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ANALYTIC CONTINUATION OF HOLOMORPHIC MAPS RESPECTING VARIETIES OF MINIMAL RATIONAL TANGENTS AND APPLICATIONS TO RATIONAL HOMOGENEOUS MANIFOLDS

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Abstract

In a series of works, one of the authors has developed with J.-M. Hwang a geometric theory of uniruled projective manifolds, especially those of Picard number 1, basing on the study of varieties of minimal rational tangents. A fundamental result in this theory is a principle of analytic continuation under very mild assumptions, called Cartan-Fubini extension, of biholomorphisms between connected open subsets of two Fano manifolds of Picard number 1 which preserve varieties of minimal rational tangents. In this article we develop a generalization of Cartan-Fubini extension for non-equidimensional holomorphic immersions from a connected open subset of a Fano manifold of Picard number 1 into a uniruled projective manifold, under the assumptions that the map sends varieties of minimal rational tangents onto linear sections of varieties of minimal rational tangents and that it satisfies a mild geometric condition formulated in terms of second fundamental forms on varieties of minimal rational tangents. Formerly such a result was known only in the very special case of irreducible Hermitian symmetric manifolds of rank at least two, and the proof relied on the existence of flattening coordinates, viz., Harish-Chandra coordinates, with respect to which the varieties of minimal rational tangents form a constant family. The proof of the main result, which is based on the deformation theory of rational curves, is differential-geometric in nature and is applicable to the general situation of uniruled projective manifolds without any assumption on the existence of special coordinate systems. As an application, we give a characterization of standard embeddings for certain pairs of rational homogeneous manifolds in terms of embeddings of varieties of minimal rational tangents.

1. Introduction

For a polarized uniruled projective manifold X, by a minimal rational curve we mean a free rational curve of minimal degree among such

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curves. A connected component $\mathcal{K}$ of the space of (unparametrized) minimal rational curves, which carries a natural topology, will be called a minimal rational component. $\mathcal{K}$ carries naturally the structure of a quasi-projective manifold. At a general point $x \in X$ we have correspondingly a moduli space $\mathcal{K}_x$ of minimal rational curves marked at x, and $\mathcal{K}_x$ is a projective manifold by minimality. By associating a minimal rational curve immersed at the marking at x to the tangent line of the curve at the marking, we obtain a rational map $\Phi_x : \mathcal{K}_x \to \mathbb{P}T_x(X)$ called the tangent map, and its strict transform $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is called the variety of minimal rational tangents at x. The union of $\mathcal{C}_x$ over general points $x \in X$ gives the fibered space $\pi : \mathcal{C} \to X$ of varieties of minimal rational tangents associated to $\mathcal{K}$. We will use the notations $\mathcal{C}_x(X)$ and $\mathcal{C}(X)$ when we need to emphasize that they are associated to a minimal rational component defined on X.

In a series of works of one of the authors with J.-M. Hwang, it was revealed that there is a rich geometry on uniruled projective manifolds, especially those of Picard number 1, embodied in the fibered spaces of varieties of minimal rational tangents. A prototypical example is given by the hyperquadric $\mathbf{Q}_n$ of dimension $n \geq 3$, which is equipped with a holomorphic conformal structure, and for which the variety of minimal rational tangents at any point is given by the projectivization of the cone of null vectors. The holomorphic conformal structure is an example of S-structures modeled after an irreducible Hermitian symmetric manifold S of rank ≥ 2. For the theory of S-structures there is the result of Ochiai [**Oc70**], according to which any biholomorphism between two nonempty connected open subsets of S preserving the S-structure can be analytically continued to a biholomorphic automorphism of S. Ochiai's result was proved using cohomological methods on Lie algebras, but it can be interpreted as a statement about analytic continuation of germs of holomorphic maps which preserve varieties of minimal rational tangents. With this interpretation, Hwang-Mok gave in [**HM01**] and [**HM04**] a far-reaching generalization of Ochiai's Theorem to a result of analytic continuation on Fano manifolds of Picard number 1, called Cartan-Fubini extension (which is related to the works of Fubini and of Cartan on second fundamental forms of smooth hypersurfaces in the projective space), as follows.

Theorem (Equidimensional Cartan-Fubini extension [**HM01**], [**HM04**])**.** *Let Z and X be two Fano manifolds of Picard number 1 with minimal rational components. Assume that $\mathcal{C}_z(Z)$ is positive-dimensional and is not a finite union of linear subspaces at a general point $z \in Z$. Let $f : U \to V$ be a biholomorphic map from a connected open subset $U \subset Z$ to $V \subset X$. If the differential df sends each irreducible component of $\mathcal{C}(Z)|_U$ to an irreducible component of $\mathcal{C}(X)|_V$ biholomorphically, then f extends to a biholomorphic map $F : Z \to X$.*

Cartan-Fubini extension lies at the heart of the theory of geometric structures modeled on varieties of minimal rational tangents. It was used in Hwang-Mok [**HM99a**, **HM04**] to give solutions of Lazarsfeld's Problem on finite holomorphic maps $f : G/P \to X$ from a rational homogeneous manifold of Picard number 1 onto a Fano manifold X. Cartan-Fubini extension on irreducible Hermitian symmetric manifolds S of Picard number 1, in the form of Ochiai's Theorem, was used in [**HM98**] as a first step towards proving rigidity under Kähler deformation of rational homogeneous manifolds G/P of Picard number 1 (cf. [**HM04**] and the references there). In this paper we generalize Cartan-Fubini extension to the non-equidimensional situation for a holomorphic immersion from a connected open subset $U \subset Z$ of a Fano manifold Z of Picard number 1 into a uniruled projective manifold X which respects varieties of minimal rational tangents in the sense that it sends varieties of minimal rational tangents onto linear sections of varieties of minimal rational tangents, i.e.,

$$df(\mathcal{C}_z(Z)) = df(\mathbb{P}(T_z Z)) \cap \mathcal{C}_{f(z)}(X)$$

for every $z \in U$. Our main result is as follows.

Theorem 1.1. *Let $(Z, \mathcal{H})$ and $(X, \mathcal{K})$ be two uniruled projective manifolds with minimal rational components. Assume that Z is of Picard number 1 and that $\mathcal{C}_z(Z)$ is positive-dimensional at a general point $z \in Z$. Let $f : U \to X$ be a holomorphic immersion defined on a connected open subset $U \subset Z$. If f respects varieties of minimal rational tangents and is non-degenerate with respect to $(\mathcal{K}, \mathcal{H})$, then f extends to a rational map $F : Z \to X$.*

We say that $f : U \to X$ is non-degenerate with respect to $(\mathcal{K}, \mathcal{H})$ whenever the image $f(U)$ is not contained in the bad locus of $\mathcal{K}$ and, at a general point $z \in U$ and a general smooth point $\alpha \in \widetilde{\mathcal{C}}_z(Z)$, $df(\alpha)$ is a smooth point of $\widetilde{\mathcal{C}}_{f(z)}(X)$ such that the second fundamental form σ of $\widetilde{\mathcal{C}}_{f(z)}(X) \subset T_{f(z)}(X)$ at $df(\alpha)$, restricted to the subspace $T_{df(\alpha)}(df(\widetilde{\mathcal{C}}_z(Z)))$ of $T_{df(\alpha)}(\widetilde{\mathcal{C}}_{f(z)}(X))$, has trivial kernel, i.e.,

$$\Big\{\zeta \in T_{df(\alpha)}(\widetilde{\mathcal{C}}_{f(z)}(X)) : \sigma(\zeta, \xi) = 0 \text{ for any } \xi \in T_{df(\alpha)}(df(\widetilde{\mathcal{C}}_z(Z)))\Big\} \\ = \mathbb{C}df(\alpha).$$

In the equidimensional case, the non-degenerate condition corresponds to the generical finiteness of the Gauss map of $\mathcal{C}_z(Z)$ for a general point $z \in U$, which was assumed in [**HM01**].

A first instance of non-equidimensional Cartan-Fubini extension was implicitly established in the special case of holomorphic immersions between connected open subsets of irreducible Hermitian symmetric manifolds S of rank ≥ 2, by a combination of the differential-geometric proof of Ochiai's Theorem of Mok [**Mk99**] and the proof of the equidimensional Cartan-Fubini extension result of Hwang-Mok [**HM01**] under a non-degeneracy assumption on the Gauss map. The starting point of the proof of Theorem 1.1 relies on a comparison of the tautological foliation $\mathcal{F}$ on the fibered space $\mathcal{C}(X)$ of varieties of minimal rational tangents on the ambient manifold X and the image of the tautological foliation $\mathcal{E}$ on the fibered space $\mathcal{C}(Z)$ under a holomorphic embedding $f : U \to X$, $U \subset Z$, which respects varieties of minimal rational tangents. More precisely, we compare over the image of $f(U)$ the 1-dimensional distribution $\mathcal{F}|_{f(U)}$ with the foliation $f_*\mathcal{E}$. The method of Mok [**Mk99**] in the Hermitian symmetric case relies on the existence of flattening coordinates for S-structures, viz., Harish-Chandra coordinates, with respect to which the varieties of minimal rational tangents form a constant family. We establish first of all the special case of Theorem 1.1 where the Fano manifolds Z and X of Picard number 1 are equipped with privileged $\mathcal{E}$-linearizing resp. $\mathcal{F}$-linearizing coordinate systems in the sense that the minimal rational curves are lines with respect to these coordinate systems. In terms of privileged coordinate systems the differential-geometric arguments using Euclidean geometry and second fundamental forms work out in analogy to the Hermitian symmetric case, and the basis for such a generalization is the fact that the positive part of the Grothendieck decomposition of the holomorphic tangent bundle constitutes a constant family along a minimal rational curve (which is a line with respect to a privileged coordinate system), a crucial fact which results from the deformation theory of (minimal) rational curves. While such coordinate systems exist for rational homogeneous manifolds of Picard number 1 through the use of minimal canonical embeddings into projective spaces, where minimal rational curves are mapped onto projective lines, and through the use of linear projections, both for the sake of completeness and for anticipated applications to non-homogeneous uniruled projective manifolds we establish Theorem 1.1 in full generality through the use of coordinate systems which are in some sense approximations of privileged coordinate systems along a given minimal rational curve.

While equidimensional Cartan-Fubini extension provides a fundamental result for the study of germs of open holomorphic immersions between uniruled projective manifolds of Picard number 1, we expect non-equidimensional Cartan-Fubini extension to provide a basic tool for the study of germs of non-equidimensional holomorphic immersions between such manifolds. Furthermore, we expect that Theorem 1.1 can be used as a basic tool to study complex-analytic subvarieties of Fano manifolds of Picard number 1 which are distinguished from the perspective of the theory of geometric structures modeled on varieties of minimal rational tangents. From this perspective, taking minimal rational curves to play heuristically the role of geodesics in Riemannian geometry, the class of complex subvarieties which are saturated (cf. [**Mk07**]) with respect to the adjunction of minimal rational tangents can be taken as the analogue of totally geodesic submanifolds in Riemannian geometry. In this vein Mok [**Mk07**] provides a first example of such characterization theorems where non-equidimensional Cartan-Fubini extension in the Hermitian symmetric case was applied to give a characterization of standard embeddings between complex Grassmannians of rank ≥ 2. In this article, by means of Theorem 1.1 we give a vast generalization of the latter characterization theorem covering a great variety of pairs of rational homogeneous manifolds of Picard number 1. Specifically, we consider rational homogeneous manifolds $X = G/P$ of Picard number 1 associated to long simple roots of simple Lie groups G and characterize the standard embedding of certain rational homogeneous submanifolds $Z \hookrightarrow X$ of Picard number 1 in terms of embeddings of varieties of minimal rational tangents, as follows.

Theorem 1.2. *Let $X = G/P$ be a rational homogeneous manifold associated to a long simple root and let $Z = G_0/P_0$ be a rational homogeneous manifold associated to a subdiagram of the marked Dynkin diagram of G/P. Assume that Z is not linear. If $f : U \to X$ is a holomorphic embedding from a connected open subset U of Z into X which respects varieties of minimal rational tangents for a general point $z \in U$, then f is the restriction of a standard embedding of Z into X.*

Here, the choice of a subdiagram induces naturally an embedding $\varphi : G_0/P_0 \to G/P$. By a standard embedding of G_0/P_0 into G/P we will mean the composite $g \circ \varphi$ for any automorphism g of $X = G/P$.

