consists of 2 points. In case the Darboux transform corresponding to a point $\Sigma_{hol}\backslash\mu^{-1}(\{1\})$ is constant it has to be contained in $\check{L} = \ker(A_\circ)$ which is only possible if $\check{L}$ is constant.

A Darboux transform corresponding to a point in $\Sigma_{mult}\backslash\Sigma_{hol}$ is never constant unless $\check{L} = \ker(A_\circ)$ is constant: such Darboux transform has to be contained in $\check{L}$, because by Theorem 4.5 it solves (4.1) and, on a non–empty open set U, (4.2) so that $\check{L}$ is constant on U and hence everywhere. (In the following section we will see that $\Sigma_{mult}\backslash\Sigma_{hol} = \emptyset$ if $\check{L}$ is constant.) q.e.d.

For some special cases, using results from the following two sections enables us to give more precise information about the number of points in Σ_{mult} that correspond to constant Darboux transforms:

a) All non–super conformal minimal tori in the 4–sphere S^4 with its standard metric belong to Case I and the fiber $\mu^{-1}(\{1\})$ consists of 4 points corresponding to constant Darboux transforms: the fact that they belong to Case I follows from Corollary 5.2. The fiber $\mu^{-1}(\{1\})$ consists of 4 points, because for all $\mu \in S^1$ the holonomy of ∇^μ is contained in SU(4) so that $\mu\colon \Sigma_{hol} \to \mathbb{C}_*$ is unbranched over the unit circle.

If $\mathrm{im}(Q_\circ)$ is constant and the immersion is neither super conformal nor Euclidean minimal with planar ends, by Proposition 6.8 its holonomy belongs to Case II and, by Lemma 6.7, the two points $\mu^{-1}(\{1\}) \subset \Sigma_{hol}$ correspond to constant Darboux transforms contained in $\mathrm{im}(Q_\circ)$. If $\ker(A_\circ)$ is non–constant, no other points in Σ_{mult} correspond to constant Darboux transforms. If $\ker(A_\circ)$ is as well constant, there is at most one other pair of points corresponding to constant Darboux transforms in Σ_{mult} (following from the fact that a constant Darboux transform has to be contained in $\ker(A_\circ)$ and that parallel sections for different μ of the connections ∇^μ on V/L defined in (6.6) are linearly independent over $\mathbb{C}$).

b) For CMC tori in $\mathbb{R}^3$, the Lagrange multiplier η can be chosen such that $\ker(A_\circ) = \mathrm{im}(Q_\circ)$, see Section 6.6. This shows that, apart from the 2 points $\mu^{-1}(\{1\})$ there are no other points in Σ_{mult} that correspond to constant Darboux transforms.

c) In case of CMC tori in S^3 with mean curvature $H^{S^3} = \cot(\alpha/2)$, formula (6.13) shows that for the right choice of η (namely $\rho = \pm 1/2$ in Section 6.7), the 4 points $\mu^{-1}(\{1\})$ and $\mu^{-1}(\{e^{i\alpha}\})$ are the only points corresponding to constant Darboux transforms in $\mathrm{im}(Q_\circ)$ and $\ker(A_\circ)$ respectively.

5. The Main Theorem and its Proof

We prove the main theorem of the paper by separately dealing with all possible cases of holonomy representations that occur for the associated family of constrained Willmore tori. For Cases I and II of Proposition 3.1 we prove the existence of a polynomial Killing field ξ, a family of sections of $\mathrm{End}_{\mathbb{C}}(V)$ which is polynomial in μ and, for all $\mu \in \mathbb{C}_*$, satisfies $\nabla^\mu \xi(\mu, .) = 0$. Because ξ commutes with all holonomies, its existence implies that Σ_{hol} and hence Σ_{mult} can be compactified by adding points at infinity. For Case IIIa of Proposition 3.1 we prove the existence of a polynomial family of ∇^μ–parallel sections of the complex rank 4–bundle (V, i) itself and for Case IIIb the existence of a nil–potent polynomial Killing field ξ. Investigating the asymptotics of such polynomial families of sections reveals that Cases IIIa and IIIb correspond to immersions that are super conformal or Euclidean minimal with planar ends.

5.1. Main theorem of the paper. The following is a more detailed formulation of the main theorem stated in the introduction. Its proof will be given in Sections 5.2 to 5.4.

Theorem 5.1. *Let $f\colon T^2 \to S^4$ be a constrained Willmore immersion from a torus into the conformal 4–sphere S^4. Then one of the following holds:*

I. The holonomy spectral curve Σ_{hol} can be compactified to a 4–fold branched covering of $\mathbb{CP}^1$ by adding points over $\mu = 0$ and $\mu = \infty$. The two ends of the spectral curve Σ_{mult} correspond to branch points of μ over $\mu = 0$ and $\mu = \infty$. The complement $\Sigma_{mult} \backslash \Sigma_{hol}$ of the holonomy spectral curve inside the multiplier spectral curve consists of at most four points.

II. The holonomy and multiplier spectral curves coincide $\Sigma_{mult} = \Sigma_{hol}$ and can be compactified to a 2–fold branched covering of $\mathbb{CP}^1$ by adding one point over $\mu = 0$ and one over $\mu = \infty$.

III. The immersion f is super conformal or minimal with planar ends in the Euclidean space $\mathbb{R}^4 = S^4 \backslash \{\infty\}$ defined by some point $\infty \in S^4$ at infinity. More precisely:
- *f belongs to Case IIIa in Proposition 3.1 if and only if it is super conformal or $f\colon T^2 \backslash \{p_1, ..., p_n\} \to \mathbb{R}^4 \cong S^4 \backslash \{\infty\}$ is an algebraic Euclidean minimal immersion with planar ends (with algebraic meaning that the closed form $*df$ has no periods so that f is the real part of a meromorphic null immersion from T^2 into $\mathbb{C}^4$).*
- *f belongs to Case IIIb in Proposition 3.1 if and only if f is a non–algebraic Euclidean minimal immersion with planar ends (that is, $*df$ has periods).*

If the normal bundle $\perp_f$ of the immersion f is topologically trivial and f is not Euclidean minimal with planar ends, it belongs to the "finite type" Cases I and II. If the normal bundle $\perp_f$ is non–trivial, then f is of "holomorphic type" and belongs to Case III.

It should be noted that for an isothermic constrained Willmore torus $f\colon T^2 \to S^4$ which belongs to Case I or II in Theorem 5.1, changing the form $\eta \in \Omega^1(\mathcal{R})$ in the Euler–Lagrange equation (2.4) changes Σ_{hol} which amounts to changing the meromorphic function μ on the multiplier spectral curve Σ_{mult}. For different choices of η, the holonomy curve might then change between Cases I and II of the theorem. This happens for example in case of minimal tori in S^3, see Section 6.7.

Corollary 5.2. *Let $f\colon T^2 \to S^4$ be a Willmore torus with $\eta \equiv 0$ that is not Euclidean minimal with planar ends. Then:*
- *f belongs to Case I if and only if $\deg(\perp_f) = 0$ and*
- *f belongs to Case III and is super conformal if and only if $\deg(\perp_f) \neq 0$.*

Proof. For immersions belonging to Case II or III, one can check as in the proof of Lemma 5.8 below that the fiber $\mathcal{V}_\infty$ over $\mu = \infty$ of the holomorphic vector bundle $\mathcal{V}$ constructed in Lemma 5.6 is a space of solutions to (4.1) of dimension greater or equal 2. By Lemma 4.3, such bundle $\mathcal{V}$ cannot exist for a Willmore torus f with $\eta \equiv 0$ that is neither super conformal nor Euclidean minimal. q.e.d.

Corollary 5.3. *If the holonomy spectral curve Σ_{hol} of a constrained Willmore torus that belongs to Case I or II in Theorem* 5.1 *coincides with Σ_{mult}, the spectral curve $\Sigma_{mult} = \Sigma_{hol}$ is irreducible, that is, has a single connected component.*

Proof. Since $\Sigma_{hol} = \Sigma_{mult}$ has only two ends, cf. Section 3.4, it can be compactified by glueing in a single point over $\mu = 0$ and ∞, respectively. Because the branch order of the meromorphic function μ at the two added points is then 3 in Case I and 1 in Case II, the total space of the branched covering $\mu\colon \Sigma_{hol} \to \mathbb{C}_*$ is connected. q.e.d.

Irreducibility of the spectral curve automatically holds

- in Case II, because then always $\Sigma_{hol} = \Sigma_{mult}$, and
- for Willmore immersions f belonging to Case I which $\eta \equiv 0$ and $AQ \equiv 0$, see Corollary 4.6.

The condition $AQ = 0$ is satisfied e.g. for all Willmore tori in the conformal 3–sphere S^3, for minimal tori in the 4–sphere with its standard metric, or for minimal tori in hyperbolic 4–space, see Chapter 10 of [**7**].

Corollary 5.4. *Let $f\colon T^2 \to S^4$ be a constrained Willmore torus whose Willmore functional is bounded by $\mathcal{W} < 8\pi$. Then:*

- *f belongs to Case I if and only if $\check{L} = \ker(A_\circ)$ is non–constant and*
- *f belongs to Case II if and only if $\check{L} = \ker(A_\circ)$ is constant.*

In the latter case $\operatorname{im}(Q_\circ)$ is also constant and $\ker(A_\circ) \neq \operatorname{im}(Q_\circ)$.

Proof. A constrained Willmore torus f with $\mathcal{W} < 8\pi$ cannot belong to Case III of Theorem 5.1, because super conformal and Euclidean minimal immersions with planar ends have Willmore energy $\mathcal{W} \geq 8\pi$. Moreover, f has trivial normal bundle and the quaternionic Plücker formula [**13**] implies that the space $H^0(V/L)$ of holomorphic sections with trivial monodromy is quaternionic 2–dimensional. All holomorphic sections of V/L with trivial monodromy are thus projections of constant sections of V. This shows that f belongs to Case II if and only if $\ker(A_\circ)$ is constant, because ∇^μ with $\mu \in \mathbb{C}_*\backslash\{1\}$ has parallel sections with trivial monodromy if and only if $\ker(A_\circ)$ is constant.

The fact that $\operatorname{im}(Q_\circ)$ is also constant in case $\ker(A_\circ)$ is constant follows by passing to the dual constrained Willmore surface $f^\perp$ because, by (2.11) and (2.12), the immersion f belongs to Case II if and only if its dual immersion $f^\perp$ belongs to Case II. Lemma 6.11 implies that $\ker(A_\circ) \neq \operatorname{im}(Q_\circ)$, because otherwise f had to be CMC in $\mathbb{R}^3$ or Euclidean minimal with planar ends which is impossible if $\mathcal{W} < 8\pi$.

q.e.d.

5.2. Hitchin trick. In the following we construct, for each case in the holonomy list of Proposition 3.1, a family of sections of either (V, i) or

$\mathrm{End}_{\mathbb{C}}(V)$ that is polynomial in the spectral parameter μ and has the property that the evaluation at every $\mu \in \mathbb{C}_*$ is ∇^μ–parallel. To do this we use the following idea from [**19**]: although ∇^μ has poles at $\mu = 0$ and $\mu = \infty$, the holomorphic families of elliptic operators

$$\begin{aligned} \partial^\mu &= (\nabla^\mu)^{(1,0)} = \nabla^{(1,0)} + (\mu - 1)A_\circ^{(1,0)} \\ \bar\partial^\mu &= (\nabla^\mu)^{(0,1)} = \nabla^{(0,1)} + (\mu^{-1} - 1)A_\circ^{(0,1)} \end{aligned} \tag{5.1}$$

obtained by taking the $(1,0)$– and $(0,1)$–parts of the associated family ∇^μ of flat connections extend to $\mu = 0$ or $\mu = \infty$, respectively. Because elliptic operators on compact manifolds are Fredholm, Proposition 3.3 applies to the holomorphic families ∂^μ and $\bar\partial^\mu$. We use this to show for each of the cases in Proposition 3.1 that the family of spaces of ∇^μ–parallel sections of either (V, i) or $\mathrm{End}_{\mathbb{C}}(V)$ gives rise to a holomorphic vector bundles over $\mathbb{CP}^1$ whose fiber over generic points $\mu \in \mathbb{C}_* \subset \mathbb{CP}^1$ coincides with the space of ∇^μ–parallel sections. A polynomial family of parallel sections can then be obtained as a meromorphic section with a single pole at $\mu = \infty$ of this holomorphic vector bundle over $\mathbb{CP}^1$.

