On the Minimum Number of Convex Quadrilaterals in Point Sets of Given Numbers of Points

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Abstract

Consider the following problem. Given $n, k \in \mathbb{N}$, $X \subset \mathbb{R}^2$, let $Q_k(X)$ denote the number of convex k-gons in X, where no three points are collinear. Put

$$f(n,k) = \inf_{|X|=n} Q_{n,k}(X),$$

and find properties of $f(n,k)$.

In this paper, only cases of $k = 4$ are dealt. So $f(n)$ is used in short for $f(n,4)$ when there can be no misunderstanding.

By using methods such as mathematical induction and structuring, etc., $f(4) = 0$, $f(5) = 1$, $f(6) = 3$, $f(7) = 9$, $f(8) = 19$ and $f(9) = 36$ can be obtained, and upper and lower bounds can be found for $f(n)$ when $n \geq 9$, which are

$$f(n) \geq n(n-1)(n-2)(n-3)/84 = \frac{2}{7}\binom{n}{4},$$

and

$$f(n) \leq t(n),$$

where $t(n)$ as an upper bound of $f(n)$ has a form as

$$t(n) = 2t(n-3) - t(n-6) + 6n - 33 + R(n) \quad (n \geq 9),$$

$R(n)$ being

$$R(n) = \begin{cases} (n-6)(7n-52)/4, & n \text{ even;} \\ (n-7)(7n-45)/4, & n \text{ odd.} \end{cases}$$

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Chapter 1

Introduction

This paper is about the number of convex quadrilaterals in a given point set of n distinct points with no three collinear. This problem is originated from the *Happy Ending Problem*, posed by Esther Klein in 1933.

The Happy Ending Problem is as follows. Given integer $n \geq 3$, let $N(n)$ be the smallest natural number such that for all $m \geq N(n)$, any point set with m distinct points with no three collinear has at least a convex n-gon. See [1].

Erdos P. and Szekeres G. further studied this problem and came up with the upper and lower bound of $N(n)$ in [1] and [2],

$$2^{n-2} + 1 \leq N(n) \leq \binom{2n-4}{n-2} + 1.$$

$N(3) = 3$, $N(4) = 5$ and $N(5) = 9$ are already known but for $n \geq 6$, no proof is found. However, [3] gives an inequality that

$$2^{n-2} + 1 \leq N(n) \leq \binom{2n-5}{n-3} + 2.$$

Now consider a related question. Let $f(n,k)$ be the infimum of the number of convex k-gons in any point set of n distinct points with no three collinear.

We only studied the $k = 4$ cases. So $f(n)$ is used to denote $f(n,4)$ for convinience.

Feng Yuefeng did some study on this problem in [4]. And the Angle Cover in Chapter 1 is from [4]'s inspiration. However, some incorrectness exists in [4]. For example, the solution for $n = 7$ and the proof for $n = 8$ have some mistakes.

And in this paper we did some improvement on his method.

Chapter 2 gives the values of $f(n)$ when $n \leq 9$ and presents the proof.

The lower and upper bound are separately put forward in Chapter 3 and Chapter 4.

Note that all the theorems and lemmas in this paper has a precondition that no three points are collinear.

Chapter 2

For $n \leq 9$

This chapter will need a notion called Angle Cover.

Definition 2.1. *A region (always a plane) is called an Angle Cover of graph G (with n points) if*

1. *No three points in G are collinear;*
2. *$\binom{n}{2}$ distinct lines that each passes through two points in G divide the region into several subregions.*
3. *Each subregion is associated with a number, called the degree of the subregion, which shows how many angles formed over G can cover it ($\angle ABC$ is said to be formed over G if $A, B, C \in G$). Then a subregion of degree k can be called as a k-subregion.*

2.1 shows an example of Angle Cover.

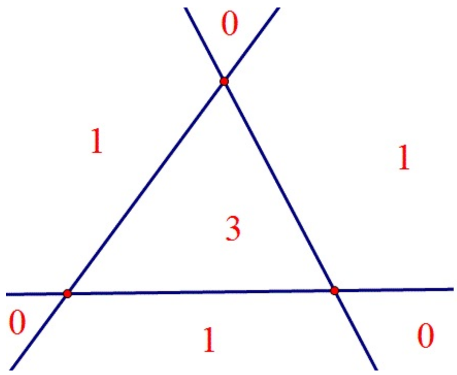

Figure 2.1: An example of Angle Cover of three points

The Angle Covers used later in this paper is in fact Angle Cover outside a convex hull (we don't consider the interior of the convex hull).

2.1 For $n < 7$

Obviously $f(4) = 0$. As for $f(5) = 1$, it can be obtained by the result from [3]. See Figure 2.2.

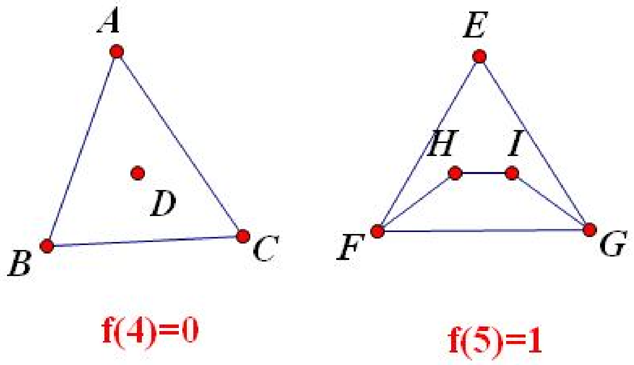

Figure 2.2: Solutions for $n = 4, 5$

Consider $n = 6$.

A point set of 6 points has $\binom{6}{5} = 6$ subsets of 5 points. Each 5-point subset has at least one convex quadrilateral. Each convex quadrilateral can at most be counted twice in this way, thus we have,

$$f(6) \geq \binom{6}{5}/2 = 3.$$

And Figure 2.3 suggests $f(6) = 3$.

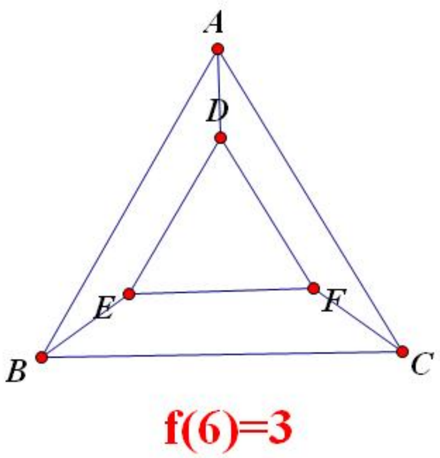

Figure 2.3: Solution for $n = 6$

2.2 For $n = 7$

First we prove $f(7) \geq 9$, which needs two lemmas.