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2. Preservation of tautological foliations and analytic continuations

2.1. Definitions and statements. Let X be a polarized uniruled projective manifold, let $\mathcal{H}$ be a connected component of parametrized free rational curves $h : \mathbb{P}^1 \to X$ of minimal degree among such curves, and let $\mathcal{K} := \mathcal{H}/\text{Aut}(\mathbb{P}^1)$ be the associated quotient space of (unparametrized) free rational curves, which inherits the structure of a quasi-projective manifold. We call $\mathcal{K}$ a minimal rational component. At a general point $x \in X$ we have correspondingly a moduli space $\mathcal{K}_x$ of minimal rational curves marked at x, and $\mathcal{K}_x$ is a projective manifold by minimality. For a member u of $\mathcal{K}_x$, represented by $f : \mathbb{P}^1 \to X, f(0) = x$, which is an immersion at the marking at x, the tangent map Φ associates u to the tangent line $[\Phi(u)] \in \mathbb{P}T_x(X)$ at the marking, i.e., $\Phi(u) = [df(T_0(\mathbb{P}^1))]$. By Kebekus [**Ke**] any minimal rational curve passing through a general point is immersed, so that the tangent map $\Phi : \mathcal{K}_x \to \mathbb{P}T_x(X)$ is holomorphic. Its image $\mathcal{C}_x := \Phi(\mathcal{K}_x)$ is called the variety of minimal rational tangents at x, and by Hwang-Mok [**HM01**, **HM04**] the map $\Phi : \mathcal{K}_x \to \mathcal{C}_x$ is a normalization. (Originally the variety of minimal rational tangents $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ was defined in Hwang-Mok [**HM99a**] as the strict transform of the tangent map which was only known to be a rational map.) The union of $\mathcal{C}_x$ over general points $x \in X$ gives the fibered space $\pi : \mathcal{C} \to X$ of varieties of minimal rational tangents associated to $\mathcal{K}$.

Let $\rho : \mathcal{U} \to \mathcal{K}$ and $\mu : \mathcal{U} \to X$ denote the universal family morphisms. The fibers of $\rho : \mathcal{U} \to \mathcal{K}$ induce a foliation $\mathcal{F}$ on $\mathcal{C}$, called the tautological foliation on $\mathcal{C}$ associated to $\mathcal{K}$. We assume that $\mathcal{C}_x$ is irreducible, of positive dimension, and non-linear for a generic point $x \in X$. Here $\mathcal{C}_x$ is said to be linear whenever it is a finite union of linear subspaces, and non-linear otherwise. Then the tautological foliation $\mathcal{F}$ is univalent at a generic point of $\mathcal{C}$([**HM04**]).

For a general reference on the deformation theory of rational curves from an algebro-geometric perspective, the reader is referred to Kollár [**Ko**]. For surveys on the theory of geometric structures modeled on varieties of minimal rational tangents at various stages of its development, the reader may consult Hwang-Mok [**HM99b**], Hwang [**Hw01**], and Mok [**Mk08**]. The article Hwang [**Hw07**] contains discussions on rational homogeneous manifolds in the frame work of a theory of geometric structures modeled on varieties of minimal rational tangents.

Now we consider two uniruled projective manifolds $(Z, \mathcal{H}), (X, \mathcal{K})$ each equipped with a minimal rational component. Denote by $\mathcal{C}(Z)$, $\mathcal{C}(X)$ the associated fibered spaces of varieties of minimal rational tangents and denote by $\mathcal{F}(Z)$, $\mathcal{F}(X)$ the associated tautological foliations. A main ingredient in the proof of Theorem 1.1 is to prove that f sends minimal rational curves passing through U to minimal rational curves in X whenever $f : U \to X$ respects varieties of minimal rational tangents (Proposition 2.1).

For a finite-dimensional vector space V and a complex-analytic subvariety $E \subset \mathbb{P}(V)$, we denote by $\widetilde{E} \subset V - \{0\}$ the pre-image $\pi^{-1}(E)$ of the canonical projection $\pi : V - \{0\} \to \mathbb{P}(V)$. We recall the definition of the second fundamental form on $\widetilde{\mathcal{C}}_x(X) \subset T_xX$ for $x \in X$. For $\eta \in \widetilde{\mathcal{C}}_x(X)$ the second fundamental form

$$\sigma_\eta : T_\eta(\widetilde{\mathcal{C}}_x(X)) \times T_\eta(\widetilde{\mathcal{C}}_x(X)) \to T_xX/T_\eta(\widetilde{\mathcal{C}}_x(X))$$

on $\widetilde{\mathcal{C}}_x(X) \subset T_xX$ at $\eta \in \widetilde{\mathcal{C}}_x(X)$ is defined by

$$\sigma_\eta(\xi, \zeta) = \nabla_\xi \hat{\zeta} \mod T_\eta(\widetilde{\mathcal{C}}_x(X))$$

for any $\xi, \zeta \in T_\eta(\widetilde{\mathcal{C}}_x(X))$, where $\hat{\zeta}$ is a local vector field with $\hat{\zeta}(\eta) = \zeta$ and ∇ is the Euclidean flat connection on the Euclidean space T_xX. For a subspace W of $T_\eta(\widetilde{\mathcal{C}}_x(X))$, define $\operatorname{Ker}\sigma_\eta(W, \cdot\,)$ by

$$\operatorname{Ker}\sigma_\eta(W, \cdot\,) := \{\zeta \in T_\eta(\widetilde{\mathcal{C}}_x(X)) : \sigma_\eta(\zeta, \xi) = 0 \text{ for any } \xi \in W\}.$$

Since $\widetilde{\mathcal{C}}_x(X)$ is a cone, $\mathbb{C}\eta$ is contained in $\operatorname{Ker}\sigma_\eta(W, \cdot\,)$ for any subspace W of $T_\eta(\widetilde{\mathcal{C}}_x(X))$.

Definition. Let $(X, \mathcal{K})$ and $(Z, \mathcal{H})$ be two polarized uniruled projective manifolds each equipped with a minimal rational component. Let $f : U \to X$ be a holomorphic immersion defined on a connected open subset $U \subset Z$. We say that

1) f *respects varieties of minimal rational tangents* if

$$df(\mathcal{C}(Z)|_U) = df(\mathbb{P}(TZ|_U)) \cap \mathcal{C}(X)|_{f(U)}$$

and

2) f is *non-degenerate with respect to* $(\mathcal{K}, \mathcal{H})$ if
   a) $f(U)$ is not contained in the bad locus of $\mathcal{K}$
   b) for a general point $z \in U$ and a general smooth point $\alpha \in \widetilde{\mathcal{C}}_z(Z)$, $df(\alpha)$ is a smooth point of $\widetilde{\mathcal{C}}_{f(z)}(X)$ such that

$$\operatorname{Ker}\sigma_{df(\alpha)}(T_{df(\alpha)}(df(\widetilde{\mathcal{C}}_z(Z))), \cdot\,) = \mathbb{C}df(\alpha).$$

Here, the bad locus of $\mathcal{K}$ is the smallest subvariety E of X such that for any $x \in X \backslash E$, any minimal rational curve passing through x is free and a general minimal rational curve passing through x is standard.

Proposition 2.1. *Let $(X, \mathcal{K})$ and $(Z, \mathcal{H})$ be two polarized uniruled projective manifolds each equipped with a minimal rational component. Assume that $\mathcal{C}_z(Z)$ is irreducible and is positive-dimensional for a general point $z \in Z$. Let $f : U \to X$ be a holomorphic immersion defined on a connected open subset $U \subset Z$. If f respects varieties of minimal rational tangents and is non-degenerate with respect to $(\mathcal{K}, \mathcal{H})$, then f preserves the tautological foliations.*

We remark that when $\mathcal{C}_{f(z)}(X) = df(\mathcal{C}_z(Z))$, the non-degenerate condition is equivalent to the generic finiteness of the Gauss maps of $\mathcal{C}_{f(z)}(X) = df(\mathcal{C}_z(Z))$. Under this assumption Proposition 2.1 was proved in [**HM99b**] Section 3.1, and, together with analytic continuation, it implies the equidimensional Cartan-Fubini type extension theorem: if $\dim Z = \dim X$ and if $f : U \to X$ preserves varieties of minimal rational tangents, then f extends to a biholomorphism $F : Z \to X$([**HM01**]).

In our non-equidimensional case, once we have a holomorphic immersion $f : U \to X$ preserving the tautological foliations, the arguments using the method of parametrized analytic continuation in [**HM01**], Sections 2–4 work word by word, except that in our case the extension F is just a rational map.

Proposition 2.2. *Let X and Z be two polarized uniruled projective manifolds each equipped with a minimal rational component. Assume that Z is of Picard number 1. Let $f : U \to X$ be a holomorphic immersion defined on a connected open subset $U \subset Z$ which preserves the tautological foliations. Then f extends to a rational map $F : Z \to X$.*

Together with Proposition 2.1, this completes the proof of Theorem 1.1. It remains to prove Proposition 2.1, whose proof relies on Proposition 2.4 in Section 2.2.

2.2. The difference of two tautological foliations. In this section we investigate the conditions which ensure the preservation of the tautological foliations for a holomorphic immersion $f : U \to X$ defined on a connected open subset $U \subset Z$ respecting varieties of minimal rational tangents. In what follows we assume for notational simplicity that f is injective.

Consider two rank-1 subbundles $\mathcal{E}$ and $\mathcal{F}$ of the tangent bundle $T\mathcal{C}(X)$ of $\mathcal{C}(X)$ restricted on $\mathcal{R} := df(\mathcal{C}(Z)|_U) \subset \mathcal{C}(X)|_{f(U)}$. The first one $\mathcal{E}$ is defined by vectors tangent to liftings of the image $f(C)$ of germs of minimal rational curves C passing through U, and the second one $\mathcal{F} = \mathcal{F}(X)$ is defined by vectors tangent to liftings of germs of minimal rational curves in X.

The subbundles $\mathcal{E}$ and $\mathcal{F}$ are tautological in the sense that for $\eta \in \mathcal{R}_{f(z)}$, $d\pi_\eta(\mathcal{E}_\eta) = d\pi_\eta(\mathcal{F}_\eta) = \mathbb{C}\eta$, where $\pi : \mathcal{C}(X) \to X$ is the projection map. Furthermore, $\mathcal{E}$ and $\mathcal{F}$ are equal if and only if f maps germs of minimal rational curves in $\mathcal{H}$ passing through U to germs of minimal rational curves in $\mathcal{K}$. We are going to express the difference of $\mathcal{E}$ and $\mathcal{F}$ at η as the Hessian of f with respect to some coordinate systems.

Fix $\alpha \in \mathcal{C}_z(Z)$ and let C be the minimal rational curve tangent to α at z. We say that a coordinate system $(z_1, \ldots, z_m)$ around z is *adapted to* α whenever there is a parametrization $(z_1(s), \ldots, z_m(s))$ of C with linear coordinates $z_i(s)$. Let $(u_1, \ldots, u_m)$ be the fiber coordinate system on TZ induced by $(z_1, \ldots, z_m)$. Then the lifting $\widehat{C}$ of C to $\mathbb{P}(TZ)$ has constant coordinates in $(u_1, \ldots, u_m)$, and thus the tangent vector to $\widehat{C}$ at α has zero coefficients in $\left\{\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_m}\right\}$ when expressed in terms of the standard basis $\left\{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_m}, \frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_m}\right\}$. For $\eta = df(\alpha)$, take a coordinate system $(x_1, \ldots, x_n)$ around $x = f(z)$ adapted to η, too.

Define the Hessian $d^2 f : T_z Z \times T_z Z \to T_{f(z)} X$ of f by

$$d^2 f(\alpha, \beta) = \sum_{i,j,k} \alpha^i \beta^j \frac{\partial^2 f^k}{\partial z_i \partial z_j} \frac{\partial}{\partial x_k}|_{f(z)} \in T_{f(z)} X$$

for $\alpha = \sum \alpha^i \frac{\partial}{\partial z_i}$ and $\beta = \sum \beta^i \frac{\partial}{\partial z_i}$. We remark that the definition of the Hessian $d^2 f$ of f depends on the choice of the coordinate systems $(z_1, \dots, z_m)$ and $(x_1, \dots, x_n)$ around z and $f(z)$.

Lemma 2.3. *Let $z \in Z$ and $\alpha \in \mathcal{C}_z(Z)$. Let $(z_1, \dots, z_m)$ be a coordinate system on U adapted to α and let $(x_1, \dots, x_n)$ be a coordinate system on V adapted to $\eta = df(\alpha)$. Then the Hessian $d^2 f(\alpha, \alpha)$ is contained in $T_{df(\alpha)}(\mathcal{C}_{f(z)}(X))$, and the Hessian $d^2 f(\alpha,\alpha) \equiv 0 \mod \eta$ if and only if $\mathcal{E}_{[\eta]} = \mathcal{F}_{[\eta]}$.*

Proof. Let $\eta^\sharp \in \mathcal{E}_{[\eta]}$ and $\eta^\flat \in \mathcal{F}_{[\eta]}$ be such that $d\pi(\eta^\sharp) = d\pi(\eta^\flat) = \eta$ for the projection $\pi : \mathcal{C}(X) \to X$. Then $\eta^\sharp - \eta^\flat \in T_{[\eta]}(\mathcal{C}_{f(z)}(X)) \subset T_{f(x)}X/\mathbb{C}\eta$.