5.3. Case by case study. A *polynomial Killing field* [**12**, **6**] for the constrained Willmore associated family ∇^μ is a polynomial family

$$\xi(\mu, p) = \sum_{l=0}^{d} \xi_l(p)\mu^l$$

of sections of $\mathrm{End}_{\mathbb{C}}(V)$ with coefficients $\xi_l \in \Gamma(\mathrm{End}_{\mathbb{C}}(V))$ whose value at every $\mu \in \mathbb{C}_*$ is ∇^μ–parallel, that is, satisfies

$$\nabla^\mu \xi(\mu, \_) = 0 \tag{5.2}$$

which is equivalent to the Lax–type equation

$$\nabla\xi = \big[(1-\mu)A_\circ^{(1,0)} + (1-\mu^{-1})A_\circ^{(0,1)}, \xi\big].$$

Lemma 5.5. *A constrained Willmore torus that belongs to* ***Case I*** *of Proposition* 3.1 *admits a polynomial Killing field* ξ *which, for generic* μ*, has four different eigenvalues.*

Proof. In Case I of Proposition 3.1, away from isolated spectral parameters $\mu \in \mathbb{C}_*$ the space of ∇^μ–parallel sections of the bundle $\mathrm{End}_{\mathbb{C}}(V)$ is 4–dimensional, because parallel endomorphisms are parametrized by the elements of one fiber $\mathrm{End}_{\mathbb{C}}(V)_p$, $p \in T^2 = \mathbb{C}/\Gamma$ that commute with all $H_p^\mu(\gamma)$, $\gamma \in \Gamma$. In order to prove the existence of the polynomial Killing field ξ we construct a holomorphic rank 4 subbundle $\mathcal{U}$ of the trivial bundle $\mathbb{CP}^1 \times \Gamma(\mathrm{End}_{\mathbb{C}}(V))$ whose fiber $\mathcal{U}_\mu$ over generic $\mu \in \mathbb{C}_* \subset \mathbb{CP}^1$ coincides with the space of ∇^μ–parallel sections. Following the idea explained in Section 5.2, we define the bundle $\mathcal{U}$ as the kernel bundle of the families of operators (5.1) acting on sections of $\mathrm{End}_{\mathbb{C}}(V)$. To make sure that the kernels of (5.1) define a bundle with the given properties, by Proposition 3.3 it remains to check that there is at least one $\mu \in \mathbb{C}_*$ for which the kernel of $\bar\partial^\mu$ (or, which is equivalent by (2.10), the kernel of $\partial^\mu = j^{-1}\,\bar\partial^{1/\bar\mu}\,j$) is 4–dimensional and hence coincides with the space of ∇^μ–parallel sections.

Assume this was not the case, i.e., for all μ the dimension of the space of $\bar\partial^\mu$–holomorphic sections of $\mathrm{End}_{\mathbb{C}}(V)$ is strictly greater than four. For generic $\mu \in \mathbb{C}_*$, the complex rank 4 bundle (V, i) is a direct sum $V = E_{h_1} \oplus ... \oplus E_{h_4}$ of ∇^μ–parallel complex line subbundles whose parallel sections have different multipliers h_1,..., h_4. In particular $\mathrm{End}_{\mathbb{C}}(V) = \oplus_{kl}\, E_{h_k} E_{h_l}^{-1}$. The fact that the space of $\bar\partial^\mu$–holomorphic sections of $\mathrm{End}_{\mathbb{C}}(V)$ is strictly greater than the 4–dimensional space of ∇^μ–parallel sections thus implies that for some $k \neq l$ there exists a non–trivial holomorphic homomorphism between the complex holomorphic line bundles E_{h_k} and E_{h_l} of degree 0. In other words, there are $k \neq l$ such that the bundles E_{h_k} and E_{h_l} are holomorphically equivalent and represent the same point in the Jacobi variety of $T^2 = \mathbb{C}/\Gamma$, the space of holomorphic equivalence classes of topologically trivial holomorphic line bundles over T^2.

Our above assumption therefore means that we are in the following situation: at the four preimages under $\mu\colon \Sigma_{hol} \to \mathbb{C}_*$ of a generic $\mu_0 \in \mathbb{C}_*$, the map $h\colon \Sigma_{hol} \to \mathrm{Hom}(\Gamma, \mathbb{C}_*)$ takes four different values corresponding to four different gauge equivalence classes of flat complex line bundles, but the induced map into the Jacobian takes the same value on at least two of the four points over μ_0. This assumption is equivalent to $U \neq \emptyset$ with

$$\begin{aligned} U \;=\; \{\sigma \in \Sigma' \mid & \text{ for all } \sigma' \text{ sufficiently close to } \sigma \text{ there exists } \sigma'' \neq \sigma' \\ & \text{with } \mu(\sigma') = \mu(\sigma'') \text{ such that } h^{\sigma'} \text{ and } h^{\sigma''} \\ & \text{give rise to isomorphic holomorphic line bundles}\}, \end{aligned}$$

where $\Sigma' = \Sigma_{hol} \backslash \{\sigma \in \Sigma_{hol} \mid \mu(\sigma) \text{ is singular value of } \mu\}$ denotes the subset of Σ_{hol} on which $\mu\colon \Sigma_{hol} \to \mathbb{C}_*$ becomes an unbranched 4–fold covering.

If $U = \emptyset$, the minimal kernel dimension of $\bar\partial^\mu$ (and ∂^μ) acting on $\Gamma(\mathrm{End}_{\mathbb{C}}(V))$ is four and Proposition 3.3 implies the existence of a holomorphic rank 4 subbundle $\mathcal{U}$ of the trivial bundle $\mathbb{CP}^1 \times \Gamma(\mathrm{End}_{\mathbb{C}}(V))$ whose fiber over a generic $\mu \in \mathbb{C}_* \subset \mathbb{CP}^1$ coincides with the space of ∇^μ–parallel endomorphisms. A polynomial Killing field ξ can then be obtained as a global meromorphic section of $\mathcal{U}$ with a single pole at $\mu = \infty$.

In order to prove the lemma it therefore remains to prove that the assumption $U \neq \emptyset$ leads to a contradiction. We do this in two steps. In both steps we use that the projection from $\mathrm{Hom}(\Gamma, \mathbb{C}_*)$ to the Jacobian of $T^2 = \mathbb{C}/\Gamma$ has the coordinate representation

(5.3)
$$\mathrm{Hom}(\Gamma, \mathbb{C}_*) \to \mathbb{C}/\Gamma' \qquad h(\gamma) = \exp(a\gamma + b\bar{\gamma}) \quad \mapsto \quad b \mod \Gamma',$$

where Γ' is the lattice $\Gamma' = \{c \in \mathbb{C} \mid -\bar{c}\gamma + c\bar{\gamma} \in 2\pi i\mathbb{Z} \text{ for all } \gamma \in \Gamma\}$. To see this, note that the gauge equivalence class of flat complex line bundles with holonomy $h(\gamma) = \exp(a\gamma + b\bar{\gamma})$ over $T^2 = \mathbb{C}/\Gamma$ is represented by $\nabla = d - a\,dz - b\,d\bar{z}$ with d the trivial connection on the trivial bundle; this representation is unique up to the action $(a, b) \mapsto (a - \bar{c}, b + c)$ of $c \in \Gamma'$ on $\mathbb{C}^2$. The induced holomorphic structure is $\bar{\partial} = \nabla'' = \bar{\partial}_0 - b\,d\bar{z}$ with $\bar{\partial}_0$ the trivial holomorphic structure. Since up to holomorphic equivalence every holomorphic structure on the trivial bundle is of this form with b unique up to $b \mapsto b + c$, $c \in \Gamma'$, the quotient $\mathbb{C}/\Gamma'$ is a model for the Jacobian of T^2.

Step 1: We show that $U \neq \emptyset$ implies $U = \Sigma'$. For this we use that U is open and closed and hence the union of connected components of Σ': that U is open is clear from its definition; its closedness follows from the fact that, by the identity theorem for holomorphic functions, the set U is the complement of the open set

$$\begin{aligned}\{\sigma \in \Sigma' \mid \quad & \text{for all } \sigma' \neq \sigma \text{ sufficiently close to } \sigma \text{ and all } \sigma'' \neq \sigma' \\ & \text{with } \mu(\sigma') = \mu(\sigma''), \text{the flat line bundles corresponding to} \\ & h^{\sigma'} \text{ and } h^{\sigma''} \text{ are not holomorphically isomorphic}\}.\end{aligned}$$

In the case that Σ' is connected this immediately shows that $U \neq \emptyset$ implies $U = \Sigma'$. It therefore remains to deal with the case that Σ' is not connected.

In order to show that $U \neq \emptyset$ implies $U = \Sigma'$ also if Σ' is not connected, we apply results from [**4**] about the asymptotic behavior of $h\colon \Sigma_{mult} \to \mathrm{Hom}(\Gamma, \mathbb{C}_*)$. In the disconnected case Σ' has two connected components, because Σ_{mult} has either one or two components, cf. Section 3.4, and the number of components of Σ' and Σ_{mult} coincide (Σ' and Σ_{hol} differ by a discrete set while Σ_{hol} and Σ_{mult} differ by a finite set, cf. Theorem 4.5). By Theorem 4.1 of [**4**], if Σ_{mult} has two components its genus is finite and each of its components is a compact surface with one point removed. In particular, $\mu\colon \Sigma_{hol} \to \mathbb{C}_*$ has a finite number of branch points so that Σ' and Σ_{mult} differ by a finite number of points and Σ' contains both ends of Σ_{mult}. The necessary information about the asymptotics of the monodromy map h at the ends of Σ_{mult} is provided by Lemma 5.2 of [**4**]: it shows that one of the two ends of Σ_{mult} can be parametrized $x \mapsto \sigma(x)$ by $x \in D_*$ in a punctured disc $D_* = \{x \in \mathbb{C}_* \mid |x| < \epsilon\}$ such that

$$h^{\sigma(x)}(\gamma) = \exp((a_0 + x^{-1})\gamma + b(x)\bar{\gamma}) \tag{5.4}$$

for all $x \in D_*$ and $\gamma \in \Gamma$ with $b(x)$ holomorphic in $x = 0$. The other end can be parametrized by $x \in D_*$ such that

$$h^{\sigma(x)}(\gamma) = \exp(a(x)\gamma + (b_0 + x^{-1})\bar{\gamma}) \tag{5.5}$$

for all $x \in D_*$ and $\gamma \in \Gamma$ with $a(x)$ holomorphic at $x = 0$.

If now $U \neq \emptyset$ and $U \neq \Sigma'$, then U coincides with one of the two connected components of Σ'. In particular, it contains one of the ends of Σ_{mult}. Moreover, the restriction of $\mu\colon \Sigma_{hol} \to \mathbb{C}_*$ to U is a 2–sheeted covering (by definition it is at least 2–sheeted and since the anti–holomorphic involution ρ interchanges the components of Σ' it cannot have more sheets). The first kind of end cannot be contained in U: by (5.3), the composition $\tilde{h}\colon \Sigma_{mult} \to \mathbb{C}/\Gamma'$ of the map $h\colon \Sigma_{mult} \to \mathrm{Hom}(\Gamma, \mathbb{C}_*)$ with the projection to the Jacobian of T^2 extends holomorphically though the first kind of end (because the function $b(x)$ in (5.4) is holomorphic at $x = 0$) and is hence well defined on the compactification $\bar{\Sigma}_1$ of the connected component Σ_1 of Σ_{mult} containing this end. If U was contained in the connected component Σ_1, then $\tilde{h}_{|\bar{\Sigma}_1}$ would (by definition of U) descend to the μ–plane and define a holomorphic map $\mathbb{CP}^1 \to \mathbb{C}/\Gamma'$. But this is impossible, because such a map had to be constant so that $\tilde{h}$ would be constant near one of the ends which could only happen if the quaternionic holomorphic line bundle V/L had vanishing Willmore energy, cf. Theorem 5.5 of [**4**].

To see that the second kind of end cannot be contained in U, denote by $x \mapsto \sigma(x)$, $x \in D_*$ the above coordinate at the end. The map $x \in D_* \mapsto \mu(\sigma(x))$ then restricts to a 2–fold covering of a punctured neighborhood of $\mu = 0$ or $\mu = \infty$. (To see that the ends of Σ_{mult} cannot correspond to single sheets of μ, note that when σ approaches one of them the multiplier h goes to infinity of $\mathrm{Hom}(\Gamma, \mathbb{C}_*) \cong T^2 \times \mathbb{R}^2$. This cannot happen on a single sheet of μ, because, firstly, the product of the four values of h over a generic point $\mu \in \mathbb{C}_*$ is the trivial representation $1 \in \mathrm{Hom}(\Gamma, \mathbb{C}_*)$, secondly, h is continuous at those ends of Σ_{hol} that correspond to points in $\Sigma_{mult} \backslash \Sigma_{hol}$, and, thirdly, one of the ends of Σ_{mult} is located at $\mu = 0$ and the other one at $\mu = \infty$, because both ends are interchanged under ρ which covers $\mu \mapsto 1/\bar{\mu}$.) This allows to introduce another coordinate y centered at the end such that $\mu = y^2$ or $\mu = y^{-2}$ respectively. By definition of U the multipliers $h^{\sigma(y)}$ and $h^{\sigma(-y)}$ are then holomorphically equivalent. Using (5.3) and (5.5), we therefore obtain that

$$\frac{1}{x(y)} = \frac{1}{x(-y)} + c \qquad \text{for some } c \in \Gamma',$$

where $y \mapsto x(y)$ denotes the coordinate change. But this is impossible, because $x(0) = 0$ and $x'(0) \neq 0$ so that the residues at $y = 0$ of both sides of this equation have opposite signs.