Lemma 2.2. *For P, an inner point of a convex hull formed by n ($n \geq 4$) points, the number of convex quadrilaterals formed by P and three vertices of the convex hull is at least $n-2$.*

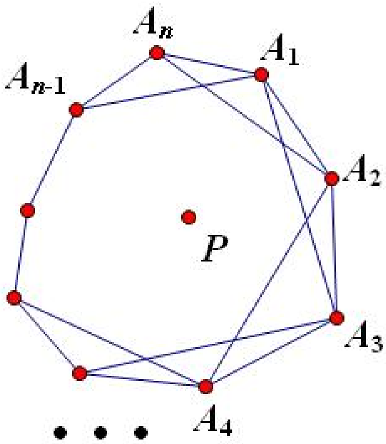

Figure 2.4: Proof for Lemma 2.2

Proof. See Figure 2.2. Consider the n triangles formed by three adjoint vertices of convex polyhedron $A_1A_2\dots A_n$, i.e. $\triangle A_1A_2A_3, \triangle A_2A_3A_4, \dots, \triangle A_nA_1A_2$, then P is at most inside two of the triangles, and thus it's outside the other $n-2$ ones. Hence P could form a convex quadrilateral with the three vertices of each of the $n-2$ triangles. And it completes the proof. $\square$

Lemma 2.3. *For two inner points, P, Q, of a convex hull of n points. The number of convex quadrilaterals formed by P, Q and two vertices of the convex hull is at least $\frac{n(n-2)}{4}$.*

Proof. See Figure 2.5. Line PQ divides the vertices of the convex hull into two groups. Let x, y separately denote the number of vertices of the two groups, and then $x + y = n$. Properties of convex hulls promise that any two vertices of the same group can form a convex quadrilateral with P and Q.

Hence, when $x, y > 1$, we have at least

$$\binom{x}{2} + \binom{y}{2} = \frac{x^2 + y^2 - n}{2}$$

convex quadrilaterals. According to the inequality of arithmetic and geometric means,

$$\frac{x^2 + y^2 - n}{2} \geq \frac{\frac{(x+y)^2}{2} - n}{2} = \frac{n(n-2)}{4}.$$

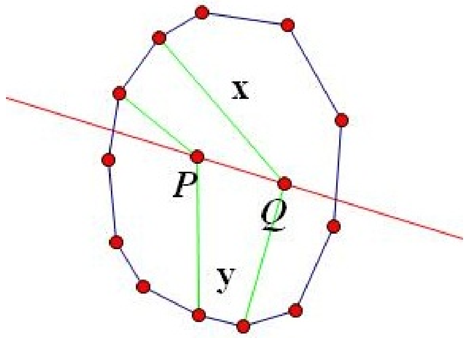

Figure 2.5: Proof for Lemma 2.3

When $x = 1$ or $y = 1$, the number of convex quadrilaterals of this kind is

$$\binom{n-1}{2} = \frac{(n-1)(n-2)}{2} > \frac{n(n-2)}{4}.$$

Thus it proves 2.3. □

Back to the proof of the proposition ($f(7) \geq 9$). Let S be the convex hull of the 7 points. Discuss the following situations.

1. S is a hexagon or a heptagon. $\binom{6}{4} = 15 > 9$ completes the proof.

2. S is a pentagon. Let A, B be two inner points of the convex hull. According to Lemma 2.2, each point at least form three convex quadrilaterals with three vertices of the convex hull. Hence $\binom{5}{4} + 2 \times 3 = 11 > 9$ completes the proof.

3. S is a quadrilateral, say $D_1D_2D_3D_4$. Let A, B, C be the three inner points of it. Similarly, according to Lemma 2.2, each inner point can form two convex quadrilaterals with three vertices of the convex hull.

   As for A, B according to Lemma 2.3, they can at least form two convex quadrilaterals with two vertices of the convex hull. Same for C, A or B, C. Then $1 + 2 \times 3 + 3 = 10 > 9$ completes the proof.

4. S is a triangle, say $A_1A_2A_3$. And B_1, B_2, B_3, B_4 are the four inner points.

   Connect each two of the four inner points and we get six lines. According to Lemma 2.3, each two of the four inner points form one convex quadrilaterals with two vertices of the convex hull.

   Then discuss the convex hull of the four inner points, B_1, B_2, B_3, B_4.

   (a) It's a quadrilateral. It's easy to consider when two sides of the quadrilateral is parallel with each other. Here we only consider other general cases.

   Draw the Angle Cover on the plane of these four points. See Figure 2.7.

   We have the following results for Angle Cover.

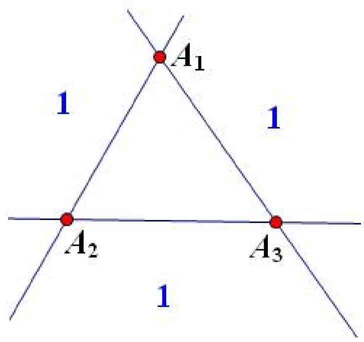

Figure 2.6: Proof for Lemma 2.4

Lemma 2.4. *In a Angle Cover in region R of graph G, let H denote the convex hull of the points of G. For any point $x \in R\backslash H$, the number of convex quadrilaterals it can form with the vertices of G equals to the degree of the subregion x lies in.*

Proof. The lemma is equivalent to the following proposition: a point outside H can form a convex hull with three of the points in G if and only if x can be covered by an angle formed by the three points. For any three points, A_1, A_2, A_3, in G (see 2.6), connect each two of them. Then the three lines divide the plane into 7 parts, one of them being $\triangle A_1A_2A_3$. The 1-subregions can all be covered by angles formed by A_1, A_2, A_3 exactly for once, while 0-subregions can't be covered then.
Obviously, if P lies in a 1-subregion, then it can form a convex polyhedron with A_1, A_2, A_3. Otherwise, no convex polyhedron can be formed.
And it completes the proof. □

Back to the proof that $f(7) \geq 9$.
See Figure 2.7. Obviously, at least one point in $\{A_1, A_2, A_3\}$ is not in 0-subregions. If two of them are in 0-subregions, the other one must be in a 4-subregion. If there is at most one point in a 0-subregion, the other two are at lest in 2-subregions. Hence by Lemma 2.4, we have at least 4 convex quadrilaterals.
Thus $6 + 1 + 4 > 9$ completes the proof in this situation.