The coordinate systems $(z_1, \dots, z_m)$ and $(x_1, \dots, x_n)$ induce the fiber coordinate systems $(u_1, \dots, u_m)$ and $(v_1, \dots, v_n)$ on $TZ|_U$ and $TX|_V$. With respect to the coordinate systems $(z_1, \dots, z_m)$ and $(x_1, \dots, x_n)$, the map f is given by

$$(f^1(z_1, \dots, z_m), \dots, f^n(z_1, \dots, z_m)).$$

Let $(z_1(s), \dots, z_m(s))$ be a parametrization of C with $\left(\frac{dz_1}{ds}, \dots, \frac{dz_m}{ds}\right)$ $|_{s=0} = (\alpha^1, \dots, \alpha^m)$ such that $z_i(s)$ are linear for all $i = 1, \dots, m$. Then the image $f(C)$ is parametrized by

$$(f^1(z_1(s), \dots, z_m(s)), \dots, f^n(z_1(s), \dots, z_m(s)))$$

and the lifting of $f(C)$ is parametrized by

$$\Bigg(f^1(z_1(s), \dots, z_m(s)), \dots, f^n(z_1(s), \dots, z_m(s)), \\ \sum_j \frac{df^1}{dz_j}\frac{dz_j}{ds}, \dots, \sum_j \frac{df^n}{dz_j}\frac{dz_j}{ds} \Bigg).$$

Thus the vector $\eta^\sharp$ tangent to the lifting of $f(C)$ at $\eta = df(\alpha)$ is given by $\sum_{j,k} \alpha^j \frac{df^k}{dz_j} \frac{\partial}{\partial x_k}|_\eta + \sum_{i,j,k} \alpha^i \alpha^j \frac{\partial^2 f^k}{\partial z_i \partial z_j} \frac{\partial}{\partial v_k}|_\eta$ because the $z_i(s)$ are linear functions. But $\eta^\flat$ is $\sum_{j,k} \alpha^j \frac{df^k}{dz_j} \frac{\partial}{\partial x_k}|_\eta$ because $(x_1, \dots, x_n)$ is adapted to η. Thus $\eta^\sharp - \eta^\flat$, after being canonically identified as a tangent vector in $T_{f(x)}X$, is equal to the Hessian $d^2 f(\alpha, \alpha)$ of f with respect to the coordinate systems $(z_1, \dots, z_m)$ and $(x_1, \dots, x_n)$. q.e.d.

Proposition 2.4. *Let $(X, \mathcal{K})$ and $(Z, \mathcal{H})$ be two uniruled projective manifolds each equipped with a minimal rational component. Assume that $\mathcal{C}_z(Z)$ is irreducible and is of positive dimension for a general point $z \in Z$. Let $f : U \to X$ be a holomorphic immersion defined on a connected open subset $U \subset Z$ respecting varieties of minimal rational tangents. Then, there exist a coordinate system around z adapted to α*

and a coordinate system around $x = f(z)$ adapted to $df(\alpha)$ with respect to which the Hessian $d^2 f(\alpha,\alpha)$ satisfies

$$\sigma_{df(\alpha)}\left(d^2 f(\alpha,\alpha),\xi\right) = 0$$

for any $\alpha \in \widetilde{\mathcal{C}}_x(X)$ and for any $\xi \in T_{df(\alpha)}(df(\widetilde{\mathcal{C}}_z(Z))$.

For the moment we assume the validity of Proposition 2.4 and proceed to complete the proof of Proposition 2.1.

Proof of Proposition 2.1. By Lemma 2.3 the Hessian $d^2 f(\alpha,\alpha)$ is a tangent vector in $T_{df(\alpha)}(\widetilde{\mathcal{C}}_x(X))$, and by Proposition 2.4 there are, a coordinate system around z adapted to α and a coordinate system around $x = f(z)$ adapted to $df(\alpha)$ such that $d^2 f(\alpha,\alpha)$ is contained in the vector subspace $\mathrm{Ker}\,\sigma_{df(\alpha)}(T_{df(\alpha)}(df(\widetilde{\mathcal{C}}_z(Z))),\,\cdot\,) \subset T_{df_d}(\widetilde{\mathcal{C}}_z(X))$. By the assumption that f is non-degenerate, $\mathrm{Ker}\,\sigma_{df(\alpha)}(T_{df(\alpha)}(df(\widetilde{\mathcal{C}}_z(Z))),\,\cdot\,) = \mathbb{C}df(\alpha)$ and thus $d^2 f(\alpha,\alpha) \equiv 0 \mod df(\alpha)$. By Lemma 2.3, the two subbundles $\mathcal{E}$ and $\mathcal{F}$ are equal, i.e., f preserves the tautological foliations. q.e.d.

In the remaining sections we will prove Proposition 2.4 by constructing special coordinate systems adapted to the tautological foliations, which will be given in Section 2.3 for the special case where X and Z are uniruled by projective lines, and in Section 2.4 for the general case.

2.3. Privileged system of $\mathcal{F}$-linearizing coordinates. Let X be a uniruled projective manifold and let $\mathcal{K}$ be a minimal rational component. Denote by $\mathcal{C}$ the fibered space of varieties of minimal rational tangents associated to $\mathcal{K}$ and denote by $\mathcal{F}$ the tautological foliation on $\mathcal{C}$ associated to $\mathcal{K}$. Let E be the bad locus of $\mathcal{K}$ and let $W = X\backslash E$.

Definition. Let $U \subset W$ be a chart with a coordinate system $(z_1,\ldots,z_n)$. We say that $(z_1,\ldots,z_n)$ is *$\mathcal{F}$-linearizing* if $C \cap U$ is an open subset of an affine line with respect to $(z_1,\ldots,z_n)$ for any minimal rational curve C such that $C \cap U \neq \emptyset$.

Let C be a minimal rational curve passing through $U \subset W$ and denote by $\alpha(x)$ a non-zero element of $T_x(C)$. We have $T_{[\alpha(x)]}(\mathcal{C}_x) = P_{\alpha(x)}/\mathbb{C}\alpha(x)$, where $P_{\alpha(x)} \subset T_x(X)$ is the fiber $(\mathcal{O}(2)\oplus[\mathcal{O}(1)]^p)_x$ with respect to a Grothendieck decomposition $TX|_C \cong \mathcal{O}(2)\oplus[\mathcal{O}(1)]^p \oplus \mathcal{O}^q$ over C. The positive part $P_{\alpha(x)}$ is independent of the choice of Grothendieck decomposition of $TX|_C$.

Definition. Let $(z_1,\ldots,z_n)$ be an $\mathcal{F}$-linearizing system of coordinates. We say that $\pi : \mathcal{C}|_U \to U$ is *tangentially constant along $\mathcal{F}$ with respect to* $(z_1,\ldots,z_n)$ if along each minimal rational curve C, $T_{[\alpha(x)]}(\mathcal{C}_x) = P_{\alpha(x)}/\mathbb{C}\alpha(x)$ with $P_{\alpha(x)} = P_C \subset \mathbb{C}^n$ for some complex vector subspace P_C of $\mathbb{C}^n$ depending only on C, when one identifies TU with $\mathbb{C}^n \times U$ by the standard trivialization with respect to $(z_1, \ldots, z_n)$. We will say for short that $(z_1, \ldots, z_n)$ is a *privileged system of $\mathcal{F}$-linearizing coordinates* on U, meaning that $(z_1, \ldots, z_n)$ is $\mathcal{F}$-linearizing and $\pi : \mathcal{C}|_U \to U$ is tangentially constant along $\mathcal{F}$ with respect to $(z_1, \ldots, z_n)$.

There are many examples of uniruled manifolds having privileged systems of $\mathcal{F}$-linearizing coordinates. An irreducible Hermitian symmetric space of compact type equipped with Harish-Chandra coordinates gives such an example. As we will see in the proof of Proposition 2.4, a privileged system of $\mathcal{F}$-linearizing coordinates will play a similar role as Harish-Chandra coordinates when we deal with a family of minimal rational curves, with an error term which we will prove to be irrelevant.

Proposition 2.5. *Let $X \subset \mathbb{P}^N$ be a projective submanifold of $\mathbb{P}^N$ uniruled by projective lines. Let $\mathcal{K}$ be a minimal rational component consisting of projective lines on X. Assume that for a general point x of X, the subvariety $\mathcal{K}_x$ of $\mathcal{K}$ consisting of projective lines passing through x is irreducible. Then, at any general point $x \in X$, there exists an open neighborhood U of x and a privileged system of $\mathcal{F}$-linearizing coordinates $(z_1, \ldots, z_n)$ on U.*

Proof. Let E be the bad locus of $\mathcal{K}$ and let $W = X \backslash E$. Let $x \in W$, let C be a projective line on X passing through x, and write $T_x(C) = \mathbb{C}\alpha$. The minimal rational curve C is standard, i.e., $TX|_C \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q$.

Let $P \subset TX|_C$ be the subbundle $\mathcal{O}(2) \oplus [\mathcal{O}(1)]^p$, which is well-defined independent of the choice of Grothendieck decomposition. We have also $T\mathbb{P}^N|_C \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^{N-1}$. Consider now P as a subbundle of $T_{\mathbb{P}^N}|_C$. Write $\mathbb{P}^N = \mathbb{P}(\mathbb{C}^{N+1})$. The projective line C is the projectivization $\mathbb{P}(E_o)$ for some 2-dimensional complex vector subspace $E_o \subset \mathbb{C}^{N+1}$. We assert that there exists a $(p+2)$-dimensional complex vector subspace $E \subset \mathbb{C}^{N+1}$ such that $E \supset E_o$ and such that $T\mathbb{P}(E)|_C$ agrees with P on C. Geometrically, our assertion means that there exists a $(p+1)$-dimensional projective subspace of $\mathbb{P}^N$ tangent to X along the projective line C.

To prove our assertion, pick any point $x \in C, x = [\mathbb{C}\eta]$, and let $E \subset \mathbb{C}^N$ be the $(p+2)$-dimensional vector subspace such that $E/\mathbb{C}\eta = P_x$. We are going to show that our assertion is valid with this choice of E. For a complex submanifold A of a complex manifold B, we denote by $N_{A|B}$ the holomorphic normal bundle of A in B. Both P/TC and $T\mathbb{P}(E)|_C/T_C = N_{C|\mathbb{P}(E)}$ are holomorphic subbundles of $T\mathbb{P}^N|_C/T_C = N_{C|\mathbb{P}^N}$. We have thus two injective holomorphic bundle homomorphisms μ, ν: $[\mathcal{O}(1)]^p \to [\mathcal{O}(1)]^{N-1}$ over C with identical images at the point $x \in C$. Obviously, μ and ν can be normalized to agree at the point

x. Since $Hom([\mathcal{O}(1)]^p, [\mathcal{O}(1)]^{N-1}) \cong \mathcal{O}^{p(N-1)}$, any global holomorphic section of the former is determined by its value at one point, implying that $P = T\mathbb{P}(E)|_C$, as asserted.

Let now $(z_1, \ldots, z_N)$ be a system of inhomogeneous coordinates for $\mathbb{P}^N$ at the point $x \in W$, chosen so that $(z_1, \ldots, z_n)$ serves as a system of holomorphic coordinates for X on some connected open neighborhood U of x in W. Any minimal rational curve C belonging to K and passing through U lies on an affine line with respect to the inhomogeneous coordinates $(z_1, \ldots, z_N)$ on an affine part of $\mathbb{P}^N$, and hence with respect to the coordinates $(z_1, \ldots, z_n)$ on $U \subset W \subset X$. It follows that $(z_1, \ldots, z_n)$ is $\mathcal{F}$-linearizing. Given any projective line C on X passing through U, $P \cong T_{\mathbb{P}(E)}\big|_C$ implies that there exists a $(p+1)$-dimensional complex vector subspace A of $\mathbb{C}^n$ such that A is parallel to $P_\alpha \subset T_x(X)$ with respect to the $\mathcal{F}$-linearizing coordinates $(z_1, \ldots, z_n)$, for any $x \in C \cap U$ and for any non-zero vector α tangent to C at x. In other words, $\pi : \mathcal{C}|_U \to U$ is tangentially constant along $\mathcal{F}$, and $(z_1, \ldots, z_n)$ serves as a privileged system of holomorphic coordinates on U with respect to $\mathcal{F}$. The proof of Proposition 2.5 is complete. q.e.d.