Step 2: We show now that the assumption $U \neq \emptyset$ (which by Step 1 implies $U = \Sigma'$) leads to a contradiction. For this we have to deal with two different cases: either for all $\sigma \in \mu^{-1}(S^1)$ the multipliers h^σ and $h^{\rho(\sigma)}$ are holomorphically equivalent or for all $\sigma_1 \in \mu^{-1}(S^1)$ at which μ is unbranched there exists σ_2 with $\sigma_1 \neq \sigma_2 \neq \rho(\sigma_1)$ and $\mu(\sigma_1) = \mu(\sigma_2)$ such that h^{σ_1} and h^{σ_2} as well as $h^{\rho(\sigma_1)}$ and $h^{\rho(\sigma_2)}$ are holomorphically equivalent.

To prove that the first case is impossible we write the multiplier h^σ at $\sigma \in \mu^{-1}(S^1)$ as

$$h^\sigma(\gamma) = \exp(a\gamma + b\bar{\gamma})$$

with $a, b \in \mathbb{C}$ unique up to the action of $c \in \Gamma'$ by $(a, b) \mapsto (a - \bar{c}, b + c)$. The multiplier of $\rho(\sigma)$ is then $h^{\rho(\sigma)}(\gamma) = \exp(\bar{b}\gamma + \bar{a}\bar{\gamma})$. Because h^σ and $h^{\rho(\sigma)}$ give rise to isomorphic holomorphic line bundles, by (5.3) we obtain that $\bar{a} = b + c$ for some $c \in \Gamma'$. But this implies $h^{\rho(\sigma)}(\gamma) = \exp((a - \bar{c})\gamma + (b + c)\bar{\gamma})$ so that $h^\sigma = h^{\rho(\sigma)}$ for all $\sigma \in \mu^{-1}(S^1)$ which is impossible because in Case I all four multipliers over a generic $\mu \in \mathbb{C}_*$ are different.

To see that the second case is impossible, we chose two non–constant continuous curves $\sigma_1(t)$ and $\sigma_2(t)$ in $\mu^{-1}(S^1) \subset \Sigma_{hol}$ with $\mu(\sigma_1(t)) = \mu(\sigma_2(t))$ for all t and $\mu(\sigma_1(0)) = \mu(\sigma_2(0)) = 1$ such that $h^{\sigma_1(t)}$ and $h^{\sigma_2(t)}$ are holomorphically equivalent. By (5.3) there are unique continuous complex functions $a_1(t)$, $a_2(t)$ and $b(t)$ with $a_1(0) = a_2(0) = b(0)$ such that

$$h^{\sigma_1(t)}(\gamma) = \exp(a_1(t)\gamma + b(t)\bar{\gamma}) \quad \text{and} \quad h^{\sigma_2(t)}(\gamma) = \exp(a_2(t)\gamma + b(t)\bar{\gamma})$$

for all t. Because $h^{\rho(\sigma_1(t))}$ and $h^{\rho(\sigma_2(t))}$ are also holomorphically equivalent there is a constant $c \in \Gamma'$ such that $a_2(t) = a_1(t) + \bar{c}$. Evaluating at $t = 0$ implies $c = 0$. As a consequence, $h^{\sigma_1(t)} = h^{\sigma_2(t)}$ for all t which is impossible. q.e.d.

In Cases II and III of Proposition 3.1, the space of ∇^μ–parallel sections with trivial monodromy for $\mu \in \mathbb{C}_*$ is at least a (complex) 2–dimensional subspace of the finite dimensional space $\mathcal{H} = \{\psi \in \Gamma(V) \mid \nabla\psi \in \Omega^1(L)\}$ of prolongations of holomorphic sections with trivial monodromy of V/L.

Lemma 5.6. *For a constrained Willmore torus that belongs to* ***Case II or III*** *of Proposition* 3.1*, there is a holomorphic vector subbundle $\mathcal{V}$ of the trivial bundle $\mathbb{CP}^1 \times \mathcal{H}$ whose fiber $\mathcal{V}_\mu$ over generic $\mu \in \mathbb{C}_* \subset \mathbb{CP}^1$ coincides with the space of ∇^μ–parallel sections with trivial monodromy. The bundle $\mathcal{V}$ has rank 2 in Case II and IIIb and rank 4 in Case IIIa.*

Proof. We prove the existence of $\mathcal{V}$ by applying Proposition 3.3 to the operators (5.1) acting on the finite dimensional space $\mathcal{H}$. For doing this it remains to check that, over generic $\mu \in \mathbb{C}_*$, the space of $\bar{\partial}^\mu$–holomorphic sections (or, which is equivalent by (2.10), the space of

∂^μ–anti–holomorphic sections) contained in $\mathcal{H}$ coincides with the space of ∇^μ–parallel sections.

This is immediately clear in Case IIIa of Proposition 3.1 because, for every $\mu \in \mathbb{C}_*$, the connection ∇^μ is then trivial and so is the induced holomorphic structure $\bar\partial^\mu = (\nabla^\mu)^{(0,1)}$. Therefore, all $\bar\partial^\mu$–holomorphic sections are ∇^μ–parallel and $\mathcal{V}$ is a holomorphic bundle of rank 4. In Case II, for generic $\mu \in \mathbb{C}_*$, the trivial bundle V over the torus has a splitting $V = E_1 \oplus E_{h_1} \oplus E_{h_2}$ into ∇^μ–parallel subbundles with the property that the connection induced by ∇^μ on the rank 2 bundle E_1 is trivial while parallel sections of the line bundles E_{h_l}, $l = 1, 2$ have nontrivial monodromy $h_l \in \mathrm{Hom}(\Gamma, \mathbb{C}_*)$. As above, every $\bar\partial^\mu$–holomorphic section of the trivial subbundle E_1 is ∇^μ–parallel. Every $\bar\partial^\mu$–holomorphic section of E_{h_l} is of the form $\psi_l f_l$, where ψ_l is a ∇^μ–parallel section of E_l and f_l is a holomorphic complex function with monodromy h_l^{-1}. But such a section $\psi_l f_l$ is never contained in $\mathcal{H}$: if the quotient of two holomorphic sections of V/L is complex, it is constant unless the Hopf field Q is trivial (which is impossible, because a torus of Case II cannot be super conformal). Thus, in Case II, the holomorphic subbundle $\mathcal{V}$ of the trivial bundle $\mathbb{CP}^1 \times \mathcal{H}$ has rank 2.

Proving the existence of $\mathcal{V}$ in Case IIIb of Proposition 3.1 is slightly more involved: for generic $\mu \in S^1 \subset \mathbb{C}_*$ there is a ∇^μ–parallel section ψ with trivial monodromy and a parallel section φ together with $t \in \mathrm{Hom}(\Gamma, \mathbb{C})$ such that $\gamma^*\varphi = \varphi + \psi\, t_\gamma$ for all $\gamma \in \Gamma$. Every $\bar\partial^\mu$–holomorphic section $\tilde\psi$ is then of the form $\tilde\psi = \psi(f_1 + jf_2) + \varphi(g_1 + jg_2)$ where f_1, f_2, g_1 and g_2 are complex holomorphic functions. Such a section has trivial monodromy if and only if g_1, g_2 are constant and $\gamma^* f_1 = f_1 - g_1\, t_\gamma$ and $\gamma^* f_2 = f_2 - g_2\, \bar t_\gamma$ for all $\gamma \in \Gamma$. By the same argument as in Case II, the section $\tilde\psi$ is in $\mathcal{H}$ if and only if f_1 and f_2 are constant and $g_1 = g_2 = 0$. This shows that in Case IIIb the bundle $\mathcal{V}$ has rank 2. q.e.d.

Lemma 5.7. *A constrained Willmore torus that belongs to* ***Case II*** *of Proposition* 3.1 *admits a polynomial Killing field ξ which, for generic μ, has a 2–dimensional kernel and two different non–trivial eigenvalues.*

Proof. The existence of ξ is proven by similar arguments as in Case I. For generic $\mu \in \mathbb{C}_*$, the trivial bundle V has a splitting $V = E_1 \oplus E_{h_1} \oplus E_{h_2}$ into ∇^μ–parallel subbundles. The space of ∇^μ–parallel sections of $\mathrm{End}_{\mathbb{C}}(V)$ is now 6–dimensional, because E_1 is a trivial subbundle. Using Proposition 3.3 we construct a subbundle $\mathcal{U}$ of the trivial bundle $\mathbb{CP}^1 \times \Gamma(\mathrm{End}_{\mathbb{C}}(V))$ whose fiber $\mathcal{U}_\mu$ over generic μ coincides with the space of ∇^μ–parallel sections. For this we have to check that generically the space of $\bar\partial^\mu$–holomorphic sections of $\mathrm{End}_{\mathbb{C}}(V)$ (or, equivalently, that of ∂^μ–anti–holomorphic sections) is also 6–dimensional. Assuming that this was not the case would force the multipliers h_1^μ and h_2^μ to be holomorphically equivalent for every μ which is impossible because, as in Case I, this would imply $h_1^\mu = h_2^\mu$ for all $\mu \in S^1$.

A global meromorphic section ξ of the holomorphic rank 6 bundle $\mathcal{U}$ with a single pole at $\mu = \infty$ is a polynomial Killing field. For the reconstruction of Σ_{hol} from ξ it is preferable to have a polynomial Killing field $\tilde{\xi}$ that, for every μ, vanishes on the trivial ∇^μ–parallel subbundle E_1. We construct such $\tilde{\xi}$ by using the holomorphic rank 2 bundle $\mathcal{V}$ defined in Lemma 5.6 as well as the corresponding bundle $\mathcal{V}^\perp$ that belongs to the dual constrained Willmore immersion $f^\perp$. The latter is the holomorphic rank 2 subbundle of $\mathcal{H}^\perp = \{\alpha \in \Gamma(V^*) \mid \nabla\alpha \in \Omega^1(L^\perp)\}$ whose fiber over generic $\mu \in \mathbb{C}_*$ is the space of $(\nabla^\perp)^\mu$–parallel sections. The existence of $\mathcal{V}^\perp$ follows from Lemma 5.6 because, by (2.11), the connections $(\nabla^\perp)^\mu$ on the bundle $(V^*, -i)$ are gauge equivalent to the connections $(\tilde{\nabla}^\perp)^\mu = \nabla + (\mu - 1)(Q_\circ^\perp)^{(1,0)} + (\mu^{-1} - 1)(Q_\circ^\perp)^{(0,1)}$ dual to ∇^μ so that all its holonomies have 1 as an eigenvalue of geometric multiplicity 2. Let ψ_1, ψ_2 be two linearly independent meromorphic sections of $\mathcal{V}$ and α^1, α^2 two linearly independent meromorphic sections of the holomorphic bundle $\tilde{\mathcal{V}}^\perp$ that is image of $\mathcal{V}^\perp$ under the gauge transformation of (2.11).

For generic $\mu \in \mathbb{C}_*$, the sections $\psi_1(\mu, .)$, $\psi_2(\mu, .)$ of (V, i) are pointwise linearly independent and span the rank 2 subbundle $E_1 = \ker(H_p^\mu(\gamma) - \mathrm{Id}) \subset (V, i)$ where $\gamma \in \Gamma\backslash\{0\}$. Similarly, for generic μ, the sections $\alpha^1(\mu, .)$, $\alpha^2(\mu, .)$ of $(V^*, -i)$ are pointwise linearly independent and span the rank 2 subbundle $\ker((H_p^\mu(\gamma))^* - \mathrm{Id}) = \mathrm{im}(H_p^\mu(\gamma) - \mathrm{Id})^\perp$ for $\gamma \in \Gamma\backslash\{0\}$, where we use the identification of Section 2.4 between $(V^*, -i)$ and the complex dual space of (V, i). The meromorphic family of 2×2–matrices $g_{kl}(\mu) = < \alpha_{\mathbb{C}}^k(\mu, .), \psi_l(\mu, .) >$ (with $\alpha_{\mathbb{C}}^k$ denoting the complex part of the α^l) is therefore invertible for generic μ and

$$\tilde{\xi}(\mu, p) := \xi(\mu, p) - \sum_{kl} \xi(\mu, p)(\psi_k(\mu, p)) \cdot g_{kl}^{-1}(\mu) \cdot \alpha_{\mathbb{C}}^l(\mu, p)$$

defines a polynomial Killing field that, for every μ, vanishes on the trivial subbundle E_1. q.e.d.

Lemma 5.8. *If a constrained Willmore torus belongs to* ***Case IIIa*** *of Proposition* 3.1 *it is super conformal or Euclidean minimal with planar ends and* $\eta \equiv 0$.

Proof. In Case IIIa of Proposition 3.1, the bundle $\mathcal{V}$ constructed in Lemma 5.6 has rank 4. Because a local holomorphic section ψ^μ of $\mathcal{V}$ defined in a neighborhood U of $\mu = \infty$ satisfies $\nabla^\mu\psi^\mu = 0$ for all $\mu \in U \cap \mathbb{C}_*$, the fiber $\mathcal{V}_\infty$ over $\mu = \infty$ is a 4–dimensional space of solutions to (4.1). Thus, by Proposition 4.1, an immersion f belonging to Case IIIa has to be super conformal or Euclidean minimal with planar ends and $\eta \equiv 0$. q.e.d.