(b) It's a triangle, say $\triangle B_1B_2B_3$. Then radials B_4B_1, B_4B_2 and B_4B_3 divide the plane into three parts, which covered the whole plane. So A_1 must be in one part and then form a convex quadrilateral with the three vertices of that region. Same for A_2, A_3. Hence in this situation $6 + 3 = 9$ completes the proof.

Above completes the proof of $f(7) \geq 9$. And according to the proof of situation 4b, we know there exist exactly 9 convex quadrilaterals in Figure 2.8, where A_1, A_2, A_3 are in 0-subregions of B_1, B_2, B_3, and that they are separately covered by $\angle B_1CB_2, \angle B_1CB_2, \angle B_3CB_1$. These 9 convex quadrilaterals can be found by discussing different situations.

Let their convex hull, $\triangle A_1A_2A_3$ be S.

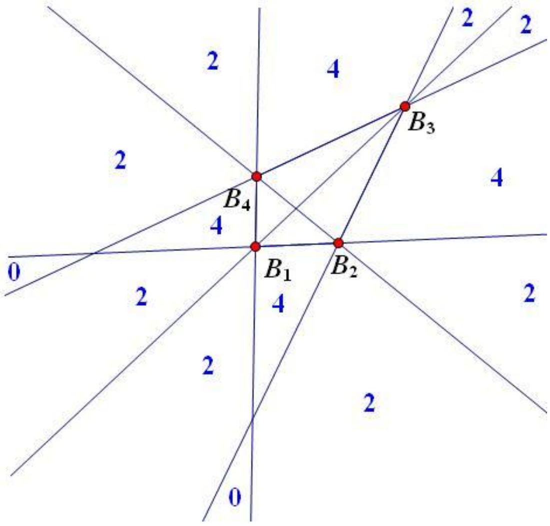

Figure 2.7: Angle Cover for general convex quadrilaterals

1. Convex quadrilaterals formed by two vertices of S and two of the four inner points. Lemma 2.3 shows there are six convex quadrilaterals of this kind. They are $CB_1A_1A_2$, $CB_2A_2A_1$, $CB_3A_3A_1$, $B_2B_1A_1A_2$, $B_2B_3A_3A_2$, and $B_3B_1A_1A_3$.
2. Convex quadrilaterals formed by one vertices of S and three of the four inner points. There are three convex quadrilaterals of this kind. They are $CB_1A_1B_2$, $CB_1A_2B_2$, and $CB_1A_3B_3$.
3. Convex quadrilaterals formed by four inner points. Zero.

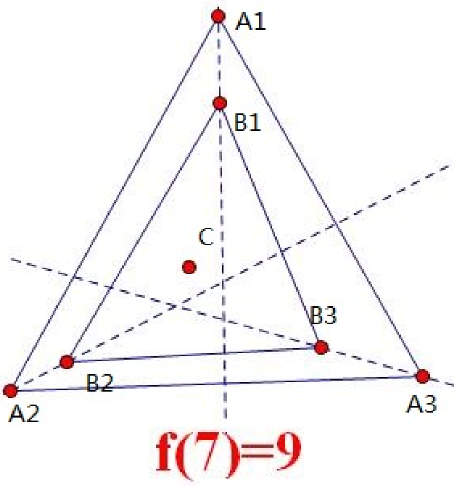

Figure 2.8: Solution for $f(7) = 9$

Hence $f(7) = 6 + 3 = 9$.

2.3 For $n = 8$

First we prove $f(8) \geq 19$, by discussing the convex hull S, of the eight points.

1. S is an octagon or a heptagon. $\binom{7}{4} = 35 > 19$ completes the proof.

2. S is a hexagon. The vertices form $\binom{6}{4} = 15$ convex quadrilaterals. For the two inner points, say A, B, each at least form four convex quadrilaterals with three vertices, similarly to the proof of $n = 7$ case. See 2.

   Hence $\binom{6}{4} + 4 \times 2 > 19$ completes the proof.

3. S is a pentagon. The vertices form $\binom{5}{4} = 5$ convex quadrilaterals.

   Let A, B, C be the three inner points. Similarly to situation 3 in the proof of $n = 7$ case, it can be proved that each inner point at least form three convex quadrilaterals with the vertices of the convex hull. While for each two inner points, they form at least four convex quadrilaterals with the vertices.

   Hence, $5 + 3 \times 3 + 4 \times 3 > 19$ completes the proof of this situation.

4. S is a quadrilateral. Then there are four inner points. Method similar to the proof of situation 3 in $n = 7$ case can prove that each of the inner point can form two convex quadrilaterals with the vertices while each two inner points can form at least two convex quadrilaterals with the vertices.

   Hence $1 + 2 \times 4 + 2 \times \binom{4}{2} = 21 > 19$ completes the proof of this situation.

5. S is a triangle. According to Lemma 2.3, each two inner points can form a convex quadrilateral with two vertices. There are $\binom{5}{2} = 10$ convex quadrilaterals of this kind.

   Then discuss the convex hull of the five inner points, S_1.

   (a) S_1 is a pentagon, say $A_1A_2A_3A_4A_5$. Use a similar methoed used in proving $n = 7$ case when discussing the situation that the convex hull is a triangle. Draw the Angle Cover. See situation 4b in $n = 7$ case.

   We need two further lemmas related to the notion Angle Cover.

   Lemma 2.5. *If the points of G form a polyhedron, the Angle Cover of G can at most have two 0-subregions.*

   Proof. A 0-subregion should satisfy the condition that the sum of degrees of two of its adjacent internal angles is less than 180°. As a matter of fact, if the condition is not satisfied, according to the Euclidean Axiom V, lines will be parallel with each other or will intersect at a point that is outside the convex hull, off the common line segment of the two angles, see Figure 2.9. Then $\angle DAB$ and $\angle CBA$ cover the region by line AB that doesn't contain E. Hence in the figure there can't be a 0-subregion off the right side of line AB. So there must be two adjacent internal angle in a 0-subregion such that the sum of their degree is less than 180°.

   See Figure 2.10. It is easy to see that the region the vertical angle of $\angle ACB$ covers is a 0-subregion, and that the region off line AB that contains this 0-subregion exist no other 0-subregions (because all the subregions are covered by $\angle 1, \angle 2$).