Example. The following are examples of projective submanifolds of $\mathbb{P}^N$ uniruled by projective lines; by Proposition 2.5, each of them has a privileged system of $\mathcal{F}$-linearizing coordinates where $\mathcal{F}$ is associated to the family of projective lines lying on it:

(1) Rational homogeneous manifolds G/P of Picard number 1 in the first canonical embedding,
(2) Smooth hypersurfaces of $\mathbb{P}^N$ of degree $1 < d \leq N-1$
(3) More generally, smooth complete intersections of dimension ≥ 2 and of degree $(d_1, \ldots, d_\ell)$ with $1 < d_1 + \cdots + d_\ell \leq N-1$

Before considering general privileged systems of $\mathcal{F}$-linearizing coordinates, we give a proof of Proposition 2.4 in the case when $X = Z$ is a Hermitian symmetric space so that it has Harish-Chandra coordinates, the simplest privileged system of $\mathcal{F}$-linearizing coordinates. For the sake of completeness we recall the proof of Proposition 2.2.1 of [**Mk99**] after simplifying it, in order to explain what we need to modify in the general case.

Proof of Proposition 2.4 *in the case where* $X = Z$ *is a Hermitian symmetric space.* Choose Harish-Chandra coordinates, which flatten the fibered spaces $\widetilde{\mathcal{C}}(Z)$ over U and $\widetilde{\mathcal{C}}(X)$ over $V = f(U)$. Denote by ∇ the Euclidean connection on U defined by Harish-Chandra coordinates. We may assume that $f(z) = z$ and $df_z : T_z Z \to T_{f(z)} X$ is the identity map.

For any $\alpha, \beta \in \widetilde{\mathcal{C}}_z(Z)$, consider a minimal rational curve C through z and tangent to α and consider the constant section $\tilde{\beta}$ of $\widetilde{\mathcal{C}}(Z)|_C$ with $\tilde{\beta}(z) = \beta$. Since $\tilde{\beta}$ is parallel, $d^2 f(\alpha, \beta) = \nabla_{df(\alpha)} df(\tilde{\beta})$ and thus $d^2 f(\alpha, \beta) \in P_\beta = T_\beta(\widetilde{\mathcal{C}}_z(Z))$. Similarly, considering the constant section

$\tilde{\alpha}$ of $\widetilde{\mathcal{C}}(Z)|_{C'}$ along a minimal rational curve C' tangent to β, we have $d^2f(\beta,\alpha) = \nabla_{df(\beta)}df(\tilde{\alpha}) \in P_\alpha = T_\alpha(\widetilde{\mathcal{C}}_z(Z))$. By the symmetry of the Hessian,

$$d^2f(\alpha,\beta) \in P_\alpha \cap P_\beta.$$

Now for $\xi \in T_\alpha(\widetilde{\mathcal{C}}_z(Z))$, put $\beta = \alpha(t) = \alpha + t\xi + t^2\zeta_t, |t| < \epsilon$. Then

$$d^2f(\alpha,\alpha(t)) \in P_\alpha \cap P_{\alpha(t)} \qquad (*)$$

and thus we get $d^2f(\alpha,\xi) \in P_\alpha$. Since $\alpha(t)$ is a curve in $\widetilde{\mathcal{C}}_z(Z)$ through α tangent to ξ and $d^2f(\alpha(t),\alpha(t)) \in P_{\alpha(t)} = T_{\alpha(t)}(\widetilde{\mathcal{C}}_z(Z))$,

$$\sigma_\alpha(d^2f(\alpha,\alpha),\xi) = \frac{d}{dt}|_{t=0}d^2f(\alpha(t),\alpha(t)) \mod P_\alpha.$$

But $\frac{d}{dt}|_{t=0}d^2f(\alpha(t),\alpha(t)) = 2d^2f(\alpha,\xi) \in P_\alpha$ and thus $\sigma_\alpha(d^2f(\alpha,\alpha),\xi) = 0$. q.e.d.

We proceed now to give a proof of Proposition 2.4 in the case of projective manifolds Z, X having privileged systems of $\mathcal{F}$-linearizing coordinates. From the argument above, it is clear that the conclusion $\sigma(d^2f(\alpha,\alpha),\xi) = 0$ remains true if in place of $(*)$ we have the weaker statement

$$\begin{cases} pr\Big(d^2f\big(\alpha,\alpha(t)\big), P_\alpha^\perp\Big) = O(t^2); \\ pr\Big(d^2f\big(\alpha,\alpha(t)\big), P_{\alpha(t)}^\perp\Big) = O(t^2). \end{cases} \qquad (**)$$

In fact, according to the proof in the case of Hermitian symmetric spaces, we need only the first half of $(**)$ and the second half of $(**)$ is redundant. The proofs for the two statements are the same in the case we consider in this section because in a privileged system of $\mathcal{F}$-linearizing coordinates, α is indistinguishable from $\alpha(t)$ and the arguments are symmetric in α and $\alpha(t)$. But it will be different in a more general case which will be dealt with next section.

In the proof of $(*)$ we make use of constant sections of $\mathcal{C}|_U$ resp. $\mathcal{C}|_V$ over U resp. V with respect to Harish-Chandra coordinates. In the case where there is a privileged system of $\mathcal{F}$-linearizing coordinates, we have the following replacement for constant sections. Given a minimal rational curve C, $x \in C$ with $T_x(C) = \mathbb{C}\alpha$, we can find a section of $\widetilde{\mathcal{C}}$ extending $\beta \in \widetilde{\mathcal{C}}_x$ along C whose deviation from being a constant section at x is a function vanishing to the order ≥ 2 at $\beta = \alpha$. More precisely, we have:

Lemma 2.6. *Let $(z_1,\ldots,z_n)$ be a privileged system of $\mathcal{F}$-linearizing coordinates on U. Let $x \in U$ and D_x be a non-empty relatively compact open subset of $\widetilde{\mathcal{C}}_x - \{0\}$. Then there exists a constant K for which the following holds true: Let $\alpha, \beta \in D_x$ and C be a minimal rational* *curve passing through x with $T_x(C) = \mathbb{C}\alpha$. Then there is a holomorphic section $\tilde{\beta}$ of $\widetilde{\mathcal{C}}$ over $L := C \cap U$ such that $\tilde{\beta}(x) = \beta$ and such that*

$$|\nabla_\alpha \tilde{\beta}(x)| \leq K|\alpha - \beta|^2,$$

where ∇ stands for the Euclidean flat connection with respect to $(z_1, \ldots, z_n)$, and norms are measured with respect to the standard Euclidean metric.

Proof. For any point $y \in U$, $T_y(U)$ will sometimes be identified with $T_x(U)$ by the standard trivialization $TU \cong U \times \mathbb{C}^n$ with respect to $(z_1, \ldots, z_n)$. Parametrize C by a complex linear map γ such that $\gamma(0) = x$ and such that $\gamma'(s) = \alpha$ for the complex parameter s. When it is necessary to identify the base point $\gamma(s)$ over which the tangent vector α lies, we will write $\alpha(\gamma(s))$. By the definition of privileged systems of $\mathcal{F}$-linearizing coordinates, $P_\alpha(\gamma(s))$ are independent of s. We may assume that α corresponds to $\frac{\partial}{\partial z_1}$, and that P_α is the linear span of $\frac{\partial}{\partial z_j}, 1 \leq j \leq p+1$.

We denote by B^k a Euclidean ball of $\mathbb{C}^k$. For the proof of Lemma 2.6, without loss of generality we may take D_x to be the intersection of $\tilde{\mathcal{C}}_x - \{0\}$ with the Cartesian product of two Euclidean balls $B^{p+1} \times B^q$ of sufficiently small radii, so that the restriction of the projection maps $\rho_x : D'_x \to \mathbb{C}^{p+1}$ into the first $p+1$ factors is an open immersion on some open subset D'_x of $\tilde{\mathcal{C}}_x - \{0\}$ which contains the closure $\overline{D}_x$. The same is valid when x is replaced by $\gamma(s)$ for s sufficiently small, say, $|s| < \epsilon$. Define now $\varphi_s : D_x \to D_{\gamma(s)}$ by $\varphi_s(\beta) := \rho^{-1}_{\gamma(s)}(\rho_x(\beta))$.

Define $\varphi : \triangle(\epsilon) \times D_x \to \tilde{\mathcal{C}}|_{\gamma(\triangle(\epsilon))}$ by $\varphi(s, \beta) = (\gamma(s), \varphi_s(\beta))$. The tangent vector α corresponds to $(1, 0, \ldots, 0)$ in the coordinates $(z_1, \ldots, z_n)$. Write $e = \rho_x(\alpha) = (1, 0, \ldots, 0) \in \mathbb{C}^{p+1}$. Write $\beta - \alpha = (\xi, \zeta)$, where $\xi := (\beta_1 - 1, \beta_2, \ldots, \beta_{p+1}) = \rho_x(\beta - \alpha) \in \mathbb{C}^{p+1}$ and $\zeta := (\beta_{p+2}, \ldots, \beta_n)$. Then

$$\varphi_s(\beta) = \rho^{-1}_{\gamma(s)}(e + \xi) \in D_{\gamma(s)}.$$

Since $\frac{\partial}{\partial x_j}, 1 \leq j \leq p+1$ spans $P_\alpha(\gamma(s))$ for every s, we conclude that

$$|\rho^{-1}_{\gamma(s)}(e + \xi)| \leq C|\xi|^2$$

for some constant C. As $\varphi'_s(\beta) = \frac{\partial}{\partial s}(\rho^{-1}_{\gamma(s)}(e+\xi))$, from Cauchy estimates we conclude that

$$|\nabla_\alpha \tilde{\beta}(x)| = \left|\varphi'_s(\beta)|_{s=0}\right| \leq K|\xi|^2$$

for some constant K. Now write $\tilde{\beta}(\gamma(s))$ for $\varphi(s, \beta)$ to finish the proof, observing that K can be chosen to be independent of α as α runs over a small open set. q.e.d.

Proof of Proposition 2.4 *in the case where X resp. Z has privileged systems of $\mathcal{F}$-linearizing resp. $\mathcal{H}$-linearizing coordinates.* Let $(z_1, \ldots, z_m)$ be a privileged system of $\mathcal{F}$-linearizing coordinates on $U \subset Z$ and let $(x_1, \ldots, x_n)$ be a privileged system of $\mathcal{H}$-linearizing coordinates on $V \subset X$. It now remains to establish the estimate $(**)$.

For a tangent vector μ of type (1,0) at $z \in U$ we identify $T_\mu(TU)$ with $T_z(U) \oplus T_z(U)$ using the privileged $\mathcal{F}$-linearizing coordinates $(z_1, \ldots, z_m)$. Similarly for $x = f(z)$ and a tangent vector μ' of type (1,0) at x we have $T_{\mu'}(TV) \cong T_x(V) \oplus T_x(V)$. We maintain furthermore the normalization $df_z = id|_{T_z Z}$ with respect to the basis $\left\{ \frac{\partial}{\partial z_1}|_z, \ldots, \frac{\partial}{\partial z_m}|_z \right\}$ and $\left\{ \frac{\partial}{\partial x_1}|_x, \ldots, \frac{\partial}{\partial x_n}|_x \right\}$ and identify $T_z(U)$ with a vector subspace of $T_x(V)$.