Lemma 5.9. *If a constrained Willmore torus belongs to* ***Case IIIb*** *of Proposition* 3.1 *it is Euclidean minimal with planar ends and* $\eta \equiv 0$.

Proof. Firstly, we prove the existence of a nilpotent polynomial Killing field ξ for the associated family ∇^μ of a constrained Willmore torus that belongs to Case IIIb. We do this by using the holomorphic rank 2 bundle $\mathcal{V}$ defined in Lemma 5.6 as well as the corresponding bundle $\mathcal{V}^\perp$ that belongs to the dual constrained Willmore immersion $f^\perp$ (which exists because, by (2.11), the connections $(\nabla^\perp)^\mu$ on the bundle $(V^*, -i)$ also have non–semisimple holonomy): we define ξ by

$$\xi(\mu, p) = \psi(\mu, p)\alpha_{\mathbb{C}}(\mu, p),$$

where ψ is a non–trivial meromorphic section of $\mathcal{V}$ and $\alpha_{\mathbb{C}}$ the complex part of a non–trivial meromorphic section α of the holomorphic bundle $\tilde{\mathcal{V}}^\perp$ that is image of $\mathcal{V}^\perp$ under the gauge transformation of (2.11). The Killing field ξ is polynomial in μ if ψ and α are chosen holomorphic on $\mathbb{C}$ with a single pole at $\mu = \infty$.

The polynomial Killing field ξ thus constructed is nilpotent: for generic $\mu \in \mathbb{C}_*$, at every point $p \in T^2$ the elements of $\mathcal{V}_\mu \subset \Gamma(\mathcal{V})$ span the rank 2 subbundle $\operatorname{im}(R^\mu_p(\gamma)) = \ker(R^\mu_p(\gamma)) \subset (V, i)$ with $R^\mu_p(\gamma) = H^\mu_p(\gamma) - \mathrm{Id}$ denoting the nilpotent part of the non–semisimple holonomy around a non–trivial cycle $\gamma \in \Gamma\backslash\{0\}$. Similarly, the elements of $(\tilde{\mathcal{V}}^\perp)_\mu \subset \Gamma(\mathcal{V}^*)$ generically span the subbundle $\operatorname{im}((R^\mu_p(\gamma))^*) = \ker((R^\mu_p(\gamma))^*)$ of $(V^*, -i)$ which, under the identification of Section 2.4 between $(V^*, -i)$ and the complex dual space to (V, i), coincides with the subbundle $\operatorname{im}(R^\mu_p(\gamma))^\perp = \ker(R^\mu_p(\gamma))^\perp$.

We show now that the existence of ξ leads to a contradiction unless f is Euclidean minimal with planar ends and $\eta \equiv 0$. Because ξ does not vanish identically, on the non–empty open set U on which $V = L \oplus \check{L}$ with $\check{L} = \ker(A_\circ)$, see Section 4, it takes the form $\xi = \sum_{l=0}^{d} \xi^l \mu^l$ with highest coefficient $\xi^d \in \Gamma_U(\mathrm{End}_{\mathbb{C}}(V))$ that is not identically zero. Comparing coefficients in $\nabla^\mu \xi = 0$ implies that $\xi^d \in \Gamma_U(\mathrm{End}_{\mathbb{C}}(V))$ satisfies

$$(5.6) \qquad [A_\circ^{(1,0)}, \xi^d] = 0 \quad \text{and} \quad \nabla \xi^d = -[A_\circ^{(1,0)}, \xi^{(d-1)}] + [A_\circ^{(0,1)}, \xi^d].$$

Assume now that $\check{L} = \ker(A_\circ)$ is non–constant. The bundles L and $\check{L}$ can then be trivialized by nowhere vanishing sections $\psi \in \Gamma(L)$ and $\check{\psi} \in \Gamma(\check{L})$ with $J\psi = \psi i$ and $\check{J}\check{\psi} = \check{\psi} i$, where $\check{J}$ is the complex structure on $\check{L}$ occurring in the mean curvature sphere $\check{S}$ of $\check{L}$, see (4.5). With respect to the frame ψ, ψj, $\check\psi$, $\check\psi j$ on U we then have

(5.7)

$$A_\circ^{(1,0)} = \begin{pmatrix} 0 & a\,dz & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and therefore } \xi^d = \begin{pmatrix} \xi_{11} & \xi_{12} & \xi_{13} & \xi_{14} \\ 0 & \xi_{11} & 0 & 0 \\ 0 & \xi_{32} & \xi_{33} & \xi_{34} \\ 0 & \xi_{42} & \xi_{43} & \xi_{44} \end{pmatrix},$$

where the form of ξ^d follows from the first part of (5.6) because a doesn't vanish on U. The fact that ξ is nilpotent implies that ξ_{11} vanishes identically.

As in the proof of Lemma 5.8, the elements in the fiber $\mathcal{V}_\infty$ over ∞ of the rank 2 bundle $\mathcal{V}$ are solutions to (4.1). Because an immersion that belongs to Case IIIb is not super conformal and $\check L$ is assumed to be non–constant so that the immersion is not Euclidean minimal with $\eta \equiv 0$, by Proposition 4.1 the space of solutions to (4.1) is at most 2–dimensional and therefore coincides with $\mathcal{V}_\infty \subset \Gamma(V)$. By construction of ξ, its highest order coefficient ξ^d vanishes on the elements in $\mathcal{V}_\infty$ and pointwise takes values in the span of the sections in $\mathcal{V}_\infty$.

As we have seen in the proof of Lemma 4.3, there is a dense open subset of U on which the solutions to (4.1) span the $-i$–eigenspace of $\check S$. This eigenspace is the complex span of $\check\psi$ and $\psi - \check\psi\frac{k}{2}b$ with b given by $\check B\psi = \check\psi j b$ for $\check B$ as in (4.5). The function b is nowhere vanishing, because $\check B$ vanishes at a point $p \in U$ if and only if η vanishes at p (because, by (4.4), $(A_\circ)_{|p}$ commutes with S_p iff $B_p = 0$ which, by (4.7), is equivalent to $\check B_p = 0$) but η has no zeros at all since on a torus the holomorphic quadratic differential $\delta\eta$ has none (if it had one, η had to vanish identically which is impossible by Lemma 4.3, because in the Willmore case with $\eta \equiv 0$ the space of solutions to (4.1) is at most 1–dimensional unless the immersion is Euclidean minimal).

That ξ^d vanishes on the sections of $\mathcal{V}_\infty$ and therefore on the span of $\check S$ thus implies

$$\xi^d = \begin{pmatrix} 0 & \xi_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \xi_{32} & 0 & 0 \\ 0 & \xi_{42} & 0 & 0 \end{pmatrix}.$$

In particular, because $\xi^d\check\psi j = 0$, the second part of (5.6) yields

$$-\xi^d\,\check\delta\check\psi j = (\nabla\xi^d)\check\psi j = -[A_\circ^{(1,0)}, \xi^{(d-1)}]\check\psi j + [A_\circ^{(0,1)}, \xi^d]\check\psi j,$$

where $\check\delta$ denotes the derivative $\check\delta = \pi_L\nabla_{|\check L}$ of $\check L$ with π_L the projection to $L \cong V/\check L$. Because the right hand side of this equation takes values in L we obtain $\xi_{32} = \xi_{42} = 0$ and, because ξ^d takes values in the $-i$–eigenbundle of $\check S$, $\xi_{12} = 0$. This is impossible as it contradicts the assumption that ξ^d does not vanish identically on U.

The assumption that $\check{L} = \ker(A_\circ)$ is non–constant thus leads to a contradiction so that $\check{L} = \ker(A_\circ)$ has to be constant. Proposition 6.8 below shows that a constrained Willmore torus which is not super conformal and has constant $\check{L} = \ker(A_\circ)$ belongs to Case III if and only if it is Euclidean minimal with planar ends. (Alternatively, one could directly prove, by similar arguments as in case of non–constant $\check{L} = \ker(A_\circ)$, that if $\check{L} = \ker(A_\circ)$ is constant the polynomial Killing field constructed above can only exists if $\eta \equiv 0$ and the immersion is Euclidean minimal with planar ends.) q.e.d.

5.4. Proof of the main theorem. For a constrained Willmore torus that belongs to Case I or II of Proposition 3.1, Lemmas 5.5 and 5.7 imply the existence of a polynomial Killing field ξ. This gives rise to a Riemann surface parametrizing the non–trivial eigenlines of ξ, see Lemma 3.5. Because for every $\mu \in \mathbb{C}_*$ the polynomial Killing field ξ commutes with all holonomies of ∇^μ, the eigenlines of ξ are also eigenlines of the holonomies and the uniqueness part of Lemma 3.5 implies that the Riemann surface constructed using ξ coincides with the holonomy spectral curve Σ_{hol}. The fact that ξ is polynomial in μ implies that Σ_{hol} has finite genus and can be compactified by adding points over $\mu = 0$ and $\mu = \infty$.

By Theorem 4.5, the holonomy spectral curve Σ_{hol} is a subset of the multiplier spectral curve Σ_{mult} and $\Sigma_{hol} \subset \Sigma_{mult}$ consists of finitely many points. Hence Σ_{mult} also has finite genus and therefore two ends, see Section 3.4. In particular, two of the points needed to compactify Σ_{hol} correspond to the two ends of Σ_{mult} while the other added points are contained in the complement $\Sigma_{mult} \backslash \Sigma_{hol}$ of the holonomy spectral curve inside the multiplier spectral curve. Because the two ends Σ_{mult} are interchanged under the anti–holomorphic involution ρ, one of them corresponds to a point over $\mu = 0$ and the other one to a point over $\mu = \infty$. To see that both ends of Σ_{mult} are branch points of μ, note that when σ tends to one of the ends the corresponding multipliers go to infinity in $\mathrm{Hom}(\Gamma, \mathbb{C}^2) \cong T^2 \times \mathbb{R}^2$ which cannot happen on a single sheet of the covering $\mu \colon \Sigma_{hol} \to \mathbb{C}_*$ only, because the holonomies $H^\mu(\gamma)$ itself can be represented as $\mathrm{SL}(4, \mathbb{C})$–matrices.

For immersions belonging to Case II of Proposition 3.1, this shows that the two points corresponding to the ends of Σ_{mult} are the only points needed to compactify Σ_{hol}, because $\mu \colon \Sigma_{hol} \to \mathbb{C}_*$ is a 2–fold branched covering. In particular, Σ_{hol} then coincides with Σ_{mult}. In Case I, the map $\mu \colon \Sigma_{hol} \to \mathbb{C}_*$ is a 4–fold branched covering and, because μ is branched at the two ends of Σ_{mult}, the complement $\Sigma_{mult} \backslash \Sigma_{hol}$ of the holonomy spectral curve inside the multiplier curve consists of at most four points.

As we have seen in Section 2.4, all super conformal tori belong to Case IIIa of Proposition 3.1. Euclidean minimal tori with planar ends belong to Case III, because the holonomy of ∇^μ for $\mu \in S^1$ is of Jordan type with eigenvalue 1 and off–diagonal part related to the translational periods of $*df$, see [**20**].

Conversely, Lemma 5.8 shows that a constrained Willmore torus that belongs to Case IIIa of Proposition 3.1 is either super conformal or Euclidean minimal with planar ends and Lemma 5.9 shows that Case IIIb is only possible for Euclidean minimal tori with planar ends.

The fact that all constrained Willmore tori with non–trivial normal bundle $\perp_f$ belong to Case III follows from Proposition 3.1. A constrained Willmore torus with trivial normal bundle $\perp_f$ that is not Euclidean minimal with planar ends belongs to Case I or II, because otherwise it had to be super conformal which is impossible, because super conformal tori have non–trivial normal bundle, see (2.3). q.e.d.

6. The Harmonic Case

We discuss a special class of constrained Willmore surfaces related to harmonic maps to S^2. It includes CMC surfaces in $\mathbb{R}^3$ and S^3, Hamiltonian stationary Lagrangian surfaces in $\mathbb{C}^2 \cong \mathbb{H}$, and Lagrangian surfaces with conformal Maslov form in $\mathbb{C}^2 \cong \mathbb{H}$. Constrained Willmore tori of this class belong to Cases II or III of Theorem 5.1. More precisely, they belong to Case II if and only if the appendant harmonic map to S^2 is non–conformal. Then, the harmonic map itself admits a spectral curve [**24**, **19**] which is shown to coincide with the constrained Willmore spectral curve studied above.

6.1. Main theorem of the section. The following theorem will be proven in Section 6.5.

Theorem 6.1. *If a conformal immersion $f\colon M \to S^4$ of a Riemann surface M admits a point $\infty \in S^4$ at infinity for which one factor of the (Euclidean) Gauss map*

$$M \to Gr^+(2,4) = S^2 \times S^2$$

of f seen as an immersion into $\mathbb{R}^4 = S^4 \backslash \{\infty\}$ is harmonic, then f is constrained Willmore. If $M = T^2$ is a torus there is a Lagrange multiplier η such that f belongs to Case II or III of Theorem 5.1*, depending on whether the harmonic factor is non–conformal or conformal. In the non–conformal case, the harmonic map spectral curve coincides with the constrained Willmore spectral curve.*

As pointed out by Fran Burstall [**9**], the property that the Gauss map of an immersion into $\mathbb{R}^4$ has a harmonic factor is equivalent to holomorphicity or anti–holomorphicity of its mean curvature vector, see Section 6.2. The first part of Theorem 6.1 has been generalized by Burstall [**9**] who proved that every immersion into a 4–dimensional space form with (anti–)holomorphic mean curvature vector is constrained Willmore.