This conclusion also suggests some properties of 0-subregions that their shape are generally the same, i.e. they are bounded by two intersectant lines; and each is associated with an only pair of adjacent internal angle that the sum of their degrees is less than 180°.

Back to the lemma that needs proof. According to the previous discussion, 0-subregions exist if and only if there exist two adjacent internal angle that the sum of their degrees is less than 180°. So the sum of the degrees of the two corresponding external angles are greater than 180°. If there exist three 0-subregions, then the external angles of at least two of them are not the same, which contradicts the rule that the sum of external angles should obey.

Thus it completes the proof of Lemma 2.5.

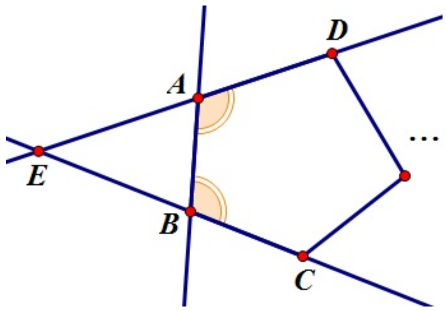

Figure 2.9: The condition of 0-subregions

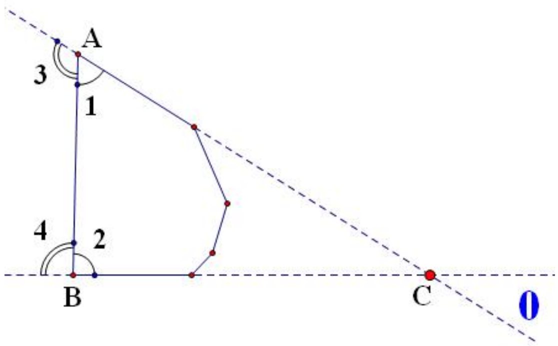

Figure 2.10: Proof for Lemma 2.5

□

Now we need another lemma.

Lemma 2.6. *In a Angle Cover by a convex pentagon, if a point out side the pentagon lies in a Great Angle (angle formed by three adjacent vertices of the pentagon, the center one being the vertex of the angle), then it can at least form three convex quadrilaterals with* *the five inner points that the quadrilaterals cover the vertex of the angle and that the vertex of the angle and the point are a pair of opposite vertices.*

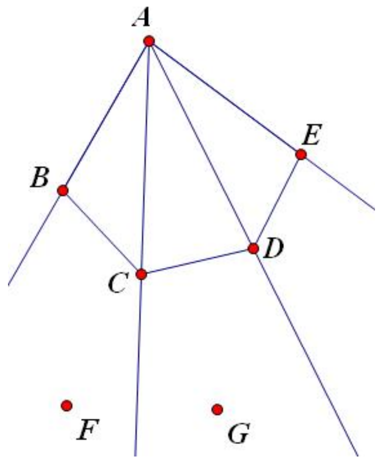

Figure 2.11: Proof for Lemma 2.6

Proof. See Figure 2.11, F, G are all covered by $\angle BAE$. Nevertheless, F is also covered by $\angle BAC$ and $\angle BAD$, so it can form 3 convex quadrilaterals, while G is also covered by $\angle BAD$, $\angle CAE$ and $\angle CAD$, which makes another four convex quadrilaterals. And all the convex quadrilaterals satisfy the conditions mentioned in the lemma, which proves Lemma 2.6. □

Back to the proposition. Draw Figure 2.12.

According to Lemma 2.5, there are at most two 0-subregions. See Figure 2.12, in the up-left region of line A_5A_3 and A_5A_2, i.e. the interior of $\angle TA_5A_2$ in the figure (when there at less than two 0-subregions, we can do some adjustment to the figure and use the same method to prove it), there exist no 0-subregions. And it is easy to prove that points in this area are at least covered by two great angles. And obviously at least one of the three outside points lies in this area. Hence by Lemma 2.6, this point can at least form 6 convex quadrilaterals with $A_1A_2A_3A_4A_5$. And according to the property that the vertex of the Great Angle and this point are opposite vertices, the three convex quadrilaterals this point forms with the vertex of every Great Angle are different from each other, i.e. the 6 convex quadrilaterals are all different.

Besides, there are $\binom{5}{4} = 5$ convex quadrilaterals in a convex pentagon. Hence we have $5 + 6 + 10 = 21 > 19$ completes the proof.

(b) S_1 is a quadrilateral, say $A_1A_2A_3A_4$, B being the inner point. Previous proof shows it itself has 3 convex quadrilaterals. Radials BA_1,

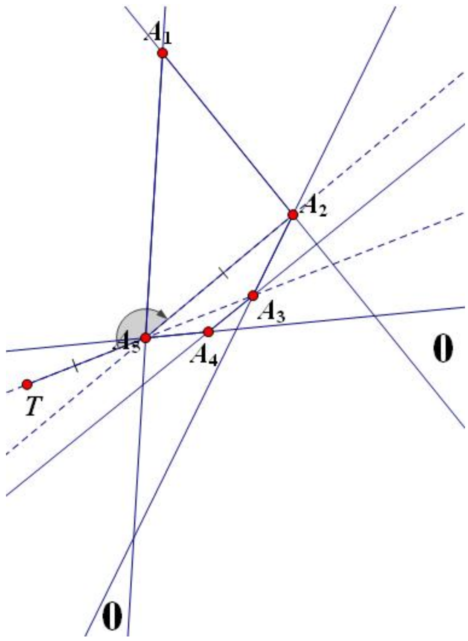

Figure 2.12: Proof for the situation when the inner convex hull is a pentagon

BA_2, BA_3 and BA_4 divide the plane into four areas, then the vertices of the outer triangle are all contained in one of the areas. Hence we have at least three convex quadrilaterals.

Last is the situation when convex quadrilaterals are formed by one point from the seven points ($A_1A_2A_3A_4$ and the outer three points) and three of the inner points. According to the proof of situation 4a in $n = 7$ case, there are at least 4 convex quadrilaterals.

Hence $10 + 3 + 3 + 4 > 19$ completes the proof.

(c) S_1 is a triangle. Similarly to Lemma 2.4 of $n = 7$, we draw Angle Cover in Figure 2.13.