Let C be a minimal rational curve in Z tangent to α at z and let C' be the minimal rational curve in X tangent to $\alpha = df(\alpha)$ at $x = f(z)$. Apply Lemma 2.6 simultaneously to U at z and V at x. Then there exist a constant K, and tangent vectors $\eta \in T_z(U)$, $\eta' \in T_x(V)$ such that

$$(\alpha, \eta) \in T_\beta(\widetilde{\mathcal{C}}(Z)|_U), \quad (\alpha, \eta') \in T_\beta(\widetilde{\mathcal{C}}(X)|_V)\ ; |\eta|, |\eta'| \leq K|\alpha - \beta|^2.$$

More precisely, writing ∇ for the Euclidean flat connection on U with respect to $(z_1, \ldots, z_m)$ and writing ∇' for the Euclidean flat connection on V with respect to $(x_1, \ldots, x_n)$, for the section $\tilde{\beta}$ of $\widetilde{\mathcal{C}}(Z)$ over $C \cap U$ and for the section $\tilde{\beta}'$ of $\widetilde{\mathcal{C}}(X)$ over $C' \cap U$, given in Lemma 2.6, we have

$$\nabla_\alpha \tilde{\beta} = \eta \text{ and } \nabla'_\alpha \tilde{\beta}' = \eta'.$$

From

$$\nabla'_{df(\alpha)}(df(\tilde{\beta})) = d^2 f(\alpha, \beta) + \eta,$$

it follows that

$$\big(\alpha, d^2 f(\alpha, \beta) + \eta\big) \in T_\beta(\widetilde{\mathcal{C}}(X)|_V).$$

Comparing with $(\alpha, \eta') \in T_\beta(\widetilde{\mathcal{C}}(X)|_V)$, we conclude that the difference projects to zero on U and hence gives a vertical vector tangent to $\widetilde{\mathcal{C}}_x(X)$ at β. Hence,

$$d^2 f(\alpha, \beta) + (\eta - \eta') \in T_\beta(\widetilde{\mathcal{C}}_x(X)).$$

In other words,

$$pr\Big(d^2 f\big(\alpha, \beta\big), P_\beta^\perp\Big) = O(|\alpha - \beta|^2).$$

By the same arguments we can take a section $\tilde{\alpha}$ of $\widetilde{\mathcal{C}}$ along the minimal rational curve tangent to β. Thus the same remains true if $P_\beta^\perp$ is replaced by $P_\alpha^\perp$. Fixing α and letting $\beta = \alpha(t)$ vary over a smooth local curve on $\widetilde{\mathcal{C}}_z(Z)$ (equivalently $\widetilde{\mathcal{C}}_x(X)$), we conclude that

$$\begin{cases} pr\Big(d^2 f\big(\alpha, \alpha(t)\big), P_\alpha^\perp\Big) = O(t^2); \\ pr\Big(d^2 f\big(\alpha, \alpha(t)\big), P_{\alpha(t)}^\perp\Big) = O(t^2). \end{cases} \qquad (**)$$

As we explained before the statement of Lemma 2.6, this completes the proof of Proposition 2.4 in the case where X resp. Z has privileged systems of $\mathcal{F}$-linearizing resp. $\mathcal{H}$-linearizing coordinates. q.e.d.

2.4. The general case. In general we do not know whether a privileged system of $\mathcal{F}$-linearizing coordinates exists. In the previous section we give a proof of Proposition 2.4 in the case where X resp. Z has privileged systems of $\mathcal{F}$-linearizing resp. $\mathcal{H}$-linearizing coordinates. As its proof shows, it suffices to prove the following Lemma, which is a generalization of Lemma 2.6.

Lemma 2.7. *Let X be a uniruled projective manifold with a minimal rational component $\mathcal{K}$. Let E be the bad locus of $\mathcal{K}$. For a point $x \in X \backslash E$ and for a standard minimal rational curve C with $\alpha \in T_x(C)$ a smooth point of $\widetilde{\mathcal{C}}_x$, there is a coordinate system $(z_1, \dots, z_n)$ on $U \subset X \backslash E$ adapted to α satisfying the following properties: For a relatively compact neighborhood D_x of α in $\widetilde{\mathcal{C}}_x \backslash \{0\}$, there is a constant K such that for each $\beta \in D_x$*

1. *there is a holomorphic section $\tilde{\beta}$ of $\widetilde{\mathcal{C}}$ over $C \cap U$ such that $\tilde{\beta}(x) = \beta$ and $|\nabla_\alpha \tilde{\beta}(x)| \leq K|\alpha - \beta|^2$, and*
2. *there is a holomorphic section $\tilde{\alpha}$ of $\widetilde{\mathcal{C}}$ over $C_\beta \cap U$, where C_β is the minimal rational curve tangent to β at x, such that $\tilde{\alpha}(x) = \alpha$ and $|\nabla_\beta \tilde{\alpha}(x)| \leq K|\alpha - \beta|^2$,*

where ∇ stands for the Euclidean flat connection with respect to $(z_1, \dots, z_n)$, and norms are measured with respect to the standard Euclidean metric.

The section $\tilde{\beta}$ in (1) can be used to show that the second half of the estimate $(**)$ is valid and the section $\tilde{\alpha}$ in (2) can be used to show that the first half of the estimate $(**)$ is valid: just take a local curve $\alpha(t)$ in $\widetilde{\mathcal{C}}_x$ with $\alpha(0) = \alpha$ and apply Lemma 2.7 to $\beta = \alpha(t)$ and to $df(\beta) = df(\alpha(t))$ as in the proof of Proposition 2.4 in the case where X resp. Z has privileged systems of $\mathcal{F}$-linearizing resp. $\mathcal{H}$-linearizing coordinates. The same arguments work verbatim in the general case once we have proved Lemma 2.7.

To construct a coordinate system $(z_1, \dots, z_n)$ around x having the properties as in Lemma 2.7, we will first choose a coordinate system on a neighborhood of x in the locus of the family $\mathcal{K}_y$ for $y \in C \backslash \{x\}$, which is analogous to polar coordinates on the Euclidean plane. More precisely, denote by $\rho : \mathcal{U} \to \mathcal{K}$, $\mu : \mathcal{U} \to X$ the universal family associated to $\mathcal{K}$. For a subset $\mathcal{D}$ of $\mathcal{K}$, the image $\mu(\rho^{-1}(\mathcal{D}))$ is the subset of X wiped out by minimal rational curves belonging to $\mathcal{D}$.

Lemma 2.8. *Let $x \in X$ be a general point and assume that the variety $\mathcal{C}_x$ of minimal rational tangents at x is of dimension $p \geq 1$. Let C be a standard minimal rational curve passing through x and let $y \in C$ be a smooth point different from x. Denote by $\kappa \in \mathcal{K}_y$ the element corresponding to the minimal rational curve C with a marking at y. Let $\mathcal{D} \subset \mathcal{K}_y$ be a sufficiently small neighborhood of κ so that the tangent map $\Phi|_{\mathcal{D}} : \mathcal{D} \to \mathbb{P}T_y(X)$ is an embedding. Denote by $w \in \rho^{-1}(\kappa)$ the point corresponding to x. Let $\mathcal{W} \subset \rho^{-1}(\mathcal{D})$ be a sufficiently small open neighborhood of w in $\rho^{-1}(\mathcal{D})$ and define $\Sigma := \mu(\mathcal{W})$ (which is a set containing x wiped out by open subsets of minimal rational curves belonging to $\mathcal{D}$). Then*

(1) *Σ is a locally closed complex submanifold of dimension $p+1$ and*
(2) *the tangent space $T_x\Sigma$ of Σ at x can be identified with the tangent space of $\widetilde{\mathcal{C}}_x$ at $\alpha \in T_x(C)$.*

Proof. (1) By construction $\Sigma := \mu(\mathcal{W})$ is a locally closed complex submanifold of dimension $p+1$.

(2) Choosing a smaller neighborhood $\mathcal{D}_y$ of κ if necessary, we may assume that the universal $\mathbb{P}^1$-bundle $\rho^{-1}(\mathcal{D}_y)$ over $\mathcal{D}_y$ is holomorphically trivial. Without loss of generality, suppose $x \in C$ corresponds to $0 \in \mathbb{P}^1$, and $y \in C$ corresponds to ∞. Let z_1 be the standard coordinate on $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ and let $(z_2, \ldots, z_{p+1})$ be a holomorphic coordinate system on $\mathcal{D}_y$ at κ. (This system is analogous to polar coordinates (r, θ) on the Euclidean plane, where z_1 plays the role of r, and $(z_2, \ldots, z_{p+1})$ plays the role of θ.) We are going to make use of $(z_1, \ldots, z_{p+1})$ as a holomorphic coordinate system of the germ of Σ at x by regarding the evaluation map $\mu : \rho^{-1}(\mathcal{D}_y) \simeq \mathcal{D}_y \times \mathbb{P}^1 \to X$ as a chart. Then the tangent space $T_x\Sigma$ of Σ at x is generated by $\left\{ \frac{\partial}{\partial z_1}|_x, \ldots, \frac{\partial}{\partial z_{p+1}}|_x \right\}$.

On the other hand, the tangent space to $\mathcal{K}_x$ at the point in $\mathcal{K}_x$ corresponding to the minimal rational curve C marked at x is $H^0(C, N_{C|X} \otimes \mathfrak{m}_x)$, where $N_{C|X}$ is the normal bundle of C in X and $\mathfrak{m}_x$ is the maximal ideal sheaf of x on C. Any section $\sigma \in H^0(C, N_{C|X} \otimes \mathfrak{m}_x)$ can be lifted to a section $\widetilde{\sigma}$ of $TX|_C$ vanishing to the order 1 at x. Write $\widetilde{\sigma}(z_1) = z_1\widetilde{\sigma}_1(z_1)$. Then the differential $d\Phi$ of the tangent map $\Phi : \mathcal{K}_x \to \mathbb{P}(T_xX)$ sends σ to $\widetilde{\sigma}_1(x)$ mod $T_x(C)$, where we identify $T_\alpha(T_xX)$ with T_xX in a canonical way for any nonzero $\alpha \in T_x(C)$. Now that $\mathcal{C}_x$ is the image of the tangent map $\Phi : \mathcal{K}_x \to \mathbb{P}(T_xX)$, these vectors $\widetilde{\sigma}_1(x) \in T_x(X)/T_x(C)$ constitute the tangent space $T_{[\alpha]}(\mathcal{C}_x)$ at $[\alpha] = [T_x(C)]$.

Assume that the choice of coordinates $(z_1, \ldots, z_{p+1})$ is such that $\frac{\partial}{\partial z_1}$ extends to a holomorphic vector field on C vanishing to the order 2 at

y, and such that $\frac{\partial}{\partial z_2}, \dots, \frac{\partial}{\partial z_{p+1}}$ each extends to a holomorphic section of TX over C vanishing to the order 1 at y. Then $z_1\frac{\partial}{\partial z_2}, \dots, z_1\frac{\partial}{\partial z_{p+1}}$ extend to holomorphic sections of TX over C, which span $H^0(C, N_{C|X} \otimes \mathfrak{m}_x)$. Thus $T_\alpha(\widetilde{\mathcal{C}}_x)$ agrees with the vector space spanned by $\left\{\frac{\partial}{\partial z_1}|_x, \dots, \frac{\partial}{\partial z_{p+1}}|_x\right\}$ after we identify $T_\alpha(T_xX)$ with T_xX in a canonical way. Therefore, $T_x\Sigma$ can be identified with $T_\alpha(\widetilde{\mathcal{C}}_x)$. q.e.d.

Proof of Lemma 2.7 Let $(z_1, \dots, z_{p+1})$ be the coordinate system of Σ around x given in the proof of Lemma 2.8, and complete $(z_1, \dots, z_{p+1})$ in an arbitrary way to a holomorphic coordinate system $(z_1, \dots, z_n)$ for X at the point x. In terms of the "polar coordinates" $(z_1, \dots, z_{p+1})$ the positive part of the Grothendieck decomposition of TX over C agrees with the vector space spanned by $\{\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_{p+1}}\}$ around x. This is precisely what was needed to prove Lemma 2.6, which is the same statement as (1).

We note that at this point, unlike the case where there exists a privileged system of $\mathcal{F}$-linearizing coordinates, there is no symmetry between α and $\beta = \alpha(t)$ in the argument here. We were verifying some estimates on $d^2 f(\alpha, \beta)$ with respect to a coordinate system chosen to be adapted to a general point x on a *fixed* minimal rational curve C. Thus, along C the vector $\beta = \alpha(t)$ can be translated within $\widetilde{\mathcal{C}}|_C$ so that the holomorphic section $\widetilde{\beta}$ defined on a fixed neighborhood U of x in C is almost constant, with an error of the order $O(|t|^2)$ which is uniform on U. To get the statement (2) we have to invert the roles of α and $\beta = \alpha(t)$.

Let $\xi \in T_\alpha(\widetilde{\mathcal{C}}_x)$. Consider ξ as a vector $\sum_i \xi_i \frac{\partial}{\partial z_i}|_x$ in T_xX. Let $f_0 : \mathbb{P}^1 \to X$ be the parametrization of C given by $f_0(z_1) = (z_1, 0, \dots, 0)$ for $z_1 \in \mathbb{C}$. After considering $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_{p+1}}$ as sections of f_0^*TX, we will use the same symbol ξ to denote the section of f_0^*TX with constant coefficients ξ_i with respect to $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_{p+1}}$. Then $\hat{\xi}(z_1) = z_1\xi(z_1)$ for $z_1 \in \mathbb{C}$ extends to a section $\hat{\xi} \in H^0(\mathbb{P}_1, f_0^*TX \otimes \mathfrak{m}_0)$. Since $H^0(\mathbb{P}^1, f_0^*TX \otimes \mathfrak{m}_0)$ is the tangent space to the space $Hol((\mathbb{P}^1, 0), (X, x))$ of holomorphic maps $\mathbb{P}^1 \to X$ sending 0 to x and $H^1(\mathbb{P}^1, f_0^*TX \otimes \mathfrak{m}_0)$ is zero, $f_0 : \mathbb{P}^1 \to X$ can be extended to a 1-parameter family of curves $f_t : \mathbb{P}^1 \to X, t \in \Delta(\epsilon)$, with $f_t(0) = x$ and $\frac{d}{dt}|_{t=0} f_t = \hat{\xi}$.