It should be noted that, conversely, every constrained Willmore torus $f\colon T^2 \to S^4$ that belongs to Case III of Theorem 5.1 (and hence is super conformal or Euclidean minimal with planar ends) admits a point $\infty \in S^4$ at infinity for which the Gauss map of f seen as an immersion into $\mathbb{R}^4 = S^4\backslash\{\infty\}$ has a conformal factor, see Section 6.2. It is also worth noting that (following from Corollary 5.4 together with Lemma 6.3) every constrained Willmore torus that belongs to Case II and has Willmore functional $\mathcal{W} < 8\pi$ admits a point $\infty \in S^4$ for which a factor of the Euclidean Gauss map is harmonic and non–conformal.

6.2. Surfaces in Euclidean 4–space $\mathbb{R}^4 = \mathbb{H}$. The immersions studied in this section come with a distinguished point $\infty \in S^4$ at infinity in the conformal 4–sphere. In the following we therefore work with a fixed trivialization $V \cong \mathbb{H}^2$ and identify $\mathbb{R}^4 = \mathbb{H}$ with $\mathbb{HP}^1\backslash\{\infty\}$ via $x \in \mathbb{H} \mapsto [(x,1)]$. The relation between the (Möbius geometric) mean curvature sphere congruence of an immersion $f\colon M \to \mathbb{H} = \mathbb{HP}^1\backslash\{\infty\}$ and its (Euclidean) Gauss map is as follows (see Chapter 7 of [**7**] for details): the Gauss map of a conformal immersion $f\colon M \to \mathbb{H}$ is represented by the *left* and *right normal vectors* N, $R\colon M \to S^2 \subset \operatorname{Im}\mathbb{H}$ of f which are defined by

$$*df = N df = -df R.$$

The mean curvature vector $\mathcal{H} = \frac{1}{2}\operatorname{tr}(\mathrm{I\!I})$ can be expressed in terms of N and R by

(6.1)
$$dN' = \frac{1}{2}(dN - N{*}dN) = -df H \quad\text{and}\quad dR' = \frac{1}{2}(dR - R{*}dR) = -H df$$

where $H = \bar{\mathcal{H}}N = R\bar{\mathcal{H}}$. The mean curvature sphere of the immersion $L = \begin{pmatrix} f \\ 1 \end{pmatrix}\mathbb{H}$ is then

$$S = \operatorname{Ad}\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}\begin{pmatrix} N & 0 \\ H & -R \end{pmatrix} \tag{6.2}$$

and the Hopf fields of f take the form

(6.3)
$$2{*}A = \operatorname{Ad}\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ w & dR'' \end{pmatrix} \quad\text{and}\quad 2{*}Q = \operatorname{Ad}\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}\begin{pmatrix} dN'' & 0 \\ w + dH & 0 \end{pmatrix}$$

with $dN'' = \frac{1}{2}(dN + N{*}dN)$, $dR'' = \frac{1}{2}(dR + R{*}dR)$ and $w = \frac{1}{2}(-dH - R{*}dH + H{*}dN'')$.

An immersion $f\colon M \to \mathbb{R}^4 = S^4\backslash\{\infty\}$ is Euclidean minimal if and only if its left and right normal vectors N and R with respect to ∞ are both anti–holomorphic, that is, if $dN' = 0$ and $dR' = 0$, see (6.1). An immersion $f\colon M \to S^4$ is super conformal, i.e., satisfies $A \equiv 0$ or

$Q \equiv 0$, if and only if with respect to one and therefore every point ∞ at infinity its left normal N is holomorphic $dN'' = 0$ or its right normal R is holomorphic $dR'' = 0$, see (6.3) and Lemma 22 of [**7**].

The following lemma gives a quaternionic characterization of harmonic maps into S^2.

Lemma 6.2. *A map $N\colon M \to S^2 \subset \mathrm{Im}(\mathbb{H})$ into the 2–sphere is harmonic if and only if the 1–form $dN' = \frac{1}{2}(dN - N{*}dN)$ or, equivalently, $dN'' = \frac{1}{2}(dN + N{*}dN)$ is closed.*

Proof. A map $N\colon M \to S^2$ is harmonic if the 2–form

$$d{*}dN = d(NdN' - NdN'')$$

$$= dN' \wedge dN' + Nd(dN') - dN'' \wedge dN'' - Nd(dN''),$$

is normal to S^2. Both $dN' \wedge dN'$ and $dN'' \wedge dN''$ are normal and one can easily check that $d(dN')$ and $d(dN'')$ are tangential. The tangential part of the form $d{*}dN$ is therefore $Nd(dN') - Nd(dN'') = 2Nd(dN') = -2Nd(dN'')$ which proves the statement. q.e.d.

Using Lemma 6.2 we show now that the Gauss map of an immersion $f\colon M \to \mathbb{R}^4 = \mathbb{H}$ into Euclidean 4–space has a harmonic factor if and only if its mean curvature vector $\mathcal{H}$ is (anti–)holomorphic. More precisely, f has a harmonic left normal N if and only if its mean curvature vector is a holomorphic section of the normal bundle, that is, ${*}\nabla^\perp\mathcal{H} = N\nabla^\perp\mathcal{H}$ with $\nabla^\perp$ denoting the normal connection of f. Analogously, harmonicity of the right normal R is equivalent to anti–holomorphicity of $\mathcal{H}$, that is, to ${*}\nabla^\perp\mathcal{H} = -N\nabla^\perp\mathcal{H}$. We only prove the second statement: by (6.1), R is harmonic if and only if ${*}dH = dHN$ or, equivalently, ${*}d\bar{H} = -Nd\bar{H}$. Using again (6.1), $HN = RH$ implies $dHN - RdH = dR''H - HdN''$ which shows that ${*}d\bar{H} = -Nd\bar{H}$ is equivalent to ${*}\nabla^\perp\bar{H} = -N\nabla^\perp\bar{H}$. By $\mathcal{H} = N\bar{H}$ this is equivalent to ${*}\nabla^\perp\mathcal{H} = -N\nabla^\perp\mathcal{H}$.

6.3. Möbius geometric characterization of the harmonic case.

The following lemma gives a characterization in terms of the Hopf fields A and Q of the property that there is a Euclidean subgeometry in which the Euclidean Gauss map has a harmonic factor.

Lemma 6.3. *Let $f\colon M \to \mathbb{H} = \mathbb{HP}^1\backslash\{\infty\}$ be a conformal immersion. Then:*

a) *The right normal vector $R\colon M \to S^2$ of f with respect to the point ∞ at infinity is harmonic if and only if f is constrained Willmore and admits a 1–form $\eta \in \Omega^1(\mathcal{R})$ such that $2{*}A_\circ = 2{*}A + \eta$ is closed and vanishes on the line corresponding to ∞.*

b) *The left normal vector* $N\colon M \to S^2$ *of* f *with respect to* ∞ *is harmonic if and only if* f *is constrained Willmore and admits* η *such that the form* $2{*}Q_\circ = 2{*}Q + \eta$ *is closed and takes values in the line described by* ∞*.*

Proof. It is sufficient to prove a), because b) follows by passing to the dual immersion $f^\perp$. By Proposition 15 of [**7**], the differential of $2{*}A$ is in $\Omega^2(\mathcal{R})$, i.e., vanishes on L and takes values in L, so that

$$d^\nabla(2{*}A) = \operatorname{Ad}\begin{pmatrix}1 & f\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ dw & 0\end{pmatrix} = \begin{pmatrix}f dw & -f dw f\\ dw & -dw f\end{pmatrix}. \tag{6.4}$$

Using (6.3) this implies $w \wedge df + d(dR'') = 0$ and, by Lemma 6.2, the right normal vector R is harmonic if and only if $*w = wN$.

Every 1-form $\eta \in \Omega^1(\mathcal{R})$ can be written as

(6.5)

$$\eta = \operatorname{Ad}\begin{pmatrix}1 & f\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ \hat\eta & 0\end{pmatrix} \quad\text{and}\quad d^\nabla\eta = \operatorname{Ad}\begin{pmatrix}1 & f\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ d\hat\eta & 0\end{pmatrix} \in \Omega^2(\mathcal{R})$$

if and only if $*\hat\eta = -R\hat\eta = \hat\eta N$. In particular, the form $2{*}A + \eta$ is closed if and only if $*\hat\eta = -R\hat\eta = \hat\eta N$ and $dw + d\hat\eta = 0$.

This proves the lemma: the form $2{*}A + \eta$ vanishes on $\infty = [(1,0)]$ if and only if $\hat\eta = -w$ and, because w always satisfies $*w = -Rw$, the form $2{*}A + \eta$ is closed if and only if $*w = wN$ which, as we have seen above, is equivalent to R being harmonic. q.e.d.

For a Willmore immersion $f\colon M \to S^4$ with $\eta \equiv 0$ that is not super conformal, the following are equivalent (see Section 11.2 of [**7**] for details):

- $\check L = \ker(A_\circ) = \ker(A)$ is constant,
- $\hat L = \operatorname{im}(Q_\circ) = \operatorname{im}(A)$ is constant, and
- f is Euclidean minimal with planar ends in $\mathbb{R}^4 = S^4\backslash\{\infty\}$ for some $\infty \in S^4$.

In particular, we then have $\infty = \check L = \hat L$. This immediately follows from the fact that, firstly, $\check L$ and $\hat L$ are invariant under S if $\eta \equiv 0$ and, secondly, if there is a point ∞ contained in all mean curvature spheres the immersion is Euclidean minimal with planar ends in $\mathbb{R}^4 = S^4\backslash\{\infty\}$ and vice versa.

6.4. Willmore bundles of rank 1 and the harmonic map spectral curve. We review the quaternionic approach to the spectral curve of harmonic maps from tori to S^2 as developed in Sections 6.1 to 6.3 of [**13**]. This allows a short proof of the prototype result mentioned in the introduction.

A flat connection ∇ on a quaternionic vector bundle W with complex structure J over a Riemann surface M is a *Willmore connection* if the

Hopf field $A = \frac{1}{4}(J\nabla J + *\nabla J)$ satisfies $d^\nabla(2*A) = 0$, or equivalently, if for every parameter $\mu \in \mathbb{C}_*$ the complex connection

$$\nabla^\mu = \nabla + (\mu - 1)A^{(1,0)} + (\mu^{-1} - 1)A^{(0,1)} \tag{6.6}$$

on the complex bundle (W, i) is flat.

The Hopf fields A and $Q = \frac{1}{4}(J\nabla J - *\nabla J)$ of a Willmore connection ∇ are holomorphic sections $A \in H^0(K\,\mathrm{End}_-(W))$ and $Q \in H^0(\mathrm{End}_-(W)K)$, see [**13**]. In case W is a quaternionic line bundle, the complex line bundle $\mathrm{End}_+(W)$ is canonically trivial and $AQ \in H^0(K^2) \cong H^0(K^2\,\mathrm{End}_+(W))$ is a holomorphic quadratic differential. In particular, if $M = T^2$ is a torus the holomorphic quadratic differential AQ is either nowhere vanishing or vanishes identically.

Lemma 6.4. *Let ∇ be a trivial connection on a trivial quaternionic line bundle W with complex structure J over a Riemann surface M and define $N\colon M \to S^2$ by $J\psi = \psi N$ for ψ a non–trivial parallel section. Then ∇ is a Willmore connection if and only if N is harmonic. In case ∇ is Willmore and $M = T^2$ is a torus, either*

- *AQ is nowhere vanishing and N is non–conformal with* $\deg(N) = 0$ *or*
- *$AQ \equiv 0$ and N is holomorphic or anti–holomorphic depending on whether $A \equiv 0$ or $Q \equiv 0$. In particular* $\deg(N) \neq 0$ *unless N is constant.*

Proof. That ∇ is Willmore if and only if N is harmonic follows from $(2*A)\psi = -\psi dN'$ and Lemma 6.2. If $M = T^2$, the holomorphic quadratic differential AQ either has no zeros at all or vanishes identically. Because $A\psi = \psi\frac{1}{2}NdN'$ and $Q\psi = \psi\frac{1}{2}NdN''$, the latter is equivalent to N being conformal. The statement about the degree $\deg(N)$ of N in the non–conformal case holds, because A and Q are then nowhere vanishing holomorphic sections of the bundles $K\,\mathrm{End}_-(W)$ and $\mathrm{End}_-(W)K$, respectively, and $\deg(\mathrm{End}_-(W)) = 2\deg(W) = 2\deg(N)$. In the conformal case the statement about the degree follows from $\deg(N) = \deg(W) = \frac{1}{2\pi}\int_M A \wedge *A - Q \wedge *Q$. q.e.d.

The following lemma implies the existence of the harmonic map spectral curve for non–conformal harmonic maps $N\colon T^2 \to S^2$.