Consider the number of points that lie in the only 2-subregion. Obviously it's less than 3. And if there two in the 2-subregion, the other point must be in a 7-region, adding the number of convex quadrilaterals by $2 + 2 + 7 = 11$. But if there is at most one point in the 2-subregion, the number of convex quadrilaterals is added by at least $2 + 3 + 3 = 8$.

Adding $f(5) = 1$ to the number, $10+8+1 = 19$ completes the proof.

Finally we have $f(8) \geq 19$.

And Figure 2.14 shows the solution. Let S be the triangle convex hull

1. Convex quadrilaterals formed by two vertices of S and two inner points. According to Lemma 2.3, we have $\binom{5}{2} = 10$ convex quadrilaterals of this kind.

2. Convex quadrilaterals formed by one vertex of S and three inner points. Three vertices of S separately lie in 2-subregion, 3-subregion and 3-subregion

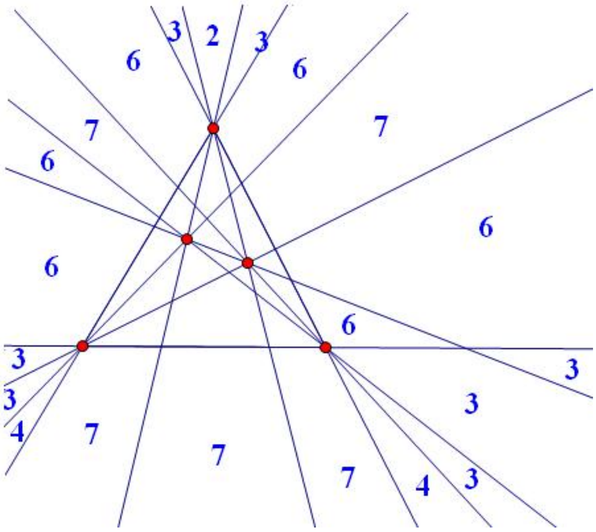

Figure 2.13: Angle Cover when S_1 is a triangle

in Figure 2.14, according to the proof of 5c. Hence we have $2 + 3 + 3 = 8$ convex quadrilaterals of this kind.

3. Convex quadrilaterals formed by four inner points. Only $f(5) = 1$.

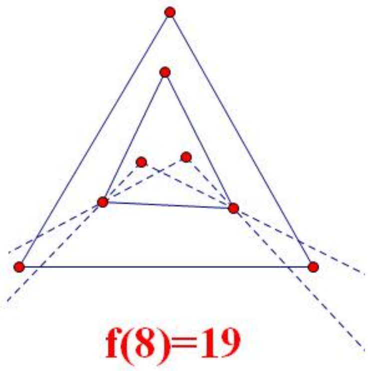

Figure 2.14: Solution for $f(8) = 19$

Hence we have $10 + 8 + 1 = 19$ convex quadrilaterals, which gives $f(8) = 19$.

2.4 For $n = 9$

Similarly to the proof of $n = 8$, by discussing different situations, we can obtain $f(9) \geq 36$. Here we just present how we may prove this.

Let S be the convex hull of the nine points.

1. S is an octagon or an enneagon. $\binom{8}{4} > 36$ completes the proof.

2. S is a heptagon. According to Lemma 2.2, the number of the convex quadrilaterals formed by one of its inner point and three vertices is greater than or equal to $7 - 2 = 5$; while the vertices alone can form $\binom{7}{4} = 35$ convex quadrilaterals. Hence $35 + 5 > 36$ completes the proof.

3. S is a hexagon. According to Lemma 2.3, the number of convex quadrilaterals formed by two inner points and two vertices is at least $3 \times 6 = 18$. Furthermore, according to Lemma 2.2, we have at least $3 \times (6 - 2) = 12$ convex quadrilaterals formed by one inner point and three vertices. Hence $18 + 12 + \binom{6}{4} > 36$ completes the proof.

4. S is a pentagon. According to Lemma 2.3, we have at least $4 \times \binom{4}{2} = 24$ convex quadrilaterals formed by two inner points and two vertices. While Lemma 2.2 promises at least $4 \times (5-2) = 12$ convex quadrilaterals formed by one inner point and three vertices. Then $24 + 12 = 36$ completes the proof.

5. S is a quadrilateral. According to Lemma 2.3, we have at least $2 \times \binom{5}{2} = 20$ convex quadrilaterals formed by two inner points and two vertices. According to Lemma 2.2, we have at least $5 \times (4 - 2) = 10$ convex quadrilaterals formed by one inner points and three vertices.

   Then discuss the convex of the five inner points, S_1.

   (a) S_1 is a pentagon, which promises 5 convex quadrilaterals. Then $20 + 10 + 1 + 5 = 36$ completes the proof.

   (b) S_1 is a quadrilateral. We have $1+2 = 3$ convex quadrilaterals formed by points in S_1. By the Angle Cover of a quadrilateral, it is easy to know the number of convex quadrilaterals formed by three of S's inner points and the one vertex of S is at least 4. Hence $20 + 10 + 1 + 3 + 4 > 36$ completes the proof.

   (c) S_1 is a triangle. Then it has a Angle Cover similar to Figure 2.13. It's easy to know the number of convex quadrilaterals formed by three of S's inner points and a vertex of S is at least 8. Hence $20 + 10 + 1 + 8 > 36$ completes the proof.

6. S is a triangle. According to Lemma 2.3, we have at least $1 \times \binom{6}{2} = 15$ convex quadrilaterals formed by two inner points and two vertices. Let S_1 be the convex hull of the 6 inner points.

   (a) S_1 is a hexagon. Its vertices form $\binom{6}{4} = 15$ convex quadrilaterals. Using the method similar to the proof of 5a in 2.3, we firstly draw the Angle Cover by the convex hexagon, and discuss the 0-subregions, and then prove that at least one of the three vertices of S lies in the two Great Angle of S_1. So like Lemma 2.6, we can prove this point can at least form 6 convex quadrilaterals with the vertices of S_1. Hence $15 + 15 + 6 = 36$ completes the proof.

   (b) S_1 is a pentagon. S_1's vertices can form 5 convex quadrilaterals. Applying the conclusion of 5a in 2.3, we have at least 6 convex quadrilaterals formed by three vertices of S_1 and one vertex of S.

   Then, let C be the inner point of S_1.

We need to improve Lemma 2.2, i.e. we are to prove that C can at least form 5 convex quadrilaterals with the vertices of S_1. The method is as follows.