Write $f : \Delta(\epsilon) \times \mathbb{P}^1 \to X$ for the map defined by $f(t, s) = f_t(s)$. Then for $s \in \mathbb{C}$,

$$\frac{\partial}{\partial t} f(t, s)\bigg|_{t=0} = s\xi.$$

Write

$$\frac{\partial}{\partial t} f(t, s) =: s\xi(t, s); \quad \frac{\partial}{\partial s} f(t, s) =: \alpha(t, s).$$

Here $\xi(0, s) = \xi$ and $\alpha(0, s) = \alpha$. We may take $\alpha(t) = \alpha(t, 0)$.

Now for any $t \in \Delta(\epsilon)$, $\xi(t,s)$ is tangent to the germ of complex submanifold Σ' wiped out by the family $\{f_t\}$ at $f(t,s)$. By Lemma 2.8, we have an identification

$$T_{f(t,s)}\Sigma' \simeq T_{\alpha(t,s)}\widetilde{\mathcal{C}}_{f(t,s)},$$

so that we may regard $\xi(t,s)$ as a vector tangent to $\widetilde{\mathcal{C}}_{f(t,s)}$. Thus the tangent vector α at $x = f(t,0)$ can be translated within $\widetilde{\mathcal{C}}|_{C_t}$ to give $\varphi(t,s) \in \widetilde{\mathcal{C}}_{f(t,s)}$ such that

$$\varphi(t,s) = \alpha(t,s) - t\xi(t,s) + O\big(|t|^2\big).$$

From the commutativity of second derivatives, it follows that

$$\begin{aligned}
\frac{\partial}{\partial t}\alpha(t,s) &= \frac{\partial}{\partial t}\frac{\partial}{\partial s}f(t,s) \\
&= \frac{\partial}{\partial s}\frac{\partial}{\partial t}f(t,s) \\
&= \frac{\partial}{\partial s}\left(s\xi(t,s)\right) \\
&= \xi(t,s) + s\frac{\partial}{\partial s}\xi(t,s).
\end{aligned}$$

Since $\xi(0,s) = \xi$ for all s, we have $\alpha(t,s) = \alpha + t\xi + O\big(|t|^2\big)$ and thus

$$\begin{aligned}
\varphi(t,s) &= (\alpha + t\xi) - t\xi(t,s) + O\big(|t|^2\big) \\
&= \alpha + t(\xi - \xi(t,s)) + O\big(|t|^2\big) \\
&= \alpha + O\big(|t|^2\big).
\end{aligned}$$

By the Cauchy estimates we have

$$\left|\frac{\partial}{\partial s}\varphi(t,s)\right| = O\big(|t|^2\big),$$

from which the statement (2) follows. q.e.d.

3. Characterization of the standard embedding between homogeneous manifolds

3.1. Homogeneous manifold associated to a subdiagram of the marked Dynkin diagram. Let G be a complex simple Lie group. Let Φ be the set of all roots of G with respect to a Cartan subalgebra $\mathfrak{h}$ and let Δ be a simple root system of G. To a subset Γ of Δ we associate a parabolic subgroup P of G whose Lie algebra is

$$\mathfrak{h} + \sum_{\alpha \in \mathbb{Z}(\Delta \setminus \Gamma)} \mathfrak{g}_\alpha + \sum_{\alpha \in \mathbb{Z}_- \Gamma} \mathfrak{g}_\alpha.$$

In particular, the whole set Δ corresponds to a Borel subgroup B of G.

The Dynkin diagram $\mathcal{D}(G)$ of G is the graph consisting of nodes and edges such that each node corresponds to a simple root in Δ and two nodes are connected by an edge if and only if the corresponding simple roots are not orthogonal. We will identify nodes with the corresponding simple roots. We call $(\mathcal{D}(G), \Gamma)$ *the marked Dynkin diagram* of the rational homogeneous manifold G/P.

A subdiagram $\mathcal{D}_0$ of $\mathcal{D}(G)$ is the Dynkin diagram $\mathcal{D}(G_0)$ of a semisimple Lie subgroup G_0 of G. When $\mathcal{D}(G_0)$ contains Γ, the homogeneous space X_0 of G_0 by the parabolic subgroup P_0 associated to Γ is called the *homogeneous manifold associated to the subdiagram* $(\mathcal{D}(G_0), \Gamma)$ of $(\mathcal{D}(G), \Gamma)$.

Assume that Γ consists of one simple root γ. Consider the first canonical embedding of $X = G/P$ into $\mathbb{P}^N$. Then we have a canonical choice of a minimal rational component $\mathcal{K}(X)$, i.e., the irreducible family of lines in $\mathbb{P}^N$ which are contained in X. Similarly we have a canonical choice of a minimal rational component $\mathcal{K}(Z)$ of $Z = G_0/P_0$. Since the ample generater of the Picard group of Z is the restriction of the ample generator of the Picard group of X, lines in $\mathcal{K}(Z)$ are lines in $\mathcal{K}(X)$ which are contained in Z. With this canonical choice of a minimal rational component, the variety of minimal rational tangents is given as follows.

Proposition 3.1 ([**HM02**])**.** *Let $X = G/P$ be a rational homogeneous manifold associated to a long simple root γ and let $Z = G_0/P_0$ be a homogenous manifold associated to a subdiagram of the marked Dynkin diagram of G/P. Let L be the semisimple part of P and let Υ be the set of simple roots which are adjacent to γ in the Dynkin diagram of G. Then:*

(1) *The variety $\mathcal{A}$ of minimal rational tangents of X at the base point is the homogeneous manifold L/R of L by the parabolic subgroup R associated to Υ.*
(2) *The variety $\mathcal{B}$ of minimal rational tangents of Z at the same base point is the homogeneous manifold associated to the subdiagram $(\mathcal{D}(G_0) \cap \mathcal{D}(L), \Upsilon)$ of the marked Dynkin diagram $(\mathcal{D}(L), \Upsilon)$ of $\mathcal{A}$.*

More concretely, the variety $\mathcal{A} = \mathcal{C}_x(X)$ of minimal rational tangents of X at $x \in X$ associated to $\mathcal{K}(X)$ is one of the following form ([**HM02**], p. 176):

I. $\mathcal{A} \subset \mathbb{P}(V)$, an irreducible Hermitian symmetric space of compact type in the first canonical embedding
II. $\mathbb{P}(E_1) \times \mathbb{P}(E_2) \subset \mathbb{P}(E_1 \otimes E_2)$, the Segre embedding of the product of two projective spaces $\mathbb{P}(E_1)$ and $\mathbb{P}(E_2)$
III. $\mathbb{P}(E) \subset \mathbb{P}(S^2E)$, the Veronese embedding of a projective space $\mathbb{P}(E)$
IV. $\mathbb{P}(E_1) \times \mathcal{A}_2 \subset \mathbb{P}(E_1 \otimes E_2)$, the Segre embedding of the product of a projective space $\mathbb{P}(E_1)$ and an irreducible Hermitian symmetric

space of compact type, $\mathcal{A}_2 \subset \mathbb{P}(E_2)$, in the first canonical embedding

V. $\mathbb{P}_a \times \mathbb{P}_b \times \mathbb{P}_c$, $\mathbb{P}_1 \times \nu(\mathbb{P}_2)$

The variety $\mathcal{B} = \mathcal{C}_z(Z)$ of minimal rational tangents of Z at $z \in Z$ associated to $\mathcal{K}(Z)$ is one of the following form:

I. $\mathcal{B} \subset \mathbb{P}(W)$, an irreducible Hermitian symmetric space of compact type in the first canonical embedding, where $\mathbb{P}(W) \subset \mathbb{P}(V)$ is a subspace and $\mathcal{B} = \mathcal{A} \cap \mathbb{P}(W)$

II. $\mathbb{P}(F_1) \times \mathbb{P}(F_2) \subset \mathbb{P}(F_1 \otimes F_2)$, the Segre embedding the product of two projective spaces $\mathbb{P}(F_1)$ and $\mathbb{P}(F_2)$, where F_i is a subspace of E_i for $i = 1, 2$

III. $\mathbb{P}(F) \subset \mathbb{P}(S^2F)$, the Veronese embedding of a projective space $\mathbb{P}(F)$, where F is a subspace of E

IV. $\mathbb{P}(F_1) \times \mathcal{B}_2 \subset \mathbb{P}(F_1 \otimes F_2)$, the Segre embedding of the product of a projective space $\mathbb{P}(F_1)$ and an irreducible Hermitian symmetric space of compact type, $\mathcal{B}_2 \subset \mathbb{P}(F_2)$, in the first canonical embedding, where F_i is a subspace of E_i for $i = 1, 2$ and $\mathcal{B}_2 = \mathcal{A}_2 \cap \mathbb{P}(F_2)$

V. $\mathbb{P}_{a'} \times \mathbb{P}_{b'} \times \mathbb{P}_{c'}$, $\mathbb{P}_1 \times \nu(\mathbb{P}_1)$, $pt \times \nu(\mathbb{P}_2)$

3.2. Transport of varieties of minimal rational tangents. Theorem 1.2 in the case where X is a Grassmannian of rank ≥ 2 is proved [**Mk07**] Section 3, applying a simple version of Theorem 1.1 [**Mk99**]. A main idea in the proof is parallel transport of varieties of minimal rational tangents along minimal rational curves, based on the deformation theory minimal rational curves (Lemma 2.8).

Let X be a uniruled manifold with a minimal rational component. A subvariety Z of X is said to be *rationally saturated* whenever (1) $\mathbb{P}(T_zZ) \cap \mathcal{C}_z(X) \neq \emptyset$ for a smooth point $z \in Z$ and (2) for every smooth point $z \in Z$ and for every minimal rational curve C on X passing through z, C must lie on Z whenever C is tangent to Z at z. Then the family of minimal rational curves contained in Z can be considered as a minimal rational component of Z, with respect to which the variety $\mathcal{C}_z(Z)$ of minimal rational tangents of Z at $z \in Z$ is equal to $\mathbb{P}(T_zZ) \cap \mathcal{C}_z(X)$.

Let Z_1, Z_2 be rationally saturated subvarieties of X. If $\mathcal{C}_z(Z_1) = \mathcal{C}_z(Z_2)$ at an intersection point $z \in Z_1 \cap Z_2$, then by Lemma 2.8, varieties of minimal rational tangents of Z_1 and Z_2 are tangent along a minimal rational curve, i.e., for any minimal rational curve C passing through z and for a generic point $y \in C$, $\mathcal{C}_y(Z_1)$ is tangent to $\mathcal{C}_y(Z_2)$ at $[T_yC]$.

In certain circumstances, such as the cases which we will consider in the proof of Theorem 1.2, this tangency implies the equality of $\mathcal{C}_y(Z_1)$ and $\mathcal{C}_y(Z_2)$, eventually leading to an identification of Z_1 and Z_2. This can be considered as an analogue of the parallel transport along a geodesic in Riemannian geometry.

Proof of Theorem 1.2. Let $X = G/P$ be a rational homogeneous manifold associated to a long simple root and let $Z = G_0/P_0$ be a rational homogenous manifold associated to a subdiagram of the marked Dynkin diagram of G/P. Let $f : U \to X$ be a holomorphic embedding from a connected open subset U of Z into X, which respects varieties of minimal rational tangents, i.e., for which $df(\mathcal{C}(Z)|_U) = df\mathbb{P}(TZ)|_U \cap \mathcal{C}(X)|_{f(U)}$ holds true.

Proposition 3.2. *Let $X = G/P$ and $Z = G_0/P_0$ and $f : U \to X$ be as in Theorem* 1.2. *Assume that Z is not linear. Then f is nondegenerate.*

By Proposition 3.2, we can apply Theorem 1.1 to get a rational extension $F : Z \to X$ of f. In the middle of the proof of Theorem 1.1, we also proved that F sends minimal rational curves to minimal rational curves (Proposition 2.1). In our case, F sends lines in Z to lines in X. So $F(Z)$ is rationally saturated.

Furthermore, since Z is of Picard number 1 and is uniruled, there is a sequence of irreducible varieties $\mathcal{U}^0 = \{z_0\} \subset \mathcal{U}^1 \subset \cdots \subset \mathcal{U}^k$ with $\dim \mathcal{U}^k = \dim Z$ such that a general point in $\mathcal{U}^{i+1}$ can be connected to a point in $\mathcal{U}^i$ by a line in Z [**HM98**], Section 4.3; [**Mk07**], Section 3. By the fact that F sends lines to lines, a general point in $\mathcal{V}^{i+1} := F(\mathcal{U}^{i+1})$ can be connected to a point in $\mathcal{V}^i := F(\mathcal{U}^i)$ by a line in X.