Lemma 6.5. *Let W be a quaternionic line bundle with complex structure J and Willmore connection ∇ over a torus T^2. In case ∇ has non–trivial Hopf fields $A \not\equiv 0$ and $Q \not\equiv 0$, there are only finitely many spectral parameters $\mu \in \mathbb{C}_*$ for which all holonomies of the flat connection ∇^μ in* (6.6) *have eigenvalue* 1.

Proof. Assume the statement was not true. Because the space $H^0(W)$ of ∇''–holomorphic sections with trivial monodromy is finite dimensional, there are then $\mu_0, ..., \mu_n \in \mathbb{C}_*$ and $\psi_{\mu_0}, ..., \psi_{\mu_n} \in H^0(W)$ with

$\nabla^{\mu_l}\psi_{\mu_l} = 0$ for $l = 0, ..., n$ such that $\psi_{\mu_1}, ..., \psi_{\mu_n}$ are linearly independent over $\mathbb{C}$ while $\psi_{\mu_0}, ..., \psi_{\mu_n}$ are linearly dependent. Because $Q \not\equiv 0$, it is impossible that a holomorphic section of W is contained in the $\pm i$ eigenspaces of J so that both $(A\psi_{\mu_1})^{(1,0)}, ..., (A\psi_{\mu_n})^{(1,0)}$ and $(A\psi_{\mu_1})^{(0,1)}, ..., (A\psi_{\mu_n})^{(0,1)}$ are also linearly independent over $\mathbb{C}$. Moreover, there are $\lambda_l \in \mathbb{C}$ such that $\psi_{\mu_0} = \sum_{l=0}^{n} \psi_{\mu_l}\lambda_l$ and, using $\nabla\psi_{\mu_l} = (1-\mu_l)(A\psi_{\mu_l}))^{(1,0)} + (1-\mu_l^{-1})(A\psi_{\mu_l})^{(0,1)}$, we obtain

$$\sum_{l=0}^{n}(A\psi_{\mu_l})^{(1,0)}(\mu_l - \mu_0)\lambda_l + \sum_{l=0}^{n}(A\psi_{\mu_l})^{(0,1)}(\mu_l^{-1} - \mu_0^{-1})\lambda_l = 0$$

which is a contradiction, because $\mu_0 \neq \mu_l$ for all $l = 1,...,n$ and $\lambda_l \neq 0$ for some l. q.e.d.

The *harmonic map spectral curve* of a non–conformal harmonic map $N\colon T^2 \to S^2$ is the hyper–elliptic Riemann surface $\mu\colon \Sigma_{harm} \to \mathbb{C}_*$ parametrizing the holonomy eigenlines of the associated family (6.6) of the corresponding trivial Willmore connection. For the spectral curve to be well defined one has to make sure that Lemma 3.5 can be applied to the holonomies of ∇^μ around non–trivial loops, i.e., that for generic μ the holonomies have two different eigenvalues. If this was not the case, all holonomies had a double eigenvalue which had to be 1 because ∇^μ is a family of $\mathrm{SL}(2,\mathbb{C})$–connections and $\nabla^{\mu=1}$ is trivial. But this is impossible by Lemma 6.5. The same lemma shows that the harmonic map spectral curve Σ_{harm} has finite genus, because the number of branch points of the projection $\mu\colon \Sigma_{harm} \to \mathbb{C}_*$ is finite: the holonomies corresponding to branch points have ± 1 as double eigenvalues so that every branch point gives rise to a ∇''–holomorphic section with trivial monodromy defined on a 4–fold cover of T^2.

6.5. Proof of Theorem 6.1. The first part of Theorem 6.1 immediately follows from Lemma 6.3. In Section 6.2 we have seen that an immersion $f\colon M \to \mathbb{H}$ with conformal left or right normal N or R is super conformal or Euclidean minimal with planar ends. Thus, an immersions $f\colon T^2 \to S^4$ of a torus whose left or right normal with respect to some point $\infty \in S^4$ is conformal belongs to Case III of Theorem 5.1. It therefore remains to show that f belongs to Case II of Theorem 5.1 if there is $\infty \in S^4$ for which one of the Euclidean normals is harmonic but non–conformal. This is proven by Proposition 6.8 below.

Firstly, we show that if a constrained Willmore immersion $f\colon T^2 \to S^4 = \mathbb{HP}^1$ is neither super conformal nor Euclidean minimal with planar ends and has constant $\hat{L} = \mathrm{im}(Q_\circ)$, the quaternionic holomorphic line bundle $W = V/L$ carries a Willmore connection ∇ which is induced by $\infty = \hat{L} = \mathrm{im}(Q_\circ)$ and compatible $D = \nabla''$ with the holomorphic structure D on V/L. (Passing to the dual constrained Willmore immersion $f^\perp$ shows that, if $\check{L} = \ker(A_\circ)$ is constant, the same is true for the holomorphic line bundle $L^{-1} = V^*/L^\perp$.)

For an arbitrary conformal immersion $f\colon M \to \mathbb{HP}^1$, the canonical projection π to V/L projects $v = (1,0) \in V \cong \mathbb{H}^2$ to a holomorphic section $\varphi = \pi(v) \in H^0(V/L)$ that vanishes on the discrete set Z of points at which f goes through $\infty = [(1,0)]$. Away from Z, the complex structure $\tilde{J} \in \Gamma(\mathrm{End}(V/L))$ satisfies $\tilde{J}\varphi = \varphi N$, where N is the left normal of $f\colon M\backslash Z \to \mathbb{H} = \mathbb{HP}^1\backslash\{\infty\}$. The point ∞ induces a compatible connection ∇ defined over $M\backslash Z$ by setting $\nabla\varphi = 0$. This connection is Willmore if and only if N is harmonic. By Lemma 6.3 this is equivalent to f being constrained Willmore with $\mathrm{im}(Q_\circ) = \infty$ for some Lagrange multiplier η.

The following lemma shows that if $M = T^2$ is a torus and N is non–conformal, the set Z is empty and the Willmore connection ∇ is globally defined.

Lemma 6.6. *Let $f\colon T^2 \to S^4 = \mathbb{HP}^1$ be a conformal immersion and denote N and R the left and right normals of f seen as an immersion into $\mathbb{H} = \mathbb{HP}^1\backslash\{\infty\}$ for a point $\infty \in \mathbb{HP}^1$ at infinity. In case N or R is harmonic, it smoothly extends through the points of T^2 at which f goes through ∞ to a harmonic map $T^2 \to S^2$. If this harmonic map is non–conformal, its degree is zero and f does not go through ∞.*

In case N or R is conformal and non–constant, its extension $T^2 \to S^2$ has non–zero degree and f can go through ∞. In the anti–holomorphic case when f is minimal with planar ends in $\mathbb{R}^4 \cong S^4\backslash\{\infty\}$ it goes through ∞ at the ends of the surface. The degree of the extended left and right normals is then $\deg(N) = d/2 - e$ and $\deg(R) = -e - d/2$, where $d = \deg(\perp_f)$ denotes the degree of the normal bundle and e the number of ends.

Proof. It is sufficient to prove the statement in the case that N is harmonic (the case that R is harmonic immediately follows by passing to the dual surface). As above, let $\varphi = \pi(v)$ be the holomorphic section of V/L obtained by projection of $v = (1,0)$. The left normal vector N of $f\colon T^2\backslash Z \to \mathbb{H} = \mathbb{HP}^1\backslash\{\infty\}$ then satisfies $\tilde{J}\varphi = \varphi N$ and therefore, by Lemma A.1, continuously extends through the set Z of points at which f goes through $\infty = [(1,0)]$. Because N is continuous on T^2 and, by assumption, harmonic on $T^2\backslash Z$, it is a continuous solution to an elliptic equation and therefore smooth on all of T^2, see e.g. [**15**]. In particular, the second part of Lemma A.1 implies that $dN''_p = 0$ for all $p \in Z$.

It remains to check that $Z = \emptyset$ if N is non–conformal. For this we equip the trivial bundle $E = M \times \mathbb{H}$ with a Willmore connection by setting $\nabla\psi = 0$ and $J\psi = \psi N$ for $\psi(p) = (p,1)$. Since N is non–conformal, Lemma 6.4 implies that AQ is nowhere vanishing and $\deg(N) = 0$. In particular, $Q\psi = \psi\frac{1}{2}NdN''$ implies the set Z is empty, because $dN''_p = 0$ for all $p \in Z$ but Q is nowhere vanishing. q.e.d.

The following lemma shows that, if $\operatorname{im}(Q_\circ)$ is constant, the $\mathrm{SL}(2,\mathbb{C})$–connections in the associated family (6.6) of the induced Willmore connection on V/L are gauge equivalent to an invariant subbundle of the $\mathrm{SL}(4,\mathbb{C})$–connections in the constrained Willmore associated family (2.9) on V.

Lemma 6.7. *Let $f\colon M \to S^4$ be a constrained Willmore immersion with constant* $\operatorname{im}(Q_\circ)$*. For every spectral parameter $\mu \in \mathbb{C}_*$, the prolongation of a parallel section of the connection* (6.6) *in the associated family of the Willmore connection on V/L induced by $\infty = \operatorname{im}(Q_\circ)$ is parallel with respect to the connection* (2.9) *on V.*

Proof. As above, denote by φ the holomorphic section of V/L defined by projection $\pi(v)$ of $v = (1,0)$ with $\infty = [(1,0)] = \operatorname{im}(Q_\circ)$. The prolongation ψ of a holomorphic section $\tilde\psi = \varphi g$ of V/L is then

$$\psi = \begin{pmatrix}1\\0\end{pmatrix} g + \begin{pmatrix}f\\1\end{pmatrix}\chi$$

with χ defined by $dg + df\chi = 0$.

A section $\tilde\psi = \varphi g$ of V/L is ∇^μ–parallel with respect to the connection (6.6) if

$$dg + \pi_N^{(1,0)}(NdN'g)\frac{\mu-1}{2} + \pi_N^{(0,1)}(NdN'g)\frac{\mu^{-1}-1}{2} = 0,$$

where $\pi_N^{(1,0)}(v) = \frac{1}{2}(v - Nvi)$ and $\pi_N^{(0,1)}(v) = \frac{1}{2}(v + Nvi)$. Now $dN' = -dfH$ yields $NdN' = dfRH$ and, because $dg + df\chi = 0$, the function χ is given by

$$\chi = \pi_R^{(0,1)}(RHg)\frac{\mu-1}{2} + \pi_R^{(1,0)}(RHg)\frac{\mu^{-1}-1}{2}. \tag{6.7}$$

By (6.3) and (6.5), the fact that $Q_\circ$ takes values in the line corresponding to ∞ implies

$$2{*}Q_\circ = \operatorname{Ad}\begin{pmatrix}1 & f\\0 & 1\end{pmatrix}\begin{pmatrix}dN'' & 0\\0 & 0\end{pmatrix}.$$

Therefore $\quad 2{*}A_\circ = \operatorname{Ad}\begin{pmatrix}1 & f\\0 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\-dH & dR''\end{pmatrix}$ and

$$A_\circ = \frac{1}{2}\operatorname{Ad}\begin{pmatrix}1 & f\\0 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\-RdH & RdR''\end{pmatrix}. \tag{6.8}$$

The derivative of the prolongation ψ of a holomorphic section $\tilde{\psi} = \varphi g$ with respect to the connection (2.9) on V is

$$\text{(6.9)} \quad \nabla^\mu \psi = \begin{pmatrix} f \\ 1 \end{pmatrix} d\chi + \begin{pmatrix} f \\ 1 \end{pmatrix} \Big(\pi_R^{(0,1)}(-RdHg + RdR''\chi)\frac{\mu - 1}{2} + \pi_R^{(1,0)}(-RdHg + RdR''\chi)\frac{\mu^{-1} - 1}{2} \Big).$$

In case that $\tilde{\psi}$ is parallel with respect to the connection (6.6), using $RHdg = -RHdf\chi = RdR'\chi = RdR\chi - RdR''\chi$, differentiation of (6.7) yields

(6.10)

$$\begin{aligned} d\chi = {} & \pi_R^{(0,1)}(RdHg - RdR''\chi)\frac{\mu - 1}{2} + \pi_R^{(1,0)}(RdHg - RdR''\chi)\frac{\mu^{-1} - 1}{2} \\ & + \pi_R^{(0,1)}(RdR\chi)\frac{\mu - 1}{2} + \pi_R^{(1,0)}(RdR\chi)\frac{\mu^{-1} - 1}{2} \\ & + (dRHg)\frac{\mu - 1}{4} + (dRHg)\frac{\mu^{-1} - 1}{2}. \end{aligned}$$

For proving that the prolongation ψ is a parallel section of V with respect to (2.9) if the section $\tilde{\psi}$ of V/L is parallel with respect to (6.6), it therefore remains to check that the second line of (6.10) vanishes. By (6.7)

$$RdR\chi = \pi_R^{(1,0)}(dRHg)\frac{\mu - 1}{2} + \pi_R^{(0,1)}(dRHg)\frac{\mu^{-1} - 1}{2}$$

we have

$$\begin{aligned} \pi_R^{(0,1)}(RdR\chi)\frac{\mu - 1}{2} &= \pi_R^{(0,1)}(dRHg)\frac{\mu - 1}{2}\frac{\mu^{-1} - 1}{2} \\ \pi_R^{(1,0)}(RdR\chi)\frac{\mu^{-1} - 1}{2} &= \pi_R^{(1,0)}(dRHg)\frac{\mu - 1}{2}\frac{\mu^{-1} - 1}{2} \end{aligned}$$

so that indeed, by $\frac{\mu-1}{2}\frac{\mu^{-1}-1}{2} = \frac{1}{4}(2 - \mu - \mu^{-1})$, the second line of (6.10) vanishes. q.e.d.