See Figure 2.15, discuss the location of C, i.e. region a, region b, or region c. Then prove each situation. Then with similar method

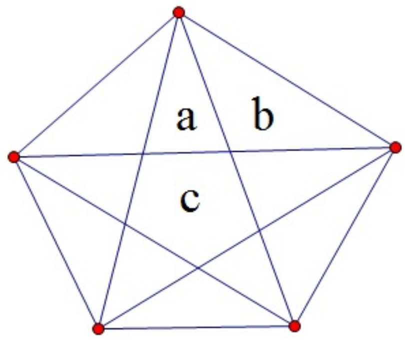

Figure 2.15: Proof for the improvement of Lemma 2.2

we can prove the number of convex quadrilaterals formed by C, two vertices of S_1 and a vertex of S is at least 5. Then $15+5+6+5+5 = 36$ completes the proof.

(c) S_1 is a quadrilateral. Then the two diagonal lines divide the quadrilateral into four areas. Let P, Q be the two inner points of S_1, and let $S_2 = \{P, Q\}$. Similarly, by discussing the location of P, Q in the quadrilateral, we can obtain the fact that P, Q and the vertices of S_1 can at least form 7 convex quadrilaterals.

Then estimate the number of convex quadrilaterals formed by one point of S_2, two vertices of S_1 and a vertex of S. Here we need a lemma almost equivalent to the Angle Cover, just for convenience.

Lemma 2.7. *A, B, C, D form a convex quadrilateral in the given order if and only if AC intersects with BD.*

Proof. Trivial. ☐

Back to the original problem. Firstly prove that we have at least 5 convex quadrilaterals formed by P, two vertices of S_1 and one vertex of S. Obviously, line segment between P and any vertex of S intersects with an edge of S_1. So according to Lemma 2.7, we have at least 3 convex quadrilaterals. Then prove that the number of convex quadrilaterals where P and a vertex of S are adjacent is at least $\frac{k}{2}$, where k is degree of the region that point lies in.

See Figure 2.16. This two figures separately give the situation when A is in a 2-subregion and a 4-subregion (situation is similar when A and P change their location). Then according to Lemma 2.7, we can complete the proof of the proposition. According to the proof of $n = 7$ and Figure 2.7, we know that at least two vertices of S

lie in 2-subregions or at least one lies in a 4-subregion. Hence according to the proposition we just proved, we have at least 2 convex quadrilaterals of this kind.

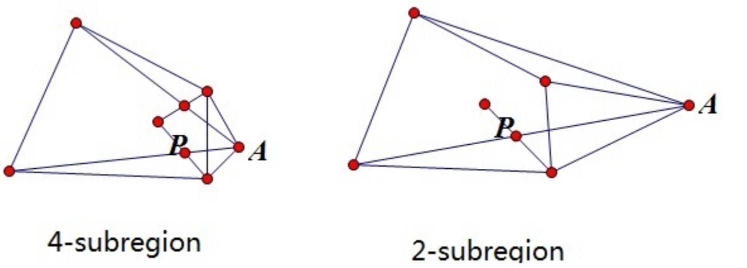

Figure 2.16: How the line segments intersect with each other when A lies in different ares

Hence we have at least 5 of this kind of convex quadrilaterals. Same for Q.

Finally estimate the number of convex quadrilaterals formed by two vertices of S_1 and two vertices of S. Again, according to the proof of $n = 7$, we at least have 4 of this kind of convex quadrilaterals.

Hence, $15 + 5 \times 2 + 7 + 4 = 36$ completes the proof of this situation.

(d) S_1 is a triangle. This situation can be solved by Angle Cover. First categorize the location of the six inner points of S. Then for each category draw the Angle Cover and discuss different possible sub-situations. Here we just draw an Angle Cover for one sub-situation.

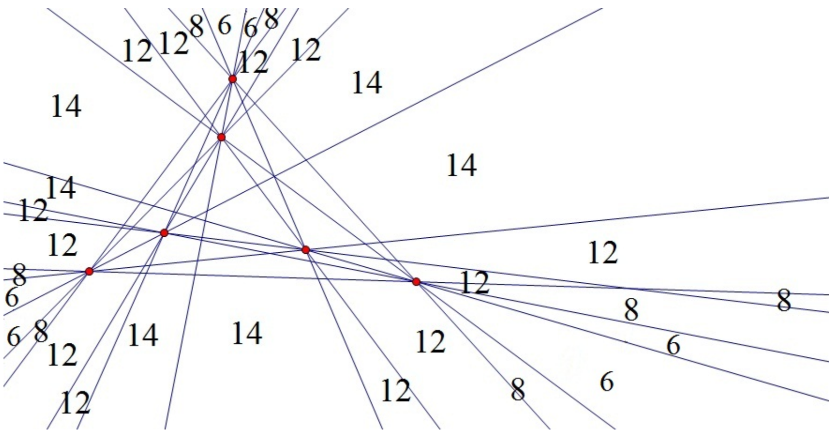

Figure 2.17: Angle Cover of six points when the convex hull is a triangle

See Figure 2.17. It's easy to know that the number of convex quadrilaterals formed by one vertex of S and three inner points of S is at least $6+6+6 = 18$. And the number of convex quadrilaterals formed only by inner points is $f(6) = 3$. Hence $18 + 15 + 3 = 36$ completes the proof.

All discussions above proves that $f(9) \geq 36$. Then see Figure 2.18. By previous proofs, we can prove that ther are only 36 convex quadrilaterals here. In fact, we will obtain the upper bound of $f(n)$. And there will be related structuring method that suggests there are only 36 convex quadrilaterals in this figure.

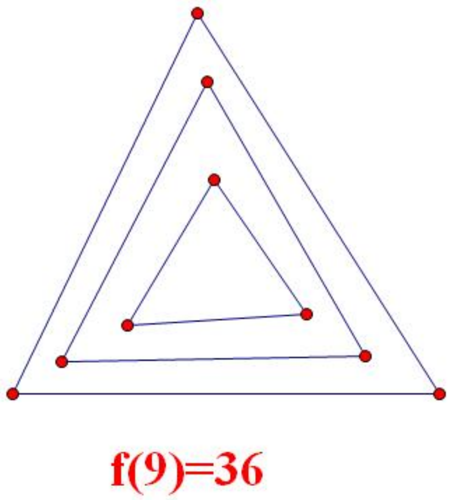

Figure 2.18: Solution for $n = 9$

Chapter 3

Lower Bound for $n > 9$

This chapter gives a lower bound of $f(n)$,

$$f(n) \geq n(n-1)(n-2)(n-3)/84 = \frac{2}{7}\binom{n}{4}.$$

To obtain this, the following theorem will be considered.