We may assume that $z_0 \in U$ and $f(z_0) = z_0$ up to the action of G. We may assume further that $df(\mathcal{C}_{z_0}(Z)) = \mathcal{C}_{z_0}(Z)$ up to the action of G by (1) of the following proposition.

Proposition 3.3. *Let $X = G/P$ be a rational homogeneous manifold associated to a long simple root and let $Z = G_0/P_0$ be a rational homogenous manifold associated to a subdiagram of the marked Dynkin diagram of G/P. Assume that Z is not linear. Let $f : U \to X$ be a holomorphic embedding from an open subset U of Z into X respecting varieties of minimal rational tangents. Then:*

(1) *For any $z \in U$, there is $g = g(z) \in G$ such that $f(z) \in gZ$ and $df_z(\mathcal{C}_z(Z)) = \mathcal{C}_{f(z)}(gZ)$.*
(2) *If there is $g_1 \in G$ such that $f(z) \in g_1Z$ and $df_z(\mathcal{C}_z(Z))$ is tangent to $\mathcal{C}_{f(z)}(g_1Z)$ at an intersection point, then we have $df_z(\mathcal{C}_z(Z)) = \mathcal{C}_{f(z)}(g_1Z)$.*

We continue the proof of Theorem 1.2. From the fact $dF(\mathcal{C}_{z_0}(Z)) = \mathcal{C}_{z_0}(Z)$, it follows that $F(\Sigma) = \Sigma$, where Σ is the locus of the family of lines in Z passing through z_0. The locus Σ is equal to $\mathcal{U}^1$ and thus $F(\mathcal{U}^1)$ is contained in Z. We will use Lemma 2.8 and induction to prove that $F(\mathcal{U}^k)$ is contained in Z. Then $F(Z) = Z$ and F is the identity map up to the action of G.

Let C be a line in Z passing through z_0 and let $y \neq x \in C$. Then by Lemma 2.8, $\mathcal{C}_y(F(Z))$ is tangent to $\mathcal{C}_y(Z)$ at $[T_yC] \in \mathcal{C}_y(F(Z)) \cap \mathcal{C}_y(Z)$.

But by Proposition 3.3 (2) we have $\mathcal{C}_y(F(Z)) = \mathcal{C}_y(Z)$. Thus $F(Z)$ and Z share the locus of the family of lines in Z passing through y, too. Hence $F(\mathcal{U}^2)$ is contained in Z. By induction, we have the desired result. q.e.d.

It remains to prove Proposition 3.2 and Proposition 3.3, which will be given in Section 3.3.

3.3. Projective geometry of varieties of minimal rational tangents of G/P and of G_0/P_0. In this section we will prove Proposition 3.2 and Proposition 3.3, which can be rephrased as statements about varieties of minimal rational tangents of X and Z as follows.

Proposition 3.4. *Let $X = G/P$ be a rational homogeneous manifold associated to a long simple root and let $Z = G_0/P_0$ be a rational homogenous manifold associated to a subdiagram of the marked Dynkin diagram of G/P. Let $\mathcal{A} := \mathcal{C}_x(X) \subset \mathbb{P}(V)$ and $\mathcal{B} := \mathcal{C}_x(Z) \subset \mathbb{P}(W)$ be the varieties of minimal rational tangents at a common base point x of X and Z, where $V := T_xX$ and $W := T_xZ$.*

(1) *The pair $(\mathcal{A}, \mathcal{B})$ is non-degenerate in the sense that*
$$\operatorname{Ker} \sigma_\beta(T_\beta\widetilde{\mathcal{B}}, \cdot\,) = \mathbb{C}\beta$$
for any $\beta \in \widetilde{\mathcal{B}}$, where $\sigma_\beta : T_\beta\widetilde{\mathcal{A}} \times T_\beta\widetilde{\mathcal{A}} \to V/T_\beta(\widetilde{\mathcal{A}})$ is the second fundamental form of the affine cone $\widetilde{\mathcal{A}}$ in V at β.

(2) *If $h \in \mathrm{Aut}_0(\mathcal{A})$ is such that $h\mathcal{B}$ and $\mathcal{B}$ are tangent at a point of intersection, then $h\mathcal{B}$ is equal to $\mathcal{B}$.*

(3) *If $\mathcal{B}' = \mathcal{C} \cap \mathbb{P}(W')$ is another linear section such that $(\mathcal{B} \subset \mathbb{P}(W))$ is projectively equivalent to $(\mathcal{B}' \subset \mathbb{P}(W'))$, then there is $h \in \mathrm{Aut}(\mathcal{A})$ such that $\mathcal{B}' = h\mathcal{B}$.*

Proof of Proposition 3.2. Proposition 3.4 (3) and (1). q.e.d.

Proof of Proposition 3.3. (1) Proposition 3.4 (2).
(2) Proposition 3.4 (3) and (2). q.e.d.

One can check the properties (1) through (3) in Proposition 3.4 one by one for the varieties in the list of the varieties of minimal rational tangents in Section 3.1. However, when X is associated to a long root, the varieties of minimal rational tangents are homogeneous manifolds and we can use Lie group theory and representation theory to prove (1) through (2) uniformly. Then (3) follows by inductive arguments.

Let $X = G/P$ be a rational homogeneous manifold associated to a long simple root γ and let $Z = G_0/P_0$ be a rational homogenous manifold associated to a subdiagram of the marked Dynkin diagram of G/P. Let L be the semisimple part of P and let Υ be the set of simple roots which are adjacent to γ in the Dynkin diagram of G, i.e., the set of simple roots which are not orthogonal to γ with respect to the Killing form.

By Proposition 3.1, the variety $\mathcal{A}$ of minimal rational tangents of X at the base point is the homogeneous manifold L/R of L by the parabolic subgroup R associated to Υ and the variety $\mathcal{B}$ of minimal rational tangents of Z at the same base point is the homogeneous manifold associated to the subdiagram $(\mathcal{D}(G_0) \cap \mathcal{D}(L), \Upsilon)$ of the marked Dynkin diagram $(\mathcal{D}(L), \Upsilon)$ of $\mathcal{A}$.

Denote by

$$\sigma_\alpha : T_\alpha\widetilde{\mathcal{A}} \times T_\alpha\widetilde{\mathcal{A}} \to V/T_\alpha\widetilde{\mathcal{A}}$$

the second fundamental form of the affine cone $\widetilde{\mathcal{A}}$ at $\alpha \in \widetilde{\mathcal{A}}$. We say that the pair $(\mathcal{A}, \mathcal{B})$ is non-degenerate if the kernel of the second fundamental form $\sigma_\beta(T_\beta\widetilde{\mathcal{B}}, \cdot)$ restricted to $T_\beta\widetilde{\mathcal{B}}$ is trivial at each point $\beta \in \widetilde{\mathcal{B}}$.

Lemma 3.5. *Let $X = G/P$ be a rational homogeneous manifold associated to a long simple root γ. Let $x \in X$ be an arbitrary point and denote by $\mathcal{A} \subset \mathbb{P}(T_xX)$ the variety of minimal rational tangents of X at the base point x. Let $\sigma : T_\alpha\mathcal{A} \times T_\alpha\mathcal{A} \to T_xX/T_\alpha\mathcal{A}$ be the second fundamental form of $\mathcal{A} \subset \mathbb{P}(T_xX)$ at $\alpha = [E_\gamma] \in \mathcal{A}$. Then, for $E_\nu, E_\eta \in T_\alpha\mathcal{A}$, we have*

$$\sigma(E_\nu, E_\eta) = \begin{cases} E_{\nu+\eta-\gamma} & \text{if } \nu+\eta-\gamma \text{ is a root,} \\ 0 & \text{otherwise .} \end{cases}$$

Here, E_ν denotes a root vector of root ν for a root ν.

Proof. This follows from the description of $\mathcal{A}$ as the closure of a vector valued cubic polynomial in [**HM99b**], Section (4.2). q.e.d.

Proof of Proposition 3.4 (1) By Lemma 3.5 it suffices to show that for any $E_\nu \in T_\alpha\mathcal{A}$, there is $E_\eta \in T_\alpha\mathcal{B}$ such that $\nu + \eta - \gamma$ is a root.

The tangent space T_xX is linearly spanned by E_ν where ν is a root with positive coefficient in γ. Let L be the semisimple part of P and let $L_1, L_2, \dots$ be its simple components. L has at most three simple components. Then $\mathcal{A} = L.E_\gamma$ and the tangent space $T_\alpha\mathcal{A}$ at $\alpha = [E_\gamma]$ is linearly spanned by root vectors E_ν of roots $\nu = \gamma + \theta$ for some roots θ of L_i for some i. The root vectors in T_xZ and in $T_\alpha\mathcal{B}$ can be expressed in the same way. But in this case, we consider only the roots in the subgroup G_0, i.e., the roots whose coefficients with respect to simple roots outside the subdiagram $\mathcal{D}(G_0)$ are zero.

Suppose that γ is not an end of the Dynkin diagram of G_0. Then γ is not an end of the Dynkin diagram of G. So the semisimple part L_0 of P_0 (and thus the semisimple part L of P) is the product of two or three simple Lie groups. Assume that L is the product of two simple Lie groups L_1, L_2. The proof for the case where it is the product of three simple Lie groups will be the same. Then $\mathcal{B}$ and $\mathcal{A}$ are Segre embeddings of two rational homogeneous spaces. Let $E_\nu \in T_\alpha\mathcal{A}$. Then $\nu = \gamma + \theta$ for some root θ of, say, L_1. Let θ' be the simple root of L_2 adjacent to γ in the Dynkin diagram of G. Then $\gamma+\theta'$ is a root and $E_{\gamma+\theta'}$ is contained in $T_\alpha\mathcal{B}$. Furthermore, $\nu+(\gamma+\theta')-\gamma=\nu+\theta'=\gamma+\theta+\theta'$ is a root because $\langle\gamma+\theta,\theta'\rangle=\langle\gamma,\theta'\rangle<0$, where $\langle\ ,\ \rangle$ is the Killing form of $\mathfrak{g}$. Hence, $\sigma(E_\nu,E_{\gamma+\theta'})\neq 0$ by Lemma 3.5.

Suppose that γ is an end of the Dynkin diagram of G_0. By the same argument as above, it suffices to consider the case where γ is also an end of the Dynkin diagram of G. Then the classification in Section 3.1 shows that $\mathcal{A}$ is either a Hermitian symmetric space in the first canonical embedding or a projective space $\mathbb{P}^a$ in the Veronese embedding. Therefore the semisimple part L of P acts on $T_\alpha\mathcal{A}$ irreducibly.

The space of root vectors $E_\nu\in T_\alpha\mathcal{A}$ of roots $\nu=\nu_1+\kappa_1$ where ν_1 is a root with $E_{\nu_1}\in T_\alpha\mathcal{B}$ and κ_1 is either zero or a root is a subspace of $T_\alpha\mathcal{A}$ which is invariant under the action of L. By the irreducibility of $T_\alpha\mathcal{A}$, any root ν with $E_\nu\in T_\alpha\mathcal{A}$ is either in $T_\alpha\mathcal{B}$ or the sum $\nu_1+\kappa_1$ of two roots ν_1 and κ_1 with $E_{\nu_1}\in T_\alpha\mathcal{B}$. By the same reasoning, any root η with $E_\eta\in T_\alpha\mathcal{B}$ is of the form $\eta=\gamma+\sigma$ for some root σ.

If $E_\nu\in T_\alpha\mathcal{B}$, then since $\mathcal{B}$ is non-linear and smooth, there is $E_\eta\in T_\alpha\mathcal{B}$ with $\sigma(E_\nu,E_\eta)\neq 0$. Assume that $\nu=\nu_1+\kappa_1$ with $E_{\nu_1}\in T_\alpha\mathcal{B}$. Then κ_1 has a positive coefficient in some simple root outside the Dynkin diagram of $\mathcal{B}$ and thus $\nu_1-\kappa_1$ is not a root. Since $\mathcal{B}$ is a non-linear smooth subvariety of T_xZ, the kernel of the Gauss map of $\mathcal{B}\subset T_xZ$ is zero. Hence, there is $E_\eta\in T_\alpha\mathcal{B}$ such that $\sigma(E_{\nu_1},E_\eta)\neq 0$, i.e., $\nu_1+\eta-\gamma=:\sigma$ is a root. Note that $\eta-\gamma$ is a root. We will show that $\sigma_1:=\nu_1+\kappa_1+\eta-\gamma$ is a root.