Proposition 6.8. *Let $f\colon T^2 \to S^4$ be a constrained Willmore torus which is neither super conformal nor Euclidean minimal with planar ends and has the property that $\check{L} = \ker(A_\circ)$ or $\hat{L} = \mathrm{im}(Q_\circ)$ is a constant point $\infty \in S^4$. Then the immersion f belongs to Case II of Theorem* 5.1 *and the harmonic map spectral curve of the harmonic left or right normal of $f\colon T^2 \to \mathbb{R}^4 = S^4\backslash\{\infty\}$ coincides with the constrained Willmore spectral curve of f.*

Proof. Passing to the dual immersion $f^\perp$ interchanges the property that $\ker(A_\circ)$ is constant with the property that $\mathrm{im}(Q_\circ)$ is constant. Moreover, f and $f^\perp$ belong to the same case in the list of Theorem 5.1, because the holonomy representations of the associated family $(\nabla^\perp)^\mu$

belonging to the dual constrained Willmore immersion $f^\perp$ are equivalent to the dual representations of the holonomy representations of ∇^μ, see (2.11).

Assuming that $\ker(A_\circ)$ is constant we obtain that we are not in Case I of Theorem 5.1, because a constant kernel $\ker(A_\circ)$ gives rise to a complex 2–dimensional space of sections with trivial monodromy of V which are parallel for all ∇^μ.

On the other hand, assuming that $\mathrm{im}(Q_\circ)$ is constant shows that we are in Case II of Theorem 5.1: because the immersion f is neither super conformal nor Euclidean minimal with planar ends, its left normal with respect to $\infty = \mathrm{im}(Q_\circ)$ is harmonic and non–conformal (see Lemma 6.3 and Section 6.2). Lemma 6.6 then implies that f does not go through $\infty = \mathrm{im}(Q_\circ)$ so that ∞ induces a compatible Willmore connection on the bundle V/L. By Lemma 6.5, for generic $\mu \in \mathbb{C}_*$ the holonomy of the associated family (6.6) of this Willmore connection has non–trivial eigenvalues. Together with Lemma 6.7 this shows that we are in Case II, that is, over generic μ the holonomy of the constrained Willmore associated family (2.9) has two non–trivial eigenvalues in addition to the two trivial eigenvalues.

In particular, all three spectral curves arising in our situation are canonically isomorphic

$$\Sigma_{harm} = \Sigma_{hol} = \Sigma_{mult}.$$

The second equality always holds for constrained Willmore tori of Case II. The first equality holds, because by Lemma 6.7 the Riemann surfaces Σ_{harm} and Σ_{hol} describing the non–trivial eigenlines of the $\mathrm{SL}(2,\mathbb{C})$– and $\mathrm{SL}(4,\mathbb{C})$–holonomies, respectively, can be defined as normalizations of the same algebraic sets which describe the non–trivial eigenvalues of the holonomies (cf. the proof of Lemma 3.5). q.e.d.

6.6. CMC surfaces in Euclidean 3–space $\mathbb{R}^3$. The left and right normal vectors of a conformal immersion $f\colon M \to \mathrm{Im}\,\mathbb{H} = \mathbb{R}^3$ coincide and $*df = Ndf = -df N$.

Lemma 6.9. *A conformal immersion $f\colon M \to \mathrm{Im}\,\mathbb{H} = \mathbb{R}^3$ has constant mean curvature if and only if its Gauss map $N\colon M \to S^2$ is harmonic.*

Proof. By Lemma 6.2, harmonicity of N is equivalent to closedness of dN' which, by (6.1), is equivalent to H being constant (note that H for surfaces in $\mathbb{R}^3$ is the real function $H = -\frac{1}{2}\operatorname{tr} < df, dN >$). q.e.d.

Because CMC immersions are constrained Willmore and isothermic, the form η for which $d^\nabla(2{*}A + \eta) = 0$ is not unique: for all $\rho \in \mathbb{R}$, the forms $2{*}A^\rho_\circ = 2{*}A + \eta_0 + \rho\omega$ and $2{*}Q^\rho_\circ = 2{*}Q + \eta_0 + \rho\omega$ are closed, where by (6.3)

$$2{*}A = \operatorname{Ad}\begin{pmatrix}1 & f\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ \frac{1}{2}H{*}dN'' & dN''\end{pmatrix} \text{ and } \omega := \operatorname{Ad}\begin{pmatrix}1 & f\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ dN'' & 0\end{pmatrix}$$

and $\eta_0 := -\frac{1}{2}H{*}\omega$. Because

$$2{*}A_\circ^\rho = \operatorname{Ad}\begin{pmatrix}1 & f\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ \rho dN'' & dN''\end{pmatrix} \text{ and}$$
$$2{*}Q_\circ^\rho = \operatorname{Ad}\begin{pmatrix}1 & f\\ 0 & 1\end{pmatrix}\begin{pmatrix}dN'' & 0\\ \rho dN'' & 0\end{pmatrix},$$

the 2–step forward and backward Bäcklund transforms for different parameters ρ are

$$\check{L}_\rho = \ker(A_\circ^\rho) = \begin{pmatrix}f\rho - 1\\ \rho\end{pmatrix} \quad \text{and} \quad \hat{L}_\rho = \operatorname{im}(Q_\circ^\rho) = \begin{pmatrix}1 + f\rho\\ \rho\end{pmatrix}$$

and the only parameter for which one of these Bäcklund transforms is constant is $\rho = 0$ when $\check{L}_0 = \hat{L}_0 = \infty$.

6.7. CMC surfaces in the 3–sphere S^3 equipped with its standard metric. Let $f\colon M \to S^3 \subset \mathbb{H}$ be a conformal immersion with ${*}df = Ndf = -dfR$. The imaginary 1–forms

$$\alpha = f^{-1}df \qquad \text{and} \qquad \beta = df f^{-1} \tag{6.11}$$

satisfy ${*}\alpha = R\alpha = -\alpha R$ and $\beta = N\beta = -\beta N$. Their Maurer–Cartan equations are

$$d\alpha + \alpha \wedge \alpha = 0 \qquad \text{and} \qquad d\beta = \beta \wedge \beta.$$

Denote by n the positive oriented normal vector of f as a surface in S^3, that is, f, n is a positive orthonormal basis of the normal bundle of f seen as an immersion into $\mathbb{H}$. Because the complex structure on the normal bundle is given by left multiplication by N and right multiplication by R we have $n = Nf = fR$. The second fundamental form of f as an immersion into $\mathbb{H}$ is $\mathrm{I\!I} = -< df, df > f - < df, dn > n$ and its mean curvature vector is $\mathcal{H} = \frac{1}{2}\operatorname{tr}\mathrm{I\!I} = H^{S^3}n - f$, where H^{S^3} denotes the scalar mean curvature of f as an immersion into S^3. Hence $H = \bar{\mathcal{H}}N = R\bar{\mathcal{H}}$ satisfies

$$H = (H^{S^3} - R)f^{-1} = f^{-1}(H^{S^3} - N). \tag{6.12}$$

A straightforward computation (using (6.11) together with (6.1) and (6.12)) shows

$$d{*}\alpha + H^{S^3}\alpha \wedge \alpha = 0 \qquad \text{and} \qquad d{*}\beta + H^{S^3}\beta \wedge \beta = 0. \tag{6.13}$$

Lemma 6.10. *A conformal immersion $f\colon M \to S^3 \subset \mathbb{H}$ with ${*}df = Ndf = -dfR$ has constant mean curvature in S^3 if and only if $N\colon M \to S^2$ is harmonic or, equivalently, $R\colon M \to S^2$ is harmonic.*

Proof. By Lemma 6.2, the map R is harmonic if and only if the form dR' is closed. Because $dR' = -(H^{S^3} - R)f^{-1}df$, this is equivalent to

$$0 = d(dR') = -dH^{S^3} \wedge \alpha - H^{S^3} d\alpha + d{*}\alpha = -dH^{S^3} \wedge \alpha$$

so that f is CMC in S^3 if and only if R is harmonic. The proof for N is analogous. q.e.d.

As for CMC surfaces in $\mathbb{R}^3$, the form η with $d^\nabla(2{*}A + \eta) = 0$ is not unique: for all $\rho \in \mathbb{R}$, the forms $2{*}A^\rho_\circ = 2{*}A + \eta_0 + \rho\omega$ and $2{*}Q^\rho_\circ = 2{*}Q + \eta_0 + \rho\omega$ are closed, where

$$2{*}A = \operatorname{Ad}\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \frac{1}{2}(1 - H^{S^3}R)dR''f^{-1} & dR'' \end{pmatrix},$$

$$\omega := \operatorname{Ad}\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ dH & 0 \end{pmatrix}$$

and $\eta_0 := \frac{1}{2}H^{S^3}S\omega$, because by

$$dH = -dRf^{-1} - (H^{S^3} - R)f^{-1}df f^{-1} = \\ -(-Hdf + dR'')f^{-1} - Hdf f^{-1} = -dR''f^{-1} \tag{6.14}$$

and $N = fRf^{-1}$ which implies $dN = \operatorname{Ad}(f)(dR + 2\alpha R)$ and therefore $dN'' = \operatorname{Ad}(f)(dR'')$, the form $w = -\frac{1}{2}(dH + R{*}dH + HNdN'')$ occurring in (6.3) is $w = \frac{1}{2}(1 - H^{S^3}R)dR''f^{-1}$. Thus

$$2{*}A^\rho_\circ = \operatorname{Ad}\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ (\rho - \frac{1}{2})dH & dR'' \end{pmatrix} \quad \text{and}$$

$$2{*}Q^\rho_\circ = \operatorname{Ad}\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} dN'' & 0 \\ (\rho + \frac{1}{2})dH & 0 \end{pmatrix}$$

and the 2–step forward and backward Bäcklund transforms for different ρ are

$$\check{L}_\rho = \ker(2{*}A^\rho_\circ) = \begin{pmatrix} (\rho + \frac{1}{2})f \\ (\rho - \frac{1}{2}) \end{pmatrix}\mathbb{H} \text{ and } \hat{L}_\rho = \operatorname{im}(2{*}Q^\rho_\circ) = \begin{pmatrix} (\rho - \frac{1}{2})f \\ (\rho + \frac{1}{2}) \end{pmatrix}\mathbb{H},$$

because $dH = -dR''f^{-1} = -f^{-1}dN''$. In particularly, for $\rho = 0$ and $\rho = \pm 1/2$ we obtain

$$\check{L}_0 = \hat{L}_0 = \begin{pmatrix} -f \\ 1 \end{pmatrix}\mathbb{H} \tag{6.15}$$

$$\check{L}_{1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathbb{H} \qquad \text{and} \qquad \hat{L}_{1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathbb{H} \tag{6.16}$$

$$\check{L}_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathbb{H} \qquad \text{and} \qquad \hat{L}_{-1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathbb{H}. \tag{6.17}$$

The parameters $\rho = \pm\frac{1}{2}$ are the only parameters for which the 2–step Bäcklund transforms $\check{L}_\rho$ and $\hat{L}_\rho$ are constant.

Minimal surfaces in S^3 are examples of isothermic surfaces that, for different choices of $\eta \in \Omega^1(\mathcal{R})$ in the Euler–Lagrange equation (2.4), belong to different cases of Theorem 5.1: for $\rho = 0$ their holonomy spectral curve belongs to Case I, see Corollary 5.2, while for $\rho = \pm\frac{1}{2}$ it belongs to Case II, because then $\ker(A_\circ^\rho)$ and $\operatorname{im}(Q_\circ^\rho)$ are constant, see Proposition 6.8.

6.8. Möbius geometric characterization of CMC surfaces in $\mathbb{R}^3$ and S^3. As we have seen in Sections 6.6 and 6.7, CMC surfaces in $\mathbb{R}^3$ and S^3 are examples of constrained Willmore surfaces for which the 1–form $\eta \in \Omega^1(\mathcal{R})$ in (2.4) can be chosen such that both $\ker(A_\circ)$ and $\operatorname{im}(Q_\circ)$ are constant. The following characterization of constrained Willmore surfaces with this property is readily verified by combining equations (6.3), (6.5) and (6.1).