Theorem 3.1. *Let $f(n)$ be the infimum of the number of convex quadrilaterals formed over a set with n distinct points where no three are collinear, then*

$$f(n) \geq \frac{n}{n-4} f(n-1), \ \ (\forall n \in \mathbb{N}^*).$$

Proof. Given $n \in \mathbb{N}^*$, suppose a point set X ($|X| = n$) gives an optimal situation where the number of convex quadrilaterals formed by n points are the smallest, i.e. $Q_k(X) = f(n)$. Then there exists a point $x \in X$ such that the number of convex quadrilaterals that has x as a vertex is no less than $\geq \frac{4}{n} f(n)$.

Consider $X \backslash \{x\}$, we have

$$f(n) \geq \frac{4}{n} f(n) + Q_k(X \backslash \{x\}) \geq \frac{4}{n} f(n) + f(n-1), \tag{3.1}$$

which gives

$$f(n) \geq \frac{n}{n-4} f(n-1).$$

And it completes the proof of Theorem 3.1.

□

Furthermore, by using the fact that $f(9) = 36$, $f(n)$ could be bounded below as

$$f(n) \geq n(n-1)(n-2)(n-3)/84 = \frac{2}{7}\binom{n}{4}.$$

Chapter 4

Upper Bound for $n > 9$

A structuring method is used in this chapter to form a point set of n points of near optimal situation.

Consider the cases for $n = 4, 5, 6, 7, 8$, see Figure 4.1.

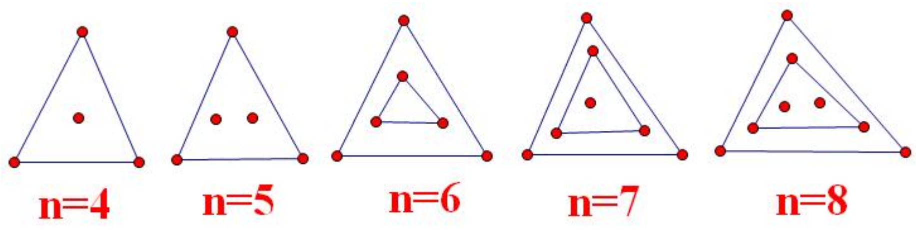

Figure 4.1: Special cases of small n's

Then we structure a point set of n distinct points, X_n. Let $t(n) = Q_k(X_n)$, and $t(n)$ can be seen as an upper bound of $f(n)$.

Following shows the method of structuring by induction and simultaneously calculates the value of $t(n)$.

Suppose the convex hull of X_n $(n < k, k \geq 10)$ is a triangle.

When $n = k$, we structure X_n based on X_{n-3}. By the hypothesis of induction, the convex hull of X_{n-3} is a triangle, say $\triangle A_1A_2A_3$. Let $P = \{A_1, A_2, A_3\}$, and let $Q = X_{n-3} \backslash P = X_{n-6}$. Then we can position three points, B_1, B_2, B_3 (let $R = \{B_1, B_2, B_3\}$) that $\triangle B_1B_2B_3$ is the convex hull of $X_{n-3} \bigcup R$, and that these three points satisfy the following conditions.

1. B_1 lies in the angle formed by the reversed extensional line of A_1A_2 and the reversed extensional line of A_1A_3, i.e. $\overrightarrow{A_2A_1} \times \overrightarrow{A_1B_1} > 0$ and $\overrightarrow{A_1B_1} \times \overrightarrow{A_3A_1} > 0$.

2. B_i is sufficiently close to A_i such that for any line l that passes through two distinct points in $X \backslash P$, we have $l \bigcap A_iB_i = \emptyset$ (here A_iB_i denotes line segment A_iB_i).

3. Line A_iB_i $(i = 1, 2, 3)$ divides the points in Q into two parts whose numbers of points are approximately the same.

See Figure 4.2.

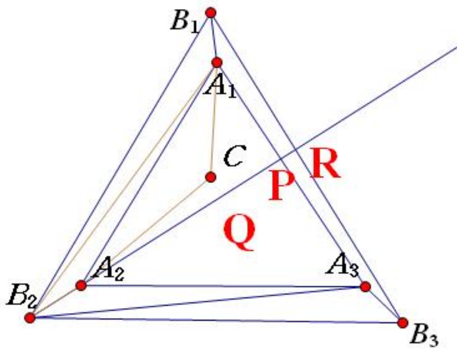

Figure 4.2: Form X_n from X_{n-3}

Now $t(n)$ can be obtained from $t(n-3)$ and $t(n-6)$.

The convex quadrilaterals in X_n can be sorted into the following six categories.

1. Formed over $P \bigcup Q$, $t(n-3)$ convex quadrilaterals;
2. Formed over $R \bigcup Q$, $t(n-3)$ convex quadrilaterals according to condition 2 mentioned above; (but the first two categories has $t(n-6)$ common convex quadrilaterals)
3. Formed over $R \bigcup P$, $f(6) = 3$ convex quadrilaterals;
4. Formed by one point in Q, two points in P and one point in R;
5. Formed by one point in Q, one point in P and two points in R;
6. Formed by two points in Q, one point in P and one point in R.

For Category 4 and Category 5, they should have the same result. Firstly calculate the 4th category. See Figure 4.3.

It does no harm in assuming A_1 and A_2 are the two points we choose from P.

If we choose $B_3 \in R$, they form no convex quadrilaterals according to condition 1.

If we choose $B_2 \in R$, $x \in Q$ can form a convex quadrilateral with A_1, A_2 and B_2 only when x is inside $\angle A_1B_2A_2$. And if we choose B_2, A_2, A_3 at the beginning, $x \in Q$ can form a convex quadrilateral with A_2, A_3 and B_2 only when x is inside $\angle A_3B_2A_2$. Hence these two situations together make $(n-6)$ convex quadrilaterals.

Similarly, we have the same result in other situations and thus Category 4 and Category 5 each has $3(n-6)$ convex quadrilaterals.

As for Category 6. Let $R(n)$ be the number of convex quadrilaterals in this category.

Suppose we choose $A_i \in P$ and $B_j \in R$, and $R(n)$ can then be obtained by discussing the following two situations.

1. $i = j$. Consider $i = j = 2$.