By the Jacobi identity,

$$[[E_{\nu_1},E_\kappa],[\overline{E}_\sigma,\overline{E}_{\kappa_1}]] \;=\; [\overline{E}_\sigma,[[E_{\nu_1},E_{\kappa_1}],\overline{E}_{\kappa_1}]]-[\overline{E}_{\kappa_1},[[E_{\nu_1},E_{\kappa_1}],\overline{E}_\sigma]].$$

By the construction, $-\eta+\gamma+\kappa_1$ has negative coefficients in some roots of the subdiagram and positive coefficients in some roots outside of the subdiagram. Thus $\nu_1+\kappa_1-\sigma=-\eta+\gamma+\kappa_1$ is not a root. Hence, $[\overline{E}_{\kappa_1},[[E_{\nu_1},E_{\kappa_1}],\overline{E}_\sigma]]$ is zero.

Since $\nu_1-\kappa_1$ is not a root and $\eta-\gamma=\sigma-\nu_1$ is a root, it follows that $[\overline{E}_\sigma,[[E_{\nu_1},E_{\kappa_1}],\ \overline{E}_{\kappa_1}]]=[\overline{E}_\sigma,[E_{\nu_1},[E_{\kappa_1},\overline{E}_{\kappa_1}]]=c[\overline{E}_\sigma,E_{\nu_1}]\neq 0$ for some constant $c\neq 0$. Therefore $[[E_{\nu_1},E_{\kappa_1}],[\overline{E}_\sigma,\overline{E}_{\kappa_1}]]$ is not equal to zero. Thus $\sigma_1=\sigma+\kappa_1$ is a root. This implies that $\sigma(E_\nu,E_\eta)\neq 0$ by Lemma 3.5. q.e.d.

Proposition 3.4 (2) is a special case of the following general result about the action of L on the family of homogeneous submanifolds of L/R.

Proposition 3.6. *Let $\mathcal{A}=L/R$ be a homogeneous manifold associated to Υ and let $\mathcal{B}=L_0/R_0$ be a homogenous manifold associated to a subdiagram of the marked Dynkin diagram of L/R. If $g\mathcal{B}$ and $\mathcal{B}$*

are tangent to each other at an intersection point for some $g \in L$, then $g\mathcal{B} = \mathcal{B}$.

Proof. Let Λ be the set of simple roots in $\mathcal{D}(L)\backslash\mathcal{D}(L_0)$ which are adjacent to $\mathcal{D}(L_0)$. Then the parabolic subgroup Q of L associated to Λ is the isotropy group of the L-action on the Chow variety of L/R at $[\mathcal{B}]$, $\mathcal{B}$ being considered as a point in the Chow variety of L/R. Hence, the L-orbit $L[\mathcal{B}]$ is isomorphic to L/Q [**Tits**].

Let $x \in \mathcal{A}$ be the point at which the isotropy group of L is R. Suppose that $g\mathcal{B}$ intersects $\mathcal{B}$ at x for some $g \in L$. Then there is h in the reductive part R^{ss} of R such that $g\mathcal{B} = h\mathcal{B}$ and thus $\{g\mathcal{B} : g \in L, x \in g\mathcal{B}\}$ is the homogeneous space of R^{ss} by the parabolic subgroups of R^{ss} associated to Λ.

For a subset Δ' of Δ, define $n_{\Delta'} : \Phi \to \mathbb{Z}$ from the root system Φ of L to $\mathbb{Z}$ by

$$n_{\Delta'}(\alpha) = \sum_{\alpha_j \in \Delta'} n_j$$

where $\alpha = \sum_j n_j \alpha_j$. Then the tangent space $T_x\mathcal{A}$ of $\mathcal{A}$ at x is $\sum_{n_\Upsilon(\alpha)>0} \mathfrak{g}_\alpha$ and the tangent space $T_x\mathcal{B}$ of $\mathcal{B}$ at x is $\sum_{n_\Upsilon(\alpha)>0,\, n_\Lambda(\alpha)=0} \mathfrak{g}_\alpha$ and the Lie algebra of R^{ss} is given by $\mathfrak{h} + \sum_{n_\Upsilon(\alpha)=0} \mathfrak{g}_\alpha$. Thus, the isotropy of the action of R^{ss} at the subspace $T_x\mathcal{B}$ of $T_x\mathcal{A}$ is the parabolic group of R^{ss} associated to Λ. In other words, the subvariety $R^{ss}[T_x\mathcal{B}]$ of the Grassmannian variety $Gr(k, T_x\mathcal{A})$ of k-dimensional subspaces of $T_x\mathcal{A}$ is the homogenous variety of R^{ss} by the parabolic subgroup associated to Λ, which is isomorphic to $\{g\mathcal{B} : g \in L, x \in g\mathcal{B}\}$. q.e.d.

Proof of Proposition 3.4 (3) We will divide the cases according to the types I through V of the varieties of minimal rational tangents $\mathcal{A}, \mathcal{B}$ in Section 3.1 and we will use induction.

Case 1. If X is a Grassmannian, then the variety of minimal rational tangents is of type II and Proposition 3.4 (3) follows from [**Mk07**], Lemma 2.

Case 2. If X is a Lagrangian Grassmannian, then the variety of minimal rational tangents is of type III. Let $\mathcal{A} = \nu(\mathbb{P}(E)) \subset \mathbb{P}(S^2E)$ and let $\mathcal{B} = \nu(\mathbb{P}(F)) \subset \mathbb{P}(S^2F)$ for some subspace F of E, where $\nu : \mathbb{P}(E) \to \mathbb{P}(S^2(E))$ is the Veronese embedding. Assume that $\mathcal{B}' = \mathcal{A} \cap \mathbb{P}(W')$ is a linear section of $\mathcal{A}$ by a subspace $\mathbb{P}(W')$ of $\mathbb{P}(S^2E)$ such that $(\mathcal{B} \subset \mathbb{P}(W))$ is projectively equivalent to $(\mathcal{B}' \subset \mathbb{P}(W'))$ via $\lambda : S^2F \to S^2E$. Then $\lambda : S^2F \to S^2E$ is an injective complex linear map which sends the set $\widetilde{\mathcal{B}} = \{\alpha \circ \alpha : \alpha \in F\}$ into the set $\widetilde{\mathcal{A}} = \{\alpha \circ \alpha : \alpha \in E\}$.

Let n be the dimension of E and let m be the dimension of F. Take a basis $\{e_1, \ldots, e_n\}$ of E such that $\{e_1, \ldots, e_m\}$ is a basis of F. For each $1 \le i \le m$, let $\eta_i \in E$ be such that $\lambda(e_i \circ e_i) = \eta_i \circ \eta_i$. Then $\lambda(S^2F)$ is a subspace of S^2E, which is generated by $\lambda(e_i \circ e_j)$, where $1 \leq i, j \leq m$. We will show that $\lambda(S^2F)$ is generated by $\eta_i \circ \eta_j$, where $1 \leq i, j \leq m$.

Fix $1 \leq i, j \leq m$. Let F_0 be the subspace of E generated by e_i and e_j and let E_0 be the subspace of E generated by η_i and η_j. Then $\lambda(\nu(\mathbb{P}(F_0)))$ and $\nu(\mathbb{P}(E_0))$ are conics in $\nu(\mathbb{P}(E))$ intersecting at two points. Thus they are equal, which implies that $\lambda(S^2(F_0)) = S^2E_0$. Thus $\eta_i \circ \eta_j$, $1 \leq i, j \leq m$ is contained in $\lambda(S^2F)$. Since $\eta_i \circ \eta_j$, $1 \leq i, j \leq m$ are linearly independent, they form a basis of $\lambda(S^2F)$. Take a linear map $\eta : E \to E$ which sends F to the subspace of E generated by $\{\eta_1, \dots, \eta_m\}$. Then λ is equal to $S^2\eta : S^2F \to S^2E$.

Case 3. If X is neither a Grassmannian nor a Lagrangian Grassmannian, the variety of minimal rational tangents is of type I, IV, or V. In this case, we remark that the variety of minimal rational tangents can be a Grassmannian or a Lagrangian Grassmannian (see [**HM02**], p.176). Note that the proof of Proposition 3.4 (3) in the case where X is either a Grassmannian or a Lagrangian Grassmannian (Case 1 and Case 2) completes the proof of Theorem 1.2 in these cases.

We start with the proof of Proposition 3.4 (3) in the case where the variety of minimal rational tangents is of type I.

Proposition 3.7. *Let $X \subset \mathbb{P}(V)$ be an irreducible Hermitian symmetric space G/P of compact type in the first canonical embedding and let $Z \subset \mathbb{P}(W)$ be an irreducible Hermitian symmetric space G_0/P_0 of compact type in the first canonical embedding, corresponding to a subdiagram of the marked Dynkin diagram of G/P. If $Z' = \mathbb{P}(W') \cap X$ is another linear section of X such that $(Z' \subset \mathbb{P}(W'))$ is projectively equivalent to $(Z \subset \mathbb{P}(W))$, then there is $g \in G$ such that $Z' = gZ$.*

Proof. When X is either a Grassmannian or a Lagrangian Grassmannian, we proved above (Case 1 and Case 2) that the varieties $\mathcal{C}_x(X)$ and $\mathcal{C}_x(Z)$ of minimal rational tangents of X and Z have the property that, for any linear section $\mathcal{C}' = \mathbb{P}^m \cap \mathcal{C}_x(X)$ of $\mathcal{C}_x(X)$ such that the inclusion $\mathcal{C}' \subset \mathbb{P}^m$ is projectively equivalent to the inclusion $\mathcal{C}_x(Z) \subset \mathbb{P}(T_xZ)$, there is $h \in \mathrm{Aut}(\mathcal{C}_x(X))$ such that $\mathcal{C}' = h\mathcal{C}_x(Z)$.

Let $Z' = \mathbb{P}(W') \cap X$ be a linear section of X such that the inclusion $Z' \subset \mathbb{P}(W')$ is projectively equivalent to the inclusion $Z \subset \mathbb{P}(W')$. Then the variety $\mathcal{C}_x(Z') \subset \mathbb{P}(T_xZ')$ of minimal rational tangents is projectively equivalent to the variety $\mathcal{C}_x(Z) \subset \mathbb{P}(T_xZ)$ of minimal rational tangents. Thus, by Theorem 1.2 in the case where X is either a Grassmannian or a Lagrangian Grassmannian, we get that Z' is the standard embedding of Z in X up to the action of G, i.e., there is $g \in G$ such that $Z' = gZ$.

Assume that X is neither a Grassmannian nor a Lagrangian Grassmannian. Then the variety $\mathcal{C}_x(X) \subset \mathbb{P}(T_xX)$ of minimal rational tangents of X is again a Hermitian symmetric space L/R in the first canonical embedding and the variety $\mathcal{C}_x(Z) \subset \mathbb{P}(T_xZ)$ of minimal rational tangents of Z at x is induced by a subdiagram of L_0/R_0. Thus, by the inductive assumption on the dimension of X, which is applied to the varieties $\mathcal{C}_x(Z)$ and $\mathcal{C}_x(X)$ of minimal rational tangents of Z and X, and by Theorem 1.2, we get that for any linear section $Z' = \mathbb{P}(W') \cap X$ of X such that the inclusion $Z' \subset \mathbb{P}(W')$ is projectively equivalent to the inclusion $Z \subset \mathbb{P}(W)$, Z' is the standard embedding of Z in X up to the action of G, i.e., there is $g \in G$ such that $Z' = gZ$. q.e.d.

Proof of Proposition 3.4 (3) (continued) It remains to prove Proposition 3.4 (3) in the case where the variety of minimal rational tangents of X is of type IV or of type V. We will give a proof in the case where it is of type IV. The proof in the case where it is of type V will be similar.

Let $\lambda : \mathbb{P}(F_1 \otimes F_2) \to \mathbb{P}(E_1 \otimes E_2)$ be an injective linear map such that $\lambda(\mathcal{B}_1 \times \mathcal{B}_2) \subset \mathcal{A}_1 \times \mathcal{A}_2$. The proof of the case II [**Mk07**], Lemma 2 works in this case, too, after replacing $\mathbb{P}(F_i)$ by its non-degenerate subvariety $\mathcal{B}_i$ for $i = 1, 2$ and noting that $\mathcal{A}_i$ intersects any line $\mathbb{P}^1$ in $\mathbb{P}(F_i)$ at least two points for $i = 1, 2$. The latter follows from the fact that $\mathcal{A}_i$ is non-degenerate in $\mathbb{P}(F_i)$ and that $\mathcal{A}_i$ is either the whole space $\mathbb{P}(F_i)$ or a non-linear subvariety. Thus, $\lambda = \eta_1 \otimes \eta_2$ for some linear maps $\eta_i : F_i \to E_i$, satisfying $\eta_i(\mathcal{A}_i) \subset \mathcal{B}_i$ for $i = 1, 2$. Applying Proposition 3.7 to each η_i, we get $\lambda = g_1 \otimes g_2$ for some $g_i \in \mathrm{Aut}(\mathcal{A}_i)$. q.e.d.

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