Lemma 6.11. *Let $f\colon M \to S^4 = \mathbb{HP}^1$ be a constrained Willmore immersion. Then:*

a) *The immersion f admits $\eta \in \Omega^1(\mathcal{R})$ such that $\ker(A_\circ) = \operatorname{im}(Q_\circ) = \infty$ if and only if $f\colon M \to \mathbb{H} = \mathbb{HP}^1\backslash\{\infty\}$ has the property that H is constant, that is, f is CMC in a 3–dimensional plane in $\mathbb{H}$ of minimal in Euclidean 4–space $\mathbb{H}$.*
b) *The immersion f admits $\eta \in \Omega^1(\mathcal{R})$ thus that $\ker(A_\circ) = 0$ and $\operatorname{im}(Q_\circ) = \infty$ if and only if $f\colon M \to \mathbb{H} = \mathbb{HP}^1\backslash\{\infty\}$ satisfies $Hf + R = c$ for some constant $c \in \mathbb{H}$. Similarly, $\ker(A_\circ) = \infty$ and $\operatorname{im}(Q_\circ) = 0$ is equivalent to $fH + N = c$ for $c \in \mathbb{H}$. The constant $c \in \mathbb{H}$ is real if and only if the surface is contained in a concentric 3–sphere in $\mathbb{H}$. In particular, f is then CMC in that 3–sphere.*

This lemma directly implies the following characterization of CMC surfaces in $\mathbb{R}^3$ and S^3.

Corollary 6.12. *A constrained Willmore immersion $f\colon M \to S^4 = \mathbb{HP}^1$ is CMC with respect to a 3–dimensional Euclidean or spherical subgeometry if and only if it is contained in a totally umbilic 3–sphere and admits a 1–form $\eta \in \Omega^1(\mathcal{R})$ satisfying (2.4) for which the 2–step Bäcklund transformation $\check{L} = \ker(A_\circ)$ is constant. The Euclidean case is then characterized by the fact that the point $\check{L}$ is contained in the totally umbilic 3–sphere.*

Proof. We have already seen in Lemma 6.11 that CMC surfaces in $\mathbb{R}^3$ and S^3 have the given properties. To prove the converse, assume that f is contained in a totally umbilic 3–sphere. This can be represented as the null lines of an indefinite quaternionic hermitian form $<\ .\ >$, see e.g. Section 10.1 of [**7**]. Then $Q_\circ = -A_\circ^*$ with $*$ denoting the adjoint with respect to $<\ .\ >$. In particular, for surfaces contained in a totally umbilic 3–sphere, $\ker(A_\circ)$ being constant is equivalent to $\operatorname{im}(Q_\circ)$ being constant and the corollary follows from Lemma 6.11. q.e.d.

The following lemma gives another characterization of CMC surfaces in $\mathbb{R}^3$ or S^3.

Lemma 6.13. *A conformal immersion $f\colon M \to S^4$ admits a point $\infty \in S^4$ at infinity such that both the left and right normal vectors N and R of f seen as an immersion into $\mathbb{R}^4 = S^4\backslash\{\infty\}$ are harmonic if and only if $f\colon M \to \mathbb{R}^4 \cong S^4\backslash\{\infty\}$ is CMC in a 3–plane or a round 3–sphere in $\mathbb{R}^4$ or minimal in Euclidean 4–space $\mathbb{R}^4$.*

Proof. By Lemma 6.2 and (6.1), harmonicity of N and R is equivalent to $*dH = -RdH = dHN$ which again is equivalent to the mean curvature vector $\mathcal{H}$ of f being a parallel section of the normal bundle of f. It is well know that a surface in $\mathbb{H} = \mathbb{R}^4$ with parallel mean curvature vector is either Euclidean minimal or CMC in a 3–plane or 3–sphere. q.e.d.

6.9. Hamiltonian stationary Lagrangian tori and Lagrangian tori with conformal Maslov form in $\mathbb{C}^2$. We discuss two classes of examples of constrained Willmore surfaces which are related to harmonic maps to S^2 but in general not CMC in $\mathbb{R}^3$ or S^3.

We identify $\mathbb{C}^2$ with $\mathbb{H}$ via $(z_1, z_2) \mapsto z_1 + jz_2$ and equip it with the standard symplectic form ω defined by $\omega(x, y) =< xi, y >$, where $< x, y >= \operatorname{Re}(\bar{x}y)$ is the usual Euclidean product on $\mathbb{H} = \mathbb{R}^4$. An immersion $f\colon M \to \mathbb{H}$ is Lagrangian with respect to ω if and only if its tangent and normal bundles T_fM and $\perp_f M$ are related via $\perp_f M = (T_fM)i$ or, equivalently, if its right normal vector is of the form $R = j\exp(i\beta)$ for a $\mathbb{R}/2\pi\mathbb{Z}$–valued function β called the Lagrangian angle.

It is shown in [**16**] that f is Hamiltonian stationary Lagrangian (i.e., a Lagrangian immersion that is stationary for the area functional under all Hamiltonian variations) if and only if β is harmonic. Because $2dR' = j\exp(i\beta)id\beta + i{*}d\beta$, harmonicity of β is equivalent to harmonicity of R, see Lemma 6.2. In particular, by Lemma 6.3, Hamiltonian stationary Lagrangian surfaces are constrained Willmore and admit a form η such that $\operatorname{im}(Q_\circ) = \infty$ for $2{*}Q_\circ = 2{*}Q + \eta$. The above formula for dR' together with (6.1) immediately implies that the special case of Lagrangian surfaces that are minimal is characterized by the property that R is constant, that is, such surfaces are complex holomorphic with respect to the complex structure given by right multiplication with $-R$.

While Lagrangian minimal surfaces can never be compact, there are compact Hamiltonian stationary Lagrangian surfaces. All Hamiltonian stationary Lagrangian tori are explicitly described in [**17**]. They are examples of constrained Willmore tori with trivial normal bundle and spectral genus $g = 0$ (the latter, because R is a harmonic map into S^2 which takes values in a great circle).

It is shown in [**10**] that Lagrangian surfaces with conformal Maslov form are characterized by the property that the left normal vector N is harmonic while the right normal vector R takes values in the great circle perpendicular to i. All Lagrangian tori with conformal Maslov form are explicitly described in [**10**]. They are examples of constrained Willmore tori with trivial normal bundle and spectral genus $g \leq 1$ (the latter, because the harmonic map N into S^2 is equivariant).

Appendix A.

The following lemma is needed in the proof of Lemma 6.6.

Lemma A.1. *Let $\varphi \in H^0(L)$ be a holomorphic section of a quaternionic holomorphic line bundle L and denote by N the C^∞–map with values in S^2 that is defined away from the zeros of φ by $J\varphi = \varphi N$. This map N continuously extends through the zeros of φ. Moreover, in case N is C^1 at a zero p of φ the Hopf field Q of L has to vanish at that point p.*

Proof of Lemma A.1. Let $z = x + iy$ be a chart centered at a zero p of φ. Then locally there is a nowhere vanishing section ψ such that

$$\varphi = \psi((x + yR)^n \lambda_n + O(n + 1)),$$

where $J\psi = \psi R$ and $\lambda_n \in \mathbb{H}\backslash\{0\}$, cf. [**21**]. In particular, $\varphi = \psi(x + yR)^n\lambda$ for a continuous function λ with $\lambda(p) \neq 0$. Now $N = \lambda^{-1}R\lambda$ implies that N continuously extends through the zero p. Moreover, away from p the Hopf field is given by $Q\varphi = \varphi\frac{1}{2}NdN''$, hence

$$Q\psi = \psi\tfrac{1}{2}(x + Ry)^n(x - Ry)^{-n}\lambda NdN''\lambda^{-1}.$$

Because the left hand side is well defined and continuous at p while $(z/\bar{z})^n$ is bounded but not continuous at zero, the form dN'' has to vanish at p in case it is continuous. This implies that $Q_p = 0$.

q.e.d.

References

[1] A. Bobenko, *All constant mean curvature tori in R^3, S^3, H^3 in terms of theta functions*. Math. Ann. **290** (1991), 209–245, MR 1109632, Zbl 0711.53007.

[2] C. Bohle, *Möbius Invariant Flows of Tori in S^4*. Thesis, TU–Berlin, 2003.

[3] C. Bohle, K. Leschke, F. Pedit & U. Pinkall, *Conformal maps from a 2–torus to the 4–sphere*. To appear in J. Reine Angew Math. http://arxiv.org/abs/0712.2311.

[4] C. Bohle, F. Pedit & U. Pinkall, *The spectral curve of a quaternionic holomorphic line bundle over a 2–torus*. Manuscr. Math. **130** (2009), 311–352, MR 2545521, Zbl 1183.58026.

[5] C. Bohle, G. P. Peters & U. Pinkall, *Constrained Willmore surfaces.* Calc. Var. Partial Differential Equations **32** (2008), 263–277, MR 2389993, Zbl 1153.53037.

[6] F. Burstall, D. Ferus, F. Pedit & U. Pinkall, *Harmonic tori in symmetric spaces and commuting Hamiltonian systems on loop algebras.* Ann. of Math. **138** (1993), 173–212, MR 1230929, Zbl 0796.53063.

[7] F. Burstall, D. Ferus, K. Leschke, F. Pedit & U. Pinkall, *Conformal Geometry of Surfaces in* S^4 *and Quaternions.* Lecture Notes in Mathematics, 1772, Springer–Verlag, Berlin, 2002, MR 1887131, Zbl 1033.53001.

[8] F. Burstall, F. Pedit & U. Pinkall, *Schwarzian derivatives and flows of surfaces.* Contemp. Math. **308** (2002), 39–61, MR 1955628, Zbl 1031.53026.

[9] F. Burstall, private communication. See also: *Twistors, 4–symmetric spaces and integrable systems*, in Progress in Surface Theory, Oberwolfach, April 29–May 5, 2007, Oberwolfach Rep. **4** (2007) 1299–1376, MR 2408014, Zbl 1177.53002.

[10] I. Castro & F. Urbano, *Lagrangian surfaces in the complex Euclidean plane with conformal Maslov form.* Tohoku Math. J. **45** (1993), 565–582, MR 1245723, Zbl 0792.53050.

[11] J. Eells & J. C. Wood, *Harmonic maps from surfaces to complex projective spaces.* Adv. Math. **49** (1983), 217–263, MR 0716372, Zbl 0528.58007.

[12] D. Ferus, F. Pedit, U. Pinkall & I. Sterling, *Minimal tori in* S^4. J. Reine Angew. Math. **429** (1992), 1–47, MR 1173114, Zbl 0746.53045.

[13] D. Ferus, K. Leschke, F. Pedit & U. Pinkall, *Quaternionic holomorphic geometry: Plücker formula, Dirac eigenvalue estimates and energy estimates of harmonic* 2*–tori.* Invent. Math. **146** (2001), 507–59, MR 1869849, Zbl 1038.53046.

[14] P. G. Grinevich & M. U. Schmidt, *Conformal invariant functionals of immersions of tori into* R^3. J. Geom. Phys. **26** (1998), 51–78, MR 1626048, Zbl 0949.53007.

[15] F. Helein, *Removability of singularities of harmonic maps into pseudo–Riemannian manifolds.* Ann. Fac. Sci. Toulouse Math. **XIII** (2004), 45–71, MR 2060029, Zbl 1061.58014.

[16] F. Helein & P. Romon, *Hamiltonian stationary Lagrangian surfaces in* $\mathbb{C}^2$. Comm. Anal. Geom. **10** (2002), 79–126, MR 1894142, Zbl 1007.53060.

[17] F. Helein & P. Romon, *Weierstrass representation of Lagrangian surfaces in four–dimensional space using spinors and quaternions.* Comment. Math. Helv. **75** (2000), 668–680, MR 1789181, Zbl 0973.53065.

[18] U. Hertrich–Jeromin, *Introduction to Möbius Differential Geometry.* Cambridge University Press, 2003, MR 2004958, Zbl 1040.53002.

[19] N. Hitchin, *Harmonic maps from a* 2*–torus to the* 3*–sphere.* J. Differential Geom. **31** (1990), 627–710, MR 1053342, Zbl 0725.58010.

[20] K. Leschke, F. Pedit & U. Pinkall, *Willmore tori in the 4–sphere with nontrivial normal bundle.* Math. Ann. **332** (2005), 381–394, MR 2178067, Zbl 1063.53058.

[21] F. Pedit & U. Pinkall, *Quaternionic analysis on Riemann surfaces and differential geometry.* Doc. Math. J. DMV, Extra Volume ICM Berlin 1998, Vol. II, 389–400, MR 1648089, Zbl 0910.53042.

[22] U. Pinkall, *Hopf tori in* S^3. Invent. Math. **81** (1985), 379–386, MR 0799274, Zbl 0585.53051.

[23] U. Pinkall & I. Sterling, *Willmore surfaces.* Math. Intelligencer **9** (1987), 38–43, MR 0883187, Zbl 0616.53049.

[24] U. Pinkall & I. Sterling, *On the classification of constant mean curvature tori*. Ann. of Math. **130** (1989), 407–451, MR 1014929, Zbl 0683.53053.

[25] M. U. Schmidt, *A proof of the Willmore conjecture*. http://arxiv.org/abs/math/0203224.

[26] I. A. Taimanov, *The Weierstrass representation of closed surfaces in R^3*. Funct. Anal. Appl. **32** (1998), 49–62, MR 1678856, Zbl 0979.53012.

[27] T. J. Willmore, *Riemannian Geometry*. Oxford University Press, Oxford, New York, 1993, MR 1261641, Zbl 0797.53002.

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