   Line A_2B_2 divides $\triangle A_1A_2A_3$ into two parts, simultaneously dividing Q into two parts.

   We are to proof that for two distinct points, say C and D, in Q can form a convex quadrilateral with A_2, B_2 if and only if they are in the same part of Q.

   Proof. See Figure 4.4. When line CD is parallel with A_2B_2, A_2, B_2, C, D certainly form a convex quadrilateral and C, D are obviously in the same part. Else, if CD is not parallel with A_2B_2 but C, D are in the same part, their crossing point can't be on line segment CD or A_2B_2 (according to condition 2), hence they form a convex quadrilateral. However, when CD are divided into two different parts of Q by line segment A_2B_2, their

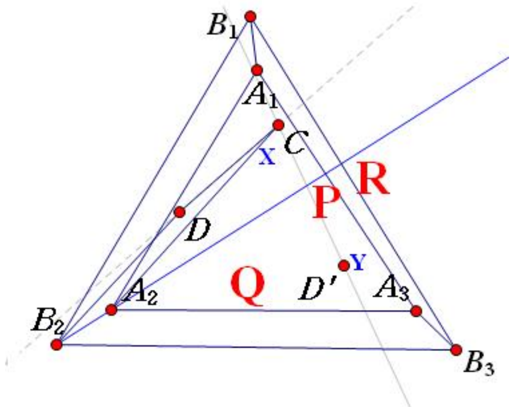

Figure 4.3: Convex Quadrilaterals formed by one point in Q, two points in P and one point in R

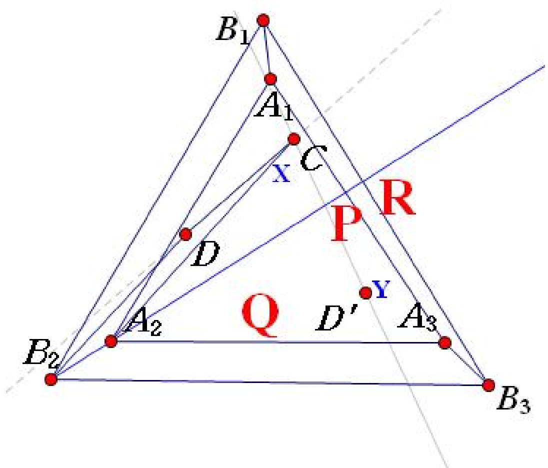

Figure 4.4: Sufficient and necessary conditions for C, D, A_2, B_2 forming a convex quadrilateral

crossing point surely lies on line segment CD and they can't form a convex quadrilateral. $\square$

Let x and y separately denote the number of points in Q's two parts. We have $x + y = n - 6$. Then the number of convex quadrilaterals in this situations is

$$\binom{x}{2} + \binom{y}{2} = \frac{x^2 + y^2}{2} - \frac{x + y}{2} = \frac{x^2 + y^2 - (n-6)}{2}.$$

The properties of inequalities promise when x is as close to y as possible, the expression gets its minimum value.

Hence

$$\binom{x}{2} + \binom{y}{2} \geq \begin{cases} \binom{(n-6)/2}{2} + \binom{(n-6)/2}{2}, & (n-6) \text{ even;} \\ \binom{(n-5)/2}{2} + \binom{(n-7)/2}{2}, & (n-6) \text{ odd.} \end{cases}$$

2. $i \neq j$. See Figure 4.5.

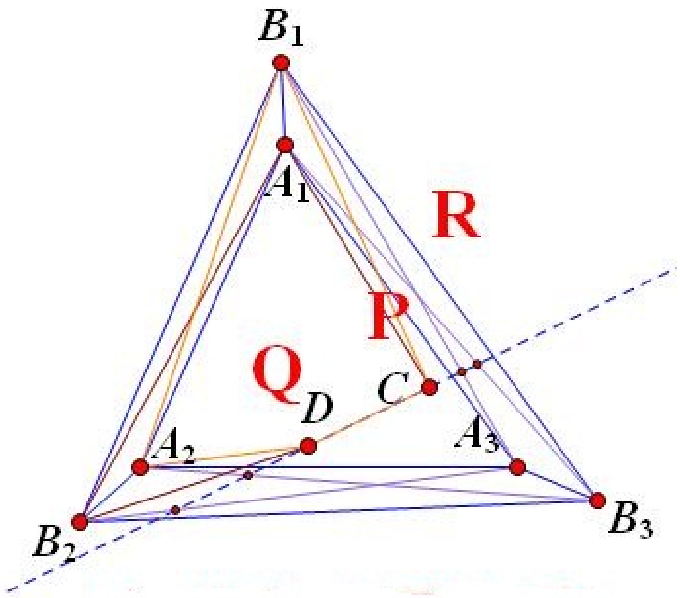

Figure 4.5: Situation where $i \neq j$

For any two distinct points, C, D, in Q. Assume line CD intersects with A_1A_3 and A_2A_3.

For $\triangle A_1A_3B_1$, because A_1A_3 intersects with line CD, we have that CD intersects with A_3B_1 according to condition 2. For the same reason, line CD intersects with A_1B_3, B_2A_3 and A_2B_3. And it won't intersect with A_1B_2,A_2B_1.

Hence C, D can only form convex quadrilaterals with A_2, B_1 or A_1, B_2. Same result for other point-pair in Q.

The total number of convex quadrilaterals in this situation is then $2\binom{n-6}{2}$.

Finally the two situations above complete the calculation of $R(n)$,

$$R(n) = \begin{cases} (n-6)(7n-52)/4, & n \text{ even;} \\ (n-7)(7n-45)/4, & n \text{ odd.} \end{cases}$$

And then we have

$$t(n) = 2t(n-3) - t(n-6) + 6n - 33 + R(n) \quad (n \geq 9). \tag{4.1}$$

Furthermore, we estimate $t(n)$ for more direct result.

We have $R(n) \leq (n-7)(7n-45)/4$ whether n is an even number of an odd number.

Hence consider n mod 3, we have the following results (for $k \geq 3$).

$$t(3k) \leq \frac{k-1}{16}(21k^3 + 245k^2 + 354k - 1908); \tag{4.2}$$

$$t(3k+1) \leq \frac{k-1}{16}k(21k^2 - 7k + 2); \tag{4.3}$$

$$t(3k+2) \leq \frac{k-1}{16}(21k^3 + 21k^2 + 16k + 4). \tag{4.4}$$

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